Integrable PDE with small dispersion

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The Korteweg-de Vries Equation (KdV)

$$q_t - 6qq_x + \varepsilon^2 q_{xxx} = 0$$

- Observation of solitary wave, in a canal in Scotland and in lab, Scott Russel, 1834

- Formulation, Korteweg and de-Vries, 1895. Also in earlier paper by Boussinesq.

- Numerical discovery of solitons and their clean interaction and separation by Kruskal and Zabusky, 1965.

- Solution of the KdV through inverse scattering, Gardner, Greene, Kruskal, Miura, 1967.

- The Lax pair and the theory of integrable systems, Lax 1968.

The Lax Pair and the integration of KdV

The infinitely many conserved quantities of KdV are the *eigenvalues of a linear operator* $L = L(t)$, that depends on the solution $q(x,t)$ of KdV and undergoes a *unitary transformation* $U = U(t)$ as time evolves.

$$U^{-1} L U = L_0$$

Differentiating with respect to time obtains

$$U^{-1} U_t U^{-1} L U + U^{-1} L_t U + U^{-1} L U_t = 0$$

Multiplying on the left by U, on the right by U^{-1},

$$U_t U^{-1} L + L_t + L U_t U^{-1} = 0$$

Letting $B = U_t U^{-1}$, thus, ($U_t = BU$, and B is the infinitesimal generator of the tansformation U)

$$\boxed{L_t = -[L, B]}$$

The pair of the operators $L,\ B$ is the Lax pair.

The Lax pair for KdV and inverse scattering ($\varepsilon = 1$)

$$\begin{cases} L = -D^2 + q \\ B = -4D^3 + 3(Dq + qD) \end{cases} \qquad D = \frac{d}{dx}, \qquad q = q(x,t)$$

The Lax equation $L_t = -[L, B]$ becomes KdV (all D cancel).

Eigenvalue problem of L. $-\psi_{xx} + q(x)\psi = \lambda\psi$

1. extended $\psi(x,k)$ asymptotics ($\lambda = k^2$, scattering)

$$\boxed{T(k)\overleftarrow{e^{-ikx}}} \quad \boxed{q(x)} \quad \boxed{\overleftarrow{e^{-ikx}} + R(k)\overrightarrow{e^{ikx}}} \longrightarrow \ x \text{ axis}$$

2. Bound state asymptotics $\lambda_j = -\kappa_j^2, \quad j = 1, 2, \cdots, n$

$$\|\psi(x, \lambda_j)\|_{L^2} = 1, \quad \psi(x, \lambda_j) \sim c_j e^{-\kappa_j x}, \quad x \to +\infty.$$

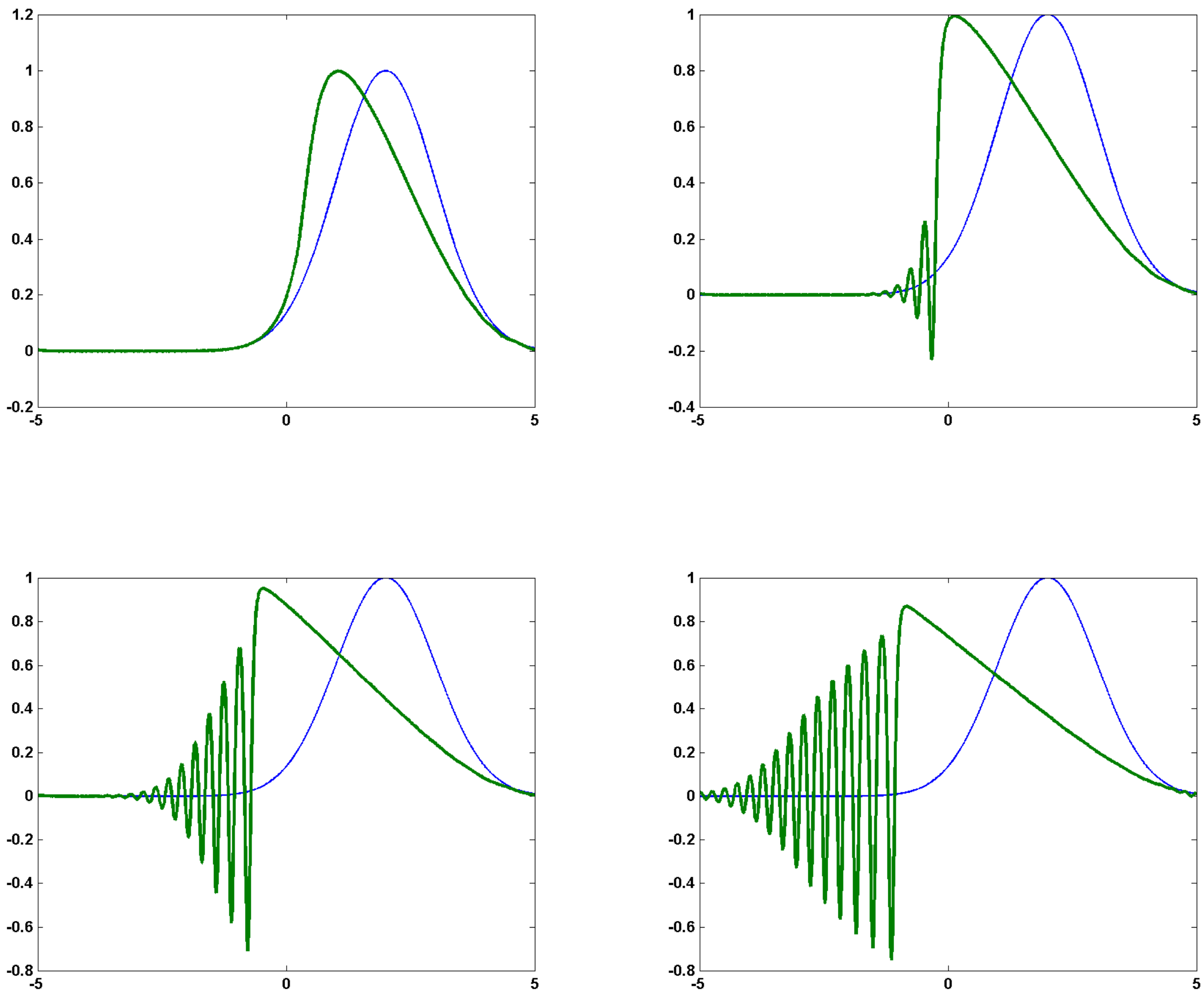
By unitarity, $\lambda_j(t) = \lambda_j(0)$.

Evolution. $R(k,t) = R(k,0)e^{8ik^3}, \quad c_j(t) = c_j(0)e^{4\kappa_j^3}.$

Gelfand, Levitan; Marcenko (1950's). Recovery of the potential q through an integral equation.

Dispersive regularization of a KdV "shock": Radiation wave.

$$q_t - 6qq_x + \varepsilon^2 q_{xxx} = 0 \qquad \varepsilon = .05$$

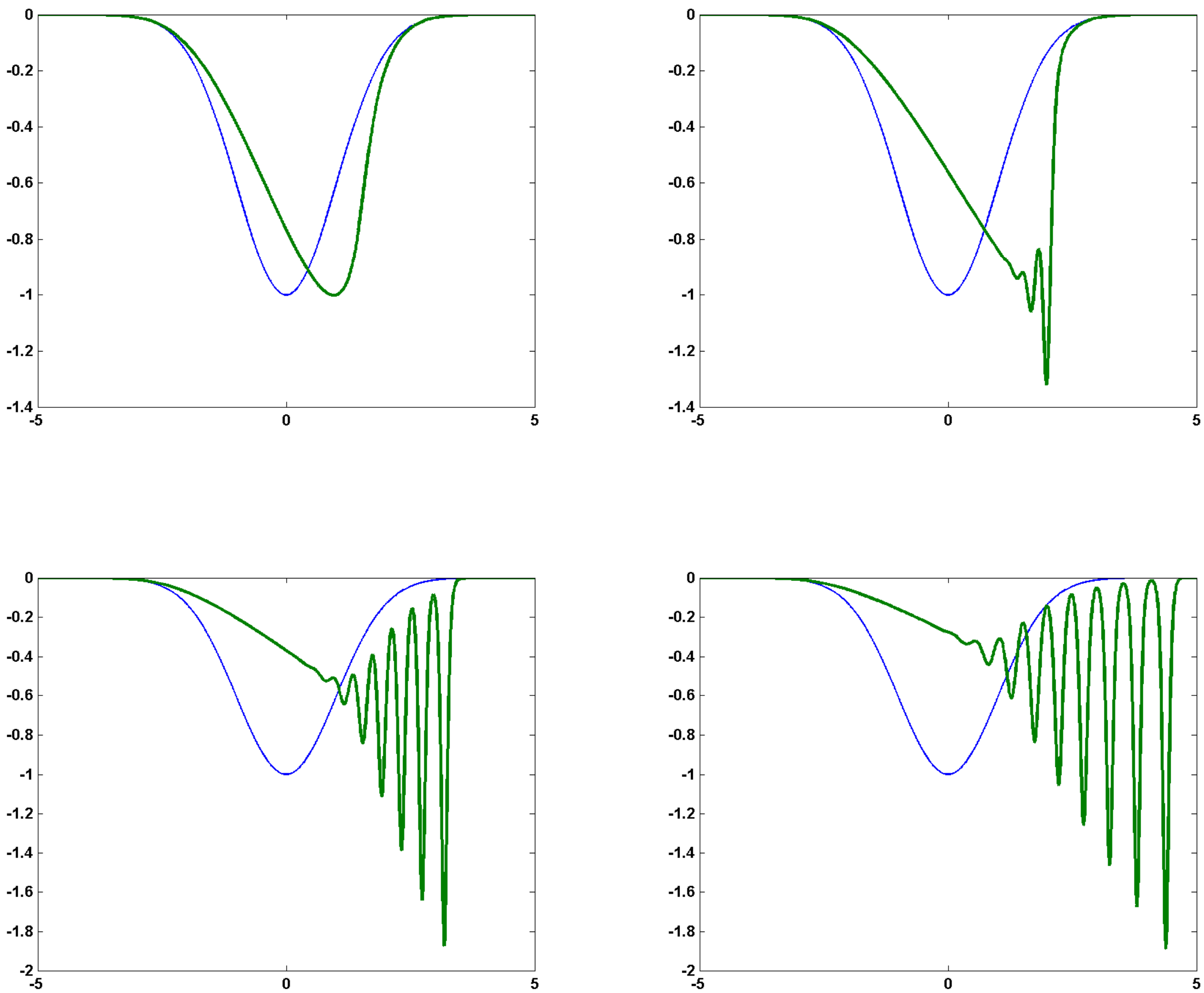


Decay as $t \to +\infty$.

Numerics. Bathi Kasturiarachi

Dispersive regularization of a KdV "shock": Multisoliton wave

$$q_t - 6qq_x + \varepsilon^2 q_{xxx} = 0 \quad \varepsilon = .05$$

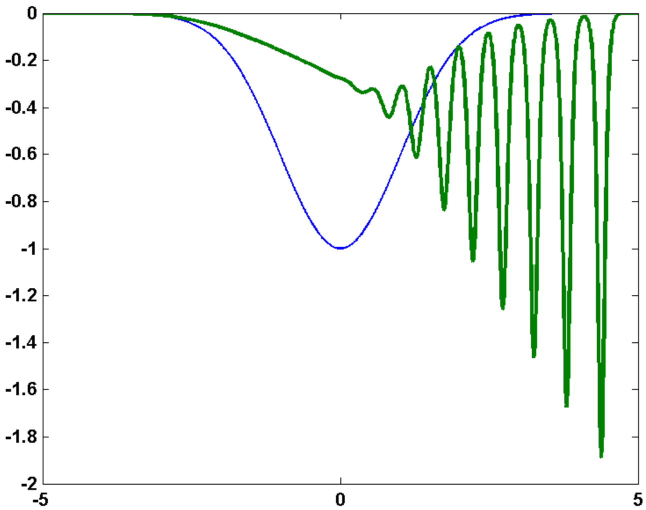


Soliton separation as $t \to +\infty$.

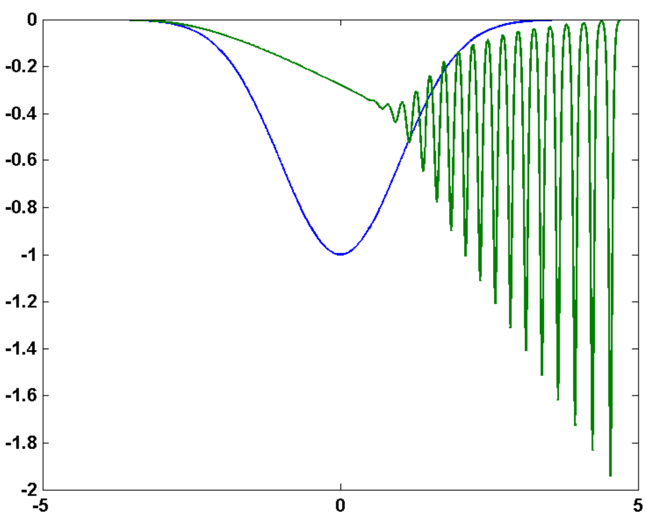
Numerics. Bathi Kasturiarachi

Scaling

$$q_t - 6qq_x + \varepsilon^2 q_{xxx} = 0$$



$\varepsilon = .05$



$\varepsilon = .025$

What if these waveforms were linear? For example $q_t + \varepsilon^2 q_{xxx} = 0$

Solution typically through a Fourier integral of the type

$$u(x,t,\varepsilon) \sim \varepsilon^{-\frac{1}{2}} \int_{-\infty}^{\infty} A(k,x,t) e^{\frac{i}{\varepsilon}\theta(k,x,t)} dk.$$

- The variables x, t are parameters of the integrand.

- The integral is calculated by the (rigorous) asymptotic method of **stationary phase / steepest descent** in the limit $\varepsilon \to 0$.

- At each x, t and due to *phase cancellation, the leading contributions to the integral arise at the critical (stationary) points of the phase function* $\theta(k,x,t)$ *with respect to the spectral variable* k.

- Any stationary point $k = k^*$ is a function of x, t.

- Different contributions at the same x, t, coming from *different stationary points*, do not interact, they merely *interfere*.

The nonlinear calculation

- As in the linear case, (x, t) are parameters.

- The game is played on the complex z plane, where $\lambda = z^2$ is the eigenvalue of the Lax operator L.

- At each value of z, a matrix function m is created with carefully chosen eigenfunctions of L as entries.

- The matrix m is analytic in the complex z plane, *except on an oriented contour*. Such a contour is determined from the initial scattering data.

- Along the contour, a *multiplicative jump occurs*, $m_+ = m_- V$.

- The square *jump matrix* $V(z)$ is determined from the initial scattering data.

- The core of the calculation is: *given the above information, determine the matrix* m. This is known as a *Riemann-Hilbert problem* (RHP).

- The RHP is a linear problem.

Challenge of small dispersion.
How can analysis make the phenomena visible?

The Steepest descent method for RHP

Steepest Descent.
Linear *vs.* Nonlinear problems

Linear PDE	Integrable NL PDE
Fourier Integral	Matrix RHP
Contour deformation	Contour deformation
	Jump matrix factoring
	Contour splitting
Large exponent	Large exponents
Real exponent $\to -\infty$	Jump matrix $\to$ Identity
Critical points	Critical arcs (bridges)
Goal: Solvable integral	Goal: Solvable matrix RHP

Strategy (g-function mechanism): Determine an *eikonal* function $g(z; x, t)$, for which contour deformation reduces the RHP to a solvable one. The function g is introduced through the change of matrix variable $m \mapsto \tilde{m}G$ where G is diagonal with entries $e^{\pm ig(z)/\varepsilon}$.

Alternative factorizations of the jump matrix generates two types of contour arcs.

The function $h = h(z; x, t) = 2g - f$ is a "sister" function of g. The function $f = f(z, x, t)$ encompasses the scattering data of the original problem.

Semiclassical limit of the focusing NLS

Goal: Asymptotic evaluation of $q(x,t,\varepsilon)$ as $\varepsilon \to 0$.

Collaborators. Alex Tovbis, Xin Zhou, Sergey Belov, Robbie Buckingham, Andreas Aristotelous

Focusing Cubic Schrödinger Equation (NLS)

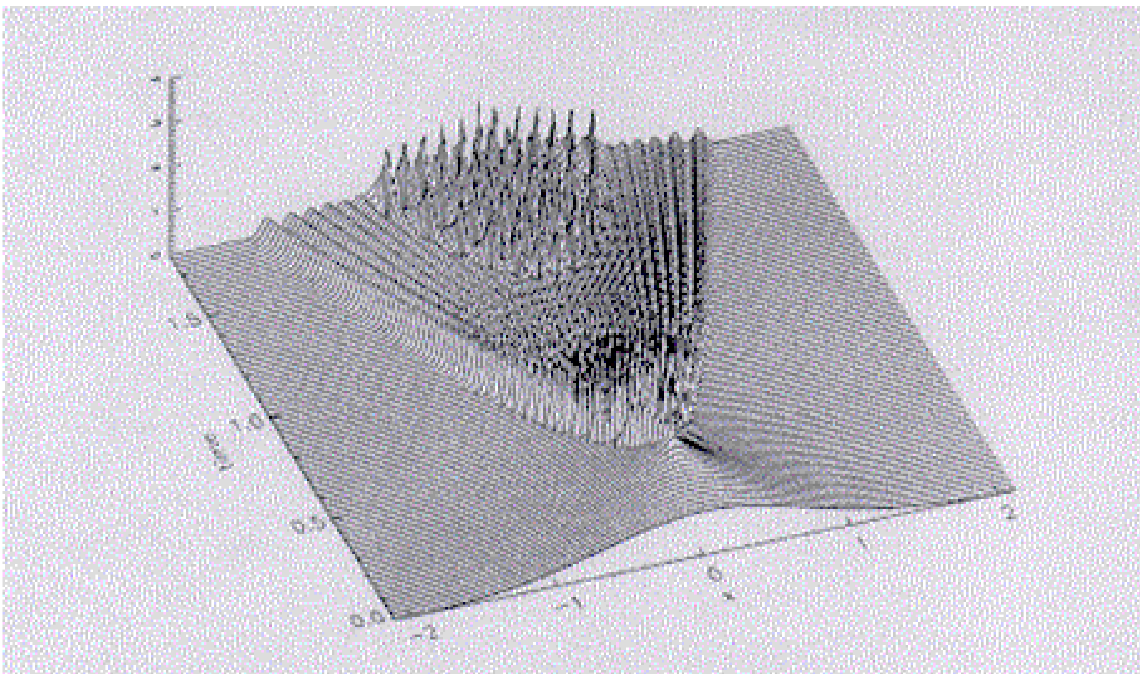
$$\begin{cases} i\varepsilon q_t + \varepsilon^2 q_{xx} + 2|q|^2 q = 0 \\ q(x,0) = A(x) e^{iS(x)/\varepsilon}. \end{cases}$$

Initial data decay as $|x| \to \infty$. Our data:

$$A(x) = -\text{sech } x, \qquad S'(x) = -\mu \tanh x$$

Integrability of NLS: Zakharov, Shabat, 1971

NLS dispersive breaking, $\varepsilon \to 0$

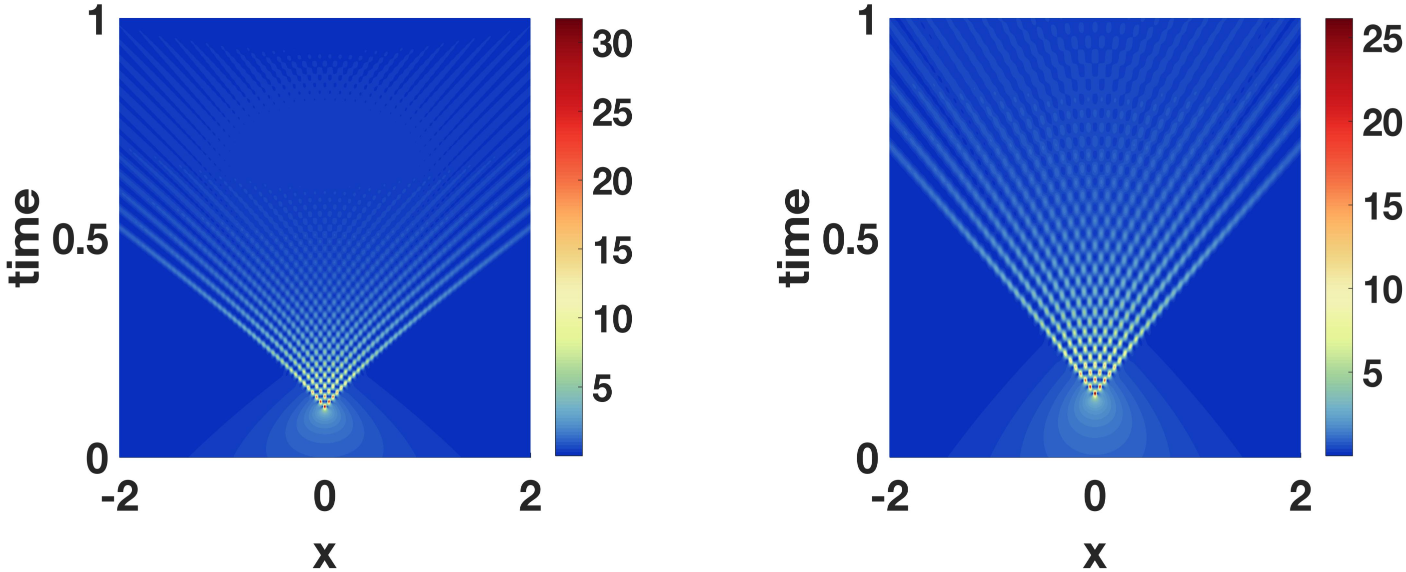


x-axis is x; y-axis is t; z-axis is $|q(x,t)|$

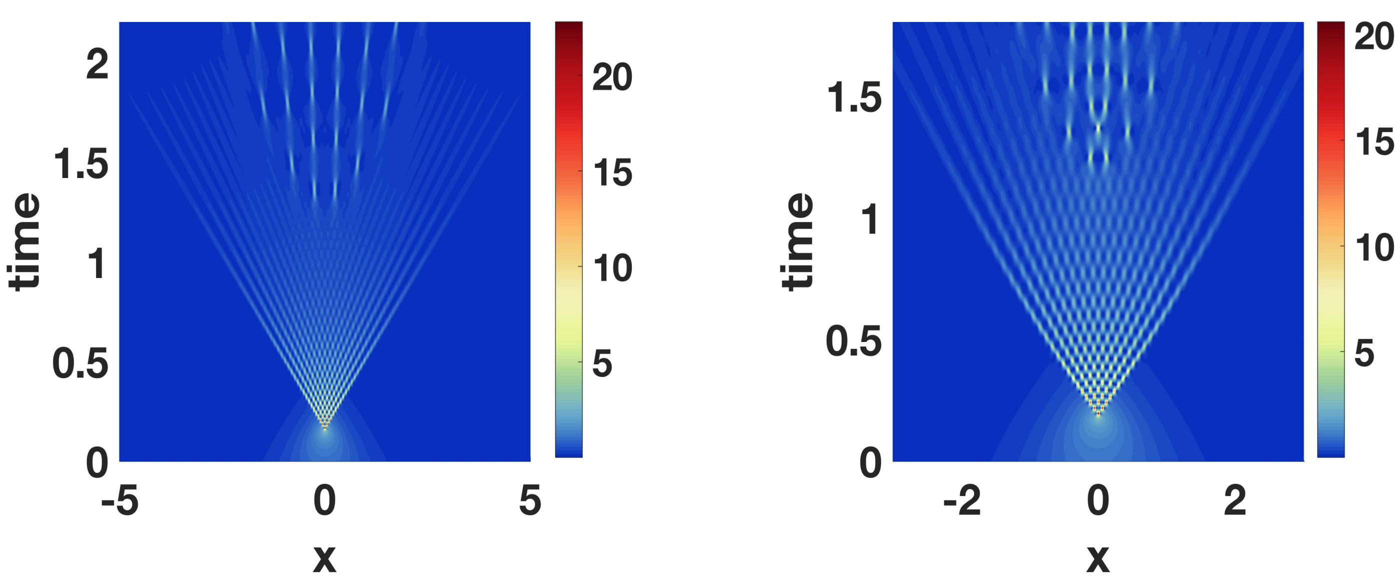
Numerics: David Cai, **Two breaks observed**

NLS dispersive breaking, $\varepsilon \to 0$

Numerics: Andreas Aristotelous

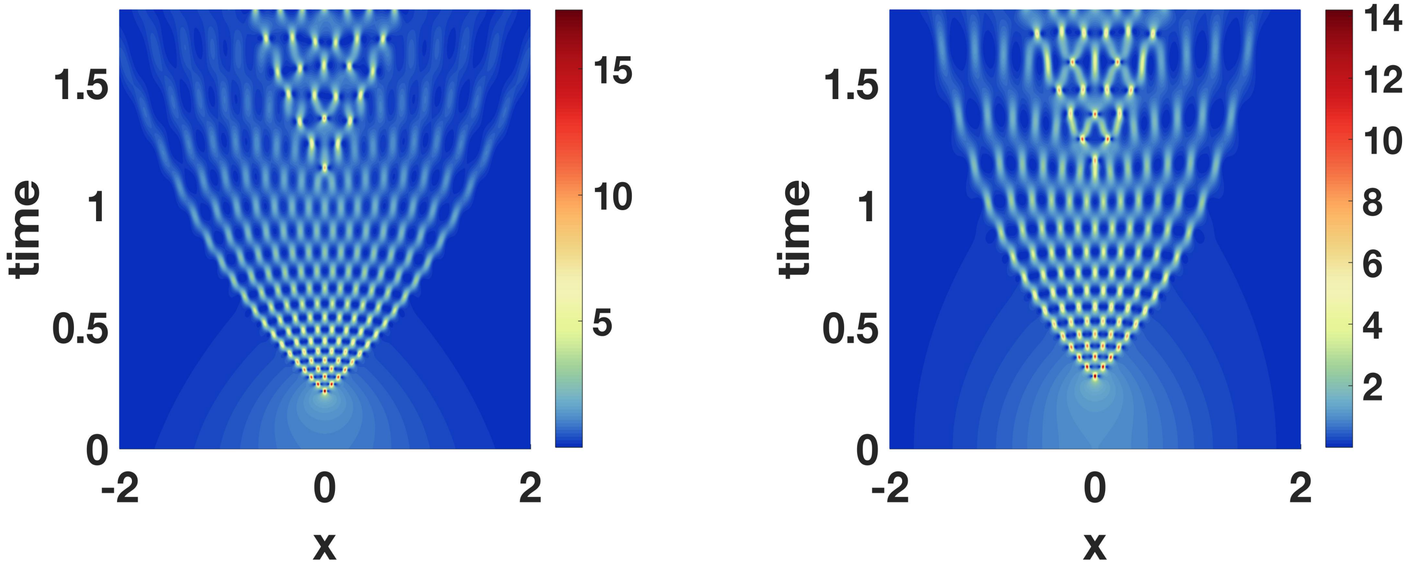


$\mu = 3$ and $\mu = 2$

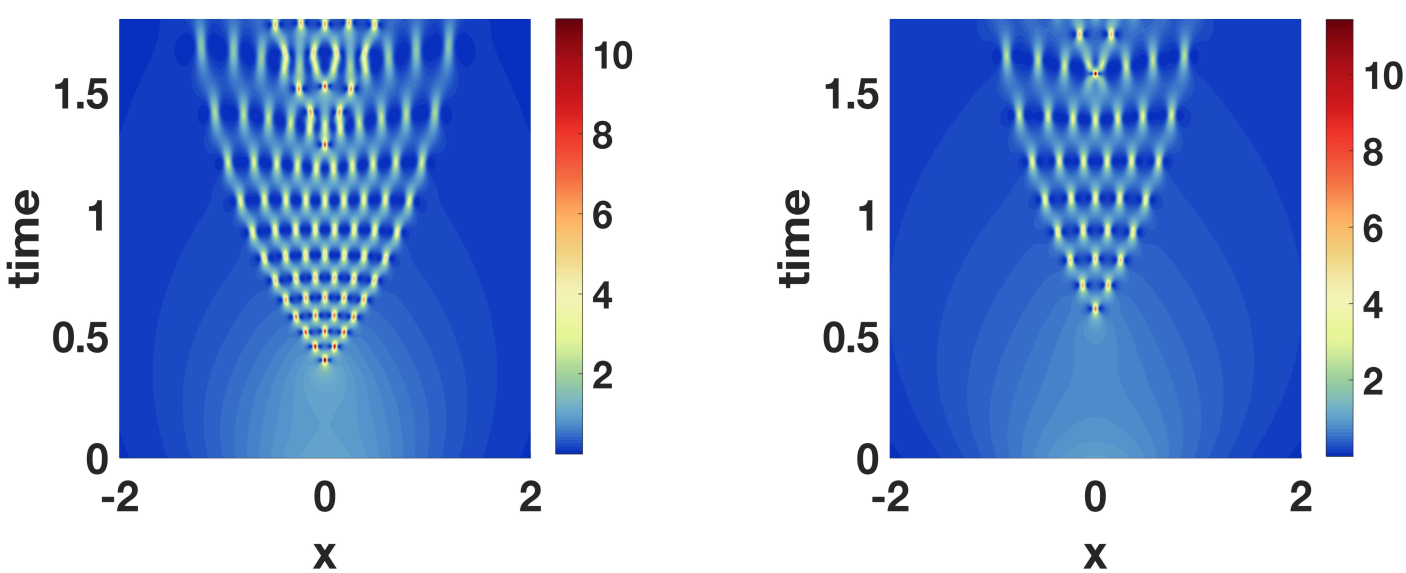


$\mu = 1.5$ and $\mu = 1$

NLS dispersive breaking, $\varepsilon \to 0$

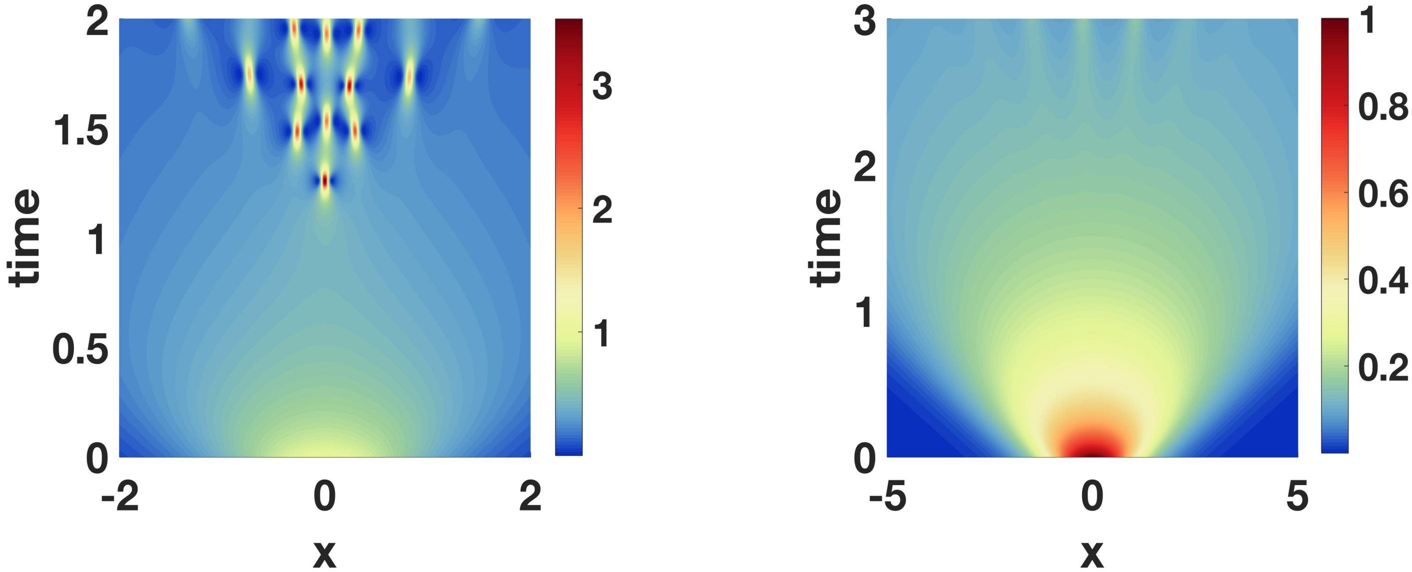


$\mu = 0.5$ and $\mu = 0$

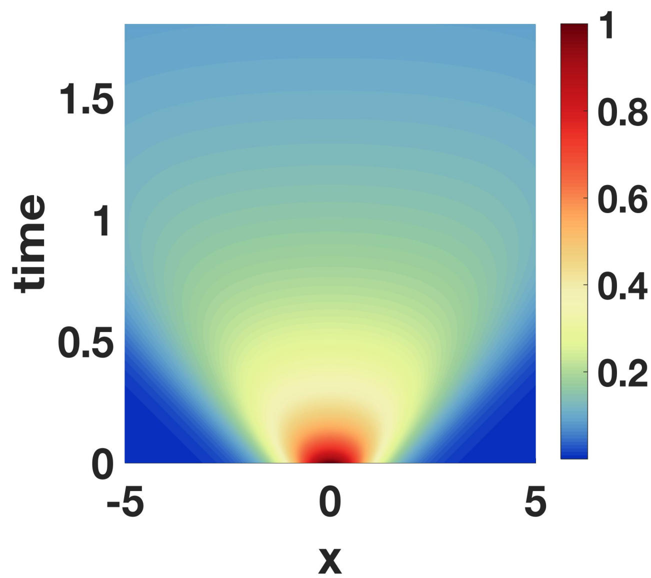


$\mu = -0.5$ and $\mu = -1$

NLS dispersive breaking, $\varepsilon \to 0$



$\mu = -1.5$ and $\mu = -2$



$\mu = -3$

Sketch of the main theorem
The case of $\mu > 0$.

There exists a breaking curve or nonlinear caustic

$$t = t_0(x), \;\; x \in \mathbb{R},$$

- *When* $0 \leq t < t_0(x)$, *the solution is controled by a point* $\alpha_0 = \alpha_0(x,t)$ *in the upper complex half plane.*

$$q_0(x,t,\varepsilon) = [Im\, \alpha_0(x,t)]\, e^{-2\frac{i}{\varepsilon}\int_0^x Re\, \alpha_0(s,t) ds}$$

- When $t_0(x) < t < t_1(x)$, the solution is controled by three points in the upper half plane $\alpha_0, \alpha_2, \alpha_4$ that depend on x and t (slow dependence) and define the radical (Riemann surface)

$$R(z) = \left(\prod_{j=0}^{2}(z - \alpha_{2j})(z - \bar{\alpha}_{2j})\right)^{1/2}$$

which plays a crucial part in the asymptotic solution

$$q_0(x, t, \varepsilon) = \Theta e^{\frac{2i}{\varepsilon}\Omega_1} \mathrm{Im}\,(\alpha_2 - \alpha_0 - \alpha_4),$$

$$\Theta = -\frac{\theta(-\frac{\widehat{W}}{2\pi\varepsilon} - u_\infty + d)\theta(u_\infty + d)}{\theta(-\frac{\widehat{W}}{2\pi\varepsilon} + u_\infty + d)\theta(-u_\infty + d)}.$$

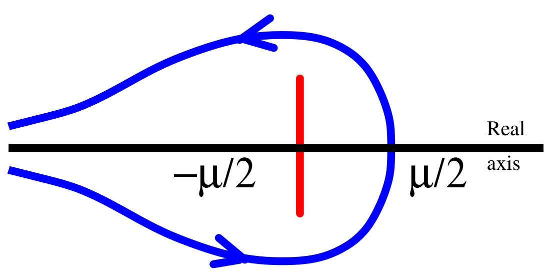
The quantities in the arguments of θ=Riemann θ-function are explicit functions of $\alpha_0, \alpha_2, \alpha_4$. Fast dependence on x, t through $\widehat{W}/2\pi\varepsilon$ and Ω_1/ε.

Early Factorization and Contour Splitting

$$m_+ = m_- \underbrace{\begin{pmatrix} 1+|r|^2 & \bar{r} \\ r & 1 \end{pmatrix}}_{jump\ matrix} = m_- \begin{pmatrix} 1 & \bar{r} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$$

RH contour: Blue, Soliton condensed poles: Red

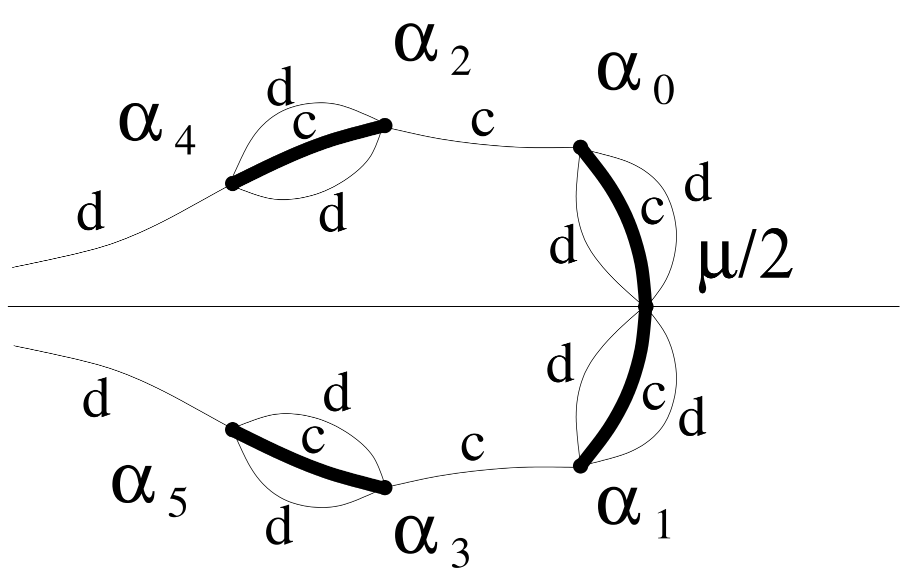
Jump matrix in upper blue half-contour $\begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix}$



Jump matrix in lower blue half-contour $\begin{pmatrix} 1 & \bar{r} \\ 0 & 1 \end{pmatrix}$

Factorization-triggered contour splits

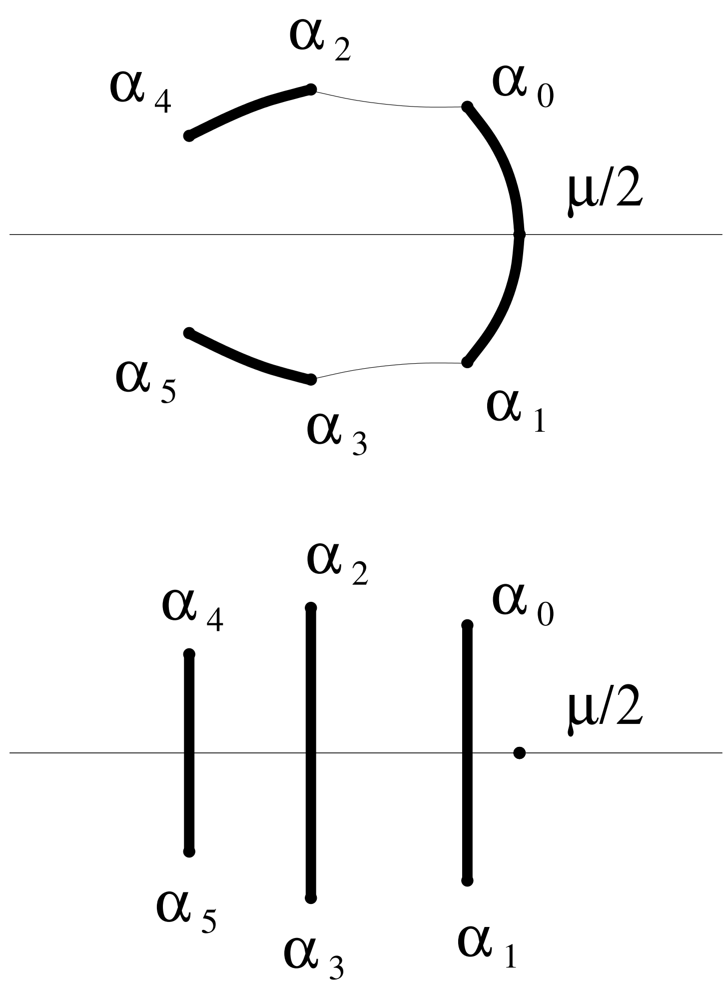
JUMP MATRIX: $\begin{cases} \text{c: } \textbf{constant} \\ \text{d: } \textbf{decay to identity}, \end{cases}$



BRIDGES: **Bold**

branchpoints α_j to be determined

MODEL PROBLEM



Branchpoints $(\alpha_0, \alpha_2, \alpha_4)$ and their number

$$\begin{cases} \text{Modulation equations } (\textit{trnscendental} \text{ not differential}), \\ \\ \text{Sign conditions.} \end{cases}$$

g-Function Mechanism. Conditions on $h(z)$

BRIDGES

$$\begin{pmatrix} \mathbf{e}^{i(h_+ - h_-)/\varepsilon} & 0 \\ -\mathbf{e}^{i(h_+ + h_-)/\varepsilon} & \mathbf{e}^{-i(h_+ - h_-)/\varepsilon} \end{pmatrix} = \begin{pmatrix} a & 0 \\ -b & a^{-1} \end{pmatrix}$$

$$=$$

$$\underbrace{\begin{pmatrix} 1 & -ab^{-1} \\ 0 & 1 \end{pmatrix}}_{\to \text{Identity}} \underbrace{\begin{pmatrix} 0 & b^{-1} \\ -b & 0 \end{pmatrix}}_{\text{constant\&bdd}} \underbrace{\begin{pmatrix} 1 & -a^{-1}b^{-1} \\ 0 & 1 \end{pmatrix}}_{\to \text{Identity}}$$

Constancy and Decay Conditions:

$$\begin{cases} h_+ + h_- = \text{Real constant}, & \textbf{on the contour} \\ \text{Im}\, h < 0, & \textbf{left and right of the contour} \end{cases}$$

g-Function Mechanism : Conditions on $h(z)$

LANDPATHS :

$$\begin{pmatrix} \mathbf{e}^{i(h_+ - h_-)/\varepsilon} & 0 \\ -\mathbf{e}^{i(h_+ + h_-)/\varepsilon} & \mathbf{e}^{-i(h_+ - h_-)/\varepsilon} \end{pmatrix} = \begin{pmatrix} a & 0 \\ -b & a^{-1} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 \\ -a^{-1}b & 1 \end{pmatrix}}_{\rightarrow\text{Identity}} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ or } \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ -ab & 1 \end{pmatrix}}_{\rightarrow\text{Identity}}$$

Constancy and Decay Conditions:

$$\begin{cases} h_+ - h_- = \text{Real constant}, & \textbf{on the contour} \\ \operatorname{Im} h > 0, & \textbf{on the contour} \end{cases}$$

Pictorial interpretation of the conditions on the phase function $h(z; x, t)$

$\operatorname{Im} h$ is $ELEVATION$

$\operatorname{Im} h > 0 \equiv \ LAND$; $\quad \operatorname{Im} h < 0 \equiv \ WATER$

ABOVE RULES PICTORIALLY

THE OPTIMAL RH CONTOUR
CANNOT GO THROUGH WATER

IT MUST BE THE UNION OF

- *BRIDGES* (main arcs, rigid),
  $\operatorname{Im} h = 0$ on *BRIDGE* and $\operatorname{Im} h < 0$ left and right

- *LANDPATHS* (complementary arcs, deformable),
  $\operatorname{Im} h > 0$.

The scalar Riemann-Hilbert problem and its solution

$$BRIDGES \begin{cases} h_+ + h_- = \text{Real constant}, & \text{on the contour} \\ \text{Im}\, h < 0, & \text{left and right of the contour} \end{cases}$$

$$LAND \begin{cases} h_+ - h_- = \text{Real constant}, & \text{on the contour} \\ \text{Im}\, h > 0, & \text{on the contour} \end{cases}$$

- The above real constants are evaluated from the condition that the function $g = (h + f)/2$ must be analytic at infinity

- The above equalities suffice to derive an integral formula for $h'(z)$ and $h(z)$ given the endpoints and sequence of bridges.

- The formulae involve the radical $\sqrt{(z-\alpha_0)(z-\bar{\alpha}_0)(z-\alpha_2)(z-\bar{\alpha}_2)(z-\alpha_4)(z-\bar{\alpha}_4)}$

Derivation of the branchpoints α_j

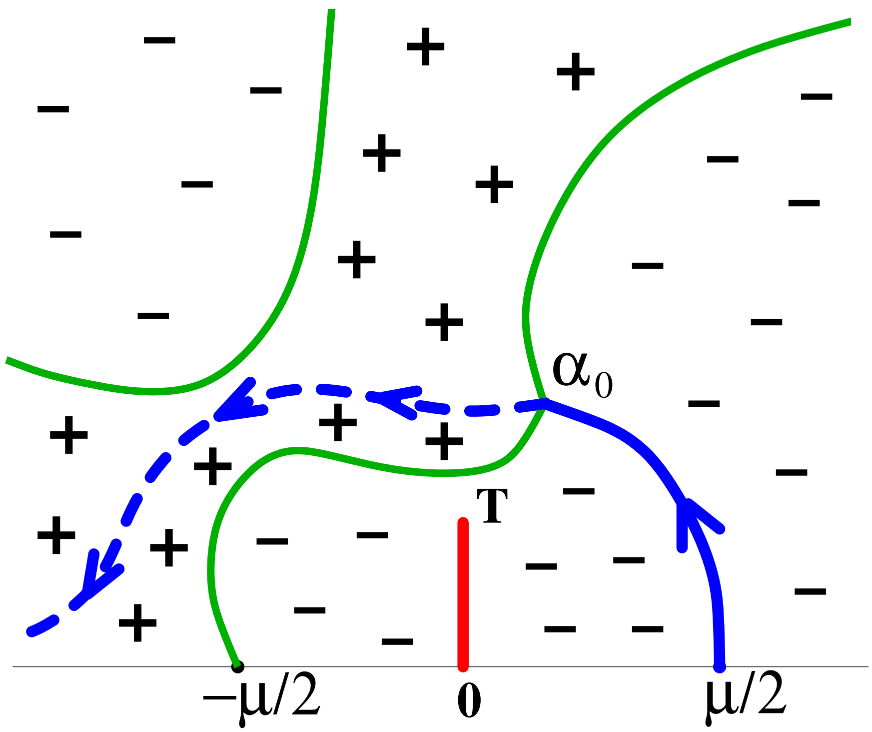
- Transcendental equations determine the branchpoints $\alpha_0, \alpha_2, \alpha_4$ from the condition that near a branch point α:

$$h(z) = c_1 + c_2(z-\alpha)^{\frac{3}{2}} + \cdots,$$

(the coefficient of $(z-\alpha)^{\frac{1}{2}}$ equals zero), where $c_1 \in \mathbb{R}$ and $c_2, \cdots$ are constants. Alternatively, from moment and integral conditions that apply. There are multiple solutions, involving different numbers of endpoints.

- Uniqueness is obtained through sign structures imposed by the inequalities.

Cartoon of Prebreak (only α_0)

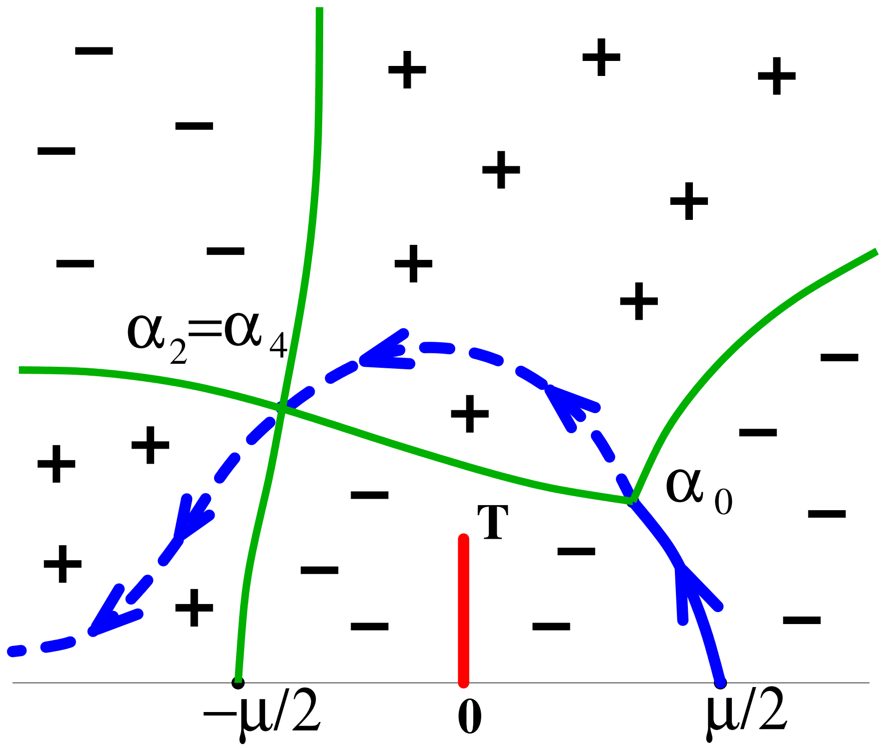


Blue: RH contour,

Full= BRIDGE; Dashed = LANDPATH

Green: $\mathrm{Im}\, h = 0$

Cartoon of Break

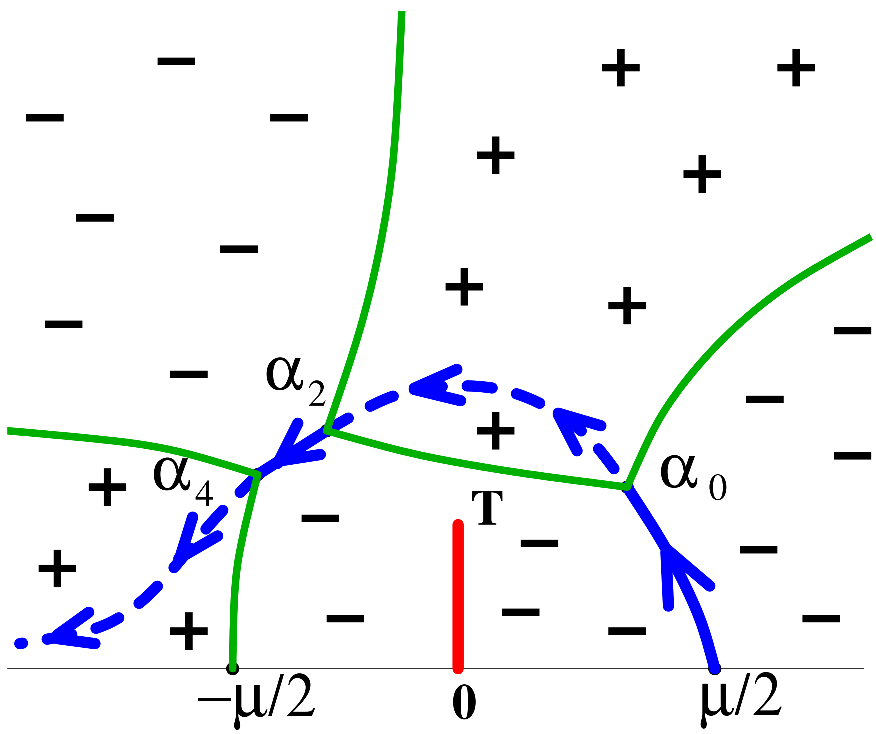


Blue: RH contour,

Full= BRIDGE; Dashed = LANDPATH

Green: Im $h = 0$

Cartoon of Postbreak

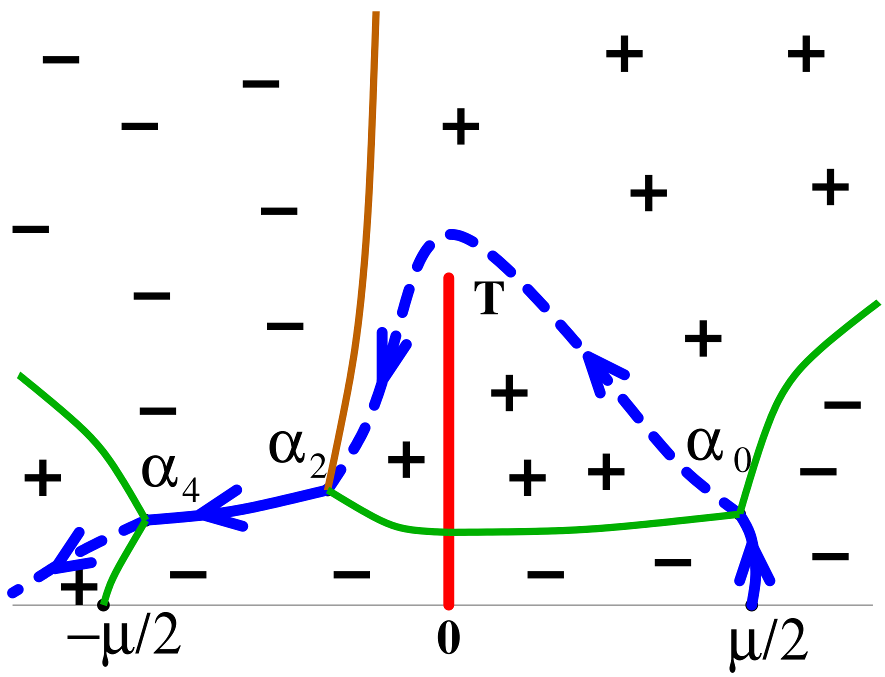


Blue: RH contour,

Full= BRIDGE; Dashed = LANDPATH

Green: Im $h = 0$

Cartoon of postbreak continued

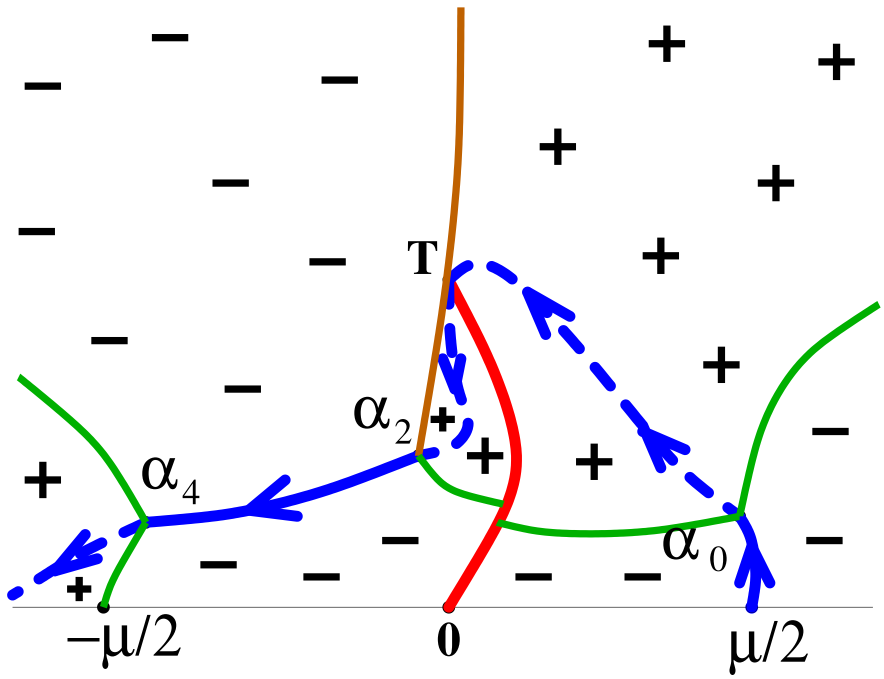


Blue: RH contour,

Full= BRIDGE; Dashed = LANDPATH

Green and brown: Im $h = 0$

Cartoon of breakdown of method



Blue: RH contour,

Full= BRIDGE; Dashed = LANDPATH

Green and brown: Im $h = 0$

Singular breaking curve in space-time: Im $h(T; x, t) = 0$

Result: If $x > \ln 2$ and t is large, then Im $h(T) > 0$ and the breakown does not happen.

Radicals and Riemann theta functions:

Behavior of sign of $\operatorname{Im} h$ at bridge endpoints leads to radicals $R(z)$:

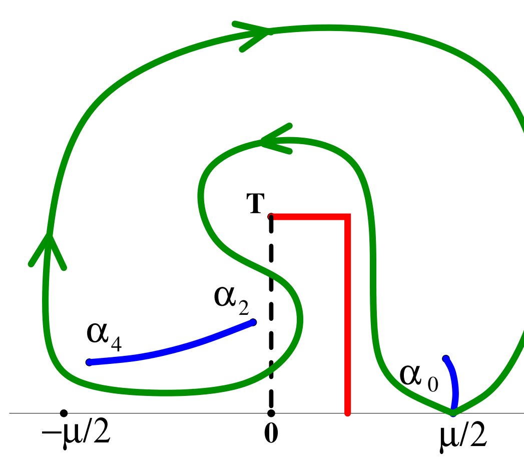
$$\sqrt{(z-\alpha_0)(z-\overline{\alpha}_0)} \qquad \text{(prebreak)}$$

$$\sqrt{(z-\alpha_0)(z-\overline{\alpha}_0)(z-\alpha_2)(z-\overline{\alpha}_2)(z-\alpha_4)(z-\overline{\alpha}_4)}$$

(postbreak)

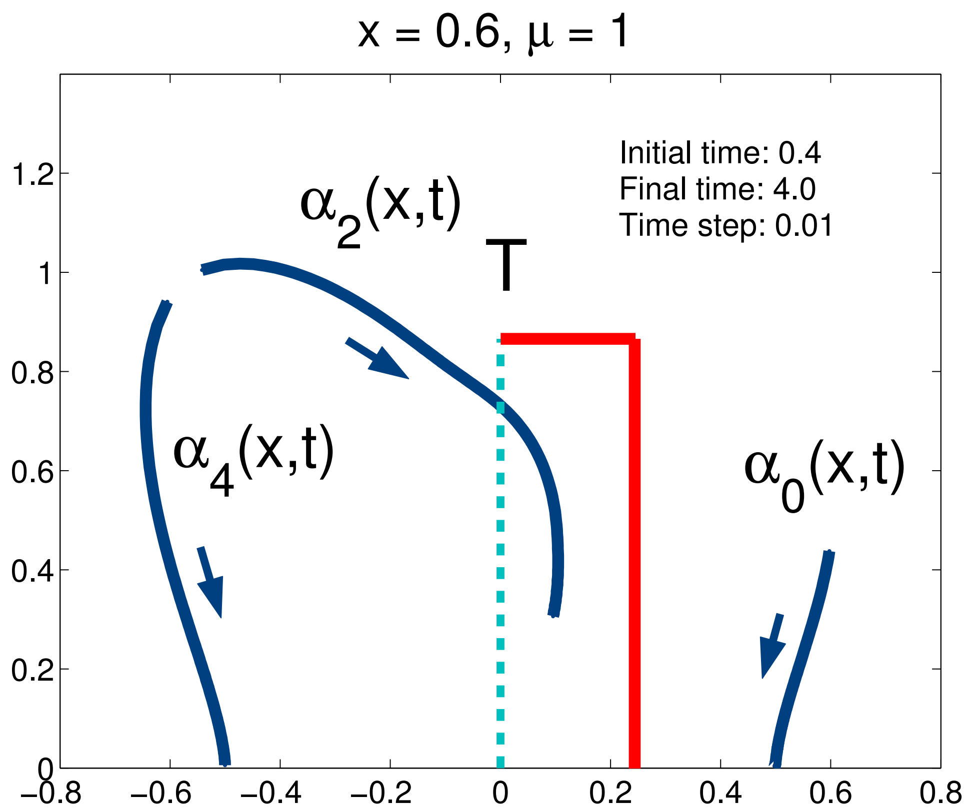
Equation for h' (z is inside Γ which must surround RH contour).

$$h'(z) = \frac{R(z)}{2\pi i}\oint_\Gamma \frac{f'(\zeta)}{(\zeta - z)R(\zeta)}d\zeta,$$



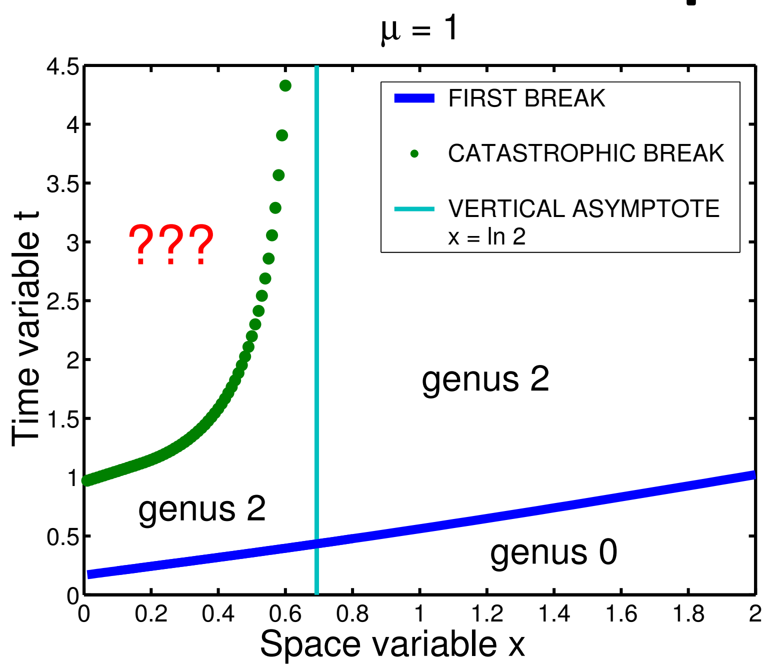
The part of contour Γ in the upper half-plane.

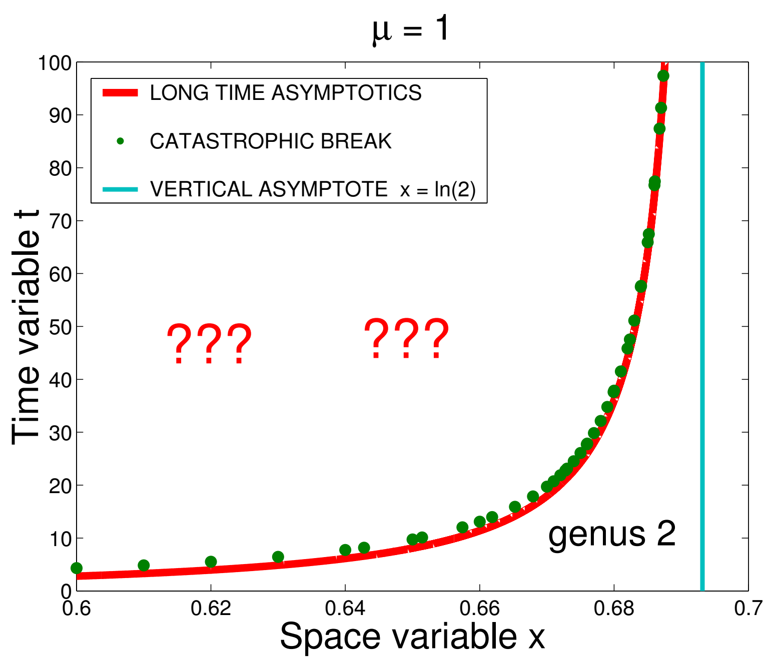
Long-time branchpoint behavior



$\alpha_0,\ \alpha_4$ approach the real axis at $\pm\frac{\mu}{2}$ exponentially fast. The distance of α_2 from the real axis goes like $t^{-1/2}$

Breakdown of method in space time





What happens beyond? What about previously?

Semiclassical Focusing NLS limit on the line, IVP

- Zakharov, Shabhat: Integration of NLS through a Lax Pair, 1971

- Satsuma, Yajima: Scattering data for $\mu = 0$, 1975

- Deift, Zhou: Steepest descent for RHPs 1990

- Deift, V., Zhou: g-function mechanism for steepest descent RHPs

- Miller, Kamvissis: Numerics reveal structure, 1998

- Tovbis, V.: Scattering data for $\mu > 0$, 2000

- Ceniseros, Tian: Numerics, 2002

- Kamvissis, Ken McLaughlin, Miller, Steepest descent analysis of pure soliton case, 2003

- Tovbis, V. , Zhou, Proof of solvability and construction of the semiclassical focusing NLS limit (global in time in pure radiation data, past first break for mixed radiation/soliton data), 2004

- Tovbis, V., Zhou, Long time semiclassical focusing NLS asymptotics, 2006

- Buckingham, V, Shock problem

- Tovbis, V., Determinant form of modulation equations, 2008

- Lyng, Miller, Mechanism for higher break ($\mu = 0$), 2007

- Bertola, Tovbis, Analysis of first break, 2013

- Belov, V., Long time behaviour of the second breaking curve

Directions/ Connections

- Connection with Orthogonal Polynomials and Random matrices

- Fundamental role of theta/ tau functions

- Higher NLS Breaks

- Nearly integrable systems

THANK YOU