# AUTOMORPHISMS OF FERMAT LIKE VARIETIES.

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## 1. Introduction

H. W. Leopoldt [12](in characteristic $p$), and P. Tzermias [16] (in characteristic zero), studied the automorphism group of the Fermat curves given as the zero locus of the homogeneous polynomial $X_0^n + X_1^n + X_2^n = 0$, where $n > 3$ and the characteristic $p$ does not divide $n$. The author [11] generalized the above result by studying the group of automorphisms of a projective non-singular model of the affine curves $1 + X_1^m + X_2^n = 0$, for $n \neq m$.

The aim of this paper is to study the group of automorphisms of similar algebraic sets in higher dimensions. By the group of automorphisms $\mathrm{Aut}(X)$ of the complete projective variety $X \subset \mathbb{P}^k$ we mean the group of biregular transformations of $X$. All varieties are defined over an algebraically closed field $k$ of characteristic $p \geq 0$.

For a strict complete intersection $X$ in $\mathbb{P}^r$ of dimension $\geq 3$ the study of the automorphism group is reduced to the study of the linear automorphisms $Lin(X)$, i.e. to automorphisms of the ambient space $\mathbb{P}^r$ that leave $X$ invariant. This is a known theorem based on the generalization of the Lefschetz theorem due to A.Grothendick and P. Deligne.

T. Shioda [15] using this method, was able to compute the group $Aut(X)$ of automorphisms for the Fermat hypersurfaces (also for the Fermat curves) given by equations

$$X_0^n + X_1^n + \cdots + X_r^n = 0,$$

where the characteristic $p$ does not divide $n$, and $n \geq 3$. It is clear that the group of automorphisms of the above equations contains the semidirect product $\mathbb{Z}_n^r \rtimes S_{r+1}$ of the abelian group $\mathbb{Z}_n^r$ and the symmetric group $S_{r+1}$. This is the whole automorphism group if $n-1$ is not a power of the characteristic. If $n-1 = p^t$ is a power of the characteristic then the automorphism group is the projective unitary group $PGU(r+1, q^{2h})$.

In this paper we simplify the computations of Shioda, and we generalize his results by studying the automorphism group of a hypersurface defined as the zero locus of a hermitian form

$$\sum_{\kappa,\lambda} X_\kappa a_{\kappa\lambda} X_\lambda^q = 0,$$

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where $q$ is a power of the characteristic and $(a_{ij})$ is an $(r+1) \times (r+1)$ matrix with ellements in $k$. We also study the case of different exponents, i.e. automorphisms of the projective closure of affine hypersurfaces of the form

$$\sum_{i=0}^{r} X_i^{n_i} + 1 = 0.$$

Namely we prove the following

**Theorem 1.1.** *Let $n = n_1 = ... = n_t > n_{t+1} \geq ... \geq n_r > 1$ be a decreasing sequence such that $n_i$ is constant for $t_k \leq i < t_{k+1}$, $r - t > 1$ and the characteristic $p$ does not divide $n_i$ for all $i$. Let $X$ be the projective hypersurface given as the zero locus of the homogeneous irreducible polynomial*

$$\sum_{i=1}^{r} X_i^{n_i} X_0^{n-n_i} + X_0^n.$$

*Denote by $d_k = t_{k+1} - t_k$. The group of automorphisms $G$ of $X$ is given by a direct sum*

$$G := \bigoplus_{k=1}^{s} G_k,$$

*where*

$$G_k \cong \begin{cases} 1 & \text{if } d_k = 1 \\ \mathbb{Z}_{n_k} & \text{if } d_k = 2 \\ \mathbb{Z}_{n_k} \rtimes S_{d_k} & \text{if } d_k > 2 \text{ and } n_k - 1 \text{ is not a } p-\text{power} \\ PGU(d_k, p^{2h_k}) & \text{if } d_k > 2 \text{ and } n_k - 1 = p^{h_k} \end{cases}$$

We also study automorphisms of intersections of Fermat hypersurfaces and prove the following theorems:

**Theorem 1.2.** *Let $n, m$ be integers not divisible by the characteristic $p$ of the algebraic closed field $k$, and let $r > 4$ or $n + m \neq 5$ for $r = 4$. The group of automorphism $Aut(X)$ of the variety $X$ defined by the homogeneous ideal $\langle \sum x_i^m, \sum x_i^n \rangle$ is the intersection of the group of automorphisms of the hypersurfaces $\sum x_i^m = 0$ and $\sum x_i^n = 0$ i.e.*

$$Aut(X) = Aut(V(\sum x_i^m = 0)) \cap Aut(V(\sum x_i^n = 0)).$$

**Theorem 1.3.** *Let $X$ be the algebraic set given as intersection of the weighted Fermat hypersurfaces*

$$X_1 = V(x_0^{q+1} + ... + x_r^{q+1}), X_2 = V(c_0 x_0^{q+1} + ... + c_r x_r^{q+1}),$$

*where $q$ is a power of the characteristic and $c_i \neq c_j$ for $i \neq j$. Let $G$ be the automorphism group of $X$ and $A := \mathbb{Z}_{q+1}^r$ be the subgroup of $G$ given*

*by* $\{x_i \mapsto \zeta^{a_i} x_i\}$ *where* $\zeta$ *is an* $q+1$ *root of unity and* $a_i$ *runs over the set* $\{0, ..., q\}$. *The group* $A$ *is a normal subgroup of* $G$ *and* $G$ *is given as an extension of groups*

$$1 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 1$$

*The group* $H$ *acts on the set* $\{c_0, ..., c_r\}$ *as a group of linear fractional trasnformations and is one of the following groups* $\mathbb{Z}_s, \mathbb{Z}_p^t, \mathbb{Z}_{p_1}^{t_1} \rtimes \mathbb{Z}_n$ *where* $s \mid r+1$ *or* $s \mid r$, $pt = r$, $p_1 t_1 n \mid r+1$ *or* $p_1 t_1 n \mid r$. *Moreover* $G$ *is a subgroup of* $Aut(X_1) = PGU(r+1, q^2)$.

## 2. Automorphisms Of Complete Intersections

In this section we reduce the study of the automorphism group of a strict complete intersection $X \subset \mathbb{P}^r$ to the study of linear automorphisms of $X$ i.e. automorphisms of the ambient space $\mathbb{P}^r$ that leave $X$ invariant. This theorem is proved in the literature for hypersurfaces using the Grothendick-Deligne version of Lefschetz theorem [2, par. 16]. For the sake of completeness we present a proof for the case of strict complete intersections which essentially follows the hypersurface proof.

**Proposition 2.1.** *Let* $i : X \to \mathbb{P}^r$ *be a closed subvariety of the projective space* $\mathbb{P}^r$, *such that the map* $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \xrightarrow{i^*} H^0(X, \mathcal{O}_X(1))$ *is an isomorphism. If* $\phi$ *is an automorphism of* $X$ *preserving* $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r}(1)$ *then* $\phi$ *can be extended to an automorphism of* $\mathbb{P}^r$.

*Proof.* The homogeneous coordinate ring $S$ of $\mathbb{P}^r$ is given as a direct sum [8, ex. 5.14a p.126]

$$S = \bigoplus_{n \geq 0} H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))$$

and $S$ is generated by $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ as a $k$-algebra (The $k$-vector space $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))$ can be identified with the vector space of homogeneous polynomials of degree $n$). Every $k$ linear automorphism of $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ can be extended to an automorphism of $S$. Every automorphism $\phi$ of $X$ that preserves $\mathcal{O}_X(1)$ induces a linear map acting on $H^0(X, \mathcal{O}_X(1))$ and since the map

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \xrightarrow{i^*} H^0(X, \mathcal{O}_X(1)),$$

is an isomorphism, $\phi$ acts on $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ as well. □

A complete intersection $X$ in $\mathbb{P}^r$ is a projective algebraic variety whose homogeneous ideal $I$, is generated by $\rho = codim(X, \mathbb{P}^r)$ polynomials.

**Theorem 2.2.** *Let* $X$ *be a complete intersection in* $\mathbb{P}^r$, *of dimension* $\geq 2$. *Then the group* $Pic(X)$ *is torsion free. If moreover* $\dim(X) \geq 3$ *then the group* $Pic(X)$ *is the free group generated by the class of* $\mathcal{O}_X(1)$.

*Proof.* [1, Th. II.1.8] and [7, Exp. XII Corolarry 3.7]. □

Theorem 2.2 can also be expressed in the following form:

**Corollary 2.3.** *Let $X$ be a complete intersection in $\mathbb{P}^r$ of dimension $\geq 3$. Every Cartier divisor on $X$ comes from an intersection of $X$ by a hypersurface of $\mathbb{P}^r$. In particular the semigroup of effective divisors on $X$ is generated by the linear systems of intersections by hyperplanes.*

*Proof.* Let $i : X \to \mathbb{P}^r$ denote the inclusion map. By definition $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r}(1)$. Since $Pic(X)$ is generated by $\mathcal{O}_X(1)$ the result follows. □

**Corollary 2.4.** *Let $X$ be a complete intersection of $\dim X \geq 3$ or a non-singular complete intersection of $\dim(X) = 2$ such that $\omega_X$ is not the identity in $Pic(X)$. Then for every automorphism $\phi$ we have that $\phi^*(\mathcal{O}_X(1)) = \mathcal{O}_X(1)$. In particular if the surface $X$ is non-singular, given by the intersection of two Fermat hypersurfaces $X_0^n + ... + X_4^n = 0$ and $X_0^m + ... + X_4^m = 0$ such that $n + m - 5 \neq 0$ then $\phi^*(\mathcal{O}_X(1)) = \mathcal{O}_X(1)$ for every automorphism $X$.*

*Proof.* If $\dim X > 3$ then by corollary 2.3 we have that the linear system $L_1$ of intersections with hyperplanes of $\mathbb{P}^r$ is complete and it is the unique base of the additive semigroup of equivalent positive divisors. Therefore $L_1$ is invariant under the action of automorphisms. (see also [13]) In the second case, since $X$ is a nonsingular intersection in $\mathbb{P}^r$, one can compute [8, Exer. 8.4 chap. II], that the canonical invertible sheaf $\omega_X := \wedge^{\dim X}\Omega_{X/k} = \mathcal{O}_X(n + m - r - 1)$.The desired result follows since $\phi^*$ preserves $\omega_X$ and $Pic(X)$ is torsion free. □

We have proved the following

**Proposition 2.5.** *If the variety $X$ given by the intersection of two Fermat hypersurfaces $X_0^n + ... + X_r^n = 0$ and $X_0^m + ... + X_r^m = 0$, is a complete intersection and $\dim X \geq 3$ then $Aut(X) = Lin(X)$. The same result holds for $\dim(X) = 2$, i.e. when $r = 4$, under the additional hypotheses, that $X$ is non-singular and that $n - m - r - 1 \neq 0$.*

## 3. Linear Automorphisms

Let $X \subseteq \mathbb{P}^r$ be a projective algebraic variety which is the zero locus of a homogeneous ideal $I$, such that there is a homogeneous basis $\{f_a\}$ of the ideal $I$ consisting of polynomials $f_a$ of degree prime to the characteristic $p$. In this section we will study the linear automorphisms of $X$, i.e. automorphisms of the form

$$\sigma(X_i) = \sum_j a_{ij}X_j, \quad (a_{ij}) \in PGL(r+1, k)$$

such that

$$\forall f \in I, \quad f(\sigma(X_0), ..., \sigma(X_r)) \in I.$$

In general, this is a complicated computation but it can be done in several cases. For example T.Shioda [15] uses this method in order to study the

automorphism group of Fermat hypersurfaces, given as the zero locus of polynomials of the form $\sum_i X_i^n$.

In order to simplify the presentation we will assume that the ideal $I$is generated by $t = 1, 2$ homogeneous polynomials. We have

$$\sigma(f_i) = \sum a_k(\sigma) f_k$$

where $a_k(\sigma)$ are homogeneous polynomials of degree $n - n_k$, where $n_k$ is the degree of $f_k$.

3.1. **The case of hypesurfaces** $(t = 1)$**.** In this case the ideal $I$ is generated by a single homogeneous polynomial $f$ of degree $n$ prime to the characteristic $p$. For a linear automorphism $\sigma$, represented by a matrix $A$, we have

$$\sigma(f) = c(\sigma) f, \tag{3.1}$$

where $c$ is a character of the automorphism group.

Assume for a while that $X$ is a nonsingular hypersurface. We consider the dual variety $X^*$ of the hypersurface. For nonsingular hypersurfaces, that are not hyperplanes, it is known that $X^*$ is also a hypersurface [4, 7.2, p.58 Zak Theorem][9], and every automorphism $\sigma$ of $X$ induces an automorphism $\sigma^*$ of $X^*$. Since $X^*$ is a hypersurface of the same dimension as $X$ $\sigma^*$ is also linear. We will give a direct proof for the following proposition not based on projective duality.

**Proposition 3.1.** *Let $X$ be an irreducible projective hypersurface of dimension $\geq 3$. Let $f(X_0, ..., X_r)$ be the defining polynomial of $X$. Every automorphism of $X$ induces a linear automorphism on $Y_i := \partial f/\partial X_i$, i.e. $\partial f/\partial X_i(\sigma(P)) = \sum_\nu \lambda_\nu \partial f/\partial X_i(P)$.*

*Proof.* Denote by $\nabla f$ the vector $(\partial f/\partial X_0, ..., \partial f/\partial X_r\ )$. Let $A = (a_{ij})$ be a matrix representing the automorphism $\sigma$ of $Y$. By differentiation of (3.1) we obtain

$$\nabla f|_{\sigma(P)} \cdot A = \nabla(fA)|_P = c(\sigma) \nabla f|_P,$$

for every $P = (X_0 : ...X_r)$. So

$$\nabla f|_{\sigma(P)} = \nabla(fA)|_P = c(\sigma) \nabla f|_P \cdot A^{-1}$$

and the action of $\sigma$ on the partial derivatives is linear as well. □

In order to compute the automorphism group of some interesting examples we will need the following

**Lemma 3.2.** *The binomial coefficient $\binom{n}{k}$ is not divisible by the characteristic $p$, if and only if $k_i \leq n_i$ for all $i$, where $n = \sum n_i p^i$, $k = \sum k_i p^i$ are the $p$-adic expansions of $n$ and $m$.*

*Proof.* [3, p. 352] □

**Example 1.** In this example we simplify Shioda's [15] calculation of the group of automorphisms of the Fermat hypersurfaces $f := \sum_{i=0}^{r} X_i^n$, defined over an algebraically closed field $k$, of characteristic $p \geq 0$, $p \not| n$. The partial derivatives of $f$ are $Y_i := nX_i^{n-1}$. If $r \geq 4$ then every automorphism of the Fermat hypersurfaces is linear. Let $(a_{ij}) \in PGL(r+1, k)$ such an automorphism.

For a point $P = (X_0 : ... : X_r)$ we compute the coefficients of $Y(A(P))$ :

$$Y_i(A(P)) := n(\sum_j a_{ij}X_j)^{n-1} = n\sum_j a_{ij}^{n-1}Y_j(X_0, ..., X_r)+$$

$$+n \sum_{\substack{\nu_0 + ... + \nu_r = n-1 \\ \nu_i < n-1}} \binom{n-1}{\nu_0, ..., \nu_r} a_{i0}^{\nu_0} \cdots a_{ir}^{\nu_r} X_0^{\nu_0} \cdots X_r^{\nu_r}. \tag{3.2}$$

By proposition 3.1 we have that modulo the defining polynomial $f$, which is of degree $n$, the right hand side of (3.2) is linear in $Y_i$, and since no polynomial of degree $n-1$ can be equal to a polynomial of degree $\geq n$, we finally arrive at

$$(\sum_j a_{ij}X_j)^{n-1} = \sum_j a_{ij}^{n-1}X_j^{n-1}, \tag{3.3}$$

so if there are more than two $a_{ij} \neq 0$ in some column then by lemma (3.2) $n-1$ is a power of the characteristic. If $A$ is the matrix $(a_{ij})$ then by $A^{(q)}$ we denote the matrix $(a_{ij}^q)$. Let $q := n-1 = p^h$ then equation (3.3) implies $A^{(q)}A^t = Id$ hence the automorphism group is $PGU(r+1, p^{2h})$.

If, on the other hand, there is only one non-zero element in every column of $(a_{ij})$, then after a permutation $(a_{ij})$ is diagonal, so the automorphism group is isomorphic to $\mathbb{Z}_n^r \rtimes S_{r+1}$.

**Example 2.** Let $A = (a_{ij})$ be an $m \times m$ matrix with ellements in the algebraic closed field $k$. We consider the hypersurface defined by the equation

$$A_q[X] := \sum_{\kappa,\lambda} X_\kappa a_{\kappa\lambda} X_\lambda^q = 0, \tag{3.4}$$

where $q$ is a power of the characteristic. By the theory of Jordan forms, the matrix $A$ can be decomposed after a linear change of coordinates in block diagonal matrices of the form

$$A \sim \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_s \end{pmatrix},$$

where $A_i$ are square matrices of the form $A_i = diag(\lambda_1, ..., \lambda_t)$ or $A_i = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$. This proves that after a linear change of coordinates the hypersurface defined in equation (3.4) can be decomposed as a product of hypersurfaces defind by Fermat polynomials ($A_i$ diagonal), and polynomials of the form

$$\sum_{\kappa=0}^{r-1} X_k(\lambda X_\kappa^q + X_{\kappa+1}^q) + \lambda X_r^{q+1} = 0. \tag{3.5}$$

Hence, $Aut(V(A_q[X] = 0)) = \oplus_i Aut(V(A_{i,q}[X] = 0))$. The automorphism group of Fermat hypersurfaces is studied in the previous example. We are left with the study of non diagonal case. Let $J$ be the non diagonal $r \times r$ matrix $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$, and $f$ be the polynomial defined by equation (3.5). The partial derivatives are given by

$$\frac{\partial f}{\partial X_i} = \begin{cases} \lambda X_i^q + X_{i+1}^q & \text{if i} < r \\ \lambda X_i^q & \text{if } i = r \end{cases}.$$

We observe that after every linear change of coordinates given by an $r \times r$ matrix $B = (b_{ij})$ the partial derivatives change linearly. Moreover, $B$ is an automorphism of $V(f = 0)$ if and only if :

(3.6)

$$(X_0, ...., X_r)B^t(B(\frac{\partial f}{\partial X_0}), ..., B(\frac{\partial f}{\partial X_r})) = (X_0, ..., X_r)B^t JB^{(q)}(X_0^q, ..., X_r^q)^t = J,$$

where $B^{(q)}$ is the matrix $(b_{ij}^q)$. Let $B^*$ be the matrix defined by $B^*J = JB^{(q)}$, i.e. $B^* = JB^{(q)}J^{-1}$. By (3.6) the matrix $B$ satisfies

$$B^tB^* = B^tJB^{(q)}J^{-1} = I \ modZ$$

,where $Z$ denotes the centre of $GL(r+1, k)$. Therefore $Aut(V(f = 0))$ is the subgroup of $PGL(r+1, k)$ consisted of matrices of the form

$$B = \left(J\left(B^{(q)}\right)^{-1}J^{-1}\right)^t.$$

**Example 3.** We consider now the hypersurface $X$ given as the zero locus of the irreducible polynomial $f := \sum_{i=1}^r X_i^{n_i}X_0^{n-n_i} + X_0^n$, defined over an algebraically closed field of characteristic $p \geq 0$. We want $f$ to be irreducible,

so at least one $n_i$ is equal to $n$. Moreover we assume that $p \not| n_i$ for all $i$. For technical reasons we also assume that $r - t > 1$.

We assume that $n_i$ form a decreasing sequence and we consider the set of indices $\{1 = t_0 < t_1 = t < \cdots < t_s\}$ such that $n_i$ is constant for $t_k \leq i < t_{k+1}$, i.e.

$$n = n_1 = ... = n_t > n_{t+1} \geq ... \geq n_r > 1.$$

The partial derivatives $Y_i = \partial_i f$ of $f$ are:

$$Y_0 = \sum_{i=t+1}^{r} (n - n_i) X_i^{n_i} X_0^{n-n_i-1} + n X_0^{n-1}$$

$$Y_i = \begin{cases} n_i X_i^{n_i-1} & \text{if} \quad 0 < i \leq t \\ n_i X_i^{n_i-1} X_0^{n-n_i} & \text{if} \quad t < i \end{cases}.$$

The action of the automorphism $\sigma = (a_{ij})$ is by proposition 3.1 linear on the partial derivatives $Y_i$. For $0 < i \leq t$

$$\sigma(Y_i) = n_i \left( \sum_{j=0}^{r} a_{ij} X_j \right)^{n_i - 1} =$$

$$= n_i \sum_{j=1}^{r} a_{ij}^{n_i-1} X_j^{n_i-1} + n_i \sum_{\substack{\nu_0 + ... + \nu_r = n_i - 1 \\ 0 \leq \nu_i < n_i - 1}} \binom{n_i - 1}{\nu_0, ...., \nu_r} a_{i0}^{\nu_0} \cdots a_{ir}^{\nu_r} X_0^{\nu_0} \cdots X_r^{\nu_r} =$$

$$= \sum \lambda_i Y_i \bmod f.$$

As in the previous example, every summand of the above sum that does not fit in a linear combination of $Y_i$ is zero since no no-zero polynomial of degree $n-1$ belongs to the principal ideal generated by $f$. By comparing coefficients we have $a_{ij} = 0$ for $j > t$ and $0 < i \leq t$.

For $t < i$ we compute

$$\sigma(Y_i) = n_i \left( \sum_{j=0}^{r} a_{ij} X_j \right)^{n_i - 1} \left( \sum_{k=0}^{r} a_{0k} X_k \right)^{n - n_i}.$$

The term $n_i a_{ij}^{n_i-1} a_{0k}^{n-n_i} X_j^{n_i-1} X_k^{n-n_i}$ that appears in the above equation can not be canceled out by linear combinations of $Y_i$, for $k \neq j, k \neq 0, j \neq 0$. This proves that $a_{ij} a_{0k} = 0$ for $k \neq j, k \neq 0, j \neq 0$. If there is a $k \neq 0$ such that $a_{0k} \neq 0$ then $a_{ij} = 0$ for $t < i$ and $j \neq 0, k$for $i \neq 0$. Therefore for all $i > t$ we have

$$\sigma(X_i) = a_{i0} X_0 + a_{ik} X_k \text{ for } i > t. \tag{3.7}$$

Since $\sigma$ is an invertible linear transformation , the set $S := \{\sigma(X_t), ..., \sigma(X_r)\}$ is of dimension $r-t$. On the other hand (3.7) implies that $S$ is at most of dimension two, a contradiction if $r-t > 2$. We consider now the case $r-t = 2$. Suppose that there is one more $i \neq k$ such that $a_{0l} \neq 0$, then $a_{ij} = 0$ for $t < i$ and $j \neq 0$. In this case

$$\sigma(X_i) = a_{i0}X_0 \text{ for } i > t.$$

and we observe that the set $S := \{\sigma(X_t), ..., \sigma(X_r)\}$ is maped on an one dimensional set, a contradiction. So if $r - t = 2$ then

$$\sigma(X_0) = a_{00}X_0 + a_{0k}X_k.$$

Let $S_1 := S \cup \{X_0\}$. The set $S_1$ is of dimension $r-t+1 > 2$ and the same argument as before implies the desired contradiction.

We have proved so far that if $r - t > 1$ then $a_{0k} = 0$ for all $k \neq 0$, i.e.

$$\sigma(X_0) = a_{00}X_0.$$

Let us write the polynomial $f$ as a sum of two polynomials $f = f_1 + f_2$, where $f_1 := \sum_{i=1}^{t} X_i^n$ and $f_2 := X_0^{n-n_{t+1}} \left( \sum_{i=t+1}^{t_2} X_i^{n_t} + \sum_{i=t_2+1}^{r} X_i^{n_i} X_0^{n_t - n_i} + X_0^{n_t} \right)$. For the arbitrary automorphism $\sigma$ we have

$$\sigma(f) = cf \Rightarrow \sigma(f_1) - cf_1 = cf_2 - \sigma(f_2). \tag{3.8}$$

Since $a_{ij} = 0$ for $i < t$ and $j > t$ the polynomial $\sigma(f_1) - f_2 \in k[X_1, ..., X_t]$. By comparing the both hand sides of (3.8) and using the fact that $\sigma(X_0) = X_0$ we have that $cf_2 - \sigma(f_2) \in k[X_0, X_{t+1}, ..., X_r]$, and finaly we arrive at

$$\sigma(f_1) = cf_1 \text{ and } \sigma(f_2) = cf_2.$$

The polynomial $f_2/X_0^{n-n_{t+1}}$ is of the same form with the original polynomial $f_1$, so proceeding inductively we have that the matrix representaion $(a_{ij})$ of $\sigma$ is a block diagonal matrix of the form:

$$\begin{pmatrix} a_{00} & 0 & \cdots & 0 \\ 0 & A_1 & 0 & \\ \vdots & & A_i & \\ 0 & & & \ddots \end{pmatrix},$$

where $A_k$ are $(t_{k+1} - t_k) \times (t_{k+1} - t_k)$ square invertible matrices. Moreover, the study of automorphisms of Fermat hypersurfaces, gives us that if in a column of the block matrix $A_k$ there are more than one non zero elements then $n_k - 1$ is a power of the characteristic. Let $d_k := i_{k+1} - i_k$. There is a direct sum decomposition of the group of automorphisms of $X$ :

$$Aut(X) := \bigoplus_{k=1}^{s} G_k,$$

where

$$G_k \cong \begin{cases} 1 & \text{if } d_k = 1 \\ \mathbb{Z}_{n_k} & \text{if } d_k = 2 \\ \mathbb{Z}_{n_k} \rtimes S_{d_k} & \text{if } d_k > 2 \text{ and } n_k - 1 \text{ is not a } p-\text{power} \\ PGU(d_k, p^{2h_k}) & \text{if } d_k > 2 \text{ and } n_k - 1 = p^{h_k} \end{cases}$$

*Remark* 3.3. With respect to the theory of projective duality it is nice to point out that in all hypersurfaces studied above there are extra automorphisms in characteristic $p$ exactly when $X$ is not reflexive. For definitions of the above terms we refer to the literature. [5][9][10]

3.2. **The case $t = 2$.** Let $X \subseteq \mathbb{P}^r$ be a strict complete intersection corresponding to the ideal $I$ which is generated by two homogeneous polynomials $f_1$ and $f_2$ of degrees $n$ and $m$ respectively. Every linear automorphism $\sigma$ defines polynomials $b_{ij}(\sigma)$ of degrees $deg f_i - deg f_j$, such that

$$\sigma(f_i) = \sum_{j=1}^{2} b_{ij}(\sigma) f_j$$

(by assumption polynomials of negative degree are zero). Let $PGL_2(k[x])$ be the group of invertible $2 \times 2$ matrices modulo diagonal matrices with coefficients in $k$. There is a group morphism

$$\rho : Aut(X) \longrightarrow PGL(2, k[x])$$

$$\sigma \longmapsto (b_{ij}(\sigma)) .$$

The automorphism $\sigma$ can be extended to an automorphism of the hypersurfaces $V(f_1)$ and $V(f_2)$ if and only if $(b_{ij}(\sigma))$ is a diagonal matrix. In case $n = m$ $\rho$ defines a representation of $Aut(X)$ in $PGL(2, k)$ and if $n > m$ then $\rho(\sigma)$ is an upper triangular matrix of the form

$$\rho(\sigma) = \begin{pmatrix} b_{11}(\sigma) & 0 \\ b_{21}(\sigma) & b_{22}(\sigma) \end{pmatrix}, \tag{3.9}$$

where $b_{ii}(\sigma) \in k$ and $b_{21}(\sigma)$ is a polynomial of degree $n - m$. Moreover, in this case every automorphism can be extended to an automorphism of the hypersurface $V(f_1)$. As in the case of hypersurfaces the automorphism $\sigma$ preserves the normal bundle $N_P X = T_P \mathbb{P}^r / T_P X$ and acts on the base of $N_P X = \langle \nabla f_1|_P, \nabla f_2|_P \rangle_k$ as follows:

$$\nabla f_2|_{\sigma(P)} \cdot \sigma = \nabla b_{21}(\sigma) f_1(P) + b_{22}(\sigma) \nabla f_2(P) + b_{21}(\sigma) \nabla f_1(P). \tag{3.10}$$

*Remark* 3.4. In case $n > m$ every automorphism of $X$ is the restriction of an automorphism of $V(f_1)$. We consider $X$ as a divisor of $V(f_1)$ and $Aut(X)$ can be interpreted as the decomposition group of $X$ in the cover

$$V(f_1) \longrightarrow V(f_1)^{Aut(V(f_1))}.$$

Hence, $Aut(X)$ is the identity unless $X$ is ramified in the above cover.

*Remark* 3.5. If $degf_1 = degf_2$ then there is a base $f_1', f_2'$ generating the ideal $I$ such that

$$Aut(X) = Aut(V(f_1')) \cap Aut(V(f_2'))$$

if and only if the representation

$$\rho : Aut(X) \longrightarrow PGL(2,k)$$

can be decomposed as a direct sum of one dimensional characters.

3.2.1. *Complete Intersections of Fermat hypersurfaces.* Denote by $f_1 = X_0^m + ... + X_r^m$, $f_2 = X_0^n + ... + X_r^n$ two Fermat polynomials, $p \not| n, m$. We will prove that the ideal $I = \langle f_1, f_2 \rangle$ is prime. This implies that $I$ is radical so the corresponding variety $X$ is a strict complete intersection.

For this we observe first that $\{f_1, f_2\}$ form a regular sequence in the polynomial ring $k[X_0, ..., X_r]$. Indeed, $f_1, f_2$ are irreducible ($r \geq 4$) so if $f_1$ is a zero divisor in the quotient ring $k[X_0, ..., X_r]/\langle f_2 \rangle$ then $f_1 \in \langle f_2 \rangle$, a contradiction, since the degree of $f_1$ in $X_0$ is strictly less than the degree of $f_2$ in $X_0$. Since $k[X_0, ..., X_r]$ is Cohen-Macaulay, the codimension of $I$ is two.

**Proposition 3.6.** *The ideal $I$ is prime.*

Proof: Since $\{f_1, f_2\}$ is a regular sequence in the Cohen-Macaulay ring $k[X_0, ..., X_r]$, proposition 18.13 in [3] implies that $R/I$ is a Cohen-Macaulay ring. We will use now the following

**Theorem 3.7.** *Let $R = k[X_0, ..., X_r]/I$ where $I = (f_1, ..., f_s)$ has codimension $c$. Let $J \subset R$ be the ideal generated by the $c \times c$ minors of the Jacobian matrix $\mathcal{J} = (\partial f_i / \partial x_j)$, taken modulo $I$. Suppose also $R$ is Cohen-Macaulay.*

- *$R$ is reduced if and only if $J$ has codimension $\geq 1$ in $R$.*
- *$R$ is a direct product of normal domains if and only if $J$ has codimension $\geq 2$ in $R$.*

*Notice that since $R$ is a graded ring, if it is a direct product of normal domains, $R$ is a domain, therefore $I$ is prime.*

Proof: [3, Th. 18.15]

*Remark.* In geometric terms the above theorem ensures that if the codimension of the singular locus is big enough, then $I$ is prime. In particular if the algebraic set corresponding to $I$ is non singular, then $codim(J) = dimX$ and the theorem holds provided that $\dim X \geq 2$. This theorem should also be compared to the classical theorem in complex algebraic geometry, which asserts that an algebraic set is irreducible if and only if the complement of the singular locus is connected [6, pag. 21].

The Jacobian matrix in our case is

$$\mathcal{J} = \begin{pmatrix} mX_0^{m-1} & mX_1^{m-1} & \cdots & mX_r^{m-1} \\ nX_0^{n-1} & nX_1^{n-1} & \cdots & nX_r^{n-1} \end{pmatrix},$$

and the $2 \times 2$ minors are of the form:

$$mnX_i^{m-1}X_j^{m-1}\prod(X_i - \zeta X_j),$$

where $\zeta$ runs over the $m-n$ roots of one. Since the characteristic $p$ does not divide $m, n$ the singular locus $X^*$ of $X$ is contained in the intersection of a finite union of lines $L_i$ with $X$. In the algebraic setting,

$$\operatorname{rad}\left(\frac{J+I}{I}\right) = I(V(X^*)) \supset I(\cup_i(L_i \cap X)) = \cap I_i,$$

where $I_i := I(L_i \cap X)$. Therefore,

$$\operatorname{codim}(\operatorname{rad}\left(\frac{J+I}{I}\right)) \geq \operatorname{codim}(\cap I_i) = \min_i(\operatorname{codim}(I_i)) \geq \dim X - 1.$$

Since the ring $k[X_0, ..., X_r]/I$ is Cohen-Macaulay, we have that $\operatorname{codim}\left(\frac{J+I}{I}\right) = \operatorname{depth}\left(\frac{J+I}{I}\right)$, and by [3, Corollary 17.8] $\operatorname{depth}(\operatorname{rad}\left(\frac{J+I}{I}\right)) = \operatorname{depth}\left(\frac{J+I}{I}\right)$. Hence, if $\dim X \geq 2$, $X$ is reduced, and if $\dim X \geq 3$ $X$ is also irreducible. □

According to proposition 2.5 every automorphism of $X$ is linear, i.e. it is induced by a matrix $A = (a_{ij}) \in PGL(r+1, k)$ such that for every $f \in I$, $fA \in I$.

We apply this to $f_1 = X_0^m + ... + X_r^m$, first. The polynomial $f_1 A = \sum_{i=1}^r \left(\sum_{j=0}^r a_{ij}X_j\right)^m$ is an element of the ideal $I$, i.e.

$$f_1 A = b_{11}(A)f_1 + b_{12}(A)f_2,$$

for two suitable polynomials $b_{11}(A), b_{12}(A)$. Since the degree of $f_1 A$ is $m$, we have that $b_{12}(A) = 0$ and $b_{11}$ is a character of $Aut(X)$. This proves that the group $Lin(X)$, of linear automorphisms of $X$, is a subgroup, of the group of automorphisms of the Fermat hypersurface given by $X_0^m + ... + X_r^m = 0$.

If $m$ is not a power of the characteristic the automorphism group $G$ of the Fermat hypersurface $X_0^m + \cdots + X_r^m = 0$ is $\mathbb{Z}_m^r \rtimes S_{r+1}$ and the group $Aut(X)$ is the subgroup of $G$ consisted of elements that keep $f_2$ in $I$, and a simple calculation shows that

$$Aut(X) = Aut(V(f_1)) \cap Aut(V(f_2)) = \mathbb{Z}_{(n,m)}^r \rtimes S_{r+1}.$$

We assume now that $m - 1 = q$ hence $Aut(V(f_1)) = PGU(r+1, q^2)$. The ideal $I$ is generated by $f_1, f_2$ which are both assumed to have degree prime to the characteristic. The normal space at a point $P = (X_0 : ... : X_r)$ is generated by the vectors $Y := (\partial f_1/\partial X_0 : .... : \partial f_1/\partial X_r)$ and $Z :=$

$(\partial f_2/\partial X_0 : ... : \partial f_2/\partial X_r)$. Let $\sigma$ be an automorphism of $X$ represented by the matrix $(a_{ij})$. Using (3.10) we obtain

(3.11)

$$\sum_{i=0}^{r}\left(\sum_{s=0}^{r} a_{is}X_s\right)^{n-1} a_{ik} = nb_{22}(\sigma)X_k^{n-1} + mb_{21}(\sigma)X_k^{m-1} + \frac{\partial b_{21}(\sigma)}{\partial X_k}f_1.$$

By comparing the coefficients of monomials of the form $X_s^i, X_k^j$ for $s \neq k$ in both sides of the above equation we obtain

$$\sum_{i=0}^{r} a_{is}^{n-1}a_{ik}X_s^{n-m-1} = \frac{\partial b_{21}(\sigma)}{\partial X_k} \textbf{ for all } s \neq k.$$

**Case A.** If $n-m-1 > 0$ then $\sum_{i=0}^{r} a_{is}^{n-1}a_{ik}X_s^{n-m-1} = \sum_{i=0}^{r} a_{it}^{n-1}a_{ik}X_t^{n-m-1}$ for $s \neq t, s \neq k, t \neq k$ therefore

$$\frac{\partial b_{21}(\sigma)}{\partial X_k} = 0 \text{ and } \sum_{i=0}^{r} a_{ik}a_{is}^{n-1} = 0 \text{ for } s \neq k.$$

All partial derivatives of the polynomial $b_{21}(\sigma)$ are vanishing, therefore $b_{21}(\sigma)$ is a polynomial of $X_0^h, ..., X_r^h$, where $h$ is a power of the characteristic $p$ and $n - m$ is divisible by $p$. Equation (3.11) can be written as

(3.12)

$$\sum_{i=0}^{r} a_{ik}^{n}X_k^{n-1} + \sum_{i=0}^{r} \sum_{\substack{\nu_0 + ... + \nu_r = n-1 \\ m-1 \leq \nu_i < n-1}} \binom{n-1}{\nu_0, ...., \nu_r} a_{i0}^{\nu_0} \cdots a_{ir}^{\nu_r} a_{ik} X_0^{\nu_0} \cdots X_r^{\nu_r}$$

$$= nb_{22}(\sigma)X_k^{n-1} + b_{21}(\sigma)X_k^{q}$$

We observe that the left hand side of the above equation does not contain any monomial summand not divisible by $X_k^q$. This gives that for any $\nu = 1, ..., n-2$ and $s \neq t, s \neq k, t \neq k$ we have

$$\binom{n-1}{\nu} \sum_{i=0}^{r} a_{is}^{\nu} a_{it}^{n-1-\nu} a_{ik} = 0.$$

We will check the value of the above sum when $k = s$.

**Case A1.** Suppose that there is a $\nu \in \{1, ..., n-2\}$ such that $\binom{n-1}{\nu} \neq 0$. Let $\nu$ be the smallest integer such that $\binom{n-1}{\nu} \neq 0$. Remember that $m - 1 = q \leq \nu$. Suppose that there is a $k = s \neq t$ such that $\sum_{i=0}^{r} a_{is}^{\nu} a_{it}^{n-1-\nu} a_{ik} = \sum_{i=0}^{r} a_{ik}^{\nu+1} a_{it}^{n-1-\nu} \neq 0$. Then $X_k^{\nu} X_t^{n-1-\nu}$ appear as a summand in the left hand side of (3.12) therefore $X_k^{\nu-q} X_t^{n-1-\nu}$ is a summand of $b_{21}(\sigma)$. We consider now equation (3.11) for the partial derivative $\partial_t$ and we obtain the analogous equation:

$$\sigma(\partial_t f_2) = nb_{22}(\sigma)X_t^{n-1} + b_{21}(\sigma)X_t^{q}. \tag{3.13}$$

We observe that $X_k^{\nu-q}X_t^{n-1-\nu+q}$ appears as a summand in the right hand side of (3.13) and the corresponding term in the left hand side must be nonzero hence $\binom{n-1}{\nu-q} \neq 0$ a contradiction. For a fixed $t \neq k$, under the assumption $\binom{n-1}{\nu} \neq 0$, we have proved that

$$\sum_{i=0}^{r} a_{is}^{\nu} a_{it}^{n-1-\nu} a_{ik} = 0$$

for all $s \neq t$ and for all $k$. Since $(a_{ij})$ is an invertible matrix we have $a_{is}^{\nu}a_{it}^{n-1-\nu} = 0$. This gives that for a fixed $i$ there is exactly one $s$ such that $a_{is} \neq 0$, therefore $(a_{ij}) = c_i\delta_{i\tau(j)}$, where $\tau \in S_{r+1}$ is a permutation. The ellements $c_i$ are $(n,m)$-roots of unity hence such an automorphism $(a_{ij})$ belongs to the intersection of $Aut(V(f_1) \cap Aut(V(f_2))$.

**Case A2.** Suppose that $\binom{n-1}{\nu} = 0$ for all $\nu = 1, ..., n-2$. Lemma 3.2 implies that $n-1$ is a power of the characteristic, hence equation (3.11) can be written as

$$\sum_{i=0}^{r} a_{ik}^{n-1} X_k^{n-1} = b_{22}(\sigma)X_k^{n-1} + b_{21}(\sigma)X_k^q,$$

so $b_{21}(\sigma)$ is apolynomial of $X_k$.Doing the same calculation in (3.11) for a different choice of $k$ we arrive at $b_{21}(\sigma) = 0$ so the representation matrix in 3.9 is diagonal, and

$$Aut(X) = Aut(V(f_1)) \cap Aut(V(f_2)).$$

**Case B.** If $n-1-m = 0$ and $m-1 = q$ then $n-1 = q+1$ and (3.11) becomes

$$\sum_{i,s,t} a_{is}^q a_{it} a_{ik} X_s^q X_t = 2b_{22}(\sigma)X_k^{q+1} + b_{21}(\sigma)X_k^q + \frac{\partial b_{21}(\sigma)}{\partial X_k} f_1. \tag{3.14}$$

Assume that $b_{21}(\sigma) \neq 0$. The polynomial $b_{21}(\sigma)$ is of degree 1, therefore $\frac{\partial b_{21}(\sigma)}{\partial X_k}$ is a constant. By comparing the coefficients of monomials in both sides of (3.14) we obtain $(s = t)$

$$\sum_{i=0}^{r} a_{is}^{q+1} a_{ik} = \begin{cases} 2b_{22}(\sigma) + 2\frac{\partial b_{21}(\sigma)}{\partial X_k} & \text{if } s = k \\ \frac{\partial b_{21}(\sigma)}{\partial X_k} & \text{if s} \neq k \end{cases}$$

and $(s \neq t)$

$$\sum_{i=0}^{r} a_{is}^q a_{it} a_{ik} = \begin{cases} \frac{\partial b_{21}(\sigma)}{\partial X_t} & \text{if } s = k \\ 0 & \text{if } s \neq k \end{cases}$$

For symmetry reasons all partial derivatives of $b_{21}(\sigma)$ are equal. Let $(b_{ij})$ be the inverse matrix of $(a_{ij})$. The last equation implies that for $s \neq k$

$$\sum_{k=0}^{r}\sum_{i=0}^{r} a_{is}^q a_{it} a_{ik} b_{kj} = \frac{\partial b_{21}(\sigma)}{\partial X_k} b_{sj}.$$

But $\sum_{k=0}^{r}\sum_{i=0}^{r} a_{is}^q a_{it} a_{ik} b_{kj} = \sum_{i=0}^{r} a_{is}^q a_{it} \sum_{k=0}^{r} a_{ik} b_{kj} = \sum_{i=0}^{r} a_{is}^q a_{it} \delta_{ij} = a_{js}^q a_{jt} = \frac{\partial b_{21}(\sigma)}{\partial X_k} b_{sj}$. Every automorphism of $X$ is an automorphism of $V(f_1)$ as well hence $\sum_{j=0}^{r} a_{js}^q a_{jt} = \delta_{st}$ and $b_{sj} = a_{js}^q$. Therefore

$$\delta_{st} = \sum_{j=0}^{r} a_{js}^q a_{jt} = \frac{\partial b_{21}(\sigma)}{\partial X_k} \left( \sum_{j=0}^{r} a_{js} \right)^q .$$

Since $(a_{js})$ is an invertible matrix there is an $s$ such that $\sum_{j=0}^{r} a_{js} \neq 0$ so for $t \neq s$ $\frac{\partial b_{21}(\sigma)}{\partial X_k} = 0$ and since $b_{21}(\sigma)$ is of degree one, we have that $b_{21}(\sigma) = 0$. This implies the desired result:

$$Aut(X) = Aut(V(f_1)) \cap Aut(V(f_2)).$$

3.3. **Intersection of Fermat hypersurfaces of the same degree.** The Fermat polynomials $\sum x_i^{q+1}$ over a field of characteristic $p$ behave like the quadratic forms $\sum x_i^2$. Indeed, we can define the bilinear form

$$\langle x, y \rangle = \sum_{i.j} x_i a_{ij} y_i^q,$$

where $x = (x_0, ..., x_r)$ and $y = (y_0, ..., y_r)$ and $(a_{ij})$ is a nonzero matrix. Let $F : x \mapsto x^q$ be the Frobenius involution in the finite field $\mathbb{F}_{q^2}$. If $F(a_{ij}) = (a_{ij})$ then $\langle \cdot, \cdot \rangle$ is a hermitian inner product with respect to the Frobenius involution i.e.

$$\langle x, y \rangle = F(\langle y, x \rangle).$$

Moreover, we observe that with respect to the theory of projective duality,the quadratic and the Fermat hypersurfaces are the only nonsingular hypersurfaces such that the dual variety is nonsingular.[14]

Since the field of definition $k$ is assumed to be algebraically closed the intersection of two hypersurfaces of the form

$$\sum_{i,j} x_i a_{ij} x_j^q = 0, \text{ and } \sum_{i.j} x_i b_{ij} x_j^q$$

can be normalized after a (not-necessary linear) change of coordinates to the intersection of the hypersurfaces

$$\sum_{i=0}^{r} x_i^{q+1} = 0 \text{ and } \sum_{i=0}^{r} c_i x_i^{q+1} = 0$$

for suitable $c_i \in k$.

*Remark* 3.8. It is known that the intersection of two quadratic surfaces is an elliptic curve with infinite many automorphisms. It seems to be interesting to study automorphism groups, of intersections of two Fermat hypersurfaces of the form:

$$\sum x_i^{q+1} \text{ and } \sum c_i x_i^{q+1}.$$

Using the Jacobian criterion 3.7 we can prove the following

**Lemma 3.9.** *The intersection of the Fermat hypersurfaces* $V(\sum x_i^{q+1})$ *and* $V(\sum c_i x_i^{q+1})$ *is a strict complete nonsingular intersection if and only if* $c_i \neq c_j$ *for* $i \neq j$.

Let $X(c_0, ..., c_r)$ be the projective variety corresponding to the ideal $I = \langle \sum x_i^{q+1}, \sum c_i x_i^{q+1} \rangle$, $c_i \neq c_j$ for $i \neq j$.

The normal space of $X(c_0, ..., c_r)$ at a point $P = (X_0 : ... : X_r)$ is generated by the vectors $Y := (X_0^q : ... : X_r^q)$ $Z := (c_0 X_0^q : ... c_r X_r^q)$. Let $\sigma$ be a linear automorphism of $X$ represented by a matrix $(a_{ij})$. By comparison of coefficients in (3.10) we obtain

$$\sum_i a_{ik} a_{i\nu}^q = (b_{11} + b_{12} c_k)\delta_{k\nu}$$

$$\sum_i a_{ik} a_{i\nu}^q c_i = (b_{21} + b_{22} c_k)\delta_{k\nu} \tag{3.15}$$

Denote by $e_k = (a_{ik})_{i=0,...,r}$ and $e'_k = (c_i a_{ik})_{i=0,...,r}$. By (3.15) we have

$$\langle e_k, e_\nu \rangle = (b_{11} + b_{12} c_k)\delta_{k\nu} \tag{3.16}$$

$$\langle e'_k, e_\nu \rangle = (b_{21} + b_{22} c_k)\delta_{k\nu}.$$

On the other hand $e'_k = \sum_i \lambda_i e_i$, and (3.16) implies

$$\langle e'_k, e_\nu \rangle = \lambda_i (b_{11} + b_{12} c_k)\delta_{ki},$$

hence $\lambda_i = \frac{b_{21} + b_{22} c_k}{b_{11} + b_{12} c_k}\delta_{ki}$ (notice that if $b_{11} + b_{12} c_k = 0$ then $b_{21} + b_{22} c_k = 0$ hence the $2 \times 2$ matrix $(b_{ij})$ is singular, a contradiction). The above expression for $\lambda_i$ allows us to write

$$e'_k = \frac{b_{21} + b_{22} c_k}{b_{11} + b_{12} c_k} e_k$$

therefore for all $i, k$

$$c_i a_{ik} = \frac{b_{21} + b_{22} c_k}{b_{11} + b_{12} c_k} a_{ik}. \tag{3.17}$$

If, for a fixed $k$, there are more than one (say $i_1, i_2$) such that $a_{i_1 k} \neq 0, a_{i_2 k} \neq 0$ then (3.17) implies that $c_{i_1} = c_{i_2}$, a contradiction. This proves that

$$a_{ij} = \delta_{i,\tau(j)} \mu_i$$

where $\tau$ is an element of the symmetric group $S_{r+1}$ and $\mu_i \in k$. Equation (3.15) implies now that

$$\sum_i a_{ik} a_{i\nu}^q = \sum_i \delta_{i,\tau(k)} \delta_{i,\tau(\nu)}^q \mu_i \mu_i^q \Rightarrow \mu_i^{q+1} = (b_{11} + b_{21} c_k).$$

We have proved so far that $H := Im\rho$ is a subgroup of $S_{r+1}$ acting on $c_i$ by a linear fractional transformation and that $ker\rho = \mathbb{Z}_{q+1}^r$.

Let $(x_{ij}), (y_{ij})$ be the images of two elements $(a_{ij}), (b_{ij}) \in G$ under $\rho$. We write

$$a_{ij} = \mu_i \delta_{i,\sigma(j)}, b_{ij} = \lambda_i \delta_{i,\tau(j)}.$$

Consider the product

$$c_{ij} := \sum_\nu a_{i\nu} b_{\nu j} = \sum_\nu \mu_i \lambda_\nu \delta_{i,\sigma(\nu)} \delta_{\nu,\tau(j)} = \mu_k \lambda_k \delta_{\sigma^{-1}(i)\tau(j)}.$$

Let $k := \sigma^{-1}(i) = \tau(j)$ for suitable $i, j$. We have

$$\mu_k^{q+1} = x_{11} + x_{12} c_k$$

and

$$\lambda_k^{q+1} = y_{11} + y_{12} c_k$$

By computing $(\mu_k \lambda_k)^{q+1}$ in two ways we obtain

$$-x_{12} y_{12} c_k^2 + (x_{12} y_{11} - x_{12} y_{22}) c_k + x_{12} y_{21} = 0. \tag{3.18}$$

For every $k$ there are $i, j$ such that $\sigma^{-1}(i) = \tau(j) = k$ and (3.18) holds for all $c_k$, hence the corresponding quadratic polynomial is identically zero. This implies that

$$x_{12} y_{12} = 0 \text{ and } x_{12}(y_{11} - y_{22}) = 0 \text{ and } x_{12} y_{21} = 0,$$

so $x_{12} = 0$ or $y_{12} = y_{21} = 0$ and $y_{11} = y_{22}$ so $(y_{ij})$ is the identity. Therefore $Aut(X)$ is a subgroup of $Aut(V(X_0^{q+1} + ... + X_r^{q+1})) = PGU(r+1, q^2)$.

Moreover $Im\rho$ is a finite subgroup of $PGL(2, k)$. The finite subgroups of $PGL(2, k)$ in positive characteristic are classified in ([17]) and there are the following possibilities $A_5, A_4, S_4, \mathbb{Z}_n, \mathbb{Z}_p^t, \mathbb{Z}_p^t \rtimes \mathbb{Z}_n, PSL(2, p^i), PGL(2, p^i)$. We have proved that all automorphisms in $Im\rho$ are upper triangular, hence

$$Im\rho \in \{\mathbb{Z}_p^t, \mathbb{Z}_s, \mathbb{Z}_{p_1}^{t_1} \rtimes \mathbb{Z}_n\}. \tag{3.19}$$

Moreover the fixed points of $Im\rho$ are $\{\infty\}, \{0, \infty\}, \{0, \infty\}$respectively. The set of $r+1$ points $\{(1, c_0), ..., (1, c_r)\}$ acted by $Im\rho$ is divided in orbits. This proves that $s \mid r+1$ or $s \mid r$, $pt = r$, $p_1 t_1 n \mid r+1$ or $p_1 t_1 n \mid r$. The proof of theorem 1.3 is now complete.

*Remark* 3.10. As in the the theory of intersections of quadratic forms, the set of isomorphism classes of $X(c_0, ..., c_r)$ are in one to one correspondence with elements $((1 : c_0), ..., (1 : c_r))$ of the configuration space

$$X\{2, r+1\} := (GL(2, k) \backslash M^*(2, r+1) / H_{r+1}) / S_{r+1}$$

where $M^*(2, r+1)$ is the space of $2 \times r+1$ matrices for which no 2-minor vanishes, and $H_n \cong (k^*)^{r+1}$ is the subgroup of $GL(r+1, k)$ consisted

of diagonal matrices. If the $r+1$ points of $\mathbb{P}^1$ are in orbits of one group mentioned in (3.19) then $Im\rho$ is not the identity.

**Example 3.11.** Automorphisms of the curves $X(c_0, c_1, c_2, c_3)$. The set of such curves corresponds to the configuration space $X\{2,4\}$ which is isomorphic to the affine line, and the isomorphism is given in terms of the $J$ invariant defined as

$$J := \frac{\left(D(12)^2D(34)^2 + D(13)^2D(24)^2 + D(14)^2D(23)^2\right)^3}{D(12)^2D(34)^2D(13)^2D(24)^2D(14)^2D(23)^2},$$

where $\mathrm{D}(ij) = c_j - c_i$. By theorem (1.3) the only possibilities for $Im\rho$ are given in the following table:

| $Im\rho$ | $(c_0, c_1, c_2, c_3)$ | $J$ | |
|---|---|---|---|
| $\mathbb{Z}_3$ | $(0, c, \omega c, \omega^2 c)$ | $0$ | $\omega^2 + \omega + 1 = 0$, $chark \neq 3$ |
| $\mathbb{Z}_4$ | $(c, \zeta c, \zeta^2 c, \zeta^3 c)$ | $2\frac{91\zeta^2+37}{\zeta^2-1}$ | $\zeta^3 + \zeta^2 + \zeta + 1 = 0$, $chark \neq 2$ |
| $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $(c, c+1, d, d+1)$ | $0$ | $chark = 2$ |

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