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Rational correspondences between moduli spaces of curves defined by Hurwitz spaces

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ABSTRACT

By associating to a curve C and a g^1_d the so-called trace curve and reduced trace curve we define two rational maps ϕ and $\hat{\phi}$ from the Hurwitz space of admissible covers of genus $g = 2k$ and degree $d = k + 1$ to moduli spaces $\overline{\mathcal{M}}_{g'}$ and $\overline{\mathcal{M}}_{\hat{g}}$. We study the induced map of the divisor class group of $\overline{\mathcal{M}}_{g'}$ and $\overline{\mathcal{M}}_{\hat{g}}$ to the divisor class group of $\overline{\mathcal{M}}_g$.

1. Introduction

Hurwitz spaces of admissible covers give rise to maps and correspondences between moduli spaces of curves. In this paper we study two examples of this. The Hurwitz space $\overline{H}_{g,d}$ in question is the space of admissible covers of even genus $g = 2k$ and degree $d = k + 1$. The general curve of genus $g = 2k$ possesses finitely many linear systems of projective dimension 1 and degree $d = k + 1$. The Hurwitz space $\overline{H}_{g,d}$ is thus a generically finite cover of the moduli space of stable curves $\overline{\mathcal{M}}_g$.

In [6] Farkas constructed for odd $g = 2k + 1$ a rational map $\overline{\mathcal{M}}_g \dashrightarrow \overline{\mathcal{M}}_{g'}$ with $g' = 1 + \binom{2k+2}{k}\frac{k}{k+1}$ by associating to a generic curve C the curve W^1_{k+2} in $\mathrm{Pic}^{k+2}(C)$ and calculated the induced action on the divisor class group. As an application he showed the upper bound $\sigma(g) < 6 + 16/(g - 1)$ for the slope $\sigma(g)$ of the movable cone of $\overline{\mathcal{M}}_g$ for odd genera g.

In this paper we deal with the even genus case $g = 2k$ and use a completely different construction to define a rational map. To a general curve C of genus g together with a g^1_d, say γ, with $d = k+1$ we associate the so-called *trace curve* $T = T_{C,\gamma}$ defined by

$$T_{C,\gamma} = \{(p, q) \in C \times C : \gamma \geq p + q\},$$

the locus of ordered pairs (p, q) contained in the fibers of γ. By extending this definition to a suitable open part of the Hurwitz space we obtain a rational map $\phi : \overline{H}_{g,d} \dashrightarrow \overline{\mathcal{M}}_{g'}$ with $g' = 5k^2 - 4k + 1$ and it fits into a diagram

$$\begin{array}{ccc} \overline{H}_{2k,k+1} & \overset{\phi}{\dashrightarrow} & \overline{\mathcal{M}}_{g'} \\ \downarrow{\scriptstyle p} & & \\ \overline{\mathcal{M}}_{2k} & & \end{array}$$

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Note that the ratio g'/g for the genera of the trace curve and the original curve is much lower than the ratio in the construction of Farkas.

The main body of this paper is devoted to calculating the induced action $p_*\phi^*$ on the divisor class group of $\overline{\mathcal{M}}_{g'}$. The trace curve carries a natural involution. Dividing the trace curve by it we obtain the reduced trace curve. This yields a similar rational map $\hat{\phi} : \overline{H}_{2k,k+1} \dashrightarrow \overline{\mathcal{M}}_{\hat{g}}$ with $\hat{g} = (5k-2)(k-1)/2$ and we calculate the induced map on the divisor class group.

The reduced trace curve has gonality $\leq k(k+1)/2$ and carries a correspondence that gives rise to an endomorphism e of its Jacobian satisfying $(e-1)(e+k-2) = 0$. It is an interesting question to determine further properties of trace curves.

As in the Farkas paper the map $p_*\phi^*$ sends the ample cone of $\overline{\mathcal{M}}_{g'}$ to the movable cone of $\overline{\mathcal{M}}_g$ and we obtain in this way a bound on the movable slope of the form $\sigma(g) < 6 + 20/g$ for g even. But, as we shall show, by viewing the Hurwitz space $\overline{H}_{2k,k+1}$ as a correspondence between $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{0,6k}$, with $\overline{\mathcal{M}}_{0,6k}$ the moduli space of stable $6k$-pointed rational curves, one can obtain the slightly better bound $\sigma(g) < 6 + 18/(g+2)$.

Besides the rational maps ϕ and $\hat{\phi}$ defined by the trace curve and its quotient we also have a rational map χ from $\overline{H}_{2k,k+1}$ to a moduli space of semi-abelian varieties defined by the Prym variety of the trace curve over the reduced trace curve and a variant given by a quotient of the Jacobian of the reduced trace curve. These maps deserve further study.

Maps between moduli spaces, like the Torelli map and the Prym map, can be important tools for a better understanding of moduli spaces. Since the rational maps and correspondences constructed here involve the geometry of algebraic curves in a natural way it is reasonable to expect the same for these correspondences.

Moduli spaces in this paper are viewed as stacks or orbifolds.

2. The trace curve of a g^1_d

Let C be a smooth projective curve of genus g and let γ be a g^1_d, that is, a linear system of degree d and projective dimension 1. To the pair (C, γ) one can associate an algebraic curve, called the *trace curve* and defined by

$$T_\gamma = T_{C,\gamma} := \{(p, q) \in C \times C \, : \gamma \geq p + q\}.$$

Here the notation $\gamma \geq p + q$ means that there is an effective divisor in γ containing the divisor $p + q$. In the following we shall assume that the linear system γ is base point free. The trace curve can have singularities. More precisely we have the following result (see [7, Lemma 5.1]).

Lemma 2.1. *For a base point free γ the trace curve T_γ is smooth except for possible singularities at points where both p and q are ramification points of γ. A ramification point p of order m of γ gives rise to an ordinary singular point (p, p) of order $m-1$. A point $(p, q) \in T_\gamma$ with $p \neq q$ and p and q both simple ramification points is a simple node of T_γ.*

It follows from the above description of the trace curve that if (p, q) is a smooth point of T_γ, then it is a ramification point of the first (resp. second) projection of T_γ to C if and only if q (resp. p) is a ramification point of γ.

We recall the following lemma from [7, Lemma 5.2].

Lemma 2.2. *Let γ be a base point free g^1_d with all branch points simple except one with arbitrary ramification. Then T_γ is irreducible.*

For general (C, γ) the trace curve $T_{C,\gamma}$ is thus a smooth irreducible curve of genus

$$g' = (g-1)(2d-3) + (d-1)^2.$$

Indeed, the class of the line bundle $\mathcal{O}(T_\gamma)$ defined by the trace curve T_γ on $C \times C$ equals $p_1^*L \otimes p_2^*L \otimes \mathcal{O}(-\Delta)$ with p_i ($i = 1, 2$) the two projections, L the line bundle defining γ and Δ the class of the diagonal, as one easily checks by restricting to horizontal and vertical fibers, hence globally on $C \times C$. The homology class of T_γ is then $d(F_1 + F_2) - [\Delta]$ with F_i the fiber of p_i. The adjunction formula implies the formula for the genus g'.

The trace curve T_γ possesses an involution ι induced by interchanging the two factors of $C \times C$. The fixed points of ι are exactly the intersection points of T_γ with the diagonal and these are the points (p, p) with p a ramification point of γ. We define the *reduced trace curve* $\hat{T}_\gamma = \hat{T}_{C,\gamma}$ as the quotient curve T_γ/ι.

We are interested in the case that $g = 2k$ is even and $d = k + 1$. The Brill–Noether number is then zero and the generic curve C of genus $2k$ has only finitely many g^1_d with $d = k + 1$, namely $N = N(k) = \binom{2k}{k+1}/k$, cf. [1, Ch. V, formula (1.2)]. For a generic γ on such a smooth curve the genus of the trace curve T_γ equals

$$g' = 5k^2 - 4k + 1,$$

while the genus of the reduced trace curve $\hat{T}_\gamma$ equals

$$\hat{g} = \frac{(5k-2)(k-1)}{2}.$$

Remark 2.3. Note that by construction the reduced trace curve possesses a morphism of degree $k(k+1)/2$ to $\mathbb{P}^1$ defined by sending $p + q$ to $\gamma(p) = \gamma(q)$. So the gonality is much lower than $[(\hat{g} + 3)/2]$.

Example 2.4. For $k = 2$ the trace curve of a curve C of genus 4 with a g^1_3 has genus 13 while the reduced trace curve has genus 4 and is isomorphic to C by sending $p + q$ to the residual point in the g^1_3.

The construction of the trace curve can be done in families. This defines a morphism $\phi : H_{g,d} \to \mathcal{M}_{g'}$ where $H_{g,d}$ is the Hurwitz scheme of simple covers of the projective line $\mathbb{P}^1$ of degree $d = k+1$ and genus $g = 2k$. Here a simple cover means that the fibers of γ always have at least $d - 1$ points. We thus get correspondences

$$\begin{array}{ccc} H_{2k,k+1} & \xrightarrow{\phi} & \mathcal{M}_{g'} \\ \downarrow{\scriptstyle p} & & \\ \mathcal{M}_{2k} & & \end{array} \qquad\qquad \begin{array}{ccc} H_{2k,k+1} & \xrightarrow{\hat{\phi}} & \mathcal{M}_{\hat{g}} \\ \downarrow{\scriptstyle p} & & \\ \mathcal{M}_{2k} & & \end{array}$$

Example 2.5. Let $k = 3$ and let C be a general curve of genus 6. According to [1, p. 218] the curve is birational to a plane sextic with four nodes. The five g^1_4 are given by the four linear systems obtained by the lines through a node and by the conics through all four nodes. The reduced trace curve $\hat{T}_\gamma$ associated to such a g^1_4 is of genus 13 and carries a fixed point free involution: if $p_1+p_2+p_3+p_4$ is a divisor from the g^1_4 and p_1+p_2 belongs to the reduced trace curve then the corresponding point is $p_3 + p_4$. This involution is fixed point free for general (C, γ). So we get a curve T'_γ of genus 7 as the quotient of the reduced trace curve. This curve is a trigonal curve and the Prym variety of the étale double cover $\hat{T}_\gamma \to T'_\gamma$ is known to be isomorphic to $\mathrm{Jac}(C)$. So up to isogeny $\mathrm{Jac}(\hat{T}_\gamma)$ is a product of $\mathrm{Jac}(T'_\gamma)$ and $\mathrm{Jac}(C)$. Our map $H_{6,4} \to \mathcal{M}_{13}$ factors through a map $H_{6,4} \to \mathcal{M}_7$ and is dominant on the trigonal locus $\mathcal{T}_7$ in $\mathcal{M}_7$. Note that both $\mathcal{M}_6$ (or $H_{6,4}$) and the trigonal locus $\mathcal{T}_7$ have dimension 15. It seems that $H_{6,4} \to \mathcal{T}_7$ is birational.

The reduced trace curve carries a correspondence:

Proposition 2.6. *For (C, γ) in $H_{2k,k+1}$ the reduced trace curve $\hat{T}$ possesses a correspondence that induces an endomorphism e of $\mathrm{Jac}(\hat{T})$ satisfying a quadratic equation $(e - 1)(e + k - 2) = 0$ in $\mathrm{End}(\mathrm{Jac}(\hat{T}))$.*

Proof. This follows from a result of Kanev, cf. [10, Prop. 5.8, p. 265]. The correspondence is given by

$$D := \{(p + q, r + s) \in \hat{T}^2 : \gamma \geq p + q + r + s\}.$$

This induces an endomorphism e of $\mathrm{Jac}(\hat{T})$ that decomposes $\mathrm{Jac}(\hat{T})$; define an abelian subvariety $A = A_e$ of $\mathrm{Jac}(\hat{T})$ as the image of the endomorphism $1 - e$. It follows from the result of Kanev loc. cit. that A is isogenous (even isomorphic) to $\mathrm{Jac}(C)$. $\square$

That $\mathrm{Jac}(\hat{T})$ contains an isogenous image of $\mathrm{Jac}(C)$ can be seen as follows. The embedding $\rho : \hat{T} \to \mathrm{Sym}^2(C)$ induces a map $\rho^* : \mathrm{Pic}^0(\mathrm{Sym}^2(C)) \to \mathrm{Pic}(\hat{T})$. Now we have an isomorphism $\mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\mathrm{Sym}^2(C))$ given by associating to the divisor class $a - b$ the divisor class $C_a - C_b$ with C_p the image of the map $C \to \mathrm{Sym}^2(C)$ that sends q to $p + q$. On the other hand we have a map $z : \mathrm{Pic}(\hat{T}) \to \mathrm{Pic}^0(C)$ by associating to $t_1 - t_2$ with $t_i = p_i + q_i$ the divisor $p_1 + q_1 - p_2 - q_2$, that is the image of $t_1 - t_2$ under $p_{1*}\sigma^*$ with $\sigma : T \to \hat{T}$ the natural map and $p_1 : T \to C$ the projection. The composition of

$$\mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\hat{T}), \qquad a - b \mapsto C_a \cdot \hat{T} - C_b \cdot \hat{T}$$

with $p_{1*}\sigma^*$ is $k - 1$ on $\mathrm{Pic}^0(C)$. Hence $\mathrm{Pic}^0(C)$ maps to an abelian subvariety of $\mathrm{Pic}^0(\hat{T})$ and the quotient is an abelian variety of dimension $\bar{g} = (5k - 1)(k - 2)/2$. We thus find a map

$$\hat{\chi} : H_{2k,k+1} \to \mathcal{A}_{(5k-1)(k-2)/2}, \quad \text{given by } (C, \gamma) \mapsto \mathrm{Jac}(\hat{T}_\gamma)/\mathrm{Jac}(C),$$

where $\mathcal{A}_{\bar{g}}$ denotes a moduli space of polarized abelian varieties of dimension $\bar{g}$.

3. The action of the correspondence on divisors

The Hurwitz space $H_{g,d}$ is a smooth irreducible scheme, cf. [9, Thm. 1.53 & Thm. 1.54], and it is compactified by the space of admissible covers $\overline{H}_{g,d}$. We can view it as a stack or orbifold, but $\overline{H}_{g,d}$ may not be normal. Its normalization is a smooth stack $\tilde{H}_{g,d}$ containing $H_{g,d}$ as an open dense subset. For more on this normalization we refer to [7].

Since $\overline{\mathcal{M}}_{g'}$ (resp. $\overline{\mathcal{M}}_{\hat{g}}$) is a smooth stack and $\tilde{H}_{g,d}$ is smooth the map ϕ viewed as a rational map $\overline{H}_{g,d} \dashrightarrow \overline{\mathcal{M}}_{g'}$ has locus of indeterminacy of codim ≥ 2. We thus get maps

$$\phi^* : \mathrm{Pic}(\overline{\mathcal{M}}_{g'}) \to \mathrm{Pic}(\tilde{H}_{g,d}), \qquad \hat{\phi}^* : \mathrm{Pic}(\overline{\mathcal{M}}_{\hat{g}}) \to \mathrm{Pic}(\tilde{H}_{g,d}).$$

Here with the Picard group we mean the Picard group with coefficients in $\mathbb{Q}$. When $g = 2k$ and $d = k + 1$ the natural map $p : \tilde{H}_{g,d} \to \overline{\mathcal{M}}_g$ is a generically finite map and we studied in [7] the behavior of the induced map $p_* : \mathrm{Pic}(\tilde{H}_{g,d}) \to \mathrm{Pic}(\overline{\mathcal{M}}_g)$.

One of the purposes of this paper is to study the composite map

$$\alpha = p_*\phi^* : \mathrm{Pic}(\overline{\mathcal{M}}_{g'}) \to \mathrm{Pic}(\overline{\mathcal{M}}_g)$$

and the similar map

$$\hat{\alpha} = p_*\hat{\phi}^* : \mathrm{Pic}(\overline{\mathcal{M}}_{\hat{g}}) \to \mathrm{Pic}(\overline{\mathcal{M}}_g)\,.$$

In [7, Prop. 3.1 & Prop. 4.1] we determined the boundary divisors in $\tilde{H}_{g,d}$ which do not map to zero under p_*. These include a divisor E_0 which maps dominantly to Δ_0, divisors $E_{j,c}$ for $1 \le j \le k$ and $0 \le c \le [j/2]$, that map dominantly to Δ_j, and divisors E_2, E_3, each mapping dominantly to a divisor in $\overline{\mathcal{M}}_g$ that intersects $\mathcal{M}_g$. The general point of E_2 and E_3 represents a curve whose stabilization is a smooth genus g curve. To study the map α (resp. $\hat{\alpha}$), it suffices to studying the trace curve (resp. reduced trace curve) for admissible covers in the smooth open substack $\tilde{H}$ of $\tilde{H}_{g,d}$

$$\tilde{H} := H_{g,d} \cup (\cup_{j,c} E_{j,c}) \cup E_0 \cup E_2 \cup E_3.$$

We shall keep the notation $p : \tilde{H} \to \overline{\mathcal{M}}_g$ for the natural map.

4. Extending the trace curve

In order to study divisors on $\tilde{H}$ it suffices to consider one-dimensional families of admissible covers with general member in $H_{g,d}$, and their associated trace curves. Therefore we study in this section the extension of the trace curve over 1-dimensional base curves B in $\tilde{H}$.

Over the Hurwitz scheme $\tilde{H}$ we have a universal curve $\mathcal{C}$. The general cover has $b = 6k$ branch points. The curve $\mathcal{C}$ fits in the following basic diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,6k+1} & \xleftarrow{\ \gamma\ } & \mathcal{C} \\ \varpi \big\downarrow & & \big\downarrow \pi \\ \overline{\mathcal{M}}_{0,6k} & \xleftarrow[\ q\ ]{} & \tilde{H} \end{array} \qquad (1)$$

where q is the map that associates to an admissible cover $C \to P$ the genus 0 curve P together with the $6k$ branch points.

We now assume that B is a 1-dimensional smooth base (disk or the spectrum of a discrete valuation ring). Over B we have the pull back of the universal curve $\mathcal{C}$ and we can restrict the basic diagram (1) to B. We shall define the trace curve $\mathcal{T}$ as the closure of the locus of points (a, b) of $\mathcal{C} \times_B \mathcal{C}$ with $a \neq b$ and a and b in the same fiber of $\gamma : \mathcal{C} \to \overline{\mathcal{M}}_{0,6k+1}$. The fiber of $\mathcal{C} \times_B \mathcal{C}$ over a point $h \in B$ consists of the products of the various components of the curve $\mathcal{C}_h$. The fiber T_h of the trace curve over h lies in the product of components of $\mathcal{C}_h$ which map, by the map γ, to the same rational component of the fiber of the map ϖ over the point $q(h)$. We shall carry out this construction locally. Note that either both a and b are smooth points of the fiber $\mathcal{C}_h$ or both are singular points. We start with the case of smooth points.

Case 1: pairs of smooth points. Assume that $a \neq b$ are smooth points of the curve $\mathcal{C}_h$ with $\gamma(a) = \gamma(b)$. We denote by σ a local coordinate on B, by u a local coordinate on $\overline{\mathcal{M}}_{0,6k+1}$, and by x, y (resp. x', y') local coordinates on $\mathcal{C}$ at a (resp. b) so that π at a (resp. b) is given by $x = \sigma$ (resp. $x' = \sigma$) and the map γ to $\overline{\mathcal{M}}_{0,b+1}$ by $y = u^m$ (resp. by $u = y'$) with $m = 1$ or $m = 2$ depending on whether a is a ramification point. (Since we assume the cover is simple at most one of the smooth points a, b is a ramification point and if so we assume it is a.) Then the equations for the trace curve (as a family over B) around the point (a, b) with $a \neq b$ are given by

$$x = \sigma, \qquad x' = \sigma, \qquad y = (y')^m.$$

For (a, b) with $a = b$ and $m = 2$ the equations of the trace curve are given by

$$x = \sigma, \qquad x' = \sigma, \qquad y + y' = 0.$$

In both cases the corresponding system defines locally a smooth family of curves with smooth central fiber.

Case 2: pairs of singular points. If the point a is a singular (nodal) point of the curve $\mathcal{C}_h$, then any b with $\gamma(a) = \gamma(b)$ is also a singular point of $\mathcal{C}_h$. We carry out now the analysis for pairs of singular points (a, b) which occur for curves representing the generic member of the divisors $E_0,\ E_2,\ E_3$ and $E_{j,c}$. The local equations of those fall in one of the following types:

Type 1: In this case $a \neq b$. The local equation at a of the map $\pi : \mathcal{C} \to B$ is $xy = \sigma^m$ (in the x, y, σ coordinate system as above) and at b it is $x'y' = \sigma^m$ (in the x', y', σ coordinate system). The local equation at a of the map $\gamma : \mathcal{C} \to \overline{\mathcal{M}}_{0,6k+1}$ is of the form $x = u,\ y = v$ and at b it is $x' = u,\ y' = v$. Then the local equations for the trace curve (as a family over B) at the point (a, b) are given by $xy = \sigma^m,\ x'y' = \sigma^m,\ x = x', y = y'$, i.e. by

$$xy = \sigma^m, \qquad x = x', \qquad y = y'.$$

The last two equations define an intersection of hyperplanes and then the first implies that the family has an A_{m-1} singularity at the point (a, b), which we may resolve by inserting a chain of (-2) curves of length $m-1$.

Type 2: In this case $a = b$. The local equation at $a = b$ of the map $\pi : \mathcal{C} \to B$ is $xy = \sigma$, (in the x, y, σ coordinate system). The local equation at $a = b$ of the map $\gamma : \mathcal{C} \to \overline{\mathcal{M}}_{0,6k+1}$ is of the form $x^m = u,\ y^m = v$. Then the local equations for the trace curve (as a family over B) at the point (a, b) are given (in the x, y, x', y', σ coordinate system) by

$$xy = \sigma, \qquad x'y' = \sigma, \qquad \frac{x^m - x'^m}{x - x'} = 0, \qquad \frac{y^m - y'^m}{y - y'} = 0.$$

Note that for $\sigma \neq 0$ the last two equations define the same locus (because of the first two equations). But for $\sigma = 0$ they define the locus of points $(x, 0, x', 0, 0)$ with $(x^m - x'^m)/(x - x') = 0$ (which is the trace curve of the map $x^m = u$ in the (x, x')-plane) plus the locus of points $(0, y, 0, y', 0)$ with $(y^m - y'^m)/(y - y') = 0$ (which is the trace curve of the map $y^m = v$ in the (y, y')-plane). The point $(0, 0, 0, 0, 0)$ is a singular point (for $m \geq 3$) of the family of trace curves. We perform a small blow up (inside the fiber product of curves) by setting: $sx' - tx = 0,\ sy - ty' = 0$. The proper transform of the trace curve by the blow up is given by the equations

$$xy = \sigma, \qquad x'y' = \sigma, \qquad sx' - tx = 0, \qquad sy - ty' = 0, \qquad \frac{s^m - t^m}{s - t} = 0.$$

The last equation gives $s = \omega^i t, i = 1, \ldots, m-1$, with ω a primitive m-th root of unity. Therefore the trace curve intersects the exceptional line at the $m-1$ points $[\omega^i, 1]$. In the neighborhood of this point the trace curve is given by the equations

$$xy = \sigma, \qquad x' = \omega^i x, \qquad y' = \omega^{-i} y, \qquad [s, t] = [\omega^i, 1], \quad i = 1, \ldots, m-1.$$

This defines locally a smooth family with nodal central fiber.

Type 3: In this case $a \neq b$. The local equation at a of the map $\pi : \mathcal{C} \to B$ is $xy = \sigma$, (in the x, y, σ coordinate system) and at b is $x'y' = \sigma$. The local equation at a of the map $\gamma : \mathcal{C} \to \overline{\mathcal{M}}_{0,6k+1}$ is of the form $x^2 = u,\ y^2 = v$ and at b it is $x'^2 = u,\ y'^2 = v$. Then the local equations for the trace curve (as a family over B) at the point (a, b) are given (in the x, y, x', y', σ coordinate system) by

$$xy = \sigma, \qquad x'y' = \sigma, \qquad x^2 - x'^2 = 0, \qquad y^2 - y'^2 = 0.$$

We blow up as before and find that the proper transform of the trace curve by the blow up is given by the equations

$$xy = \sigma, \qquad x'y' = \sigma, \qquad sx' - tx = 0, \qquad sy - ty' = 0, \qquad s^2 - t^2 = 0.$$

The last equation gives $s = \pm t$. Therefore the trace curve intersects the exceptional line at the two points $[\pm 1, 1]$. In the neighborhood of these points the trace curve is given by the equations

$$xy = \sigma, \qquad x' = \pm x, \qquad y' = \pm y, \qquad [s, t] = [\pm 1, 1].$$

This defines locally a smooth family with nodal central fiber.

Type 4: In this case $a \neq b$. The local equation at a of the map $\pi : \mathcal{C} \to B$ is $xy = \sigma$, (in the x, y, σ coordinate system) and at b is $x'y' = \sigma^m$. The local equation at a of the map $\gamma : \mathcal{C} \to \overline{\mathcal{M}}_{0,6k+1}$ is of the form $x^m = u,\ y^m = v$ and at b it is $x' = u,\ y' = v$. Then the local equations for the trace curve (as a family over B) at the point (a, b) are given (in the x, y, x', y', σ coordinate system) by

$$xy = \sigma, \qquad x'y' = \sigma^m, \quad x' = x^m, \quad y' = y^m.$$

This defines locally a smooth family of curves with nodal central fiber.

Conclusion. By performing the small blow-ups at the pairs of points of type 2 and type 3 we created a (singular) nodal model $\mathcal{T}'$ over B. By resolving the singularities (of type A_m) we obtain a smooth model $\tau : \tilde{\mathcal{T}} \to B$ of the trace curve, a nodal family of curves with smooth total space.

5. The geometry of the trace curve

In our study of the divisors in $\tilde{H}$ we shall need to know the shape of the trace curve near a point of the divisors $E_0, E_{j,c}, E_2$ and E_3 in $\tilde{H}$. We may assume that the limit point is a generic point of a component of one of these divisors. The reader can find the description of the generic admissible cover over any of these divisors in our paper [7]. For each of these cases we explicitly carry out the construction done in the preceding section.

In the following Figs. 1–4, on the left we show the fiber $\mathcal{C}_h$ of $\mathcal{C} \to \tilde{H}$ over a generic point h of the boundary components E_0, E_2, E_3 and $E_{j,c}$ respectively. On the right we show the corresponding fiber $\tilde{\mathcal{T}}_h$ of the smooth model of the family of the trace curves constructed as in the preceding section and its first projection $\eta : \tilde{\mathcal{T}}_h \to \mathcal{C}_h$. The fiber $\mathcal{C}_h$ over h maps by the map γ to a stable genus 0 curve with two components intersecting at a point Q. If $Q_1, \ldots, Q_r$ are the preimages of Q with ramification format $\mu = (m_1, \ldots, m_r)$ then the local equations of the map $\gamma : \mathcal{C} \to \overline{\mathcal{M}}_{0,6k+1}$ around the points Q_j are given by $x_j^{m_j} = u$ and $y_j^{m_j} = v$. The local equations of the family $\mathcal{C} \to B$ around the points Q_j are given by $x_j y_j = \sigma^{m(\mu)/m_j}$, where $m(\mu)$ is the least common multiple of $m_1, \ldots, m_r$, see [8, Section 4].

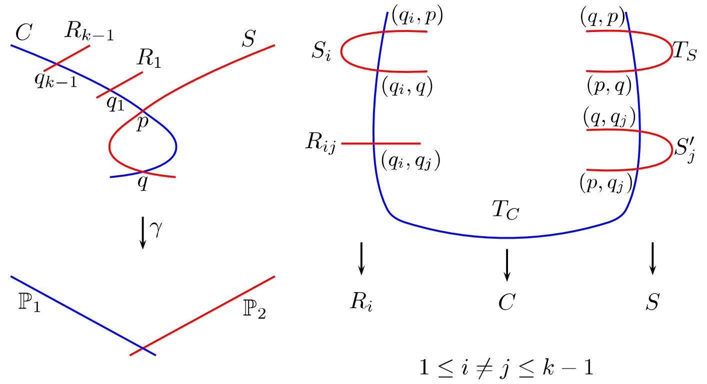


Fig. 1. The fiber of $\mathcal{C}$ over a point of E_0 and the corresponding trace curve.

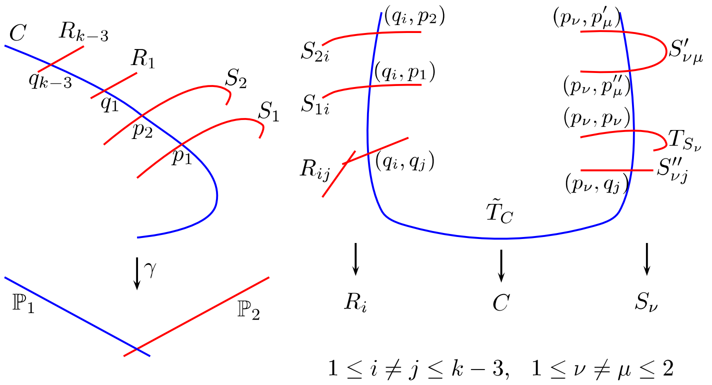


Fig. 2. The fiber of $\mathcal{C}$ over a point of E_2 and the corresponding trace curve.

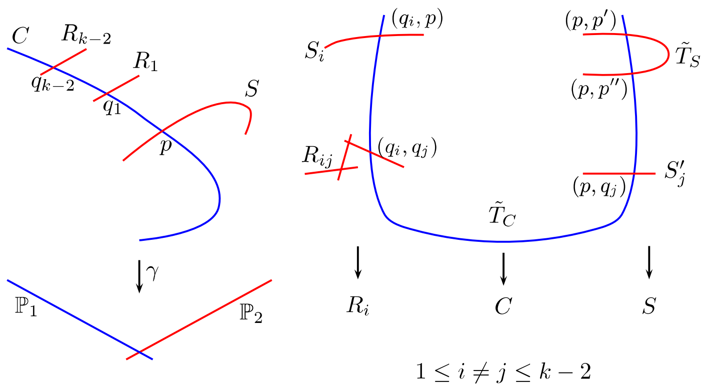


Fig. 3. The fiber of $\mathcal{C}$ over a point of E_3 and the corresponding trace curve.

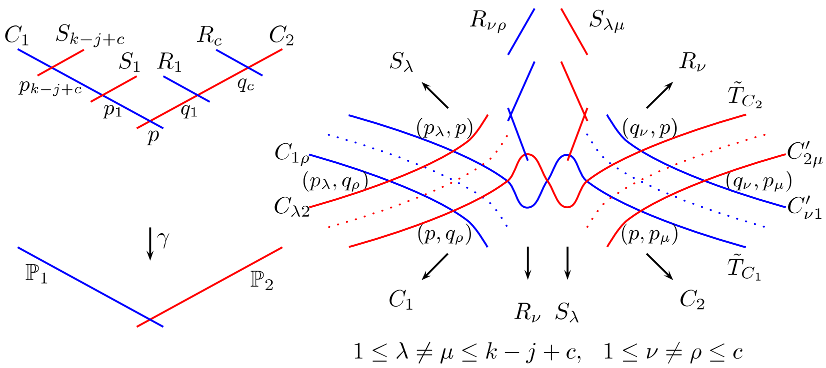


Fig. 4. The fiber of $\mathcal{C}$ over a point of $E_{j,c}$ and the corresponding trace curve.

Fig. 1 corresponds to the case where h is a general point of E_0. The admissible cover on the left is described as follows: it consists of a main component C, which is a curve of genus $2k-1$, and rational curves $R_1, \dots, R_{k-1}$ and S. This maps to a rational curve consisting of two components $\mathbb{P}_1$ and $\mathbb{P}_2$. The map from the curve C to $\mathbb{P}_1$ has degree $k+1$. The components R_i map isomorphically to $\mathbb{P}_2$ and the map from S to $\mathbb{P}_2$ has degree 2. At all the intersection points of the above components, the admissible cover has ramification degree 1 (here with ramification degree we mean the degree of the local map on the components). The rational curve $\mathbb{P}_1$ contains $6k-2$ branch points and $\mathbb{P}_2$ contains 2 branch points.

The local equations around the points $q_1, \dots, q_{k-1}, p, q$ have the form $x = u$, $y = v$, $xy = \sigma$. The trace curve on the right has the following properties:

(1) All the singular pairs (a, b) of points are of type 1 with $m = 1$.
(2) The curves T_C and T_S are the trace curves of the maps $C \to \mathbb{P}_1$ and $S \to \mathbb{P}_2$ respectively. The curves S_i (resp. S'_j) are produced by taking pairs of points from the components R_i and S (resp. S and R_j). The curves R_{ij} are produced by taking pairs of points from the components R_i and R_j.
(3) The curves R_{ij}, S_i, T_S and S'_j are all rational curves.
(4) The map $S_i \to R_i$ is 2:1 and the maps $R_{ij} \to R_i$, and T_s, $S'_j \to S$ are all isomorphisms.

Fig. 2 corresponds to the case where h is a general point of E_2. The admissible cover on the left is described as follows: it consists of a curve C of genus $2k$ and rational curves $R_1, \dots, R_{k-3}$, S_1 and S_2. The map from the curve C to $\mathbb{P}_1$ has degree $k+1$. The components R_i map isomorphically to $\mathbb{P}_2$ and the maps from S_1 and S_2 to $\mathbb{P}_2$ have degree 2. The admissible cover has ramification degree 1 at the points $q_1, \dots, q_{k-3}$ and ramification degree 2 at the points p_1 and p_2. The rational curve $\mathbb{P}_1$ contains $6k-2$ branch points while $\mathbb{P}_2$ contains 2 branch points.

The local equations around the points $q_1, \dots, q_{k-3}$ have the form $x = u$, $y = v$, $xy = \sigma^2$ and around the points p_1, p_2 have the form $x^2 = u$, $y^2 = v$, $xy = \sigma$. The trace curve on the right has the following properties:

(1) The pairs (q_i, q_j) are of type 1, with $m = 2$; the pairs (p_1, p_1) and (p_2, p_2) are of type 2, with $m = 2$; the pairs (p_1, p_2) and (p_2, p_1) are of type 3; the pairs (q_i, p_ν) and (p_ν, q_j) are of type 4, with $m = 2$.
(2) The curve $\tilde{T}_C$ is the normalization of the trace curve of the map $C \to \mathbb{P}_1$. The curves T_{S_ν} are the trace curves of the maps $S_\nu \to \mathbb{P}_2$. The curves S_{i1} (resp. S_{i2}) are produced by taking pairs of points from the components R_i and S_1 (resp. R_i and S_2). The curves $S''_{\nu j}$ are obtained by taking pairs of points from the components S_ν and R_j. The curves $S'_{\nu\mu}$ are obtained by taking pairs of points from the components S_ν and S_μ. The curves R_{ij} are obtained by taking pairs of points from the components R_i and R_j.
(3) The curves S_{1i}, S_{2i}, R_{ij}, $S'_{\nu\mu}$, T_{S_ν} and $S''_{\nu j}$ are all rational curves.
(4) The maps $S_{1i}, S_{2i} \to R_i$ are 2 : 1 and the maps $R_{ij} \to R_i$ and $S'_{\nu\mu}, T_{S_\nu}, S''_{\nu j} \to S_\nu$ are all isomorphisms. The (-2) curve which joins R_{ij} with $\tilde{T}_C$ contracts to the points q_i.

Fig. 3 corresponds to the case where h is a general point of E_3. The admissible cover on the left is described as follows: the curves $R_1, \dots, R_{k-2}$ and S are rational curves. The curve C is of genus $2k$. The components R_i map isomorphically to $\mathbb{P}_2$ and the map from S to $\mathbb{P}_2$ have degree 3. The map from the curve C to $\mathbb{P}_1$ has degree $k+1$. The admissible cover has ramification degree 1 at the points $q_1, \dots, q_{k-2}$ and ramification degree 3 at the point p. The $\mathbb{P}_1$ contains $6k-2$ branch points and the $\mathbb{P}_2$ contains 2 branch points.

The local equations around the points $q_1, \dots, q_{k-2}$ have the form $x = u$, $y = v$, $xy = \sigma^3$ and around the point p have the form $x^3 = u$, $y^3 = v$, $xy = \sigma$. The trace curve on the right has the following properties:

(1) The pairs (q_i, q_j) are of type 1, with $m = 3$; the pair (p, p) is of type 2, with $m = 3$; the pairs (q_i, p) and (p, q_j) are of type 4, with $m = 3$.
(2) The curves $\tilde{T}_C$ and $\tilde{T}_S$ are the normalizations of the trace curves of the maps $C \to \mathbb{P}_1$ and $S \to \mathbb{P}_2$ respectively. The curves S_i (resp. S'_j) are produced by taking pairs of points from the components R_i and S (resp. S and R_i). The curves R_{ij} are produced by taking pairs of points from the components R_i and R_j.
(3) The curves S_i, R_{ij}, $\tilde{T}_S$ and S'_j are all rational curves.
(4) The map $S_i \to R_i$ is 3:1, the map $R_{ij} \to R_i$ is an isomorphism, the map $\tilde{T}_S \to S$ is 2:1 and the map $S'_j \to S$ is an isomorphism. The chain of (-2) curves of length 2 which joins the R_{ij} with $\tilde{T}_C$ contracts to the point q_i.

Fig. 4 corresponds to the case where h is a general point of $E_{j,c}$. The admissible cover on the left is described as follows: The curves $R_1, \dots, R_c$ and $S_1, \dots, S_{k-j+c}$ are rational curves. The curve C_1 has genus $2k-j$ and the curve C_2 has genus j. The curve R_ν (resp. S_λ) maps isomorphically to $\mathbb{P}_1$ (resp. $\mathbb{P}_2$). The map from the curve C_1 to $\mathbb{P}_1$ has degree $k+1-c$ and the map from the curve C_2 to $\mathbb{P}_2$ has degree $j+1-c$. The $\mathbb{P}_1$ contains $6k-3j$ branch points and the $\mathbb{P}_2$ contains $3j$ branch points.

The local equations around the points $p_1, \dots, p_{k-j+c}$ and $q_1, \dots, q_c$ have the form $x = u$, $y = v$, $xy = \sigma^{j+1-2c}$ and around the point p have the form $x^{j+1-2c} = u$, $y^{j+1-2c} = v$, $xy = \sigma$. The trace curve on the right has the following properties:

(1) The pairs (p_λ, q_ρ), (q_ν, p_μ), (p_λ, p_μ) and (q_ν, q_ρ) are of type 1, with $m = j+1-2c$; the pair (p, p) is of type 2, with $m = j+1-2c$; the pairs (p_λ, p), (q_ν, p), (p, p_μ) and (p, q_ρ) are of type 4, with $m = j+1-2c$.

(2) The curves $\tilde{T}_{C_1}$ and $\tilde{T}_{C_2}$ are the normalizations of the trace curves of the maps $C_1 \to \mathbb{P}_1$ and $C_2 \to \mathbb{P}_2$, respectively and they intersect at $j-2c$ points. The curves $C_{1\rho}$ (resp. $C'_{\nu 1}$) are produced by taking pairs of points from the components C_1 and R_ρ (resp. R_ν and C_1). The curves $C'_{2\mu}$ (resp. $C_{\lambda 2}$) are produced by taking pairs of points from the components C_2 and S_μ (resp. S_λ and C_2). The curves $R_{\nu\rho}$ (resp. $S_{\lambda\mu}$) are produced by taking pairs of points from the components R_ν and R_ρ (resp. S_λ and S_μ).

(3) The curves $S_{\lambda\mu}$, $R_{\nu\rho}$ are rational curves. The curves $C_{1\rho}$ and $C'_{\nu 1}$ are isomorphic to C_1 and the curves $C_{\lambda 2}$ and $C'_{2\mu}$ are isomorphic to C_2.

(4) The maps $C_{1\rho} \to C_1, C'_{2\mu} \to C_2, S_{\lambda\mu} \to S_\lambda$ and $R_{\nu\rho} \to R_\nu$ are isomorphisms, the map $C_{\lambda 2} \to S_\lambda$ is $(j+1-c) : 1$ and the map $C'_{\nu 1} \to R_\nu$ is $(k+1-c) : 1$. The vertical chain of (-2) curves of length $j-2c$ which ends to $S_{\lambda\mu}$ (resp. $R_{\nu\rho}$) intersects $\tilde{T}_{C_1}$ (resp. $\tilde{T}_{C_2}$) at the point (p_λ, p_μ) (resp. (q_ν, q_ρ)). The chain of (-2) curves which ends to $S_{\lambda\mu}$ (resp. $R_{\nu\rho}$) contracts to the point p_λ (resp. q_ν).

Example 5.1. If $k = 1$ then the trace curve T for an admissible cover $C \to P$ representing a point of $H_{g,d}$ or a generic point of one of the divisors E_0, E_2, E_3 or $E_{j,c}$ equals the curve C and the reduced trace curve equals the curve P. For $k = 2$ we get as reduced trace curve the curve C.

The reduced trace curve is constructed as the quotient of the trace curve by the action of the involution. This involution extends to the smooth model $\tau : \tilde{\mathcal{T}} \to B$ constructed in the preceding section. Since the action is fixed point free outside the diagonal we need to consider this action only at the points of the diagonal.

In Case 1, pairs of smooth points (a, b) with $a = b$, the trace curve has equations $x = \sigma,\ x' = \sigma,\ y + y' = 0$. The involution acts by interchanging x with x' and y with y'. By taking invariant coordinates we observe that the quotient is smooth at this point. In Case 2, pairs of singular points, the only type which involves points on the diagonal is of type 2. At these points the involution acts by interchanging x with x', y with y' and u with v. When m is even we have a fixed point $[u, v] = [1, -1]$. The local equations at this point are $[u, v] = [1, -1], xy = \sigma, x + x' = 0, y + y' = 0$. By taking invariant coordinates we observe that the quotient has an A_1 singularity. There are $[m/2]$ branches on the reduced trace curve but when m is even we have to resolve the A_1 singularity in the middle by inserting a (-2) curve.

6. Generic finiteness of the trace curve map

We now prove that the rational map $\phi : \overline{H}_{g,d} \dashrightarrow \overline{\mathcal{M}}_{g'}$ is generically finite.

Proposition 6.1. *Let C be a general smooth curve of genus $g \geq 4$ and γ a base point free g^1_d with $g > 2d - 4$. Then the trace curve T_γ determines C uniquely: if C' is another curve with trace curve T' isomorphic to T then C' is isomorphic to C.*

Proof. Suppose that C' is another smooth curve of genus g with a pencil γ' such that T_γ and $T_{\gamma'}$ are isomorphic, say $\psi : T_\gamma \xrightarrow{\sim} T_{\gamma'}$. Let p_1 (resp. p'_1) denote the first projection of T (resp. $T_{\gamma'}$). Then $p_{1*}\,\psi^{-1}\,(p'_1)^*$ defines a homomorphism $j : \mathrm{Jac}(C') \to \mathrm{Jac}(C)$. We claim that j restricted to a suitable translate of C' is birational to its image. Since C is general its Jacobian is simple (see e.g. [12,13]), hence j is either zero or an isogeny. If j is zero this means that for general points x and y in C' the divisor $p_{1*}\psi^{-1}((p'_1)^*x)$ is linearly equivalent to $p_{1*}\psi^{-1}((p'_1)^*y)$ and this gives then a pencil of degree $d-1$ on C; since by assumption the Brill–Noether number $g-(r+1)(g-(d-1)+r) = 2d-g-4$ is negative this does not exist on C. Thus j is an isogeny and for a suitable translate of C' the map j will be birational. This image is then a curve of geometric genus g in $\mathrm{Jac}(C)$ and by a theorem of Bardelli and Pirola for a generic Jacobian of genus $g \geq 4$ all curves of genus g lying on it are birationally equivalent to C (see [3]). □

Example 6.2. Let C be a generic curve of genus 4. It has two g^1_3's, say γ_1 and γ_2. Then the reduced trace curve $\hat{T}$ is isomorphic to C via the map $r \mapsto p + q$ if $p + q + r \sim \gamma$. But the trace curves T_{γ_1} and T_{γ_2} (of genus 13) are in general not isomorphic since the maps $T_{\gamma_1} \to C$ and $T_{\gamma_2} \to C$ are branched at different points. So the map $\phi : H_{4,3} \to \mathcal{M}_{13}$ is of degree $(12)!$, while $\hat{\phi} : H_{4,3} \to \mathcal{M}_4$ coincides with the natural map p.

7. Intersection theory on $\overline{\mathcal{M}}_{0,b}$

We recall some basic facts about the divisor theory of the moduli space of b-pointed genus 0 curves $\overline{\mathcal{M}}_{0,b}$ (see [11] and [8, Section 2]). The boundary of $\overline{\mathcal{M}}_{0,b}$ is the union of irreducible divisors, each of which corresponds to a decomposition of $B = \{1, \ldots, b\}$ as $B = \Lambda \sqcup \Lambda^c$ into two disjoint subsets with $2 \leq \#\Lambda \leq b-2$. We write the corresponding divisor as S_b^Λ modulo the relation $S_b^\Lambda = S_b^{\Lambda^c}$. We sometimes normalize the Λ by requiring that

$$\#(\Lambda \cap \{1, 2, 3\}) \leq 1.$$

The map $\varpi : \overline{\mathcal{M}}_{0,b+1} \to \overline{\mathcal{M}}_{0,b}$ is equipped with b sections $s_j : \overline{\mathcal{M}}_{0,b} \to \overline{\mathcal{M}}_{0,b+1}$ with $j = 1, \ldots, b$.

The boundary divisors of $\overline{\mathcal{M}}_{0,b+1}$ are related to those of $\overline{\mathcal{M}}_{0,b}$ as follows:

$$\varpi^* S_b^\Lambda = S_{b+1}^\Lambda \cup S_{b+1}^{\Lambda \cup \{b+1\}},$$

with $\Lambda \subset \{1, \ldots, b\}$. Note that if $\Lambda \subset \{1, \ldots, b\}$ is normalized, then so are Λ and $\Lambda \cup \{b+1\}$ as subsets of $\{1, \ldots, b+1\}$. So all the boundary components of $\overline{\mathcal{M}}_{0,b+1}$ are coming from $\overline{\mathcal{M}}_{0,b}$ except the components $S_{b+1}^{\{j,b+1\}}$ $(j = 1, \ldots, b)$ that correspond to the image of the b sections s_j.

With $\Lambda \subset \{1, \ldots, b\}$, the generic element of the divisor S_b^{Λ} represents a stable curve with two rational components. Therefore the map $S_{b+1}^{\Lambda} \to S_b^{\Lambda}$ (resp. $S_{b+1}^{\Lambda\cup\{b+1\}} \to S_b^{\Lambda}$) is generically a $\mathbb{P}^1$-fibration. We have

$$S_b^{\Lambda_1} \cap S_b^{\Lambda_2} \neq \emptyset \iff \#(\Lambda_1 \cup \Lambda_2) \in \{\#\Lambda_1, \#\Lambda_2, \#\Lambda_1 + \#\Lambda_2, b\}.$$

Definition 7.1. With $b = 6k$ we define on $\overline{\mathcal{M}}_{0,b}$ for $2 \leq j \leq 3k - 1$ the divisors

$$T_b^j = \sum_{\Lambda \subset B,\, \#\Lambda = j} S_b^{\Lambda} \quad \text{and} \quad T_b^{3k} = \frac{1}{2} \sum_{\Lambda \subset B,\, \#\Lambda = 3k} S_b^{\Lambda}.$$

One easily determines the image of $\tilde{H}$ under the morphism $q : \tilde{H} \to \overline{\mathcal{M}}_{0,6k}$.

Lemma 7.2. *The image of $\tilde{H}$ under q is contained in*

$$\mathcal{M}_{0,b} \cup T_b^2 \cup (\cup_{j=1}^{k} T_b^{3j}).$$

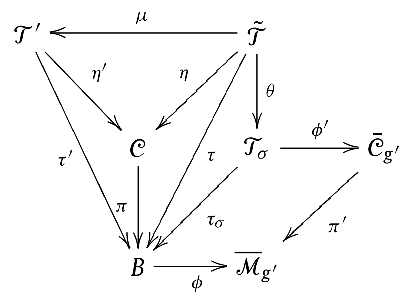
Recall that the b sections s_i define tautological classes ψ_i.

Definition 7.3. We define a divisor class on $\overline{\mathcal{M}}_{0,b}$ by

$$\psi := \sum_{i=1}^{b} \psi_i = \sum_{j=2}^{b/2} \frac{(b-j)j}{b-1} T_b^j.$$

8. Applying Grothendieck–Riemann–Roch

In this section we shall apply the Grothendieck–Riemann–Roch theorem to the family of trace curves over our 1-dimensional base B and the relative dualizing sheaf. We have the diagram



(2)

the notation of which we now explain. The curve $\mathcal{T}'/B$ is the singular trace curve in which we have performed the small blow-ups at the pairs of points of type 2 and type 3. It is a family of nodal curves. The curve $\tilde{\mathcal{T}}$ is the smooth model of $\mathcal{T}'$ and $\theta : \tilde{\mathcal{T}} \to \mathcal{T}_\sigma$ is the stabilization map. The space $\mathcal{T}'$ has singularities of type A_m and the cover $\eta' : \mathcal{T}' \to \mathcal{C}$ is a finite cover of degree k. The space $\tilde{\mathcal{T}}$ contains chains of (-2)-curves which are obtained by resolving the singularities of $\mathcal{T}'$. The map $\eta : \tilde{\mathcal{T}} \to \mathcal{C}$ is a generically finite cover of degree k.

We wish to calculate $\phi^*\lambda_{\pi'}$, where $\lambda_{\pi'}$ is the Hodge class of $\bar{\mathcal{C}}_{g'}$ over $\overline{\mathcal{M}}_{g'}$. Note that $(\phi')^*\lambda_{\pi'} = \lambda_{\tau_\sigma}$ and since $\theta : \tilde{\mathcal{T}} \to \mathcal{T}_\sigma$ is a contraction we have $\lambda_{\tau_\sigma} = \lambda_\tau$, cf. [8, Lemma 3.2], so $\phi^*\lambda_{\pi'} = \lambda_\tau$.

Applying Grothendieck–Riemann–Roch to $\tau : \tilde{\mathcal{T}} \to B$ gives

$$12\lambda_\tau = \tau_*(\omega_\tau^2) + \delta_\tau,$$

where δ_τ is the push forward of the singularity locus of the fibers and ω_τ denotes the relative dualizing sheaf of τ, cf. [15]. In order to carry this out we need to calculate $\tau_*(\omega_\tau^2)$ and δ_τ. We begin with the latter.

Proposition 8.1. *For $k \geq 3$ we have*

$$\delta_\tau = (k^2 + k)\, E_0 + (2k^2 - 10k + 18)\, E_2 + (3k^2 - 13k + 16)\, E_3 + \sum_{j,c} d_{j,c} E_{j,c}$$

with

$$d_{j,c} = \left[\binom{c}{2} + \binom{k-j+c}{2} \right] (j + 1 - 2c) + 2(c+1)(k-j+c) + j.$$

Moreover

$$\delta_\tau = \begin{cases} 2E_0 + E_{1,0} & k = 1 \\ 6E_0 + 2E_3 + 3E_{1,0} + 2E_{2,0} + 6E_{2,1} & k = 2. \end{cases}$$

Proof. This formula is obtained by analyzing the pictures in Section 5. For example, the contribution of E_2 consists of a contribution $2(k-3)(k-4)$ of the R_{ij}, a contribution $2(k-3)$ of the S_{1i} and S_{2i}, a contribution 4 of the $S'_{\nu\mu}$, a contribution 2 of the T_{S_ν}, a contribution of $2(k-3)$ of the $S''_{\nu j}$, giving in total $2(k-3)(k-4)+4(k-3)+6=2k^2-10k+18$. The other coefficients are obtained in a similar way. For example, for the case of $E_{j,c}$ we find a contribution $2\binom{k-j+c}{2}$ from the chains ending with $S_{\lambda\mu}$; similarly $2\binom{c}{2}$ from those ending with $R_{\nu\rho}$, a contribution $2\,c(k-j+c)$ from the intersections $C_{1\rho}\cdot C_{\lambda 2}$ and $C_{\nu 1}\cdot C_{2\mu}$, a contribution $2c+2(k-j+c)$ from the intersections $\tilde{T}_{C_2}\cdot C_{\nu 1}$, $\tilde{T}_{C_2}\cdot C_{\lambda 2}$, $\tilde{T}_{C_1}\cdot C_{2\mu}$, $\tilde{T}_{C_1}\cdot C_{1\rho}$ and finally $j-2c$ from the intersections of $\tilde{T}_{C_1}$ with $\tilde{T}_{C_2}$. $\square$

Remark 8.2. Note that the formula for $k\geq 3$ remains valid if we interpret E_2 and E_3 (resp. E_2) as zero for $k=1$ (resp. for $k=2$).

Now we turn to the calculation of $\tau_*(\omega_\tau^2)$. A first remark is that (in additive notation)

$$\omega_\tau=\eta^*\omega_\pi+R_\eta,$$

with $R_\eta=\mu^*R_{\eta'}$, where $R_{\eta'}$ is the ramification locus of the finite map $\eta':\mathcal{T}'\to\mathcal{C}$. This is the same as the closure of the ramification locus of the map τ' (or τ) restricted to the locus of B which represents smooth curves. Note that $R_{\eta'}$ is supported outside of the singular locus of $\mathcal{T}'$ and so it defines a Cartier divisor on $\mathcal{T}'$. The formula above is derived by applying μ^* to the formula $\omega_{\tau'}=(\eta')^*\omega_\pi+R_{\eta'}$; the latter holds because it holds outside of the singularities of the spaces $\mathcal{T}$ and $\mathcal{C}$. Since $\mu^*\omega_{\tau'}=\omega_\tau$ the formula follows.

We calculate

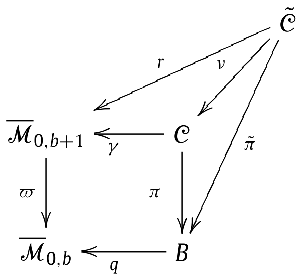
$$\omega_\tau^2=(\eta^*\omega_\pi)^2+2\,\eta^*\omega_\pi\cdot R_\eta+R_\eta^2$$

and observe

$$\tau_*((\eta^*\omega_\pi)^2)=\pi_*\eta_*((\eta^*\omega_\pi)^2)=k\,\pi_*(\omega_\pi^2),$$

because η is a generically finite map of degree k.

Note that $\mathcal{C}$ is a singular space but all the above cycles represent Cartier divisors, so the intersection product makes sense. In the calculation we use the following diagram (3) with $\tilde{\mathcal{C}}$ the smooth model of $\mathcal{C}$ and with $b=6k$ in $\overline{\mathcal{M}}_{0,b}$ and $\overline{\mathcal{M}}_{0,b+1}$.



(3)

If c is a cycle on $\mathcal{C}$ we have $\pi_*c=\tilde{\pi}_*\nu^*c$ because $\nu_*\nu^*c=c$. Since now $\nu^*\omega_\pi=\omega_{\tilde{\pi}}$ we get

$$\tau_*((\eta^*\omega_\pi)^2)=k\,\tilde{\pi}_*\omega_{\tilde{\pi}}^2. \tag{4}$$

We also have

$$\tau_*(\eta^*\omega_\pi\cdot R_\eta)=\pi_*\eta_*(\eta^*\omega_\pi\cdot R_\eta)=\pi_*(\omega_\pi\cdot\eta_*R_\eta).$$

The trace curve is ramified over $\mathcal{C}$ at the points (p,q) in the fiber over $p\in\mathcal{C}$ where q is a ramification point of the map γ. This implies

$$\nu^*\eta_*R_\eta=r^*\hat{S}-2R_r,$$

with $\hat{S}=\sum_{i=1}^{b}S_{b+1}^{\{i,b+1\}}$ the sum of the image of the sections of the map ϖ and R_r is the closure of the ramification of r over the smooth locus. This yields

$$\tau_*(\eta^*\omega_\pi\cdot R_\eta)=\tilde{\pi}_*[\omega_{\tilde{\pi}}\cdot(r^*\hat{S}-2R_r)]. \tag{5}$$

The right hand sides of (4) and (5) can be calculated in a way similar to the calculations in our paper [8]. In order to calculate $\tau_*(R_\eta^2)$ we will use that the map η' (and η) is a simple cover and therefore if $V=\eta_*R_\eta$ is the branch locus of η' (or η) then $\eta^*V=2R_\eta+R'_\eta$ with $R_\eta\cdot R'_\eta=0$. Therefore,

$$\begin{aligned}\tau_*(R_\eta^2)&=\frac{1}{2}\,\tau_*(R_\eta\cdot\eta^*V)=\frac{1}{2}\,\pi_*(\eta_*R_\eta\cdot V)\\&=\frac{1}{2}\,\pi_*(V^2)=\frac{1}{2}\,\tilde{\pi}_*(\nu^*V^2)=\frac{1}{2}\,\tilde{\pi}_*[(r^*\hat{S}-2R_r)^2].\end{aligned} \tag{6}$$

Since we are dealing with the divisors E_0, E_2, E_3 and $E_{j,c}$ only we may adapt the earlier definition of the divisor class ψ on $\overline{\mathcal{M}}_{0,b}$ by setting

$$\psi := \sum_{i=1}^{b} \psi_i = \frac{2(b-2)}{b-1}\, T_b^2 + \sum_{j=1}^{k} \frac{3j(b-3j)}{b-1}\, T_b^{3j}. \tag{7}$$

The following formulas are a consequence of Lemma 3.1 in our [8]:

$$\begin{aligned} q^* T_b^2 &= E_0 + 2E_2 + 3E_3, \\ q^* T_b^{3j} &= \sum_{c=0}^{[j/2]} (j+1-2c) E_{j,c}, \quad j = 1, \ldots, k. \end{aligned} \tag{8}$$

Carrying out the calculations as in [8] for the right hand sides of the Eqs. (4)–(6) the following formulas can be deduced from [8, Lemma 4.2]:

Lemma 8.3. *We have the following equalities*

$$\tilde{\pi}_*(r^*\omega_\varpi \cdot R_r) = q^*\psi, \qquad \tilde{\pi}_*(\omega_{\tilde{\pi}} \cdot R_r) = \frac{1}{2}\, q^*\psi, \qquad \tilde{\pi}_*(R_r^2) = -\frac{1}{2}\, q^*\psi,$$

$$\tilde{\pi}_*(\omega_{\tilde{\pi}}^2) = (3/2) q^*\psi - (k+1)\, q^* \left(T_b^2 + \sum_{j=1}^{k} T_b^{3j} \right).$$

As a check please note that for $k = 1$ the formula for $\tilde{\pi}_*(\omega_{\tilde{\pi}}^2)$ gives $5\kappa_1 = E_0 + 7E_{1,0}$, in agreement with $5\kappa_1 = \delta_0 + 7\delta_1$ (see [14, Eqn (8.5)]).

We will need the following lemma.

Lemma 8.4. *If Γ is a cycle on $\overline{\mathcal{M}}_{0,b+1}$ then $\tilde{\pi}_* r^*\, \Gamma = (k+1)\, q^* \varpi_*\, \Gamma$.*

Proof. Let $P = \overline{\mathcal{M}}_{0,b+1} \times_{\overline{\mathcal{M}}_{0,b}} B$ be the fiber product. The induced map $\xi : \tilde{\mathcal{C}} \to P$ is generically a $(k+1) : 1$ map. Let $r_1 : P \to \overline{\mathcal{M}}_{0,b+1}$ and $r_2 : P \to B$ be the projections. Then $\xi_* \xi^* = k+1$. We have $\tilde{\pi}_* r^*\, \Gamma = \tilde{\pi}_* \xi^* r_1^*\, \Gamma = r_{2*} \xi_* \xi^* r_1^*\, \Gamma = (k+1)\, r_{2*} r_1^*\, \Gamma = (k+1)\, q^* \varpi_*\, \Gamma$. □

Lemma 8.5. *We have*

$$\tilde{\pi}_*(r^*\hat{S}^2) = -(k+1) q^*\psi, \qquad \tilde{\pi}_*(R_r \cdot r^*\hat{S}) = -q^*\psi, \qquad \tilde{\pi}_*(\omega_{\tilde{\pi}} \cdot r^*\hat{S}) = k\, q^*\psi.$$

Proof. By the adjunction formula we have $\varpi_*(\hat{S}^2) = -\psi$. Therefore, $\tilde{\pi}_*(r^*\hat{S}^2) = (k+1) q^* \varpi_*(\hat{S}^2) = -(k+1) q^*\psi$. For the second formula, if we denote the ramification sections of $\tilde{\pi}$ by $\rho_i : B \to \tilde{\mathcal{C}}$ then $r \circ \rho_i = s_i \circ q$ with s_i the sections of ϖ. We have $\tilde{\pi}_*(r^*\hat{S} \cdot R_r) = \sum_i \tilde{\pi}_*(\rho_{i*}\rho_i^* r^*\hat{S}) = \sum_i \tilde{\pi}_* \rho_{i*}(q^* s_i^* \hat{S}) = -\sum_i q^*\psi_i = -q^*\psi$. For the third, we have $\omega_{\tilde{\pi}} = r^*\omega_\varpi - R_r$; so $\tilde{\pi}_*(\omega_{\tilde{\pi}} \cdot r^*\hat{S}) = \tilde{\pi}_*((r^*\omega_\varpi - R_r) \cdot r^*\hat{S})$ and by the second formula this equals $\tilde{\pi}_*(r^*\omega_\varpi \cdot r^*\hat{S}) - q^*\psi$. It thus suffices to show

$$\tilde{\pi}_*(r^*\omega_\varpi \cdot r^*\hat{S}) = (k+1)\, q^*\psi.$$

But we have

$$\varpi_*(\omega_\varpi \cdot \hat{S}) = \varpi_* \left(\sum_i s_{i*} s_i^* \omega_\varpi \right) = \sum_i \varpi_* s_{i*} \psi_i = \sum_i \psi_i = \psi.$$

Now apply Lemma 8.4. □

As a corollary of Eqs. (5) and (6), and Lemmas 8.3 and 8.5 we get the following formulas.

Corollary 8.6. *For $k \geq 1$ we have*

$$\tau_*(R_\eta^2) = -\frac{1}{2}\, (k-1)\, q^*\psi, \qquad \tau_*(\eta^*\omega_\pi \cdot R_\eta) = (k-1)\, q^*\psi$$

and

$$\tau_*(\eta^*\omega_\pi^2) = \frac{3k}{2} q^*\psi - k(k+1)\, q^* \left(T_b^2 + \sum_{j=1}^{k} T_b^{3j} \right).$$

Substituting these formulas in $\tau_*(\omega_\tau^2)$ we find

Proposition 8.7. *For $k \geq 1$ we have*

$$\tau_*(\omega_\tau^2) = \frac{-6k^3 + 31k^2 - 29k + 6}{6k - 1}(E_0 + 2E_2 + 3E_3) + \sum_{j=1}^{k} \sum_{c=0}^{[j/2]} a_{j,c} E_{j,c},$$

with $a_{j,c}$ given by

$$(j + 1 - 2c)\left(\frac{27}{2}\frac{j(2k-1)(2k-j)}{6k-1} - k(k+1)\right).$$

By substituting the formulas of Propositions 8.1 and 8.7 in the expression of $12\,\phi^*\lambda_{\pi'}$ given by the Grothendieck–Riemann–Roch theorem we have

Theorem 8.8. *The pull back of the Hodge class $\lambda_{g'}$ of $\overline{\mathcal{M}}_{g'}$ under ϕ equals*

$$12\,\phi^*\lambda_{g'} = \frac{2}{6k-1}(t_0\,E_0 + t_2\,E_2 + t_3\,E_3) + \sum_{j=1}^{k} \sum_{c=0}^{[j/2]} t_{j,c}\,E_{j,c}$$

with the coefficients t_0, t_2, t_3 and $t_{j,c}$ defined by

$$t_0 = 18\,k^2 - 15\,k + 3, \qquad t_2 = 30\,k - 3, \qquad t_3 = 6\,k^2 + 11\,k + 1$$

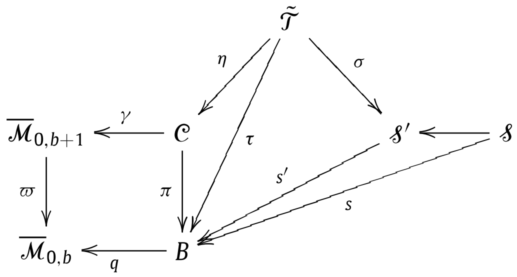
and $t_{j,c} = a_{j,c} + d_{j,c}$.

Example 8.9. Take $k = 1$ and interpret E_2 and E_3 as zero. Since ϕ is the map $p : \tilde{H} \to \overline{\mathcal{M}}_2$ we get the formula for the Hodge bundle on $\tilde{H}_{2,2}$; it says $\lambda_{\tilde{H}_{2,2}} = (E_0 + E_{1,0})/5$. This fits with the formula given in [8, Thm. 1.1], cf. also [7, Prop. 8.1].

9. The reduced trace curve

We carry out the analogous calculations for the reduced trace curve and calculate the pull back of the Hodge class on $\overline{\mathcal{M}}_{\hat{g}}$ under $\hat{\phi}$.

We denote the family of the reduced trace curves over our 1-dimensional base B by $\mathcal{S}'$ and the smooth model (obtained by resolving the A_1 singularities coming from the isolated fixed points) by $\mathcal{S}$. We have the quotient map $\sigma : \tilde{\mathcal{T}} \to \mathcal{S}'$. Note that ω_s is trivial in a neighborhood of an A_1 resolution and the pull back of $\omega_{s'}$ to $\mathcal{S}$ is ω_s. We have the diagram



(9)

Lemma 9.1. *We have $s_*(\omega_s^2) = \frac{1}{2}\tau_*(\omega_\tau^2) - \frac{3}{4}q^*(\psi)$.*

Proof. Since the singularities of $\mathcal{S}'$ are of type A_1 we can neglect them for this calculation and work on $\mathcal{S}'$. We have $\omega_\tau = \sigma^*\omega_{s'} + R_\sigma$ with R_σ the ramification divisor of σ, hence

$$\omega_s^2 = \frac{1}{2}\sigma_*\sigma^*(\omega_{s'}^2) = \frac{1}{2}\sigma_*[(\sigma^*\omega_{s'})^2] = \frac{1}{2}\sigma_*[(\omega_\tau - R_\sigma)^2]$$

and thus

$$s_*(\omega_s^2) = \frac{1}{2}\tau_*[(\omega_\tau - R_\sigma)^2] = \frac{1}{2}\tau_*(\omega_\tau^2) - \tau_*(\omega_\tau \cdot R_\sigma) + \frac{1}{2}\tau_*(R_\sigma^2).$$

We denote by R_γ the closure of the ramification of γ over the smooth locus. Note that $\nu^*R_\gamma = R_r$. We have $R_\sigma \cdot R_\eta = 0$ because $\eta(R_\sigma) = R_\gamma$ and $\eta(R_\eta) = \gamma^{-1}(\hat{S}) - R_\gamma$. We have

$$\tau_*(\omega_\tau \cdot R_\sigma) = \tau_*[(\eta^*\omega_\pi + R_\eta) \cdot R_\sigma] = \tau_*[\eta^*\omega_\pi \cdot R_\sigma] = \pi_*(\omega_\pi \cdot R_\gamma) = \tilde{\pi}_*(\omega_{\tilde{\pi}} \cdot R_r) = \frac{1}{2}q^*\psi,$$

by Lemma 8.3. If ι denotes the involution on $\tilde{\mathcal{T}}$ then $\eta^*R_\gamma = R_\sigma + \iota^*R_\eta$ and $R_\sigma \cdot \iota^*R_\eta = 0$. Furthermore we have

$$\tau_*(R_\sigma^2) = \tau_*(R_\sigma \cdot \eta^*R_\gamma) = \pi_*[(R_\gamma)^2] = \tilde{\pi}_*[(R_r)^2] = -\frac{1}{2}q^*\psi. \quad \square$$

Lemma 9.2. *The push forward δ_s of the locus of singularities of the fibers of s for $k \geq 3$ is given by*

$$\delta_s = \frac{k^2+k}{2} E_0 + (k^2 - 5k + 12) E_2 + \frac{3k^2 - 13k + 16}{2} E_3 + \sum_{j=1}^{k} \sum_{c=0}^{[j/2]} s_{j,c} E_{j,c},$$

with $s_{j,c}$ given by

$$s_{j,c} = (k-j+c)(c+1) + \left(\binom{k-j+c}{2} + \binom{c}{2}\right)(j+1-2c) + \left[\frac{j+1}{2}\right] + \begin{cases} 1 & j \text{ odd} \\ 0 & \text{else.} \end{cases}$$

and for $k = 1$ and $k = 2$ by

$$\delta_s = \begin{cases} E_0 + E_{1,0} & k = 1 \\ 3E_0 + E_3 + 3E_{1,0} + E_{2,0} + 3E_{2,1} & k = 2. \end{cases}$$

Proof. We use the local description of the reduced trace curve given in Section 5. The contribution of E_0 to δ_s is $\binom{k-1}{2} + 2(k-1) = (k^2+k)/2$. From E_2 we find the contribution $2\binom{k-2}{2} + 2(k-3) + 2 + 4 = k^2 - 5k + 10$, where the last term 4 comes from the points (p_ν, p_ν) for $\nu = 1, 2$ that give an A_1-singularity on the reduced trace curve. For $E_{j,c}$ note that for j odd the 'middle' intersection point of $\tilde{T}_{C_1}$ and $\tilde{T}_{C_2}$ gives rise to an A_1-singularity on the trace curve. The other contributions are obtained similarly. □

Remark 9.3. By interpreting for $k = 1$ (resp. $k = 2$) the divisors E_2 and E_3 (resp. E_2) as zero, the formula for $k \geq 3$ works for all $k \geq 1$. As a check on the formula note that for $k = 1$ the reduced trace curve of $C \to P$ equals P and we thus easily see that we have coefficients 1 for both E_0 and $E_{1,0}$. Similarly, for $k = 2$ the reduced trace curve equals C and we thus can easily read off from the left hand side of the Figs. 1–4 the multiplicities.

By substituting the formulas for $\tau_*(\omega_\tau^2)$ and $q^*\psi$ in $s_*(\omega_s^2)$ in Lemma 9.1 and adding δ_s we get an expression for $12\,\hat{\phi}^*\lambda_{\hat{g}}$.

Theorem 9.4. *For $k \geq 3$ the pull back of the Hodge class $\lambda_{\hat{g}}$ of $\overline{\mathcal{M}}_{\hat{g}}$ under $\hat{\phi}$ is given by*

$$12\,\hat{\phi}^*\lambda_{\hat{g}} = \frac{2}{6k-1}(u_0 E_0 + u_2 E_2 + u_3 E_3) + \sum_{j=1}^{k} \sum_{c=0}^{[j/2]} u_{j,c} E_{j,c}$$

with $u_0 = 9k^2 - 12k + 3$, $u_2 = 15k$, $u_3 = 3k^2 - 8k + 5$ and

$$u_{j,c} = s_{j,c} - \frac{(j+1-2c)}{2(6k-1)}\left((27k-27)j^2 - 54(k^2-k)j + (k^2+k)(6k-1)\right).$$

Example 9.5. Take $k = 2$ and interpret E_2 as zero. The formula says that

$$12\hat{\phi}^*\lambda_{\hat{g}} = \frac{30}{11} E_0 + \frac{2}{11} E_3 + \frac{48}{11} E_{1,0} + \frac{74}{11} E_{2,0} + \frac{54}{11} E_{2,1}.$$

Comparing this with the formula for the Hodge class of $\tilde{H}_{4,3}$ (cf. [8]) we see that it fits.

10. Pulling back boundary divisors

We shall need to know the pull backs of the boundary divisors δ'_j in $\overline{\mathcal{M}}_{g'}$ (resp. $\hat{\delta}$ in $\overline{\mathcal{M}}_{\hat{g}}$) under the rational maps $\phi : \tilde{H} \to \overline{\mathcal{M}}_{g'}$ (resp. $\hat{\phi} : \tilde{H} \to \overline{\mathcal{M}}_{\hat{g}}$).

Proposition 10.1. *For $k \geq 3$ the pull back $\phi^*(\delta'_0)$ equals*

$$(4k-2)E_0 + 4E_2 + 2E_3 + \sum_{j=2}^{k} \sum_{c=1}^{[j/2]} (2(k-j+c)(c+1) + j)\, E_{j,c} + \sum_{j=2}^{k} j\, E_{j,0};$$

furthermore, $\phi^*(\delta'_1) = (2k-1)E_{1,0}$ and

$$\phi^*(\delta'_j) = \begin{cases} (2k-2j)\, E_{j,0} & j = 2, \ldots, k \\ 0 & \text{else.} \end{cases}$$

Proof. To prove this formula for the pull back of δ'_0 (resp. δ'_j, $j \geq 1$) we count in a 1-dimensional family of semi-stable models of trace curves $\tilde{\mathcal{T}} \to B$ the number of non-disconnecting nodes (resp. of disconnecting nodes that split the curve in a component of genus j and another component of genus $g' - j$). The semi-stable model of the trace curve over a generic point of E_0 has $2(2k-1)$ non-disconnecting nodes. The semi-stable model of the trace curve over a generic point of E_2 has 4 non-disconnecting nodes. Over a generic point of E_3 has 2 non-disconnecting nodes and finally, over a generic point of $E_{j,c}$ the situation is: for $c \geq 1$ it has $2(c+1)(k-j+c)+j$ non-disconnecting nodes; for $c = 0$ and $j \geq 2$ it has j non-disconnecting nodes and $2(k-j)$ disconnecting nodes of type j, while for $c = 0$ and $j = 1$ it has $2k-1$ disconnecting nodes of type 1. □

In a similar way we derive the following proposition.

Proposition 10.2. *For $k \geq 3$ the pull back $\hat{\phi}^*(\hat{\delta}_0)$ equals*

$$(2k-2)\,E_0 + 2\,E_2 + \sum_{j=2}^{k}\sum_{c=1}^{[j/2]}\left((k-j+c)(c+1) + \left[\frac{j+1}{2}\right] + \epsilon\right) E_{j,c} + \sum_{j=3}^{k}\left(\left[\frac{j+1}{2}\right] + \epsilon\right) E_{j,0},$$

with $\epsilon = 0$ if j =even, $\epsilon = 1$ if j =odd and $\epsilon = -1$ if $j = 2,\ c = 1$; furthermore, $\hat{\phi}^*(\hat{\delta}_1) = (k-1)\,E_{1,0}$, $\hat{\phi}^*(\hat{\delta}_2) = (k-1)\,E_{2,0}$ and $\hat{\phi}^*(\hat{\delta}_j) = (k-j)\,E_{j,0}$ for $j = 3, \ldots, k$ while $\hat{\phi}^*(\hat{\delta}_j) = 0$ for $j > k$.

11. Push forward to $\overline{\mathcal{M}}_g$

In [7] we have calculated the push forwards of the boundary classes under $p : \tilde{H} \to \overline{\mathcal{M}}_g$. The result is as follows. Let

$$N = N(k) = \frac{1}{k+1}\binom{2k}{k}$$

then

$$\frac{2}{(6k)!}\,p_*E_0 = N\,\delta_0,$$

and

$$\frac{2}{(6k)!}p_*E_2 = \frac{2(k-2)N}{(2k-1)}[(18\,k^2+51\,k-9)\lambda - (3\,k^2+4\,k-1)\delta_0] + \sum_{j=1}^{k} c_j\delta_j\,,$$

$$\frac{2}{(6k)!}\,p_*E_3 = \frac{3N}{(2k-1)}\,[(12\,k^2+46\,k-8)\,\lambda - (2k^2+4k-1)\delta_0] - \sum_{j=1}^{k}\frac{3N}{2k-1}\,b_j\,\delta_j,$$

where the c_j and b_j are given in [7, Thm. 1.1 & Section 8] and with

$$e_{j,c} = \frac{(j+1-2c)^2}{(j+1)(2k-j+1)}\binom{j+1}{c}\binom{2k-j+1}{k+1-c}$$

we have finally

$$\frac{1}{(6k)!}\,p_*E_{j,c} = e_{j,c}\,\delta_j.$$

By substituting the above formulas in the expression of $\phi^*\lambda_{g'}$ given in Theorem 8.8 we get the following theorem.

Theorem 11.1. *For $k \geq 3$ the push forward $\frac{1}{(6k)!}\,p_*\phi^*\lambda_{g'}$ on $\overline{\mathcal{M}}_g$ equals*

$$\frac{N(18\,k^3+31\,k^2-69\,k+11)}{2k-1}\,\lambda - \frac{N(3\,k^3-5\,k+1)}{2k-1}\,\delta_0$$
$$-\sum_{j=1}^{k}\left(\frac{-(10\,k-1)}{4(6\,k-1)}c_j + \frac{N\,(6\,k^2+11\,k-1)}{4(12\,k^2-8\,k+1)}\,b_j + \frac{1}{12}\sum_{c=0}^{[j/2]} e_{j,c}(a_{j,c}+d_{j,c})\right)\delta_j\,.$$

By Theorem 9.4 we have a similar theorem for the map defined by the reduced trace curve.

Theorem 11.2. *For $k \geq 3$ the push forward $\frac{1}{(6k)!}\,p_*\hat{\phi}^*\lambda_{\hat{g}}$ equals*

$$\frac{N\,(18k^3+19k^2-117k+20)}{2(2k-1)}\,\lambda - \frac{N\,(k-2)(3k^2+4k-1)}{2(2k-1)}\delta_0$$
$$-\sum_{j=1}^{k}\left(\frac{(-5k-1)}{4(6k-1)}c_j + \frac{N(3k^2-8k+5)}{4(6k-1)(2k-1)}b_j - \frac{1}{12}\sum_{c=0}^{[j/2]} e_{j,c}u_{j,c}\right)\delta_j.$$

Proposition 10.1 yields the following result.

Proposition 11.3. *The action induced by the correspondence of the boundary divisors δ_j' for $j = 0, \ldots, [g'/2]$ of $\overline{\mathcal{M}}_{g'}$ is given by:*

$$p_*\phi^*\delta_0' = w_\lambda\,\lambda + w_0\,\delta_0 + \sum w_j\,\delta_j,$$

where

$$w_\lambda = \frac{6(6k)!\,N(6\,k-1)}{2\,k-1}(2\,k^2+3\,k-8), \qquad w_0 = -\frac{2(6k)!N}{2\,k-1}(6\,k^3-3\,k^2-10\,k+2)$$

and $w_1 = (6k)!(2\,c_1 - 3\,N\,b_1/(2\,k-1))$ *and for* $2 \le j \le k$

$$w_j = j\,e_{j,0} + \sum_{c=1}^{[j/2]} e_{j,c}\,(2(k-j+c)(c+1)+j) + 2\,(6k)!\,c_j - (6k)!\frac{3N}{2k-1}\,b_j;$$

furthermore, $p_*\phi^*\delta_1' = (2\,k-1)\,e_{1,0}\,\delta_1$ *and* $p_*\phi^*\delta_j' = (2\,k-2j)\,\delta_j$ *for* $j = 2, \ldots, k$ *and* $p_*\phi^*\delta_j' = 0$ *for* $j > k$.

Similarly, Proposition 10.2 yields the following result.

Proposition 11.4. *The action induced by the correspondence of the boundary divisors δ_j' for $j = 0, \ldots, [\hat{g}/2]$ of $\overline{\mathcal{M}}_{g'}$ is given by:*

$$p_*\phi^*\hat{\delta}_0 = v_\lambda\,\lambda + v_0\,\delta_0 + \sum v_j\,\delta_j,$$

where

$$v_\lambda = \frac{6N(6k)!(6k-1)}{2k-1}\,(k+3)(k-2), \qquad v_0 = -\frac{N(6k)!}{2k-1}\,(6k^3-6k^2-15k+3)$$

and $v_1 = (6k)!\,2\,c_1$ *and for* $2 \le j \le k$

$$v_j = \left(\left[\frac{j+1}{2}\right]+\epsilon\right)e_{j,0} + \sum_{c=1}^{[j/2]} e_{j,c}\left((k-j+c)(c+1)+\left[\frac{j+1}{2}\right]+\epsilon\right) + 2\,(6k)!\,c_j,$$

with ϵ as defined in Proposition 10.2; furthermore, $p_*\hat{\phi}^*\hat{\delta}_1 = (k-1)\,e_{1,0}\,\delta_1$, $p_*\hat{\phi}^*\hat{\delta}_2 = (k-1)\,e_{2,0}\,\delta_2$ *and* $p_*\hat{\phi}^*\hat{\delta}_j = (k-j)\,\delta_j$ *for* $j = 3, \ldots, k$ *and* $p_*\hat{\phi}^*\hat{\delta}_j = 0$ *for* $j > k$.

12. Slopes

We consider again the correspondence

$$\overline{\mathcal{M}}_{2k} \xleftarrow{p} \tilde{H} \xrightarrow{\phi} \overline{\mathcal{M}}_{g'}$$

It acts on the Picard group via $D \mapsto p_*\phi^*D$. We now show that it maps ample divisors of $\overline{\mathcal{M}}_{g'}$ to moving divisors of $\overline{\mathcal{M}}_g$. A moving divisor is a divisor D such that the base locus of all the linear systems $|mD|$ with $m \ge 1$ is of codimension at least 2.

Lemma 12.1. *If D' is an ample divisor on $\overline{\mathcal{M}}_{g'}$ then the divisor $D := p_*\phi^*D'$ is a moving divisor. In other words, the correspondence sends the ample cone of $\overline{\mathcal{M}}_{g'}$ to the moving cone of $\overline{\mathcal{M}}_g$.*

Proof. Let $\tilde{\mathcal{M}}_g$ be the locus of $\overline{\mathcal{M}}_g$ where the map $p : \tilde{H} \to \overline{\mathcal{M}}_g$ is finite. Since $\overline{H}_{g,d}$ is irreducible the complement of $\tilde{M}_g$ in $\overline{\mathcal{M}}_g$ is of codimension ≥ 2. We shall show that the common base locus $\mathbf{B}(D) := \cap_{m\ge1}\mathrm{Base}(|mD|)$ is a subset of the above complement. Indeed, if $x \in \tilde{\mathcal{M}}_g$ we shall show that $x \notin \mathbf{B}(D)$. Let $p^{-1}(x) = \{h_1, \ldots, h_{N_0}\}$ and let $A = \{\phi(h_1), \ldots, \phi(h_{N_0})\}$. As we may assume that mD' is very ample for appropriate m, we can choose a divisor Z in $|mD'|$ with $Z \cap A = \emptyset$. But then $p_*\phi^*Z$ is an element of $|mD|$ that does not contain x. □

We write δ_j' $(j = 0, \ldots, [g'/2])$ for the boundary divisors of $\overline{\mathcal{M}}_{g'}$ and put $\delta' = \sum_{j=0}^{[g'/2]} \delta_j'$. The span of the ample cone of $\overline{\mathcal{M}}_{g'}$ in the (λ', δ')-plane is well-known: a divisor $D' = x\,\lambda' - y\,\delta'$ is ample if and only if $x > 11\,y$, cf. [4, Thm. 1.3]. Given a divisor $D' = x\,\lambda' - y\,\delta'$ we wish to determine the slope s of the induced divisor $p_*\phi^*D'$ in terms of the slope $s' = x/y$ of D'. We write

$$p_*\phi^*\lambda' = \alpha_\lambda\lambda - \alpha_0\delta_0 - \sum_{j=1}^{k}\alpha_j\delta_j,$$

$$p_*\phi^*\delta_0' = \beta_\lambda\lambda + \beta_0\delta_0 + \sum_{j=1}^{k}\beta_j\delta_j,$$

and

$$p_*\phi^*\delta_j' = \gamma_j\delta_j \quad \text{for } j = 1, \dots, k$$

while $p_*\phi^*\delta_\nu' = 0$ for $\nu > k$ with the coefficients determined in Theorems 11.1 and 11.3. The divisor $p_*\phi^*D'$ can be written as

$$p_*\phi^*D' = (x\,\alpha_\lambda - y\,\beta_\lambda)\,\lambda - (x\,\alpha_0 + y\,\beta_0)\,\delta_0 - \sum_{j=1}^{k}(x\alpha_j + y\beta_j + y\gamma_j)\,\delta_j.$$

Thus the slope is given by

$$s(p_*\phi^*D') = \frac{x\,\alpha_\lambda - y\,\beta_\lambda}{x\,\alpha_0 + y\,\beta_0} = \frac{\alpha_\lambda s' - \beta_\lambda}{\alpha_0 s' + \beta_0}$$

provided that

$$x\,\alpha_0 + y\,\beta_0 \le x\,\alpha_j + y\,\beta_j + y\,\gamma_j, \quad j = 1, \dots, k.$$

In our case, assuming that the above conditions hold, then we have

$$s(p_*\phi^*D') = 6 + \frac{(31\,s' - 132)k^2 + (-39\,s' + 186)\,k + 5\,s' - 24}{(3\,s' - 12)\,k^3 + 6\,k^2 + (-5\,s' + 20)\,k + s' - 4}.$$

From this one can deduce for even $g > 4$ the following estimate for the moving slope

$$\sigma(g) < 6 + \frac{20}{g}.$$

For the reduced trace curve one can do similar things. The result is the formula

$$s(p_*\hat{\phi}^*D') = 6 + \frac{(31\,s - 132)\,k^2 + (264 - 63\,s)\,k - 36 + 8\,s}{(3\,s - 12)\,k^3 + (-2\,s + 12)\,k^2 + (-9\,s + 30)\,k + 2\,s - 6}$$

and this results in a similar bound $\sigma(g) \le 6 + 20/g$. We refrain from giving details because, using the same Hurwitz space but now as a correspondence between $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{0,6k}$ will result in a better slope, as we show in the next section.

13. Another correspondence

The diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,6k} & \xleftarrow{\;q\;} & \overline{H}_{2k,k+1} \\ & & \downarrow{\scriptstyle p} \\ & & \overline{\mathcal{M}}_{2k} \end{array}$$

provides us with the action p_*q^* on divisor classes. It is well-known that the divisor class

$$\kappa = \psi - \delta = \sum_{j=2}^{k} \frac{(j-1)(b-j-1)}{b-1} T_b^j$$

is ample on $\overline{\mathcal{M}}_{0,b}$, cf. [2]. As above this gives us by p_*q^* a moving divisor of good slope. We calculate now the class of $p_*q^*\kappa$. With $\alpha(k, j)$ as defined in Theorem 1.1 of [7] we get by combining relations (8) and the formulas in Section 11 that

$$p_*q^*T_b^{3j} = \sum_{c=0}^{[j/2]}(j + 1 - 2c)e_{j,c}\,\delta_j = (6k)!\,\alpha(k, j)\,\delta_j.$$

We also get

$$p_*q^*T_b^2 = (6k)!\frac{b(b-1)N}{2\,(b-3)}\,[3(2k+5)\lambda - (k+1)\delta_0] - (6k)!\sum_{j=1}^{k}\left(-c_j + \frac{9N}{4k-2}b_j\right)\,\delta_j.$$

We therefore have

$$\begin{aligned} p_*q^*\kappa \; &= \frac{b!bN}{2}\,[3(2k+5)\lambda - (k+1)\delta_0] \\ &\quad - \frac{b!}{b-1}\,\sum_{j=1}^{k}\left[(b-3)\,\left(-c_j + \frac{9N}{4k-2}b_j\right) - (3j-1)(b-3j-1)\,\alpha(k, j)\right]\,\delta_j. \end{aligned}$$

Theorem 13.1. *The moving slope $\sigma(g)$ of $\overline{\mathcal{M}}_g$ for even g satisfies the inequality*

$$\sigma(g) \leq 6 + \frac{18}{g+2}.$$

Proof. Indeed, if we write $p_*q^*\kappa$ as $a\lambda - \sum_{i=0}^{k} b_i\delta_i$ the ratio a/b_0 is $3(2k+5)/(k+1) = 6 + 18/(g+2)$, while a/b_i for $i > 0$ is much smaller as one sees by analyzing the expressions involved. □

Observe also, that since $q^*\kappa$ is an ample class, all effective divisors in a multiple of this class intersect the positive dimensional fibers of the generically finite map p. We therefore conclude that the common base locus $\cap_{m\geq 1}\mathrm{Base}(|m\,(p_*q^*\kappa)|)$ is exactly the locus of points in $\overline{\mathcal{M}}_g$ over which the corresponding fiber of the map p has positive dimension. It will be interesting to have a description of this common base locus.

14. The Prym variety of the trace curve

By associating to a point of $H_{2k,k+1}$ the Prym variety of $T/\hat{T}$ (resp. the quotient of the Jacobian of the reduced trace curve by the Jacobian of C) we can define a morphism $\chi : H_{2k,k+1} \to \mathcal{A}_{(5k^2-k)/2}$, (resp. to $\hat{\chi} : H_{2k,k+1} \to \mathcal{A}_{(5k-1)(k-2)/2}$), where $\mathcal{A}_n$ denotes a moduli space of polarized abelian varieties of dimension n. The polarization is induced by the theta divisor on the Jacobian of the trace curve. These maps are interesting and deserve further study.

Suppose that this map χ (resp. $\hat{\chi}$) extends to a rational map $\chi : \tilde{H} \to \tilde{\mathcal{A}}$, a toroidal compactification that contains the canonical rank 1 partial compactification $\mathcal{A}^{(1)}$ defined by Mumford. Then the pull back under χ of the Hodge class is equal to $\phi^*\lambda_{g'} - \hat{\phi}^*\lambda_{\hat{g}}$. This expression is given by combining Theorems 8.8 and 9.4. Let D be the divisor that is the closure of the inverse image of (open) boundary component of largest degree under the map of $\tilde{\mathcal{A}}$ to the Satake compactification $\mathcal{A}^{\star}$. Let L be the Hodge bundle (corresponding to modular forms of weight 1) Then the pull back of D is given by $\phi^*(\delta_0') - \hat{\phi}^*(\hat{\delta}_0)$. Propositions 10.1 and 10.2 give expressions for this. Thus we can calculate $p_*\chi^*(aL - bD)$ in terms of λ, δ_0 and δ_j with $j = 1, \ldots, k$. Our expressions show that nef (ample) divisors $aL - bD$ with $a = 12b$ give rise to (moving) divisors of slope $6 + 20/g$.

15. The Eisenbud–Harris divisor

The map $p : H_{2k,k+1} \to \mathcal{M}_g$ is branched along a divisor that was introduced and studied by Eisenbud and Harris in [5]. As a side product of our calculations we now can calculate in an easy way the class of (the closure of) this divisor. We give only the coefficients of λ and δ_0 but the remaining coefficients can be calculated similarly.

Since $\tilde{H}$ maps to $\overline{\mathcal{M}}_{0,b}$ and to $\overline{\mathcal{M}}_g$ via q and p we can calculate the canonical class in two ways:

$$K_{\tilde{H}} = q^*K_{\overline{\mathcal{M}}_{0,b}} + R_q \quad \text{and} \quad K_{\tilde{H}} = p^*K_{\overline{\mathcal{M}}_g} + R_p$$

with R_q and R_p the ramification divisors. For R_q we have, by relations (8), the formula

$$R_q = E_2 + 2E_3 + \sum_{j,c}(j-2c)E_{j,c},$$

while R_p has four components, namely

$$R_p = E_0 + E_2 + E_3 + G,$$

with p_*G the Eisenbud–Harris divisor. Since we have formulas for p_* applied to the divisors E_0, E_2, E_3 and $E_{j,c}$ and we have a formula for $p_*q^*K_{\overline{\mathcal{M}}_{0,b}}$ we can calculate p_*G. Indeed, we get

$$R_p = q^*K_{\overline{\mathcal{M}}_{0,b}} + R_q - p * K_{\overline{\mathcal{M}}_g}.$$

Plugging in the formula

$$K_{\overline{\mathcal{M}}_{0,b}} = \frac{-2}{b-1}T_b^2 + \sum_{i=3}^{3k}\left(\frac{i(b-i)}{b-1} - 2\right)T_b^i$$

and applying p_* we find $p_*G = p_*(R_p) - p_*(E_0 + E_2 + E_3)$ and thus get

$$p_*G = \frac{-2}{b-1}p_*q^*T_b^2 + p_*E_3 - p_*E_0 + \sum_{j,c}\left[\left(\frac{3j(b-3j)}{b-1} - 1\right)(j+1-2c) - 1\right]p_*E_{j,c} - N_0K_{\overline{\mathcal{M}}_g}$$

with $N_0 = (6k)!N$. We now substitute $K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\sum_{j=2}^{k}\delta_j$ and find

$$p_*G = \frac{N_0}{2k-1}\left[(6k^2 + 13k + 1)\lambda - k(k+1)\delta_0\right] + \cdots$$

in agreement with Theorem 2 of [5].

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