Theoretical Micromagnetics

Lecture Series

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Let an electric current I flow in a closed loop C

It gives rise to a magnetic moment (a vector)

$$\boldsymbol{\mu} = \frac{I}{2} \oint_C \boldsymbol{r} \times d\boldsymbol{s}.$$

If we assume one electron (charge e, mass m_e) orbiting around the loop

$$\boldsymbol{\mu} = \gamma \boldsymbol{L},$$

where

- $\gamma = g_e|e|/2m_e$ is the gyromagnetic ratio.
- $\boldsymbol{L}$ is the angular momentum of the electron.

Fixed length vector

An electron is orbiting around the atom in a fixed orbit (due to quantization) and $\boldsymbol{\mu}$ is a vector of fixed length.

A magnetic moment in an external field

Energy of $\boldsymbol{\mu}$ in an external magnetic field $\boldsymbol{B}$

$$E = -\boldsymbol{\mu} \cdot \boldsymbol{B}.$$

Since $|\boldsymbol{\mu}| = \mu$ is fixed, the only parameter is the angle ψ between $\boldsymbol{\mu}$ and $\boldsymbol{B}$,

$$E = -\mu B \cos \psi.$$

Torque

Changes of the energy due to the angle ψ generate a torque

$$\tau = -\frac{\partial E}{\partial \psi} = -\mu B\, \sin \psi,$$

in vector form,

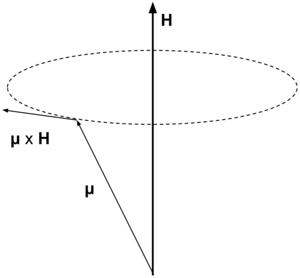
$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B}.$$

Equation of motion

For the angular momentum we have $d\mathbf{L}/dt = \boldsymbol{\tau}$

The equation of motion for the magnetic moment is

$$\frac{d\boldsymbol{\mu}}{dt} = \gamma\,\boldsymbol{\mu} \times \boldsymbol{B}$$
$$= \gamma_0 \boldsymbol{\mu} \times \boldsymbol{H}, \qquad \boldsymbol{B} = \mu_0 \boldsymbol{H}, \quad \gamma_0 = \gamma \mu_0.$$



Let the magnetic field $\boldsymbol{H} = H\hat{\boldsymbol{e}}_z = (0, 0, H)$

The equations for the components $\boldsymbol{\mu} = (\mu_x, \mu_y, \mu_z)$ are

$$\begin{cases} \dot{\mu}_x & = \gamma_0 H\,\mu_y \\ \dot{\mu}_y & = -\gamma_0 H\,\mu_x \\ \dot{\mu}_z & = 0 \end{cases} \Rightarrow \begin{cases} \dot{\mu}_x & = \omega_L\,\mu_y \\ \dot{\mu}_y & = -\omega_L\,\mu_x \\ \dot{\mu}_z & = 0 \end{cases}$$

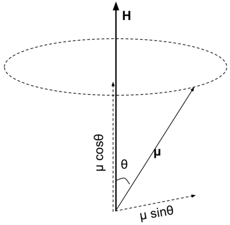
where $\omega_L = \gamma_0 H$.

Precession

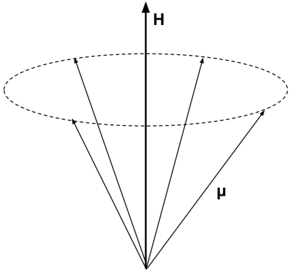
Solution of the equations of motion

$$\mu_x = \mu \sin\theta \, \cos(\omega_L t)$$
$$\mu_y = -\mu \sin\theta \, \sin(\omega_L t)$$
$$\mu_z = \mu \cos\theta \; (=\text{const.})$$

where $\mu = |\boldsymbol{\mu}|$ is constant and θ is the constant angle between $\boldsymbol{\mu}$, $\mathbf{H}$, and ω_L is called the *Larmor frequency*.



- The component of $\boldsymbol{\mu}$ parallel to $\mathbf{H}$ remains constant.
- The projection of $\boldsymbol{\mu}$ on the plane perpendicular to $\mathbf{H}$ is (μ_x, μ_y) and it rotates.
- The moment $\boldsymbol{\mu}$ performs precession around $\mathbf{H}$.



Example (A complex variable)

Recast the equation for a magnetic moment $\boldsymbol{\mu}$ in an external field $\mathbf{H}$ using a complex variable,

$$\mu_\perp = \mu_x + i\mu_y.$$

The equations of motion for μ_x, μ_y are combined in the form

$$\dot{\mu}_\perp = -i\,\omega_L\,\mu_\perp$$

with solution

$$\mu_\perp = \mu \sin\theta\, e^{-i\omega_L t}.$$

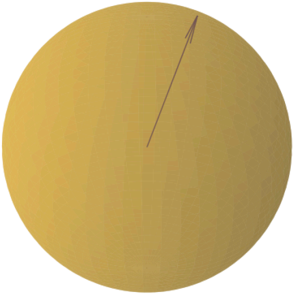
[*Exercise.* Try to carry out the above calculation.]

Exercise (Precession of a magnetic moment in an external field)

Solve numerically the equation of motion for a magnetic moment in an external field. Check whether you obtain the above analytical solution.

A field with values on the sphere, $u(s) \in \mathbb{S}^2$

Assume a field $u = u(s)$ defined in the real space, $s \in \mathbb{R}$, and taking values on the unit sphere, $u \in \mathbb{S}^2$.
Such a field is realised by a vector $\boldsymbol{u} \in \mathbb{R}^3$ with unit length $\boldsymbol{u}^2 = 1$. For any parameter s of the problem we have



$$\boldsymbol{u}^2 = 1 \Rightarrow \frac{d}{ds}(\boldsymbol{u}\cdot\boldsymbol{u}) = 0 \Rightarrow \boldsymbol{u}\cdot\frac{d\boldsymbol{u}}{ds} = 0.$$

Any derivative of $\boldsymbol{u}$ is perpendicular to $\boldsymbol{u}$ (it belongs to the tangent plane of the sphere at $\boldsymbol{u}$).

$$\frac{d\boldsymbol{u}}{ds} = \boldsymbol{u}\times\boldsymbol{f}, \quad \text{for some } \boldsymbol{f}.$$

Lecture 1b. Spins - exchange interaction

Ferromagnets

are materials that present non-zero net magnetization (at zero field). This is due to neighbouring moments that interact and tend to be aligned.
The magnetic moments are primarily due to the **spin of electrons.**

Neighbouring spins are aligned due to exchange interaction.

At the level of two individual spins $\mathbf{S}_1, \mathbf{S}_2$, the energy due to exchange interaction is modelled as

$$-J\,\mathbf{S}_1 \cdot \mathbf{S}_2, \qquad J : \text{exchange constant.}$$

For $J > 0$ and *perfectly aligned spins* the exchange energy has a minimum.

- The exchange interaction induces magnetic order.

Antiferromagnets, Weak Ferromagnets, etc

are materials that present magnetic order (at zero field).

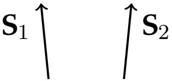
Two spins

In the energy $E = -J\mathbf{S}_1 \cdot \mathbf{S}_2$, each spin plays the role of an external field for the other one. Therefore, the equations of motion are

$$\dot{\mathbf{S}}_1 = J\,\mathbf{S}_1 \times \mathbf{S}_2, \quad \dot{\mathbf{S}}_2 = J\,\mathbf{S}_2 \times \mathbf{S}_1.$$

Note that, the change of the one spin affects the dynamics of the other one. That means that we have a system of coupled equations.

We could loosely imagine that $\boldsymbol{S}_1$ is precessing around $\boldsymbol{S}_2$, while $\boldsymbol{S}_2$ is precessing around $\boldsymbol{S}_1$.



Exercise (Dynamics of two spins)

Study the dynamics of two exchange-coupled spins $\mathbf{S}_1, \mathbf{S}_2$.

A spin chain

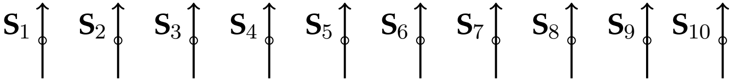
Consider a chain of N spins $\mathbf{S}_i$, $i = 1, 2, \cdots, N$.

The energy of the system is

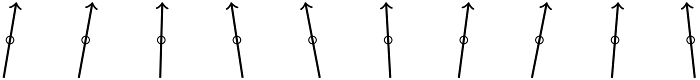
$$E = -J \sum_{i=1}^{N-1} \mathbf{S}_i \cdot \mathbf{S}_{i+1}.$$

Each spin $\mathbf{S}_k$ interacts with two neighbours at $k+1, k-1$.

Chain of aligned spins.



Chain of spins (not fully aligned).



Dynamics of a spin chain

The equation of motion for every $\mathbf{S}_k(t)$ is

$$\dot{\mathbf{S}}_k = J\,\mathbf{S}_k \times (\mathbf{S}_{k+1} + \mathbf{S}_{k-1}), \qquad k = 2, 3, \cdots, N-1.$$

or

$$\dot{\mathbf{S}}_k = \mathbf{S}_k \times \boldsymbol{f}_k, \qquad \boldsymbol{f}_k = -\frac{\partial E}{\partial \mathbf{S}_k}.$$

Exercise (Exchange-coupled spins)

Consider a spin chain and the corresponding system of equations. (a) Specify possible initial conditions and (b) solve the initial value problem numerically.

Consider the following.

- *All spins $\mathbf{S}_i$ should have the same fixed length $|\mathbf{S}_i| = s$.*
- *Try the uniform configuration $\mathbf{S}_i = \mathbf{s}$ for all i, where $\mathbf{s}$ is a constant vector (for example $\mathbf{s} = s\hat{\mathbf{e}}_3$).*
- *Try perturbations of the above uniform configuration.*

Conservation of energy. Single spin.

The energy of a spin in a magnetic field is

$$E = -g_e \mu_B \mathbf{S} \cdot \mathbf{H} \sim -\mathbf{S} \cdot \mathbf{H}.$$

The energy is conserved under $\dot{\mathbf{S}} = \mathbf{S} \times \mathbf{H}$.

We have that $\mathbf{S} = \mathbf{S}(t)$, therefore

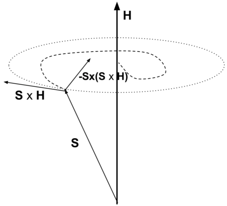
$$\frac{dE}{dt} = \frac{dE}{d\mathbf{S}} \cdot \frac{d\mathbf{S}}{dt} \sim -\mathbf{H} \cdot (\mathbf{S} \times \mathbf{H}) = 0.$$

That means, that during spin precession, the energy is conserved.

Damping effect. Single spin.

Try a friction force

$$\frac{d\mathbf{S}}{dt} = \text{friction}.$$



The term $\mathbf{S} \times (\mathbf{S} \times \mathbf{H})$ is

- Perpendicular to $\mathbf{S}$ (i.e., keeps $\mathbf{S}$ on the sphere $\mathbf{S}^2 = 1$).
- Perpendicular to $\mathbf{S} \times \mathbf{H}$ (i.e., drags $\mathbf{S}$ to $\mathbf{H}$).

Write the equation for a spin $\mathbf{S}$ under damping

$$\frac{d\mathbf{S}}{dt} = -\mathbf{S} \times (\mathbf{S} \times \mathbf{H}).$$

Note the vector identity

$$\mathbf{S} \times (\mathbf{S} \times \mathbf{H}) = (\mathbf{S} \cdot \mathbf{H})\mathbf{S} - (\mathbf{S} \cdot \mathbf{S})\mathbf{H} = (\mathbf{S} \cdot \mathbf{H})\mathbf{S} - s^2\mathbf{H}.$$

Energy loss due to damping

Spin dynamics with damping

Under both conservative and damping terms we have the equation

$$\frac{d\mathbf{S}}{dt} = \mathbf{S} \times \mathbf{H} - \alpha \mathbf{S} \times (\mathbf{S} \times \mathbf{H}).$$

where α is a damping constant.

The energy decreases under damping

$$\begin{aligned}\frac{dE}{dt} &= \frac{dE}{d\mathbf{S}} \cdot \frac{d\mathbf{S}}{dt} \sim -\mathbf{H} \cdot [(\mathbf{S} \times \mathbf{H}) - \alpha \mathbf{S} \times (\mathbf{S} \times \mathbf{H})] \\ &= \alpha \mathbf{H} \cdot [(\mathbf{S} \cdot \mathbf{H})\mathbf{S} - \mathbf{S}^2 \mathbf{H}] = (\mathbf{S} \cdot \mathbf{H})^2 - \mathbf{S}^2 \mathbf{H}^2 \leq 0.\end{aligned}$$

Question

What will happen if a spin is placed under $\mathbf{H}$ with damping when $t \to \infty$? What will the energy then be?

Dynamics of a spin chain under damping

Spin dynamics with damping

Under both conservative and damping terms we have the equation

$$\frac{d\mathbf{S}_k}{dt} = \mathbf{S}_k \times \boldsymbol{f}_k - \alpha \mathbf{S}_k \times (\mathbf{S}_k \times \boldsymbol{f}_k).$$

where α is a damping constant.

Exercise

How does the energy of a chain of spins behave under damping?

Question

What do we expect to happen with the spin configuration when $t \to \infty$?