Disorder and topology. The cases of Floquet and of chiral systems

Gian Michele Graf
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Partial Differential Equations in Physics and Materials Science
Heraklion
May 10-16, 2018

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based on joint works with A. Elgart, J. Schenker; J. Shapiro; C. Tauber

Outline

Some physics background first

- How it all began: Quantum Hall systems
- Topological insulators
- Bulk-edge correspondence
- The periodic table of topological matter

The case of the Quantum Hall Effect

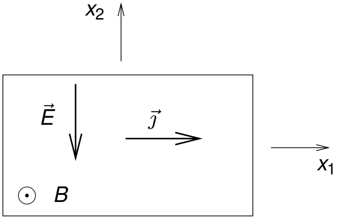
Chiral systems

- An experiment
- A chiral Hamiltonian and its indices

Time periodic systems

- Definitions and results
- Some numerics

The experiment (von Klitzing, 1980)



B: magnetic field
$\vec{E}$: electric field
$\vec{\jmath}$: current density

Electron gas of
- density n
- Fermi energy μ

Hall-Ohm law

$$\vec{\jmath} = \underline{\sigma}\vec{E}\ , \qquad \underline{\sigma} = \begin{pmatrix} \sigma_{\mathrm{D}} & \sigma_{\mathrm{H}} \\ -\sigma_{\mathrm{H}} & \sigma_{\mathrm{D}} \end{pmatrix}$$

σ_{H}: Hall conductance
σ_{D}: dissipative conductance, ideally $= 0$

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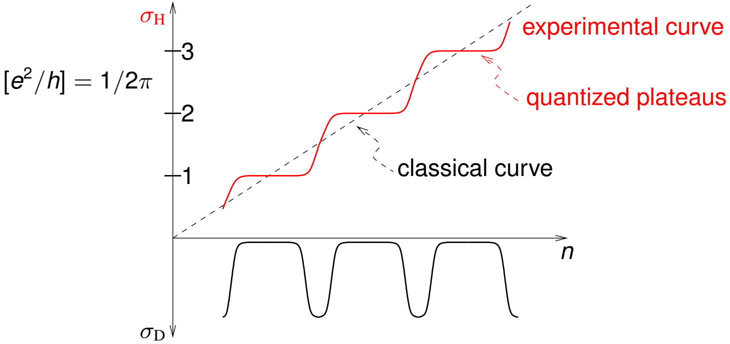

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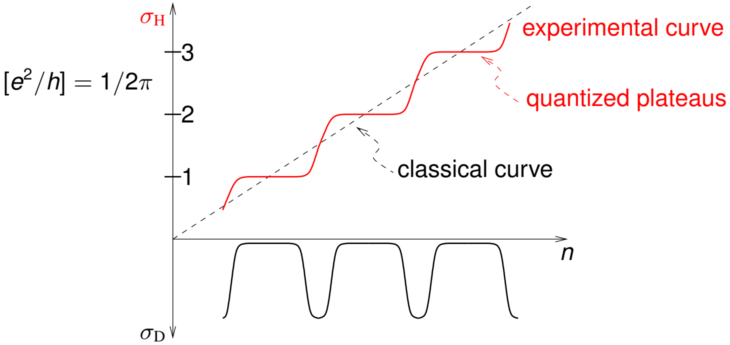
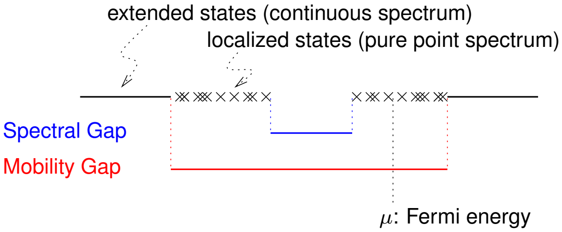


Width of plateaus increases with disorder

Spectral vs. Mobility Gap

The spectrum of a single-particle Hamiltonian



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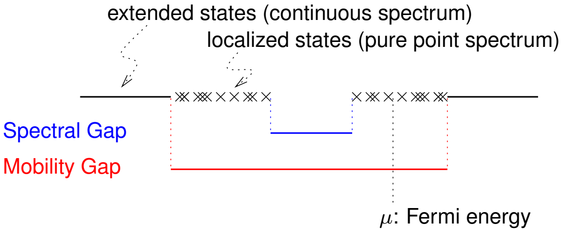

- (integrated) density of states $n(\mu)$ is constant for μ in a Spectral Gap, and strictly increasing otherwise

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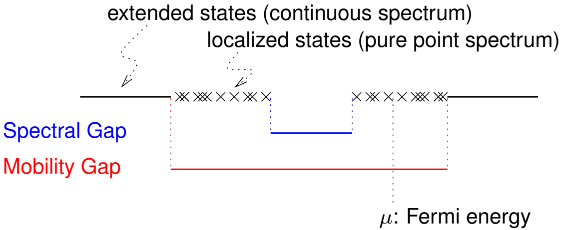

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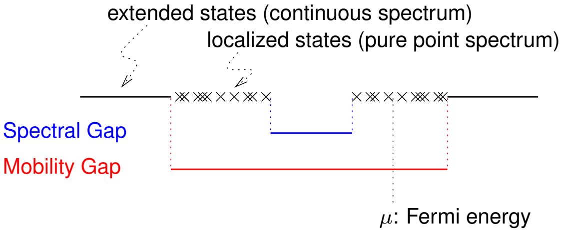

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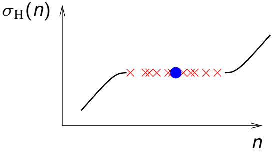
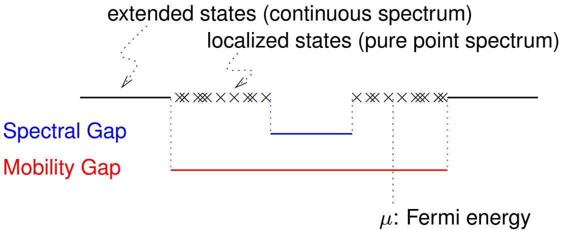


Plateaus arise because of a Mobility Gap only!

The role of disorder

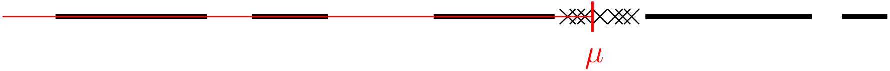
The spectrum of a single-particle Hamiltonian



- For a periodic (crystalline) medium:
  - Method of choice: Bloch theory and vector bundles (Thouless et al.)
  - Gap is spectral
- For a disordered medium:
  - Method of choice: Non-commutative geometry (Bellissard; Avron et al.)
  - Fermi energy may lie in a mobility gap (better) or just in a spectral gap

Mobility gap, technically speaking

Hamiltonian H on $\ell^2(\mathbb{Z}^d)$
$P_\mu = E_{(-\infty,\mu)}(H)$: Fermi projection



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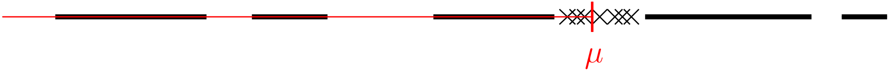

Assumption. Fermi projection has strong off-diagonal decay:

$$\sup_{x'} \mathrm{e}^{-\varepsilon|x'|} \sum_x \mathrm{e}^{\nu|x-x'|} |P_\mu(x,x')| < \infty$$

(some $\nu > 0$, all $\varepsilon > 0$)

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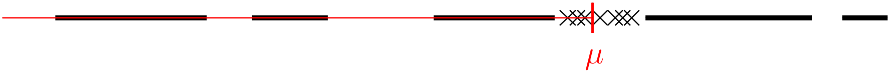

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- Proven in (virtually) all cases where localization is known.
- Trivially false for extended states at $E = \mu$.

Topological insulators: Definition stated

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  For independent electrons: Spectral gap at Fermi energy μ

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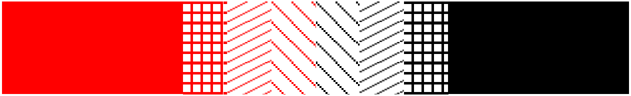
- Refinement: The Hamiltonians enjoy a symmetry which is preserved under deformations. (Classification trivially more restrictive, yet potentially richer: Hamiltonians along deformation may not enjoy symmetry even if endpoints do. Thus finer classes.)

Bulk-edge correspondence

Recall: In the space of Hamiltonians, a topological insulator can not be deformed in an ordinary one, while keeping the gap open and respecting symmetries

Bulk-edge correspondence

Deformation as interpolation in physical space:



topological insulator interpolating material ordinary insulator

- Gap must close somewhere in between. Hence: Interface states at Fermi energy.

Bulk-edge correspondence

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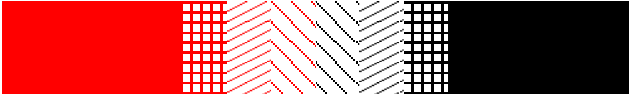

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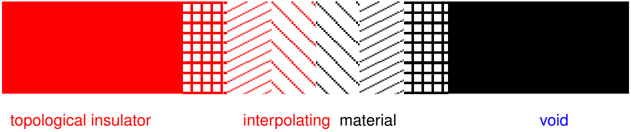

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- Bulk-edge correspondence: Termination of bulk of a topological insulator implies edge states.

Bulk-edge correspondence

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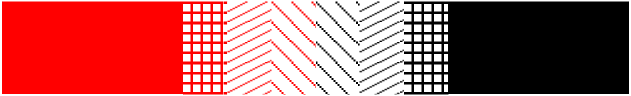

- Gap must close somewhere in between. Hence: Interface states at Fermi energy.
- Ordinary insulator $\rightsquigarrow$ void: Edge states
- Bulk-edge correspondence: Termination of bulk of a topological insulator implies edge states. (But not conversely!)

Bulk-edge correspondence

In a nutshell: Termination of bulk of a topological insulator implies edge states

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- Goal: State the (intrinsic) topological property distinguishing different classes of insulators.

More precisely:

Bulk-edge correspondence

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More precisely:

- Express that property as an Index relating to the Bulk, resp. to the Edge.

Bulk-edge correspondence

In a nutshell: Termination of bulk of a topological insulator implies edge states

- Goal: State the (intrinsic) topological property distinguishing different classes of insulators.

More precisely:

- Express that property as an Index relating to the Bulk, resp. to the Edge.
- Bulk-edge duality: Can it be shown that the two indices agree? Can it be shown even in presence of just a mobility gap?

Some physics background first

- How it all began: Quantum Hall systems
- Topological insulators
- Bulk-edge correspondence
- The periodic table of topological matter

The case of the Quantum Hall Effect

Chiral systems

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The periodic table of topological matter

Symmetry Class	Θ	Σ	Π	d = 1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Notation:

Θ time-reversal Σ charge conjugation Π combined

The periodic table of topological matter

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CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

First version: Schnyder et al.; then Kitaev based on Altland-Zirnbauer; based on Bloch theory

The periodic table of topological matter

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By now: Non-commutative (bulk) index formulae have been found in many cases (Prodan, Schulz-Baldes)

Special cases to be considered

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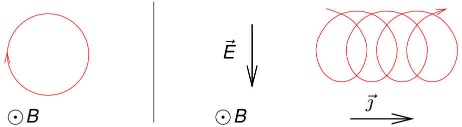
. . . and one more

IQHE as a Bulk effect

Paradigm: Cyclotron orbit drifting under a electric field $\vec{E}$

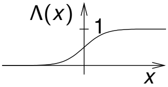


Hamiltonian H_B in the plane. Kubo formula (linear response to $\vec{E}$)

$$\sigma_{\mathrm{B}} = \mathrm{i}\,\mathrm{tr}\, P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]\big]$$

where

P_μ: Fermi projection
$\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$ switches



IQHE as a Bulk effect (remarks)

Kubo formula (Bellissard et al., Avron et al.)

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extends the formula for the periodic case (Thouless et al., Avron)

$$\sigma_{\rm B} = -\frac{\mathrm{i}}{(2\pi)^2} \int_{\mathbb{T}} d^2k\, \mathrm{tr}(P(k)[\partial_1 P(k), \partial_2 P(k)])$$

where $\mathbb{T}$: Brillouin zone (torus); $P(k)$ Fermi projection on the space of states of quasi-momentum $k = (k_1, k_2)$; $\partial_i = \partial/\partial k_i$

Remarks.

$$2\pi\sigma_{\rm B} = \mathrm{ch}(P)$$

the Chern number of the vector bundle over $\mathbb{T}$ and fiber range $P(k)$

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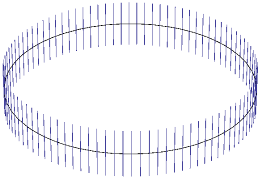
the Chern number of the vector bundle over $\mathbb{T}$ and fiber range $P(k)$

Alternative treatment of disorder (Thouless): Large, but finite system (square); $(k_1, k_2) \rightsquigarrow (\varphi_1, \varphi_2)$ phase slips in boundary conditions

Aside: What is the Chern number?

A (real) vector bundle over the circle (actually, a line bundle)



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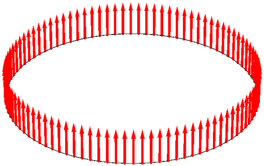
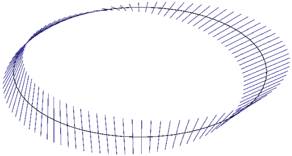


The line bundle is trivial, because it allows for a nowhere vanishing global section.

What is the Chern number?

Another vector bundle over the circle



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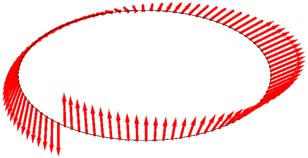
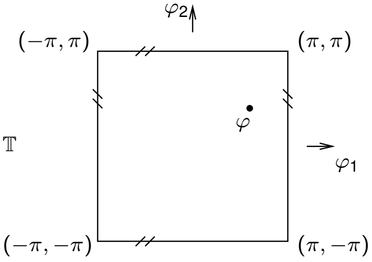


The line bundle is not trivial: No nowhere vanishing global section.

Complex bundles $(E, \mathbb{T})$ on the 2-torus



- $\mathbb{T} \ni \varphi = (\varphi_1, \varphi_2)$

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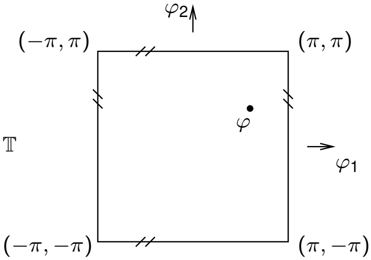
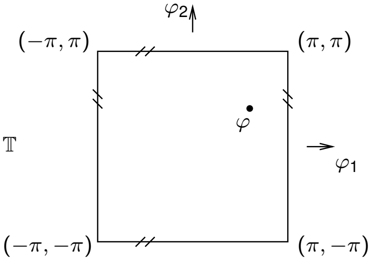


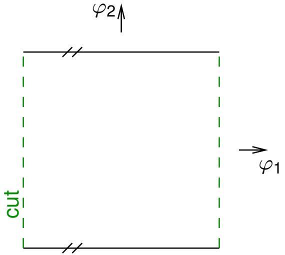
- $\mathbb{T} \ni \varphi = (\varphi_1, \varphi_2)$
- Fibers E_φ

Complex bundles $(E, \mathbb{T})$ on the 2-torus



- $\mathbb{T} \ni \varphi = (\varphi_1, \varphi_2)$
- Fibers E_φ
- Frame bundle $F(E)$ has fibers $F(E)_\varphi \ni v = (v_1, \dots v_N)$ consisting of bases v of E_φ.
- Does $F(E)$ admit a global section?

Classification by a Chern number



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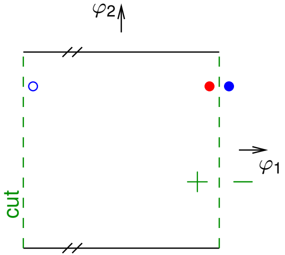

Lemma. On the cut torus the frame bundle admits a section
$\varphi \mapsto v(\varphi) \in F(E)_\varphi$

- Boundary values $v_+(\varphi_2)$ and $v_-(\varphi_2)$ at the point $(\pi, \varphi_2) \equiv (-\pi, \varphi_2)$ of the cut

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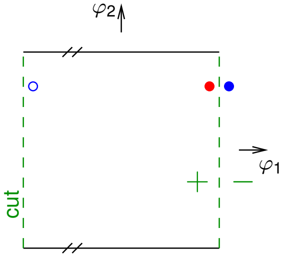

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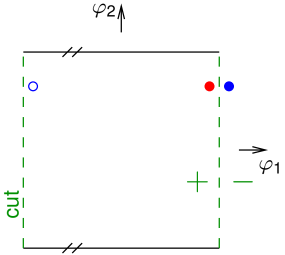

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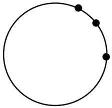

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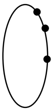
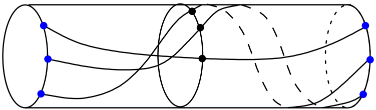


Eigenvalues of $T(\varphi_2)$ for a all $\varphi_2 \in [-\pi, \pi] \equiv S^1$ as a whole



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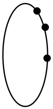

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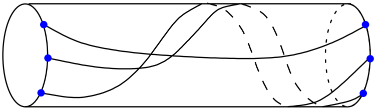

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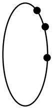

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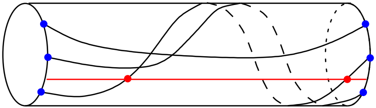

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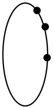

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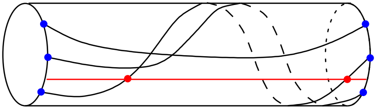

winding number=
signed number of crossings of fiducial line
$N = -2$

Hall conductance (bulk)

Definition: Bulk Index is the Chern number $\mathrm{ch}(P)$ of the Bloch bundle P defined by the Fermi projection

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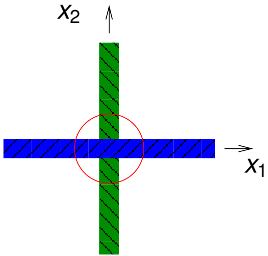
End of aside. Back to the disordered case

IQHE as a Bulk effect (remarks)

$$\sigma_{\mathrm{B}} = \mathrm{i}\,\mathrm{tr} P_\mu \big[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2] \big]$$

where $\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$ switches. Supports of $\vec{\nabla}\Lambda_i$:



Remark. The trace is well-defined. Roughly: An operator has a well-defined trace if it acts non-trivially on finitely many states only. Here the intersection contains only finitely many sites.

Equality of conductances

There is a definition of the Edge Hall conductanceσ_{E} for the case of a spectral gap, which needs to be amended in the case of a mobility gap.

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Theorem (Elgart, G., Schenker). Ergodic setting not assumed. Same is true in the case of a mobility gap.

An experiment: Amo et al.

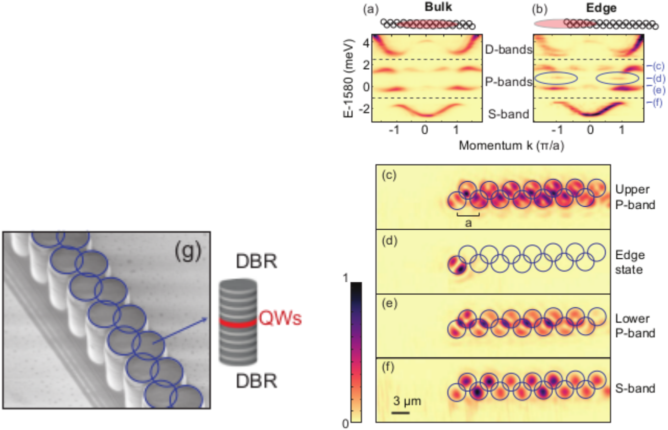

Figure: Zigzag chain of coupled micropillars and lasing modes

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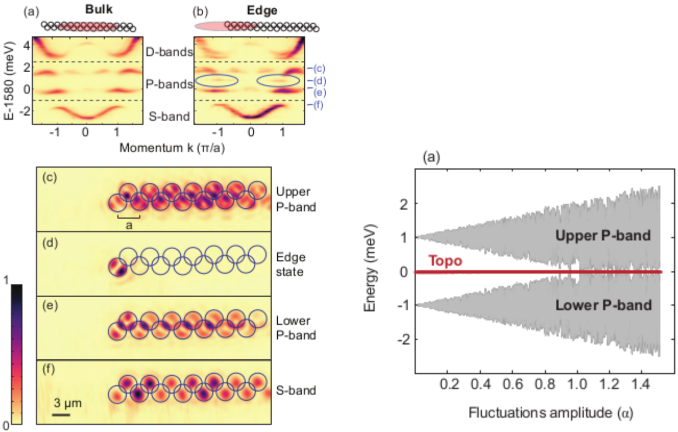
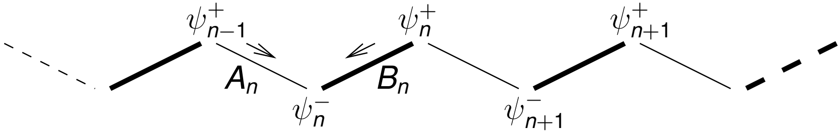


Figure: Lasing modes: bulk and edge

The Su-Schrieffer-Heeger model (1 dimensional)

Alternating chain with nearest neighbor hopping



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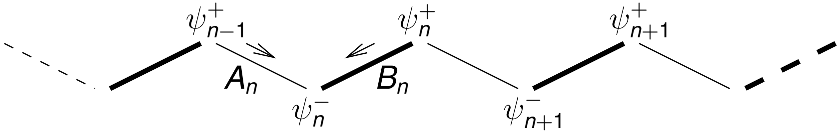

Hilbert space: sites arranged in dimers

$$\mathcal{H} = \ell^2(\mathbb{Z}, \mathbb{C}^N) \otimes \mathbb{C}^2 \ni \psi = \begin{pmatrix} \psi^+_n \\ \psi^-_n \end{pmatrix}_{n\in\mathbb{Z}}$$

Hamiltonian

$$H = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$$

with S, S^* acting on $\ell^2(\mathbb{Z}, \mathbb{C}^N)$ as

$$(S\psi^+)_n = A_n\psi^+_{n-1} + B_n\psi^+_n\,, \qquad (S^*\psi^-)_n = A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n$$

($A_n, B_n \in \mathrm{GL}(N)$ almost surely)

Chiral symmetry

$$\Pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\{H, \Pi\} \equiv H\Pi + \Pi H = 0$$

hence

$$H\psi = \lambda\psi \quad \Longrightarrow \quad H(\Pi\psi) = -\lambda(\Pi\psi)$$

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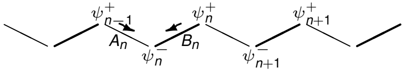


- Eigenvalue equation $H\psi = \lambda\psi$ is $S\psi^+ = \lambda\psi^-$, $S^*\psi^- = \lambda\psi^+$, i.e.

$$A_n\psi^+_{n-1} + B_n\psi^+_n = \lambda\psi^-_n\,, \qquad A^*_{n+1}\psi^-_{n+1} + B^*_n\psi^-_n = \lambda\psi^+_n$$

is one 2nd order difference equation, but two 1st order for $\lambda = 0$

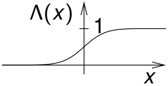
Bulk index

Let

$$\Sigma = \operatorname{sgn} H$$

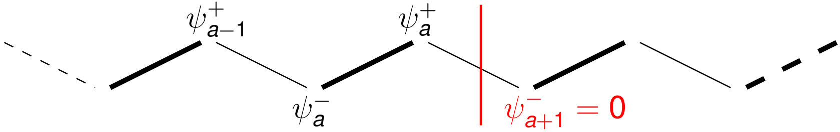
Definition. The Bulk index is

$$\mathcal{N} = \frac{1}{2} \operatorname{tr}(\Pi \Sigma [\Lambda, \Sigma])$$



with $\Lambda = \Lambda(n)$ a switch function (cf. Prodan et al.)

Edge Hamiltonian and index



Edge Hamiltonian H_a defined by restriction to $n \leq a$ (Dirichlet boundary condition $\psi^-_{a+1} = 0$). Chiral symmetry preserved.

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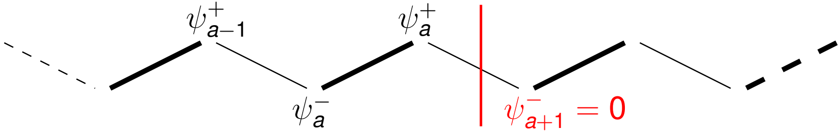

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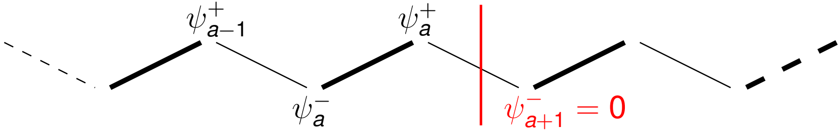

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$$\mathcal{N}^\pm_a := \dim\{\psi \mid H_a\psi = 0, \Pi\psi = \pm\psi\}$$

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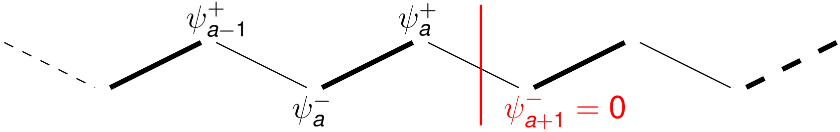

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Definition. The Edge index is

$$\mathcal{N}_a^\sharp := \mathcal{N}_a^+ - \mathcal{N}_a^-$$

and can be shown to be independent of a. Call it $\mathcal{N}^\sharp$.

Bulk-edge duality

Theorem (G., Shapiro). Assume $\lambda = 0$ lies in a mobility gap. Then

$$\mathcal{N} = \mathcal{N}^\sharp$$

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$$\gamma_1 \geq \cdots \geq \gamma_N$$

The assumption is satisfied if $\gamma_i \neq 0$; then $\mathcal{N}^\sharp = \sharp\{i \mid \gamma_i > 0\}$.

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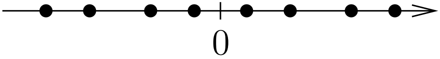
- at energy $\lambda \neq 0$ (simple spectrum)



  - Spectrum is simple because measure on transfer matrices is irreducible
  - so $\gamma = 0$ is not in the spectrum; localization follows

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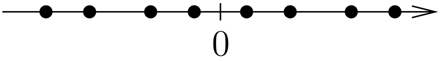

- At $\lambda = 0$ chains decouple: $\mathbb{C}^N \oplus 0$ and $0 \oplus \mathbb{C}^N$ are invariant subspaces

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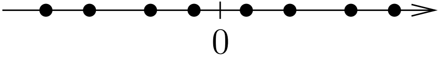
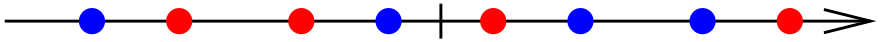
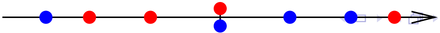


- of the upper ($+$) and lower ($-$) chains, at energy $\lambda = 0$



- at energy $\lambda = 0$ (phase boundary)



Floquet topological insulators

$H = H(t)$ (bulk) Hamiltonian in the plane with period T

$$H(t + T) = H(t)$$

(disorder allowed, no adiabatic setting)

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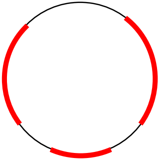
Assumption: Spectrum of $\widehat{U}$ has gaps:



spec $\widehat{U} \subset S^1$

Bulk index

Special case first: $U(t)$ periodic, i.e.

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$$\mathcal{N}_{\mathrm{B}} = \frac{1}{2}\int_0^T dt\, \mathrm{tr}\big(U^*\partial_t U\big[U^*[\Lambda_1, U], U^*[\Lambda_2, U]\big]\big)$$

with $U = U(t)$ and switches $\Lambda_i = \Lambda(x_i)$, $(i = 1, 2)$

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Remark. Extends the formula for the periodic case (Rudner et al.)

$$\mathcal{N}_{\mathrm{B}} = \frac{1}{8\pi^2} \int_0^T dt \int_{\mathbb{T}} d^2k \, \mathrm{tr}(U^* \partial_t U [U^* \partial_1 U, U^* \partial_2 U])$$

with $U = U(t, k)$ acting on the space of states of quasi-momentum $k = (k_1, k_2)$

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$H_{\mathrm{E}}(t)$ restriction of $H(t)$ to right half-space $x_1 > 0$

$\widehat{U}_{\mathrm{E}}$ corresponding fundamental propagator

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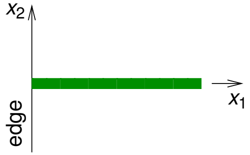
$$\mathcal{N}_{\rm E} = \operatorname{tr}(\widehat{U}_{\rm E}^*[\Lambda_2, \widehat{U}_{\rm E}]) = \operatorname{tr}(\widehat{U}_{\rm E}^* \Lambda_2 \widehat{U}_{\rm E} - \Lambda_2)$$

Remarks.

- The trace is well-defined



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In general: $\widehat{U}_{\mathrm{E}} \neq 1$

Edge index

$$\mathcal{N}_{\mathrm{E}} = \mathrm{tr}(\widehat{U}_{\mathrm{E}}^{*}[\Lambda_2, \widehat{U}_{\mathrm{E}}]) = \mathrm{tr}(\widehat{U}_{\mathrm{E}}^{*}\Lambda_2\widehat{U}_{\mathrm{E}} - \Lambda_2)$$

Remarks.

- The trace is well-defined

- $\mathcal{N}_{\mathrm{E}}$ is charge that crossed the line $x_2 = 0$ during a period.
- $\mathcal{N}_{\mathrm{E}}$ is independent of Λ_2 and an integer.

General case: Pair of Hamiltonians

$$\widehat{U} \neq 1$$

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Pair of periodic Hamiltonians $H_i(t)$, $(i = 1, 2)$ with

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Define Hamiltonian $H(t)$ with period $2T$ by

$$H(t) = \begin{cases} H_1(t) & (0 < t < T) \\ -H_2(-t) & (-T < t < 0) \end{cases}$$

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$$H(t) = \begin{cases} H_1(t) & (0 < t < T) \\ -H_2(2T - t) & (T < t < 2T) \end{cases}$$

Then

$$U(t) = \begin{cases} U_1(t) & (0 < t < T) \\ U_2(2T - t) & (T < t < 2T) \end{cases}$$

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Theorem (G., Tauber) $\mathcal{N} = \mathcal{N}_{\mathrm{E}}$

Duality in time and space

Let the interface Hamiltonian $H_{\mathrm{I}}(t)$ be a bulk Hamiltonian with

$$H_{\mathrm{I}}(t) = \begin{cases} H_1(t) \\ H_2(t) \end{cases} \quad \text{on states supported on large } \pm x_1$$

(still assuming $\widehat{U}_1 = \widehat{U}_2 =: \widehat{U}_\bullet$)

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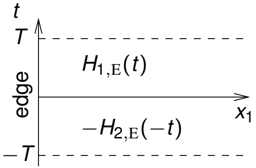
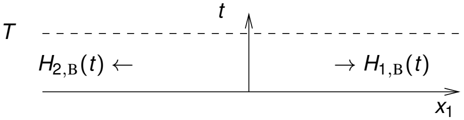
Interface index

$$\mathcal{N}_{\mathrm{I}} = \mathrm{tr}(\widehat{U}_\bullet^* \widehat{U}_{\mathrm{I}}[\Lambda_2, \widehat{U}_\bullet^* \widehat{U}_{\mathrm{I}}])$$





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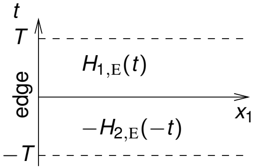
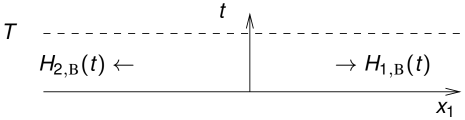
$$\mathcal{N}_{\rm I} = \mathrm{tr}(\widehat{U}_\bullet^* \widehat{U}_{\rm I}[\Lambda_2, \widehat{U}_\bullet^* \widehat{U}_{\rm I}])$$



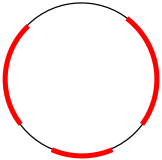


Theorem (G., Tauber) The indices for the two diagrams agree:

$$(\mathcal{N} =)\mathcal{N}_{\rm E} = \mathcal{N}_{\rm I}$$

Back to single Hamiltonian

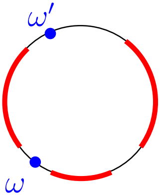
$\widehat{U} \neq 1$



spec $\widehat{U} \subset S^1$

Back to single Hamiltonian

$\widehat{U} \neq 1$



Let $\alpha \in \mathbb{R}$ and $\omega = \mathrm{e}^{\mathrm{i}\alpha}$. For $z \notin \omega\mathbb{R}_+$ (ray) define the branch

$$\log_\alpha z = \log |z| + \mathrm{i} \arg_\alpha z$$

by $\alpha - 2\pi < \arg_\alpha z < \alpha$.

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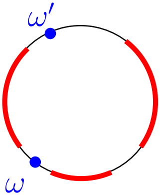

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Comparison Hamiltonian H_α: For $\omega \notin \mathrm{spec}\,\widehat{U}$ set

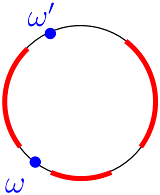
$$-\mathrm{i}H_\alpha T := \log_\alpha \widehat{U}$$

So,

- $\widehat{U}_\alpha = \widehat{U}$
- $U_{\alpha+2\pi}(t) = U_\alpha(t)\mathrm{e}^{2\pi\mathrm{i}t/T}$
- $\mathcal{N}_{\mathrm{B},\alpha+2\pi} = \mathcal{N}_{\mathrm{B},\alpha} =: \mathcal{N}_\omega$

Back to single Hamiltonian

$\widehat{U} \neq 1$



Let $\alpha \in \mathbb{R}$ and $\omega = e^{i\alpha}$. For $z \notin \omega\mathbb{R}_+$ (ray) define the branch

$$\log_\alpha z = \log|z| + i \arg_\alpha z$$

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Comparison Hamiltonian H_α: For $\omega \notin \operatorname{spec} \widehat{U}$ set

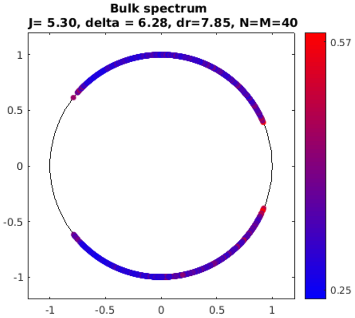
$$-iH_\alpha T := \log_\alpha \widehat{U}$$

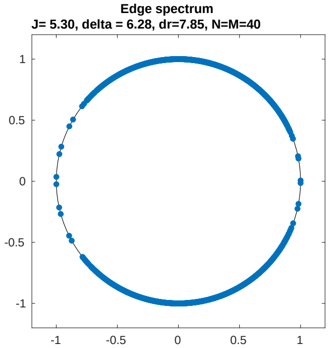
Theorem (Rudner et al.; G., Tauber) For ω, ω' in gaps

$$\mathcal{N}_{\omega'} - \mathcal{N}_\omega = i \operatorname{tr} P\big[[P, \Lambda_1], [P, \Lambda_2]\big]$$

where $P = P_{\omega,\omega'}$ is the spectral projection associated with $\operatorname{spec} \widehat{U}$ between ω, ω' (counter-clockwise)

Bulk and Edge spectrum





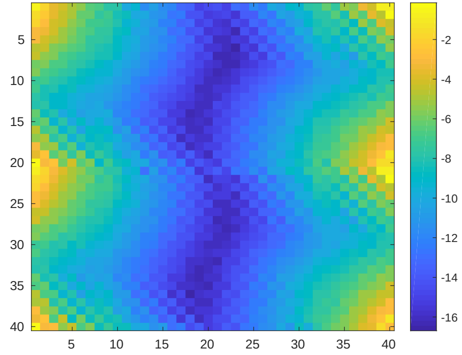
Bulk (left) and Edge spectrum (right); color: participation ratio

Computing the edge index

Edge index based $\mathcal{N}_{\mathrm{E},\alpha}$ based on the pair (H, H_α) (with $\alpha = \pi$)

$$\mathcal{N}_{\mathrm{E},\alpha} = \operatorname{tr} A \qquad A = \widehat{U}_{\mathrm{E}}^* \Lambda_2 \widehat{U}_{\mathrm{E}} - \widehat{U}_{\alpha,\mathrm{E}}^* \Lambda_2 \widehat{U}_{\alpha,\mathrm{E}}$$

The diagonal integral kernel $A(x,x)$ as $\log |A(x,x)|$



Boundary conditions:

- Vertical edges: Dirichlet
- Horizontal edges: Periodic

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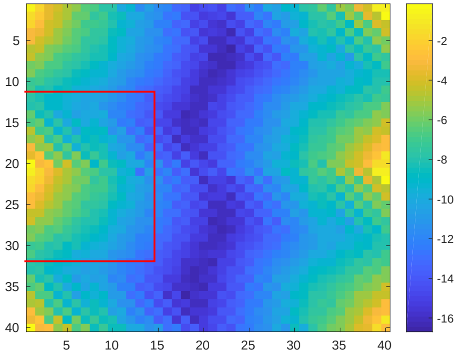

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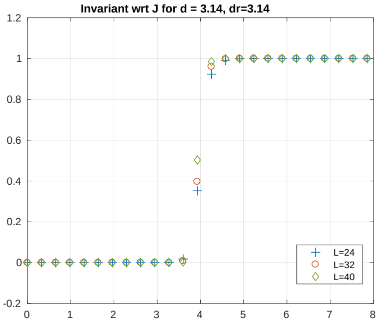
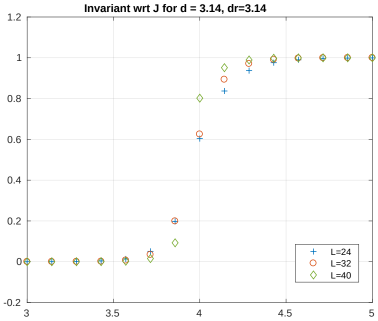
The transition





Edge index (left) and zoom (right)

Integer detected with 1 part in 10^{12}

Summary

- Quantum Hall Effect as the first type of topological insulator
- Essential role of disorder (spectral vs. mobility gap)
- Symmetry as a new twist
- Bulk-edge duality
- Chiral symmetry
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Thank you for your attention!