Divisors on Symmetric Products of Curves

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DIVISORS ON SYMMETRIC PRODUCTS OF CURVES

ALEXIS KOUVIDAKIS

Abstract. For a curve with general moduli, the Neron-Severi group of its symmetric products is generated by the classes of two divisors $x$ and $\theta$. In this paper we give bounds for the cones of effective and ample divisors in the $x\theta$-plane.

1. Introduction

Let $C$ be a smooth irreducible curve of genus $g$, $J(C)$ its Jacobian variety and $C_d$ its $d$-fold symmetric product. Fixing a point $P_0$ on $C$, we define the maps

$$u_d: C_d \to J(C) \quad \text{by} \quad u_d(D) = \Theta(D - dP_0),$$

$$i_{d-1}: C_{d-1} \to C_d \quad \text{by} \quad i_{d-1}(D) = D + P_0.\$$

On $J(C)$ we denote by $\Theta$, the theta divisor and by $\theta$, its class in the Neron-Severi group. On $C_d$ we denote by $\theta_d$, or simply (again) $\theta$, the class of $u_d^*(\Theta)$ and by $x_d$, or simply $x$, the class of $i_{d-1}(C_{d-1})$ in $C_d$. For curves $C$ with general moduli, it is known that the Neron-Severi group of the symmetric product $C_d$ is generated by $\theta$ and $x$, see [A-C-G-H]. In this paper we give estimates for the cones of the effective and ample divisors on $C_d$, in the $\theta$, $x$-plane.

By standard theory, we know the following things about the map $u_d$:

1. Abel's Theorem. The fiber of the map $u_d$ containing the divisor $D$, is exactly the set of divisors belonging to the complete series of $D$.

2. Jacobi's Inversion Theorem. The map $u_g: C_g \to J(C)$ is onto.

3. Poincare's Formula. We denote by $W_d$ the image of $C_d$ in the Jacobian by the map $u_d$ and by $w_d$ its class. For $0 \leq d \leq g$ we have

$$w_d = \frac{\theta g - d}{(g - d)!}.\$$

In particular, for $d = g - 1$ we have $w_{g-1} = \theta$.
4. If \( d \leq g \), the map \( u_d \) is birational to its image, reflecting the fact that 
\( h^0(C, D) = 1 \), for \( D \) general point in \( C_d \).
If \( g + 1 \leq d \leq 2g - 2 \), the generic fiber of \( u_d \) has dimension \( d - g \).
If \( 2g - 1 \leq d \), the map is a \( \mathbb{P}^{d-g} \)-fibration with tautological class \( x \).

**Lemma 1** (Intersections on \( C_d \)). For \( 0 \leq r \leq d \leq g \) we have on \( C_d \)

\[
\theta^r x^{d-r} = \frac{g!}{(g-r)!}.
\]

**Proof.** Indeed, \( x = \text{class}(i_{d-1}(C_{d-1})) = \text{class}\{D + P_1, D \in C_{d-1}, P_1 \text{ a fixed point in } C\} \) and so,

\[
x^{d-r} = \text{class}\{D + P_1 + \cdots + P_{d-r}, D \in C_r, P_1, \ldots, P_{d-r} \text{ fixed in } C\}.
\]

Therefore \( u_{d*}x^{d-r} = w_r \). Since the map \( u_d \) is generically 1-1, projection formula implies that \( \theta^r x^{d-r} \) on \( C_d \), is equal to \( \theta^r w_r \) on \( J(C) \). Poincaré’s Formula and the fact that \( \theta^g = g! \) complete the proof. Q.E.D.

We denote by \( H^{2n}_{an}(C_d, \mathbb{Q}) \) the algebraic part of \( H^{2n}(C_d, \mathbb{Q}) \). Given an algebraic cycle \( Z \) in \( C_d \), we define the maps

\[
A_k : H^{2m}_{an}(C_d, \mathbb{Q}) \to H^{2m}_{an}(C_{d+k}, \mathbb{Q})
\]

by

\[
A_k(Z) = \{E \in C_{d+k}, E - D \geq 0 \text{ for some } D \in Z\}
\]

and

\[
B_k : H^{2m}_{an}(C_d, \mathbb{Q}) \to H^{2m-2k}_{an}(C_{d-k}, \mathbb{Q})
\]

by

\[
B_k(Z) = \{E \in C_{d-k}, D - E \geq 0 \text{ for some } D \in Z\}.
\]

We have the standard formulas, see [A-C-G-H, p. 367].

**Lemma 2** (Push-pull formulas for symmetric products).

\[
A_k(x^{\alpha} \theta^\beta) = \sum_{i=0}^{k} {\binom{\beta}{i}} \left( \frac{g - \beta + i}{i} \right) \left( \frac{d + k - \alpha - 2\beta}{k - i} \right) i! x^{\alpha+i} \theta^{\beta-i}
\]

and

\[
B_k(x^{\alpha} \theta^\beta) = \sum_{j=0}^{k} {\alpha \choose k-j} \left( \frac{\beta}{j} \right) \left( \frac{g - \beta + j}{j} \right) j! x^{\alpha-k+j} \theta^{\beta-j}.
\]

2. AMPLE AND EFFECTIVE DIVISORS

We will use often the following criterion for ampleness, see [Ha]:

**Lemma 3** (Nakai-Moishezon). Let \( D \) be a Cartier divisor on a variety \( X \). Then \( D \) is ample on \( X \), if and only if, for every subvariety \( Y \) in \( X \) of dimension \( r \), we have that \( D^rY > 0 \).

In particular, when \( X \) is a smooth surface this says that the cone of effective and the cone of ample divisors are dual under the intersection pairing.

If \( X \) is a smooth surface, we have the following numerical criterion, for checking effectivity of a divisor \( D \) on \( X \), see [Ha]:
Lemma 4. Let \( X \) be a smooth surface, \( H \) an ample divisor on \( X \) and \( D \) a divisor with \( D^2 > 0 \) and \( DH > 0 \).

Then, for \( n \) sufficiently large, \( nD \) is an effective divisor.

Proof. We prove first that for \( n \) large enough we have \( h^2(X, nD) = 0 \): Indeed, \( h^2(X, nD) = h^0(X, K_X - nD) \), and if we assume that \( h^2(X, nD) > 0 \) for all \( n \), then \( K_X - nD \) is effective and so, \( (K_X - nD)H > 0 \) (Lemma 3) i.e. \( K_X H > nDH \). Since \( DH > 0 \), taking \( n \) big enough—namely \( n > K_X H \)—we get \( K_X H > n \). A contradiction. Therefore there exists an \( n \) with \( h^2(X, nD) = 0 \) and the Riemann-Roch theorem completes the proof. Q.E.D.

In the higher dimensional case i.e. if \( X \) is a smooth variety of dimension \( d \), the Riemann-Roch theorem gives that

\[
h^0(X, D) - h^1(X, D) + h^2(X, D) - \cdots + (-1)^d h^d(X, D) = \frac{c^d_1}{d!} + (\text{terms containing strictly lower powers of } c_1).
\]

If \( H \) is again an ample divisor on \( X \) and \( D \) a divisor satisfying \( D^r H^{d-r} > 0 \) for all \( 0 \leq r \leq d \), then it is not known if

\[
h^0(X, nD) > 0 \quad \text{for } n \text{ big enough}
\]

In order to have such a conclusion, we have to impose extra condition, for example, the restriction of \( \mathcal{O}(D) \) on \( H \) to be an ample divisor. We have

Theorem 1. Let \( X \) be a \( d \)-dimensional variety, \( H \) an ample effective divisor on \( X \) and \( D \) a divisor with \( D^d > 0 \) and \( \mathcal{O}(D)|_H \) is an ample line bundle on \( H \).

Then,

\[
h^0(nD) > 0 \quad \text{for } n \text{ big enough}.
\]

Proof. We use the following lemmas:

Lemma 5 (Serre). Let \( X \) be a proper scheme over a Noetherian ring. If \( \mathcal{N} \) is an invertible sheaf on \( X \), then the following conditions are equivalent:

(i) \( \mathcal{N} \) is ample.

(ii) For each coherent sheaf \( \mathcal{E} \) on \( X \), there exists an integer \( n_0 \) depending on \( C \) such that for each \( i \geq 1 \) and each \( n \geq n_0 \)

\[
\mathcal{H}^i(X, \mathcal{E} \otimes \mathcal{N}^n) = 0.
\]

Lemma 6. Let \( S \) be a proper scheme over a Noetherian ring, \( \mathcal{L} \) an ample line bundle on \( S \) and \( \mathcal{F} \) a line bundle generated by global sections. Then there exists an \( n_0 \) s.t.

\[
\forall n \geq n_0 \quad \mathcal{H}^i(S, \mathcal{L}^n \otimes \mathcal{F}^k) = 0 \quad \forall k \geq 0, \quad \forall i \geq 1.
\]

Acknowledgment. I would like to thank J.-F. Burnol, for showing me Lemma 6 above and the following proof.

Proof. We have an exact sequence \( \bigoplus_{i=1}^r \mathcal{O}_s \rightarrow \mathcal{F} \rightarrow 0 \). Tensoring by \( \mathcal{F} \) we get \( \bigoplus_{i=1}^r \mathcal{F} \rightarrow \mathcal{O}_S \rightarrow 0 \). Hence (Koszul):

\[
0 \rightarrow (\mathcal{F})^r \rightarrow \bigoplus_{r} (\mathcal{F})^r-1 \rightarrow \bigoplus_{r} (\mathcal{F})^r-2 \rightarrow \cdots \rightarrow \bigoplus_{r} \mathcal{F} \rightarrow \mathcal{O}_S \rightarrow 0.
\]
And so, tensoring by \( \mathcal{F}^r \) we get
\[
0 \rightarrow \mathcal{O}_S \rightarrow \bigoplus_{r \geq 0} \mathcal{F} \rightarrow \bigoplus_{r \geq 0} (\mathcal{F})^2 \rightarrow \cdots \rightarrow \bigoplus_{r \geq 0} (\mathcal{F})^{r-1} \rightarrow (\mathcal{F})^r \rightarrow 0.
\]

By Lemma 5 we can choose \( n_0 \) so that \( H^i(S, \mathcal{L}^n \otimes \mathcal{F}^k) = 0 \) for all \( i \geq 1, n \geq n_0, k = 0, 1, \ldots, r \). Then this \( n_0 \) works: suppose the claim is true for \( k \leq l \). Tensoring the above exact sequence by \( \mathcal{L}^n \otimes \mathcal{F}^{l+1-r} \), all sheaves have zero \( H^i \)'s, \( i \geq 1, n \geq n_0 \), except maybe the last one which is \( H^i = \mathcal{L}^n \otimes \mathcal{F}^{l+1} \). Therefore the last one has also zero \( \mathcal{H}^i \)'s, \( i \geq 1, n \geq n_0 \). Q.E.D.

Going back to the proof of Theorem 1, we use first the following fact: “A line bundle \( L \) is ample on \( H \) iff \( L_{\text{red}} \) is ample on \( H_{\text{red}} \).” By this fact, we can replace \( H \) by \( mH \) without changing hypothesis on \( D \), and so, we can assume that \( H \) is in fact very ample on \( X \). We denote by \( L \) the line bundle \( \mathcal{O}(D) \).

We have that \( L_1 = L \otimes \mathcal{O}_H \) is ample on \( H \). Let \( H_1 \) be the restriction of \( \mathcal{O}(H) \) to \( H \); then, \( H_1 \) is generated by global sections. Also \( L_1 \) is ample on \( H \) and, by the above Lemma 6, there exists an integer \( n_0 \) s.t.
\[
h^i(H, L^n_1 \otimes H_1^k) = 0 \quad \text{for all } i \geq 1, n \geq n_0, k \geq 0.
\]

On the other hand \( H \) is ample on \( X \) and so, given the coherent sheaf \( L^n \) there exists by Lemma 5 an integer \( m_n \) s.t. \( H^i(X, L^n \otimes H^{m_n}) = 0 \). Consider now the exact sequence:
\[
0 \rightarrow \mathcal{O}_X(L^n \otimes H^{l-1}) \rightarrow \mathcal{O}_X(L^n \otimes H^l) \rightarrow \mathcal{O}_H(L^n_1 \otimes H_1^l) \rightarrow 0.
\]

The corresponding long exact sequence gives
\[
0 \rightarrow \mathcal{H}^0(X, L^n \otimes H^{l-1}) \rightarrow \mathcal{H}^0(X, L^n \otimes H^l) \rightarrow \mathcal{H}^0(H, L^n \otimes H_1^l) \\
\rightarrow \mathcal{H}^1(X, L^n \otimes H^{l-1}) \rightarrow \mathcal{H}^1(X, L^n \otimes H^l) \rightarrow \mathcal{H}^1(H, L^n \otimes H_1^l) \\
\rightarrow \mathcal{H}^2(X, L^n \otimes H^{l-1}) \rightarrow \mathcal{H}^2(X, L^n \otimes H^l) \rightarrow \mathcal{H}^2(H, L^n \otimes H_1^l) \\
\cdots \\
\rightarrow \mathcal{H}^k(X, L^n \otimes H^{l-1}) \rightarrow \mathcal{H}^k(X, L^n \otimes H^l) \rightarrow \mathcal{H}^k(H, L^n \otimes H_1^l) \\
\cdots
\]

and so, we get for each \( n \geq n_0 \) that
\[
h^i(X, L^n \otimes H^{l-1}) = h^i(X, L^n \otimes H^l) \quad \text{for all } i \geq 2, l \geq 1.
\]

Therefore, for each \( n \geq n_0 \) we have that
\[
h^i(X, L^n) = h^i(X, L^n \otimes H) = \cdots = h^i(X, L^n \otimes H^{m_n}) = 0 \quad \text{for all } i \geq 2.
\]

For each \( n \geq n_0 \) the Riemann-Roch theorem gives
\[
h^0(X, L^n) = h^1(X, L^n) + \frac{c_1(L^n)}{d!} + \text{ (terms containing strictly lower powers of } c_1)\]

and so, since \( c_1(L) > 0 \), we get that there exists an \( n \) big enough s.t. \( h^0(X, L^n) > 0 \). Q.E.D.
3. THE CLASS OF THE DIAGONAL AND OF $\Gamma_n(g_d^r)$’S

We recall some theory from [A-C-G-H]. Consider the diagonal map

$$\phi_a = \phi: C_d - 2 \times C \to C_d$$

defined by

$$\phi(D, p) = D + 2p.$$ 

The image of this map is the diagonal $\Delta_d$ in $C_d$. A special case of Proposition 5.1 on p. 358 in [A-C-G-H] gives that

Lemma 7 (MacDonald). The class $\delta_d$ of the diagonal $\Delta_d$ in $C_d$ is given by

$$(5) \quad \delta_d = 2((d + g - 1)x - \theta).$$

We denote now by $g_d^r$ a base point free linear system of degree $d$ and dimension $r$ on $C$. Given such a $g_d^r$, then for each $n$ with $r < n \leq d$, we can construct in $C_n$ the following cycle

$$\Gamma_n(g_d^r) = \{D \in C_n \text{ s.t. } D \leq E \text{ for some } E \in g_d^r\}.$$ 

The standard way to calculate the class $\gamma_n(g_d^r)$ of the above cycle is given by the following lemma, see [A-C-G-H, Lemma 3.2, p. 342].

Lemma 8. For integers $d \geq n > r$ the class $\gamma_n(g_d^r)$ in $C_n$ is given by

$$\gamma_n(g_d^r) = \sum_{k=0}^{n-r} \binom{d - g - r}{k} \frac{x^k \theta^{n-r-k}}{(n-r-k)!}. $$

In the particular case where $n = r + 1$ and so, $\Gamma_{r+1}(g_d^r)$ is a divisor in $C_{r+1}$, we can find the class as following:

We denote by $C^{\times(r+1)}$ the $(r+1)$th Cartesian product of $C$, by $f_1, \ldots, f_{r+1}$ the class of the coordinate planes and by $\delta_C$ the class of the sum of the diagonals in the product. Also we define $\gamma_C = \pi^*(\gamma_{r+1}(g_d^r))$, where $\pi: C^{\times(r+1)} \to C_{r+1}$ the canonical map. We have the following relations:

$$\pi^*x = f \overset{\text{def}}{=} f_1 + \cdots + f_{r+1}, \quad \pi^*\delta_{r+1} = 2\delta_C, \quad \delta_{r+1} = 2((g + r)x - \theta).$$

Given a $g_d^r$ on $C$ we have a canonical map $\phi: C \to P^r = P$ and an induced (product) map $\Phi: C^{\times(r+1)} \to P^{\times(r+1)}$. We denote by $\delta_P$ the class of the sum of the diagonals in $P^{\times(r+1)}$. Observe that

$$\Phi^*(\delta_P) = \delta_C + \gamma_C.$$ 

We have $\delta_P = f_1^P + \cdots + f_{r+1}^P$, where $f_i^P$’s are the classes of the coordinate planes in $P^{\times(r+1)}$. Therefore, $\Phi^*(\delta_P) = \Phi^*(f_1^P + \cdots + f_{r+1}^P) = d(f_1 + \cdots + f_{r+1}) = df$ and so, $\gamma_C = df - \delta_C$.

Now, $\pi^*(\gamma_{r+1}(g_d^r)) = \gamma_C = df - \delta_C = d\pi^*x - \frac{d}{2} \pi^*(\delta_{r+1})$ and so, $\gamma_{r+1}(g_d^r) = dx - \delta_{r+1}/2$. Using relation (5) we conclude that

$$(6) \quad \gamma_{r+1}(g_d^r) = \theta - (g - d + r)x.$$ 

4. FIRST BOUNDS FOR THE CONES

We examine the case $d \leq g$. If $D$ is an effective divisor on $C_d$, then $u_d^*\theta^{d-1} \cdot D = \theta^{d-1} \cdot u_d \cdot D$, where $u_d$ the Abel-Jacobi map. Since $\theta$ is an ample
divisor on $J(C)$, we get by Lemma 3, that $\theta^{d-1} \cdot u_d \cdot D \geq 0$, where equality holds iff $u_d \cdot D = 0$. This gives the first naive bound for the effective cone in $C_d$:

Suppose that $D$ is divisor with class $a\theta - bx$, $a, b > 0$ i.e. it “lies” in the fourth quarter of the $\theta, x$-plane. We define slope $m$ of $D$ to be $m = \frac{b}{a}$.

If $D$ is effective, then by the above discussion we have $(\theta - mx)\theta^{d-1} \geq 0$ which implies

$$\frac{g!}{(g-d)!} - m\frac{g!}{(g-d+1)!} \geq 0 \quad \text{i.e.} \quad m \leq g - d + 1.$$  

If $D$ is ample, Lemma 3 implies that $(\theta - mx)^d > 0$. Equivalently

$$\sum_{k=0}^{d} \binom{d}{k} \theta^k m^{d-k} x^{d-k} > 0,$$

i.e.

$$(7) \quad \sum_{k=0}^{d} \binom{d}{k} m^{d-k} \frac{g!}{(g-k)!} > 0.$$  

Since for $m = 0$ this is positive, we must have $m < \left( \text{min. posit. root of } (7) \right)$.

For a divisor $D$ with class $ax - b\theta$, $a, b > 0$, i.e. it “lies” in the second quarter of the $\theta, x$-plane we define the slope $\overline{m}$ of $D$ to be $\overline{m} = \frac{a}{b}$. Similar argument gives that if $D$ is effective then $\overline{m} \geq g - d + 1$, and if $D$ is ample then $\overline{m}$ satisfies a similar relation as in (7). For example if $D$ is ample then for $d = 2$ we have $m < g - \sqrt{g}$ and $\overline{m} > g + \sqrt{g}$.

5. Effective and Ample Cones for $C_2$

By the previous discussion we have the first bounds for the effective and ample cones for $C_2$. On the other hand using the Lemma 4 we know that every

![Figure 1. Bounds for the cones in $C_2$](image-url)
class between the thick lines in Figure 1 is effective. Since the ample and effective cones are dual, it is enough to describe the effective cone. We make the observation:

**Observation.** If $D$ is an irreducible effective divisor with slope $m$ (resp. $\overline{m}$) between

$$g - \sqrt{g} < m \leq g - 1 \quad (\text{resp. } g + \sqrt{g} > \overline{m} \geq g - 1),$$

then it is unique with this property.

Indeed, if $D'$ is another irreducible effective divisor with slope $m'$ (resp. $\overline{m}'$) in the above range then we get $DD' < 0$, a contradiction.

If we are able to find such a divisor, then this describes the cone. In the second quarter such a divisor exists, namely the diagonal: Recall that $\delta_2 = 2((g + 1)x - \theta)$ and so $\overline{m} = g + 1$ i.e. $g + \sqrt{g} > \overline{m} > g - 1$. Therefore the slope of the effective cone in the second quarter is given by

$$m_{ef} = g + 1$$

and (by duality) of the ample cone by

$$m_a = 2g.$$  

(Note that a divisor with slope $m_a$, has positive intersection with all the effective divisors in the first or the fourth quarter.)

Also, if $C$ is of genus 2, then it is hyperelliptic and the corresponding $g_3^1$ gives an effective class in $C_2$ belonging in the fourth quarter with slope $m = 1$, see formula (6). By the above observation we get that $m_{ef} = 1$ and by duality $m_a = 0$. For general genus, we have the following:

**Theorem 2.** Let $C$ be a curve of genus $g \geq 3$ with general moduli. For the slopes of the cones in $C_2$, in the fourth quarter of the $\theta$, $x$-plane, we have

1. If $g$ is a square, then $m_{ef} = m_a = g - \sqrt{g}$.

2. If $g$ is not a square and $g > 3$, then $g - \lfloor \sqrt{g} \rfloor + 1 \geq m_{ef} \geq g - \sqrt{g}$ and so

$$g - \sqrt{g} \geq m_a \geq g - \frac{g}{\sqrt{g} - 1}.$$  

3. If $g = 3$, then $m_{ef} = \frac{4}{3}$ and $m_a = \frac{g}{3}$.

**Proof.** We use degenerations to special curves: From the formula (6) we have that the class of $\gamma_2(g_d^1)$ in $C_2$ is given by

$$\gamma_2(g_d^1) = \theta - (g - d + 1)x.$$  

Therefore if $d < \sqrt{g} + 1$, the slope of the divisor is

$$m^0 = g - d + 1 \geq g - \sqrt{g}.$$  

Take a smooth curve $C^0$ with a $g_d^1$, $d < \sqrt{g} + 1$. If in addition, we choose the curve to be “general” having such a $g_d^1$, then the corresponding divisor $\Gamma_2(g_d^1)$ is irreducible in $C_2$. Since $C^0$ is special, the $H^*_a(C^0, Q)$ may not be generated by $x$, $\theta$. Consider in $H^*_a(C^0, Q)$ the plane $\Pi$ spanned by $x$, $\theta$. By the previous analysis, since $m^0 > g - \sqrt{g}$ the intersection of the effective cone with the plane $\Pi$ is given by the slopes $m_{ef}^0 = g + 1$ and $m_{ef}^0 = g - d + 1$.

Since $\mathcal{M}_g$ is connected, we can find a flat family of smooth curves $\mathcal{C} \to \Delta$, $\Delta$ a disk, with central fiber $C^0$ and the other fiber curves with general moduli.
Let $\Phi: \mathbb{G}_2 \to \Delta$ be the flat family with fibers the 2-symmetric products of the fibers of $\mathbb{G} \to \Delta$.

Suppose that the general curve has an irreducible divisor with slope $m$, $g - 1 \geq m > g - \sqrt{g}$. By the above observation, this is unique and so, it gives rise to an effective divisor in $\mathbb{G}_2 \setminus \Phi^{-1}(0)$. Degenerating to the central fiber $C^0_2$, we get an effective divisor $D^0$. Since the degeneration "preserves" the algebraic equivalence we have that $D^0$ belongs to $\Pi$ and so, since it is effective, the slope $m$ satisfies $g - d + 1 \geq m \geq g - \sqrt{g}$.

Therefore if $g$ is a square i.e. $g = k^2$, then choosing $d = k - 1 = \sqrt{g} - 1$ we get that $m$ has to satisfy $m \leq g - \sqrt{g}$.

Since for slopes smaller than $g - \sqrt{g}$ we are in the effective cone by Lemma 4, we conclude that

$$m_{ef} = m_a = g - \sqrt{g}.$$ 

If $g$ is not a square, choosing $d = \lceil \sqrt{g} \rceil$ we have

$$g - \sqrt{g} \leq m_{ef} \leq g - \lceil \sqrt{g} \rceil + 1.$$ 

The estimation of $m_a$ comes from the duality of the ample and effective cone. Of course as $g \to \infty$ we have $m_{ef} \sim m_a \sim g - \sqrt{g}$. Q.E.D.

The case $g = 3$. In this case we can obtain for the general curve an irreducible divisor with slope $\frac{4}{3}$. (See Figure 2.) It is known that a general smooth curve $C$ of genus 3, can be represented as a smooth plane quartic. We construct the following divisor in $C_2$: For each point $P$ in $C$ consider the tangent at that point. This intersects the curve at two additional points $Q$, $R$. Take now in $C_2$ the divisor $D$ consisting of all the sums $Q + R$ (with $P$ moving on $C$). In order to calculate the class of $D$, we find its intersections with $x$ and $\delta_2$.

Intersection with $x$. The degree of the dual curve is 12. Fixing a point $Q$ on $C$, we ask for the number of tangents to the curve passing through $Q$ (excluding the tangent to the curve at $Q$). These are 10. So $Dx = 10$.

Intersection with $\delta_2$. This is twice the number of bitangents to $C$, so $D\delta_2 = 28 \times 2 = 56$.

Therefore if $D \sim a\theta - bx$ then, $Dx = 10$ i.e. $3a - b = 10$ and $D\theta = 56$ i.e. $2(a\theta - bx)(4x - \theta) = 56$ so $6a - b = 28$. This gives $a = 6$, $b = 8$, i.e. $m_{ef} = \frac{4}{3}$, $m_a = \frac{6}{3}$.

6. ONE SIDE SLOPE OF EFFECTIVE CONE IN $C_d$

Theorem 3. The boundary of the effective cone in $C_d$, in the second quarter of the $\theta$, $x$-plane, is given by the class $\delta_d$ of the diagonal $\Delta_d$

$$\delta_d = 2((d + g - 1)x - \theta)$$ i.e. $m_{ef} = d + g - 1$. 

Proof (Induction on $d$). For $d = 2$ it has been proved. Assuming it is true for $d$, we prove it for $d + 1$, i.e. we prove that in $C_{d+1}$, $m_{ef} = d + g$.

Suppose that there exists an irreducible divisor $D$ on $C_{d+1}$ with class $mx_{d+1} - \theta_{d+1}$ and $m < d + g$ (we add subindices for avoiding confusion). Fixing a point in $C$, we define the canonical embedding $i_k: C_k \rightarrow C_{d+1}$. Note that $i_k^*(x_{d+1}) = x_k$ and $i_k^*(\theta_{d+1}) = \theta_k$. The image of $C_d$ is an ample divisor on $C_{d+1}$, see [A-C-G-H, p. 310] and so, $i_2^*(D)$ is effective nonzero on $C_d$. Therefore the slope must satisfy $m \geq d + g - 1$.

Since $\Delta_{d+1}$, $D$ are irreducible the intersection $\Delta_{d+1}D$ is nonempty effective. Indeed, $D$, $\Delta$ are not disjoint. Otherwise,

$$m(d + g)x_{d+1}^2 - (m + d + g)\theta_{d+1}x_{d+1} + \theta_{d+1}^2 = 0.$$ 

Applying $i_2^*$ we get $m(d + g)x_2^2 - (m + d + g)\theta_2x_2 + \theta_2^2 = 0$. Then formula (1) implies $md = (d + 1)g$. Since $i_2^*(\Delta_{d+1})$ contains $\Delta_2$, $\Delta_2$ must be disjoint from $i_2^*(D) = D_2$. Therefore, $\Delta_2D_2 = 0$ i.e. $(mx_2 - \theta_2)\theta_2 = 0$ i.e. $m = 2g$, and the above relation becomes $2gd = (d + 1)g$ i.e. $d = 1$ a contradiction.

Therefore $B_1(D\Delta_{d+1})$, see §1 for definition of $B_1$, lies in the effective cone of $C_d$, i.e.

(11)

$$\text{slope of } B_1(D\Delta_{d+1}) \geq d + g - 1.$$ 

Now,

$$\text{class}(D\Delta_{d+1}) = m(d + g)x_{d+1}^2 - (m + d + g)\theta_{d+1} + \theta_{d+1}^2.$$ 

By Lemma 2 we have

$$B_1(x_{d+1}^2) = 2x_d, \quad B_1(x_{d+1}\theta_{d+1}) = \theta_d + gx_d, \quad B_1(\theta_{d+1}^2) = 2(g - 1)\theta_d.$$ 

Therefore,

$$B_1(D\Delta_{d+1}) = (2md + mg - dg - g^2)x_d - (m + d + g + 2)\theta_d.$$ 

Since $m \geq d + g - 1$, both coefficients are positive and so,

$$\text{slope of } B_1(D\delta_{d+1}) = \frac{2md + mg - dg - g^2}{m + d + g + 2}.$$ 

Relation (11) implies that

$$\frac{2md + mg - dg - g^2}{m + d + g + 2} \geq d + g - 1$$

i.e. $m(d + 1) \geq dg + d^2 + d + 3g - 2$. Since $m < d + g$ we get $(d + g)(d + 1) > dg + d^2 + d + 3g - 2$ or $1 > g$, a contradiction. Q.E.D.

7. Bounds for the effective cone in $C_r$, $r \geq 3$

We start with $C_3$. Let $D$ be a divisor in $C_3$ with class $\theta - mx$, $m \geq 0$, i.e. it "lies" in the fourth quarter. Since $x$ is the class of $i_2(C_2)$ in $C_3$ and

$$\frac{d}{dm}(\theta - mx)^3 = -3(\theta - mx)^2x,$$

we conclude that

$$\frac{d}{dm}(\theta - mx)^3 = 0 \text{ in } C_3 \iff (\theta - mx)^2 = 0 \text{ in } C_2,$$

i.e. when $m = g + \sqrt{g}$ or $m = g - \sqrt{g}$. 


The graph of the \((\theta - mx)^3\) considered as function of \(m\) is given by Figure 3.

**Theorem 4.** We denote by \(r_1\) the root of \((\theta - mx)^3 = 0\) closest to 0. Then for \(m < r_1\), the class \(\theta - mx\) is effective.

**Proof.** Note that this equation has three positive roots, see Figure 3 above. Approximately as \(g \to \infty\), \(r_1\) goes to \(g - \sqrt{3g}\). Say \(D\) a divisor with class \(\theta - mx\) where \(m < r_1\). By Theorem 2 the restriction of \(D\) to \(C_2\) is ample. Since \(D^3 > 0\), Theorem 1, applied to \(H = C_2\), gives the result. Q.E.D.

It is difficult to continue the above method for higher \(r\)'s, since we do not have a good estimate for the ample cone in \(C_3\). For these cases we have

**Theorem 5.** Let \(C\) be a smooth curve of genus \(g \geq 1\) with general moduli. For \(r \geq 3\) we have the following estimates for the effective and ample cone of \(C\) in the fourth quarter of the \(\theta, x\)-plane

1. If \(r \geq g + 1\) then the boundary of the effective and ample cone is given by the \(\theta\) line.
2. If \(3 \leq r \leq g\) then we have the following bounds for the effective cone.

   **Bound from inside:** for each rational \(m\) with
   \[0 \leq m \leq \max\left(\left\lfloor \frac{r}{g} \right\rfloor, g - (r - 1)\sqrt{g} \right)\]
   there is an effective divisor with slope \(m\).

   **Bound from outside:** for each \(m\) with \(m > g - \left\lfloor \sqrt{r - 1\sqrt{g}} \right\rfloor\) there is no effective divisor with slope \(m\).

**Remarks.** 1. For \(r = g\) the slope of the boundary of the effective cone is equal to 1. Indeed in this case \(\left\lfloor \frac{r}{g} \right\rfloor = 1 = g - \left\lfloor \sqrt{r - 1\sqrt{g}} \right\rfloor\).

2. For \(r \leq \sqrt{g}\) we have that
   \[\max\left(\left\lfloor \frac{r}{g} \right\rfloor, g - (r - 1)\sqrt{g} \right) = g - (r - 1)\sqrt{g}.
   \]

For \(r \geq \sqrt{g}\) we have that
   \[\max\left(\left\lfloor \frac{r}{g} \right\rfloor, g - (r - 1)\sqrt{g} \right) = \left\lfloor \frac{r}{g} \right\rfloor.
   \]

**Proof.** The proof of the first part of the theorem is easy: For \(r \geq g + 1\) we have that \(\theta' = \nu^*\left(\frac{\theta'}{r}\right) = 0\) and so, the class \(\theta\) is not ample. On the other hand, since \(\theta\) is ample on the Jacobian and \(x\) is ample on \(C\), see [A-C-G-H, p. 310], projection formula and Lemma 3, imply that the class \(\theta + \varepsilon x\) is ample, for each \(\varepsilon > 0\). Therefore the bound for the ample cone is given by the line \(\theta\). Also any divisor with class \(\theta - \sigma x\) cannot be effective since there is an \(\varepsilon > 0\) small enough with \((\theta + \varepsilon x)^r - 1(\theta - \sigma x) < 0\); a contradiction by Lemma 3. Therefore the bound of the effective cone is given by the \(\theta\) line too.

To prove the second part of the theorem we use again degenerations to special curves. We start with a lemma:
Lemma 9. If a curve $C$ has a “general” $g_d^{r-1}$, $r \geq 3$ (i.e. without base points and not composed with an involution), then for any $d \geq r$ the divisor $\Gamma_r(g_d^{r-1})$ in $C_r$ is irreducible.

Proof. This is an application of the fact that the monodromy acts as the full symmetric group on the generic divisor of $g_d^r$. Q.E.D.

Let us now do some calculations. Recall from relation (6), that the class of $\gamma_r(g_d^{r-1})$ is given by $\theta - (g - d + r - 1)x$. Formula (1) gives that

$$x^k \theta^{r-k} = \frac{g!}{(g - r + k)!} \quad \text{and} \quad x^{k+1} \theta^{r-k-1} = \frac{g!}{(g - r + k + 1)!}.$$ 

Using Lemma 8 we have for $d + 1 \geq 2r$ that

$$\gamma_r(g_d^{r-1}) = \sum_{k=0}^{r-1} \binom{d - r - g + 1}{k} \frac{x^k \theta^{r-k-1}}{(r - k - 1)!}.$$ 

Therefore intersection number $I = \gamma_r(g_d^{r-1}) \cdot \gamma_r(g_d^{1})$ is

$$I = \sum_{k=0}^{r-1} \binom{d - r - g + 1}{k} \frac{g!}{(g - r + k)!(r - k - 1)!} - (g - d + r - 1) \sum_{k=0}^{r-1} \binom{d - r - g + 1}{k} \frac{g!}{(g - r + k + 1)!(r - k - 1)!} - g \sum_{k=0}^{r-1} \binom{d - r - g + 1}{k} \frac{g}{r - k - 1} - (g - d + r - 1) \sum_{k=0}^{r-1} \binom{d - r - g + 1}{k} \frac{g}{r - k - 1} = A(d^2 - 2d(r - 1) - (r - 1)(g - r - 1)) \quad (A \text{ a positive constant}).$$ 

This implies that

$$I \leq 0 \Leftrightarrow r - 1 - \sqrt{r - 1}\sqrt{g} \leq d \leq r - 1 + \sqrt{r - 1}\sqrt{g}.$$ 

Take now a smooth curve $C^0$ having a “general” $g_{d_0}^{r-1}$ with $d_0 = r + 1 + [\sqrt{r - 1}\sqrt{g}]$ (note that $d_0 \geq r$). We claim that the irreducible divisor $D = \Gamma_r(g_{d_0}^{r-1})$ with class $\gamma_r(g_{d_0}^{r-1}) = \theta - (g - [\sqrt{r - 1}\sqrt{g}])x$ gives the bound for the effective cone in the $\theta, x$-plane for $(C^0)_r$. Indeed, note first that the divisor $D$ is covered by a family of curves $\mathfrak{C}_{d_0 - r + 2}$ with class $\gamma_r(g_{d_0}^{1})$. These curves correspond to the various $g_d^{1}$'s obtained by the one-parameter family of hyperplane sections of the image of the curve in $\mathbb{P}^{r-1}$, through collections of $r - 2$ fixed points on this curve. Suppose that there exists another irreducible effective divisor $D'$ with slope $m'$ strictly greater than the slope of $D$. Then relation (13) implies that $D' \cdot \gamma_r(g_{d_0 - r + 2}^{1}) < 0$ and so, since $D'$ is irreducible we get that all the members of $\mathfrak{C}_{d_0 - r + 2}$ are contained in $D'$. But since the divisor
$D'$ is "covered" by the family $\mathcal{B}_{d_0-r+2}$ this implies that $D$ is contained in $D'$. A contradiction. The rest of the proof for the bound from outside goes, using degenerations after a possible base change, as the proof of Theorem 2.

To prove the case for the bound from inside, we use the maps $A_k$ defined in the introduction of this paper. From formula (2) we have

$$A_{r-2}(x_r) = (r-1)x_2 \quad \text{and} \quad A_{r-2}(\theta_r) = \theta_2 + g(r-2)x_2.$$  \hspace{1cm} (14)

Since in $C_2$ the slope $m$ with $m < g - \sqrt{g}$ is "effective", pulling back by the $A_{r-2}$ map we get in $C_r$ that

$$A_{r-2}(\theta_r - (g - \sqrt{g})x_r) = \theta_2 - (g - (r-1)\sqrt{g})x_2.$$  \hspace{1cm} (15)

On the other hand we have that for $d = g+r-1 - \lfloor \frac{r}{2} \rfloor$, a general curve has a $g^r_d-1$ (this is the minimum $d$ for which the Brill-Noether number $\rho$ is nonnegative). Using formula (6) we obtain the following class of an effective divisor in $C_r$:

$$\gamma_r(g^r_d-1) = \theta - \lfloor \frac{r}{2} \rfloor x$$  \hspace{1cm} (16)

and this concludes the proof. Q.E.D.

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