Existence and regularity of solution for a Stochastic Cahn-Hilliard/Allen-Cahn equation with unbounded noise diffusion

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The Stochastic equation

We consider the Cahn-Hilliard/Allen-Cahn equation with multiplicative space-time noise:

$$
\begin{aligned}
&u_t = -\varrho\Delta\Big(\Delta u - f(u)\Big) + \Big(\Delta u - f(u)\Big) + \sigma(u)\dot{W} \quad \text{in} \quad \mathcal{D}\times[0,T),\\
&u(x,0) = u_0(x) \quad \text{in} \quad \mathcal{D},\\
&\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} \ = \ 0 \quad \text{on} \quad \partial\mathcal{D}\times[0,T).
\end{aligned} \tag{1}
$$

$\mathcal{D}$ is a rectangular domain in $\mathbb{R}^d$ with $d = 1, 2, 3$,

$$\varrho > 0 \quad \text{difusion constant,}$$

$$f = F', \quad F(u) = (1-u^2)^2 \quad \text{is a double equal-well potential.}$$

$\dot{W}$ is a space-time white noise in the sense of Walsh, Lecture Notes in Math. 1986.

σ is Lipschitz with sub-linear growth such that

$$|\sigma(u)| \leq C(1 + |u|^{\alpha}), \tag{2}$$

for some $\alpha \in (0, 1]$.

Main Results

We give sufficient conditions on the initial condition u_0 so that:

1. a unique local (maximal) solution exists when $d = 1, 2, 3$, for any $\alpha \in (0, 1]$,
2. when $\alpha < \frac{1}{3}$, i.e. when the supremum of α coincides with the inverse of the polynomial order of the nonlinear function f, a global solution exists with Lipschitz path-regularity for $d = 1$.

Approach motivated by the works: Cardon-Weber, Bernoulli 2001, Cardon-Weber, Millet, J. Theor. Probab.,2004, Da Prato, Debussche, Nonlin. Anal., 1996, Antonopoulou, Karali, DCDSB, 2011.

The physical model

Surface diffusion, and, adsorption/desorption micromechanisms:

$\hookrightarrow$ in surface processes,

$\hookrightarrow$ on cluster interface morphology,

see Katsoulakis, Vlachos, IMA Vol. Math. Appl. 2003.

Stochastic time-dependent Ginzburg-Landau type equations with additive Gaussian white noise source:

$\hookrightarrow$ Cahn-Hilliard (Model B),

$\hookrightarrow$ Allen-Cahn (Model A),

appear in the classical theory of phase transitions;

see the universality classification of Hohenberg and Halperin, J. Rev. Mod. Phys. 1977.

A simplified mean field model of statistical mechanics

$\hookrightarrow$ SPDE (1): Cahn-Hilliard and Allen-Cahn with noise.

1. The Cahn-Hilliard operator: mass conservative phase separation and surface diffusion.
2. The Allen-Cahn operator: adsorption and desorption and serves as a diffuse interface model for boundary coarsening.
3. Interacting particle systems are Markov processes set on a lattice corresponding to a solid surface, of Ising-type; see Giacomin, Lebowitz, Presutti, Math. Surveys Monogr. 1999.

   Assuming that the particle-particle interactions are attractive

   $\hookrightarrow$ system's Hamiltonian is nonnegative (attractive potential),

   $\hookrightarrow$ so, the diffusion constant $\varrho > 0$.

Weak formulation

For simplicity we set $\varrho = 1$ and consider $\mathcal{D}$ the unitary cube.

We say that u is a weak solution of the equation (1) if it satisfies:

$$\begin{aligned}
&\int_{\mathcal{D}} \Big(u(x,t) - u_0(x)\Big)\phi(x)\, dx = \\
&\int_0^t \int_{\mathcal{D}} \Big(- \Delta^2\phi(x)u(x,s) + \Delta\phi(x)[f(u(x,s)) + u(x,s)] \\
&\qquad\qquad - \phi(x)f(u(x,s))\Big)\, dxds \\
&+ \int_0^t \int_{\mathcal{D}} \phi(x)\sigma(u(x,s))\, W(dx,ds),
\end{aligned} \tag{3}$$

for all $\phi \in \mathcal{C}^4(\mathcal{D})$ with $\frac{\partial\phi}{\partial\nu} = \frac{\partial\Delta\phi}{\partial\nu} = 0$ on $\partial\mathcal{D}$.

Measure $W(dx, ds)$: is a space-time white noise, induced by the one-dimensional $(d+1)$-parameter Wiener process

$$W := \{W(x,t): \ t \in [0,T], \ x \in \mathcal{D}\},$$

(with d space variables and 1 time variable).
We define, $\forall \ t \geq 0$ the filtration generated by W as:

$$\mathcal{F}_t := \sigma\big(W(x,s): \ s \leq t, \ x \in \mathcal{D}\big).$$

Integral representation of solution

Using a Green's function, the solution of (3) is a mild solution:

$$\begin{aligned} u(x,t) = & \int_{\mathcal{D}} u_0(y) G(x,y,t) \, dy \\ & + \int_0^t \int_{\mathcal{D}} \big[\Delta G(x,y,t-s) - G(x,y,t-s)\big] f(u(y,s)) \, dyds \\ & + \int_0^t \int_{\mathcal{D}} G(x,y,t-s) \sigma(u(y,s)) \, W(dy,ds). \end{aligned} \tag{4}$$

The Green's function

A proper Green's function is induced by the linear part of the SPDE: i.e. by the operator

$$\mathcal{T} := -\Delta^2 + \Delta$$

on $\mathcal{D}$ with the homogeneous Neumann conditions
$\hookrightarrow$ domain $\mathcal{D}$ rectangular
$\hookrightarrow$ trigonometric $L^2(\mathcal{D})$-orthonormal basis of eigenfunctions (explicitly given),
and the associated Green's function is

$$G(x, y, t) = \sum_k e^{-(\lambda_k^2 + \lambda_k)t} \, \epsilon_k(x) \, \epsilon_k(y), \tag{5}$$

for $t > 0$, $x, y \in \mathcal{D}$.

IMPORTANT: $\mathcal{T} = -\Delta^2 + \Delta$ is uniformly strongly parabolic in the sense of Petrovskĭı.

$\hookrightarrow$ Cahn-Hilliard operator is dominant.

Hölder estimates for the Green in space-time

$$|G(x,y,t)| \le c_1\, t^{-\frac{d}{4}} \exp\Big(- c_2\, |x-y|^{\frac{4}{3}}\, t^{-\frac{1}{3}}\Big),$$
$$|\partial_x^k G(x,y,t)| \le c_1\, t^{-\frac{d+|k|}{4}} \exp\Big(- c_2\, |x-y|^{\frac{4}{3}}\, t^{-\frac{1}{3}}\Big),$$
$$|\partial_t G(x,y,t)| \le c_1\, t^{-\frac{d+4}{4}} \exp\Big(- c_2\, |x-y|^{\frac{4}{3}}\, t^{-\frac{1}{3}}\Big),$$

and

$$\int_0^t \int_{\mathcal{D}} |G(x,z,t-r) - G(y,z,t-r)|^2\, dzdr \le C|x-y|^{\gamma},$$
$$\int_0^s \int_{\mathcal{D}} |G(x,z,t-r) - G(x,z,s-r)|^2\, dzdr \le C|t-s|^{\gamma'},$$
$$\int_s^t \int_{\mathcal{D}} |G(x,z,t-r)|^2\, dzdr \le C|t-s|^{\gamma'}.$$

The cut-off SPDE

In order to prove the existence of the solution u to (4) we construct a cut-off SPDE:
Let $\chi_n \in C^1(\mathbb{R}, \mathbb{R}^+)$ be a cut-off function satisfying

$$|\chi_n| \leq 1, \quad |\chi_n'| \leq 2 \ \forall n > 0,$$

and

$$\chi_n(x) = \begin{cases} 1 & \text{if} \quad |x| \leq n, \\ 0 & \text{if} \quad |x| \geq n+1. \end{cases}$$

For fixed $n > 0$, $x \in \mathcal{D}$, $t \in [0, T]$ and $q \in [3, +\infty)$, we consider the following cut-off SPDE:

$$\begin{aligned} u_n(x,t) = &\int_{\mathcal{D}} u_0(y) G(x,y,t)\, dy \\ &+ \int_0^t \int_{\mathcal{D}} \left[\Delta G(x,y,t-s) - G(x,y,t-s)\right] \\ &\qquad \chi_n(\|u_n(\cdot,s)\|_q)\, f(u_n(y,s))\, dyds \\ &+ \int_0^t \int_{\mathcal{D}} G(x,y,t-s)\, \chi_n(\|u_n(\cdot,s)\|_q)\, \sigma(u_n(y,s))\, W(dy,ds). \end{aligned}$$

Theorem

Let σ be globally Lipschitz and satisfy the assumption (2) with $\alpha \in (0,1)$, and $u_0 \in L^q(\mathcal{D})$. Then, under certain assumptions on q, d, α, and β, the cut-off SPDE admits a unique solution u_n, in every time interval $[0, T]$, such that $u_n \in \mathcal{H}_T$, where

$$\mathcal{H}_T := \Big\{ u(\cdot, t) \in L^q(\mathcal{D}) \text{ for } t \in [0, T] : \\ u \text{ is } (\mathcal{F}_t)\text{-adapted and } \|u\|_{\mathcal{H}_T} < \infty \Big\},$$

for

$$\|u\|_{\mathcal{H}_T} := \sup_{t\in[0,T]} \Big(E[\|u(\cdot, t)\|_q^\beta] \Big)^{\frac{1}{\beta}}.$$

Main steps of proof:

Here, we split the solution in 3 parts with $\mathcal{L}_n(u_n)(x, t)$ being the noise term, as

$$u_n(x, t) = \int_{\mathcal{D}} u_0(y) G(x, y, t)\, dy + \mathcal{M}_n(u_n)(x, t) + \mathcal{L}_n(u_n)(x, t).$$

Using the Green's estimates, and the Lipschitz property of the diffusion σ, we prove that:

1. For fixed $n \geq 1$, $\exists T_0(n)$ sufficiently small and independent of u_0: for $T \leq T_0(n)$

$$\mathcal{M}_n + \mathcal{L}_n, \quad \text{is a contraction mapping from } \mathcal{H}_T \text{ into } \mathcal{H}_T.$$

   Thus for $T \leq T_0(n)$, the map $\mathcal{M}_n + \mathcal{L}_n$ has a unique fixed point in $\left\{u \in \mathcal{H}_T : \; u(\cdot, 0) = u_0\right\}$

   $\hookrightarrow$ in $[0, T]$, for $T \leq T_0(n)$, $\exists!$ solution u_n.

2. If $T > T_0(n)$, set

$$\bar{u}_0(x) = u_n(x, T_0(n))$$

   as new initial condition and $\bar{W}(t, x) = W(T_0(n) + t, x)$ related to the filtration $(\mathcal{F}_{T_0(n)+t}, t \geq 0)$ independent of $\mathcal{F}_{T_0(n)}$.

   $\hookrightarrow$ in the interval $[0, 2T_0(n)]$ take $u_n(x, t) := \bar{u}_n(x, t - T_0(n))$ etc. by induction up to some time $NT_0(n) \geq T$.

Comments

1. We define the stopping time

$$T_n = \min\{\inf\left\{t \geq 0 \ : \ \|u_n(.,t)\|_q \geq n\right\}, n\},$$

$t \hookleftarrow t_{\mathrm{rand}} := \min\{t, T_n\}$,

2. then use that for any $s \leq t_{\mathrm{rand}}$

$$\chi_n(\|u_n(.,s)\|_q) = 1,$$

so, the process $(u_n(.,t), t < T_n)$ is a solution to (4) (mild solution of the initial SPDE).

3. Further, we define $T^* := \limsup_n T_n$;
then uniqueness of the cut-off solution
$\hookrightarrow$ existence of a solution $u(.,t)$ of (4) in $[0, T^*)$
(u defined by the limit value of the truncated processes),

4. and get that u is a local maximal solution to (4) in $[0, T^*)$, i.e.

$$\sup\{\|u(.,t)\|_q : t < T^*\} = \infty \ \text{ a.s.}$$

Global existence and uniqueness of solution

1. Dimensions restrict the result to $d = 1$, due to the Gagliardo-Nirenberg inequality for the nonlinearity,
2. and the nonlinearity (polynomial of order 3) results to a restriction for the noise diffusion growth:

$$\alpha < \frac{1}{3}.$$

The method

1. We write $v_n := u_n - \mathcal{L}_n(u_n)$ as an element of $L^2(\mathcal{D})$ using the othonormal eigenfunction basis of the negative Neumann Laplacian on the cube with spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots ,$$

and then cut-off the Fourier series at the first m modes $\hookrightarrow v_n^m$, for which we prove an L^2 estimate on the limit $m \to \infty$

$$\|v_n(\cdot,t)\|_2^2+\int_0^t \|\Delta v_n(\cdot,s)\|_2^2 ds \leq C(T)\Big[1+\|u_0\|_2^4+\|\mathcal{L}_n(u_n)\|_{L^\infty}^6\Big](*),$$

for

$$\|\mathcal{L}_n(u_n)\|_{L^\infty} := \sup_{t\in[0,T]}\sup_{x\in\mathcal{D}} |\mathcal{L}_n(u_n)(x,t)|,$$

where remind

$$\mathcal{L}_n(u_n) := \int_0^t \int_{\mathcal{D}} G(x,y,t-s)\,\chi_n(\|u_n(\cdot,s)\|_q)\,\sigma(u_n(y,s))\; W(dy,ds)$$

2. We use the Hölder estimates of Green's in space and time together with Burkholder-Davies-Gundy inequality for the space-time noise integral, i.e.

$$E(|\mathcal{L}_n(u_n)|^{2p}) := E(|\int_0^t \int_{\mathcal{D}} G(x,y,t-s)\,\chi_n(\|u_n(\cdot,s)\|_q)\,\sigma(u_n(y,s))\; W(dy,ds)|^{2p})$$
$$\leq C_p E(|\int_0^t \int_{\mathcal{D}} G(x,y,t-s)\,\chi_n(\|u_n(\cdot,s)\|_q)\,\sigma(u_n(y,s))\; dy,ds|^{p}),$$

to prove
(a) in expectation a p-moment space-time Hölder estimate for $\mathcal{L}_n(u_n)$ (very technical),
(b) and to derive finally for $\tilde{q}$: $q \geq \tilde{q} > \frac{2\alpha d}{4-d}$

$$E\left(\|\mathcal{L}_n(u_n)\|_{L^\infty}^{2p}\right) \leq C_p(T) \min\left\{n^{2\alpha p}, \sup_{t\in[0,T]} E\left(\|u_n(.,t)\|_{\tilde{q}}^{2\alpha p}\right)\right\}.$$

We then use the above in (*) by replacing $u_n = v_n + \mathcal{L}_n(u_n)$ and derive:
if $u_0 \in L^q(\mathcal{D})$ and $p \in [2,\infty)$

$$E\left(\sup_{t\in[0,T]} \|u_n(\cdot,t)\|_2^p\right) \leq C_p(T)\Big[1 + \|u_0\|_2^p + (1 + \|u_0\|_2^p) \\ E\left(\sup_{t\in[0,T]} \|u_n(.,t)\|_2^{3\alpha p}\right)\Big],$$

which by a boot-strap argument gives the restriction $\alpha < \frac{1}{3}$ and the moment estimates in L^2

$$E\left(\sup_{t\in[0,T]} \|u_n(\cdot,t)\|_2^p\right) \leq C_p(T)\left[1 + \|u_0\|_2^{\frac{p}{1-3\alpha}}\right].$$

3. Our aim for global existence, is to prove moment estimates in L^q, this restricts dimensions in $d = 1$, when estimating a deterministic part of u_n, i.e.

$$\mathcal{M}_n(u_n) := \int_0^t \int_{\mathcal{D}} \big[\Delta G(x,y,t-s) - G(x,y,t-s)\big] \times \chi_n(\|u_n(\cdot,s)\|_q)\, f(u_n(y,s))\, dyds$$

for which we prove for $q \geq 3$ and $\beta \geq 2$, by using the L^2 moments

$$E\Big(\sup_{0\leq t\leq T} \|\mathcal{M}_n(u_n)(\cdot,t)\|_q^{\beta}\Big) \leq C,$$

$\hookrightarrow$ moments in L^q for u_n.

4. Thus, defining the stopping time

$$T_n := \inf\Big\{t \geq 0 : \ \|u_n(\cdot,t)\|_q \geq n\Big\},$$

Chebyshev inequality gives

$$P(T_n < T) \leq n^{-\beta} E\Big(\sup_{t\in[0,T]} \|u_n(\cdot,t)\|_q^{\beta}\Big) \leq Cn^{-\beta},$$

for $\beta \geq 2$, so, by the Borel-Cantelli Lemma

$$P(\limsup_{n\to\infty}\{T_n < T\}) = 0$$

and thus,

$$T_n \to \infty \ \text{ a.s. } \ \text{as} \ \ n \to \infty.$$

The uniqueness follows from the uniqueness of u_n, since

$$u(\cdot,t) := u_n(\cdot,t) \ \text{ on } \ [0, T_n],$$

and since $T_n \to \infty$ a.s.

Extension of results for:

$$u_t = -\varrho\Delta\Big(\Delta u - f(u)\Big) + \hat{q}\Big(\Delta u - f(u)\Big) + \sigma(u)\dot{W},$$

when

$$\varrho > 0, \quad \hat{q} \geq 0,$$

and for more general domains with a smooth in space, space-time noise

$$dxW(dt),$$

by the Antonopoulou, Karali, DCDSB, 2011, approach.

NOTE: the non-smooth in space noise in the definition of Walsh needs the domain

$$\mathcal{D} \quad \text{a cartesian product in } \mathbb{R}^d \quad \hookrightarrow \quad \text{rectangle.}$$

While by the Cardon-Weber, Bernoulli, 2001 method, we derive path regularity as follows:

Theorem

For $d = 1$, and $\alpha \in (0, \frac{1}{3})$, if $u_0 \in L^\infty(\mathcal{D})$:

(i) If u_0 is continuous, then the global solution of (4) has a.s. continuous trajectories.

(ii) If u_0 is β-Hölder continuous for $0 < \beta < 1$, then the trajectories of the global solution to (4) are a.s. $\min\{\beta, \left(2 - \frac{d}{2}\right)\}$-continuous in space and $\min\{\frac{\beta}{4}, \left(\frac{1}{2} - \frac{d}{8}\right)\}$-continuous in time.

And for $d = 2, 3$, the same result for the maximal solutions in $\mathcal{D} \times [0, T^*)$.

Comments

1. The integral form of the solution u given by (4) is split as follows:

$$u(t,x) = G_t u_0(x) + \mathcal{I}(x,t) + \mathcal{J}(x,t),$$

for

$$\mathcal{I}(x,t) = \int_0^t \int_{\mathcal{D}} [\Delta G(x,y,t-s) - G(x,y,t-s)] f(u(y,s))\, dyds,$$

and

$$\mathcal{J}(x,t) = \int_0^t \int_{\mathcal{D}} G(x,y,t-s)\sigma(u(y,s))\, W(dy,ds).$$

2. Considering the initial condition involving term

   2.1 If u_0 is continuous, then the function $G_t u_0$ is continuous.

   2.2 If u_0 belongs to $C^\delta(\mathcal{D})$ for $0 < \delta < 1$, then $(x,t) \to G_t u_0(x)$ is δ-Hölder continuous in x and $\frac{\delta}{4}$-Hölder continuous in t.

   2.3 If u_0 is bounded, then u belongs a.s. to $L^\infty(0,T;L^q(\mathcal{D}))$ for any $q < \infty$ large enough.

3. For some $a \in (0,1)$ define the operators $\mathcal{F}$ and $\mathcal{H}$ on $L^\infty(0,T;L^q(\mathcal{D}))$ as follows:

$$\mathcal{F}(v)(t,x) := \int_0^t \int_{\mathcal{D}} G(x,z,t-s)(t-s)^{-a} v(z,s)\, dzds,$$

$$\mathcal{H}(v)(z,s) := \int_0^s \int_{\mathcal{D}} \left[\Delta G(z,y,s-s') - G(z,y,s-s')\right] (s-s')^{a-1} f(v(y,s'))\, dyds'.$$

4. So, first

$$\mathcal{I}(x,t) = c_a \mathcal{F}(\mathcal{H}(u))(x,t),$$

where $c_a := \pi^{-1}\sin(\pi a)$.

Using the estimates of the Green's function, we prove that

$$\mathcal{H} \text{ maps } L^\infty(0,T;L^q(\mathcal{D})) \text{ into itself.}$$

5. Moreover,

$$\mathcal{J}(x,t) = c_a \mathcal{F}(\mathcal{K}(u_n))(x,t) \text{ on the set } \{T \leq T_n^*\},$$

for

$$\mathcal{K}(u_n)(x,t) = \int_0^t \int_{\mathcal{D}} G(x,y,t-s)(t-s)^{a-1} 1_{\{s \leq T_n^*\}} \sigma(u_n(y,s)) W(dy,ds),$$

$$\text{for } T_n^* := \min\{\inf\{t \geq 0 : \|u_n\|_q \geq n\}, \quad T^*\}.$$

Using the Hölder estimates of Green's function, we prove

$$E\left(\|\mathcal{K}(u_n)\|^{2p}_{L^\infty(\mathcal{D}\times[0,T])}\right) < \infty, \quad \forall p \geq 1,$$

and

$$E\left(\|\mathcal{K}(u_n)\|^{2p}_{q}\right) \leq E\left(\|\mathcal{K}(u_n)\|^{2p}_{L^\infty(\mathcal{D}\times[0,T])}\right) < \infty, \quad \forall p \geq 1.$$

So, we deduce that

$$\mathcal{J}(u) \in \mathcal{C}^{\lambda,\mu}(\mathcal{D} \times [0, T]) \text{ a.s. on the set } \{T \leq T_n^*\},$$

and as $n \to \infty$

$$a.s. \;\; \mathcal{J}(u) \in \mathcal{C}^{\lambda,\mu}([0, T] \times \mathcal{D}) \text{ for } \lambda < \frac{1}{2} - \frac{d}{8} \text{ and } \mu < 2 - \frac{d}{2}.$$