Singularity formation in black hole interiors

Grigorios Fournodavlos

DPMMS, University of Cambridge

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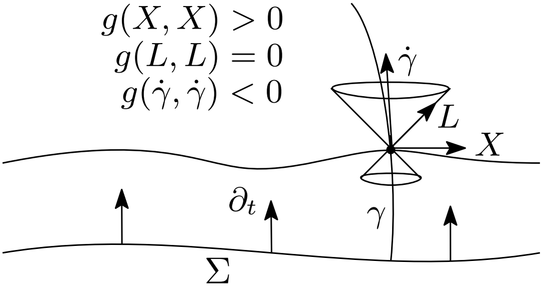
Outline

1. *The Einstein equations*

The Einstein equations

Geometric background: A Lorentzian manifold $(\mathcal{M}^{1+3}, g)$ of signature $(-,+,+,+)$, endowed with the topology $\mathcal{M}^{1+3} \cong \mathbb{R} \times \Sigma^3$.



The Einstein equations

The Einstein equations (EE) stipulate that g satisfies:

$$\mathrm{R}_{ab}(g) - \frac{1}{2} g_{ab} \mathrm{R}(g) = 8\pi T_{ab}, \qquad a, b = 0, 1, 2, 3,$$

where $\mathrm{R}_{ab}(g)$ is the Ricci curvature of g, $\mathrm{R}(g)$ its scalar curvature and T_{ab} the energy-momentum tensor of a matter field (electromagnetic, fluid etc.), satisfying the conservation laws:[1]

$$\nabla^a T_{ab} = 0, \qquad b = 0, 1, 2, 3.$$

In vacuum, $T_{ab} = 0$, the EE reduce to

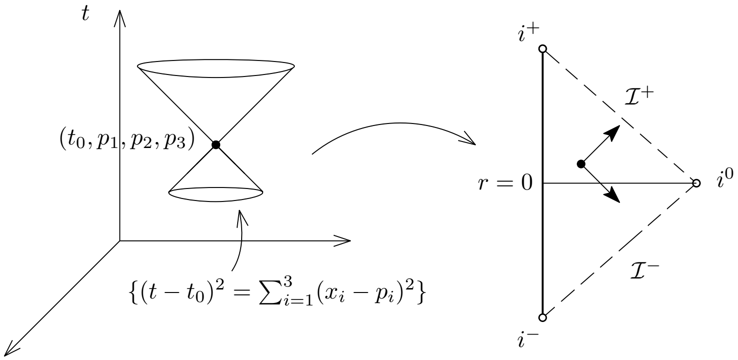
$$\mathrm{R}_{ab}(g) = 0, \qquad a, b = 0, 1, 2, 3. \qquad \text{(EVE)}$$

[1] By virtue of the second Bianchi identity.

The Einstein equations

Examples

- The flat solution: Minkowski $(\mathbb{R}^{1+3}, m)$,
  $m = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$

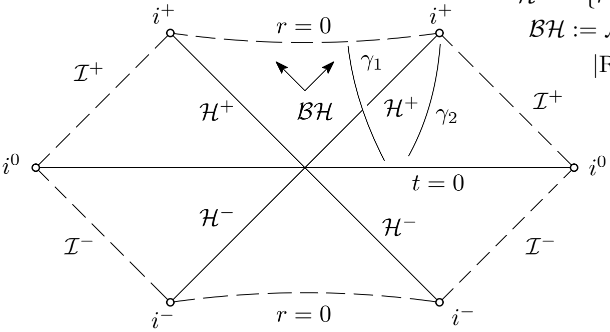


The Einstein equations

Examples

- The Schwarzschild solution $(\mathcal{M}^{1+3}, g)$, $\mathcal{M}^{1+3} \cong \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}^2$: (vacuum & spherically symmetric)[2]

$$g = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\mathbb{S}^2, \quad M > 0,$$



$\mathcal{H}^\pm = \{r = 2M\}$

$\mathcal{BH} := \mathcal{M}^{1+3} \setminus J^-(\mathcal{I}^+)$

$|\mathrm{Riem}| \sim r^{-3}$

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[2] Rigidity, Birkhoff's theorem.

The Einstein equations

Initial value problem

- In wave coordinates (x_0, x_1, x_2, x_3):

$$\Box_g x_i = 0, \qquad \Box_g = (g^{-1})^{ab}(\partial_{ab} - \Gamma^k_{ab}\partial_k)$$
$$\Gamma^i := (g^{-1})^{ab}\Gamma^i_{ab} = 0$$

the EVE take the hyperbolic form

$$\Box_g g_{ab} = Q(g^{-1}, \partial g)$$

Theorem (Choquet-Bruhat & Geroch '69)

Any asymptotically fact initial data set $(\Sigma, \overline{g}, K)$ for the EVE, gives rise to a unique maximal development.

The Einstein equations

Initial data sets

- Minkowski: $\Sigma = \mathbb{R}^3$, $\overline{g} = dx_1^2 + dx_2^2 + dx_3^2$, $K = 0$.
- Schwarzschild: $\Sigma = \mathbb{R} \times \mathbb{S}^2$, $\overline{g} = (1 - \frac{2M}{r})^{-1} dr^2 + r^2 d\mathbb{S}^2$, $K = 0$.
- Constraint equations (in vacuum):

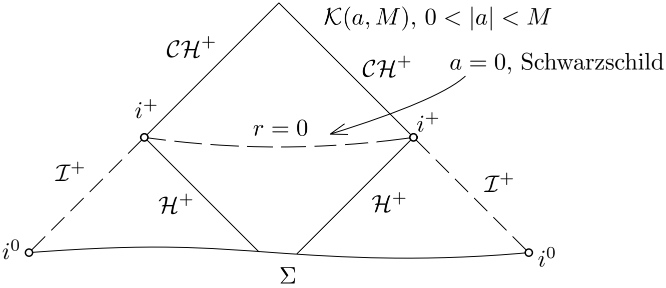
$$\begin{cases} \mathrm{R}(\overline{g}) - |K|^2 + (\mathrm{tr}_{\overline{g}} K)^2 = 0 \\ \overline{\nabla}^a K_{ab} - \overline{\nabla}_b \mathrm{tr}_{\overline{g}} K = 0, \qquad b = 1, 2, 3 \end{cases}$$

2. *Large time behaviour*

Large time behaviour

Global existence vs Blow up

- The flat solution is stable: Christodoulou-Klainerman '93, Lindblad-Rodnianski '10,...
- The Schwarzschild exterior is linearly stable Dafermos-Holzegel-Rodnianski '16, within the Kerr family of solutions, $\mathcal{K}(a, M)$, $|a| < M$.



Large time behaviour

Global existence vs blow up

- No general blow up mechanism.
- Inside a black hole region, the space-time "breaks down" (Penrose '65). However, the nature of the breakdown is not given (geodesically incomplete). The presence of such a region is a stable phenomenon.

Large time behaviour

Cosmic censorship hypothesis

Theoretical tests of GR proposed by Penrose '69:

- Strong cosmic censorship hypothesis: Einsteinian maximal developments, for reasonable matter models, arising from *generic*, asymptotically flat initial data are inextendible as suitably regular Lorentzian manifolds.
- Weak cosmic censorship hypothesis: All future singularities should be contained in black holes (no naked singularities). In other words, future null infinity $\mathcal{I}^+$ should be complete.
- Examples: The Minkowski and Schwarzschild solutions satisfy both weak and strong cosmic censorship, while Kerr satisfies weak cosmic censorship, but violates strong cosmic censorship.

3. *Spherical symmetry*

Consider the energy-momentum tensor of a massless-scalar field:

$$T_{ab} = \partial_a \varphi \partial_b \varphi - \frac{1}{2} g_{ab} \partial^k \varphi \partial_k \varphi$$

The Einstein equations are complemented with the equations of motion $\nabla^a T_{ab} = 0$:

$$\mathrm{R}_{ab}(g) - \frac{1}{2} g_{ab} \mathrm{R}(g) = 8\pi (\partial_a \varphi \partial_b \varphi - \frac{1}{2} g_{ab} \partial^k \varphi \partial_k \varphi)$$

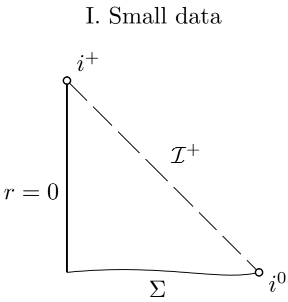
$$\Box_g \varphi = 0$$

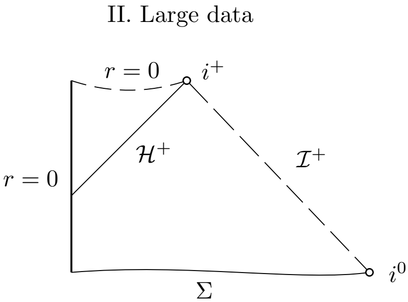
Spherical symmetry

Resolution of the scalar field model

Theorem (Christodoulou 90's)

Spherically symmetric solutions to the Einstein-scalar field equations, arising from asymptotically flat, 1-ended initial data fall in either of the three cases:

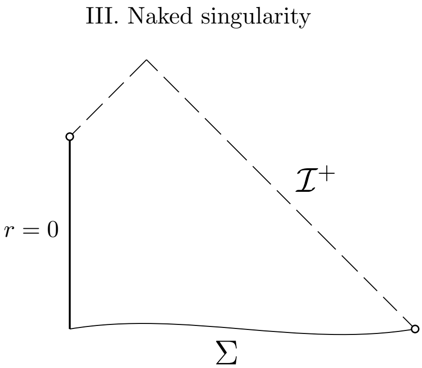




Spherical symmetry

Resolution of the scalar field model

and



However, the third case is unstable (non-generic).

4. *Beyond spherical symmetry 'near'-Schwarzschild*

Beyond spherical symmetry ‘near’ Schwarzschild

- Exterior region: Conjecturally stable, within the Kerr family of solutions. Only linearised stability has been proven.

- Interior region: Conjectural picture is unknown. Singularity formation or violation of strong cosmic censorship (as in Kerr)? Spacelike or null inner boundary? Monotonic blow up or chaotic? (BKL)

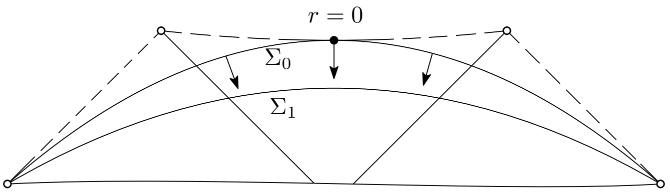
Beyond spherical symmetry 'near' Schwarzschild

Construction of singular solutions

Idea: Prescribe initial data for the EVE at Σ_0, such that

$$\bar{g} = \bar{g}_S + r^\alpha \bar{h}, \qquad K = K_S + r^\alpha u, \qquad \bar{h}, u \text{ smooth}, \ \alpha > 0,$$

and solve backwards-in-time, without any symmetry assumptions.



Theorem (F. '16)

For $\alpha \gg 1$, such a construction is possible, generating solutions to the EVE which exhibit a Schwarzschild type singularity at a collapsed sphere.

Beyond spherical symmetry ‘near’ Schwarzschild

Non-linear dynamics in polarized axisymmetry

Assume g admits a spacelike Killing vector field ∂_φ with $\mathbb{S}^1$ orbits, which is also hypersurface orthogonal:

$$g = \sum_{a,b=0,1,2} h_{ab} dx_a dx_b + e^{2\gamma} d\varphi^2$$

In this symmetry class, the EVE reduce to the system (Weinstein '90):

$$\begin{aligned} \Box_g \gamma &= 0 \\ \mathrm{R}_{ab}(h) &= \nabla_{ab}\gamma + \nabla_a \gamma \nabla_b \gamma \end{aligned}$$

Note: Under the additional conformal change $\tilde{h} := e^{2\gamma} h$, the above equations transform into the Einstein-scalar field model.

Beyond spherical symmetry 'near' Schwarzschild

Non-linear dynamics in polarized axisymmetry

The Schwarzschild solution:

$$h_S = -(\frac{2M}{r} - 1)^{-1}dr^2 + (\frac{2M}{r} - 1)dt^2 + r^2 d\theta^2,$$
$$\gamma_S = \log r + \log \sin\theta$$

On the other hand, the Kerr metric is axisymmetric, but not polarized. Restricting to polarized axisymmetric perturbations of Schwarzschild:

- Exterior region: Conjectural asymptotic convergence to Schwarzschild (possibly of different M), see Klainerman-Szeftel '17.
- Interior region: (work on progress w/ Alexakis) Strong evidence for stable spacelike singularity formation, where $|\mathrm{Riem}| \sim r^{-3}$.