SECTIONS OF COMPLEX CONVEX BODIES

ABSTRACT

The Fourier analytic approach to sections of convex bodies has been developed recently, and the main idea is to express different parameters of a body in terms of the Fourier transform and then apply methods of Fourier analysis to solve geometric problems. The original Fourier approach applies to convex bodies in $\mathbb{R}^n$.

This thesis is focused at extending this approach to the complex case, where origin symmetric complex convex bodies are convex subsets of $\mathbb{C}^n$. If considered as convex bodies in $\mathbb{R}^{2n}$ complex convex bodies acquire the property of invariance with respect to certain rotations. This crucial observation arises from the nature of the norm of the bodies. Also complex hyperplanes correspond to only few of $(2n-2)$-dimensional subspaces of $\mathbb{R}^{2n}$. These facts motivated the study of the complex analogue of certain problems on sections of real convex bodies.

In chapter 2 we present the solution of the complex Busemann-Petty problem which surprisingly gives a different answer than the original Busemann-Petty problem.

In chapter 3 we introduce a modification of the complex Busemann-Petty problem and give necessary conditions to obtain a positive answer to the problem in all dimensions.

Chapter 4 is dedicated to the study of a generalization of the complex Busemann-Petty, where the volume is replaced by any measure. This generalization forms the complex analogue to Zvavitch's result on the complex Busemann-Petty problem for arbitrary measures.

In chapter 5 we study the extremal sections of complex l_p-balls for $0 < p \leq 2$ by complex hyperplanes.

Lastly, in Chapter 6, we prove that the complex unit ball, $B_p(\mathbb{C}^n)$, of $l_p,\ p > 2$ is not a k-intersection body, if $k < 2n - 4$.

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Contents

Chapter 1

Introduction

1.1 Problems on sections of convex bodies in $\mathbb{R}^n$

In this thesis we study complex analogs of several problems from the theory of convex geometry in $\mathbb{R}^n$.

In 1956 Busemann and Petty [BP], published an article posing a series of problems on the theory of convex bodies. One of them, known as *The Busemann-Petty problem* asks the following question: Suppose that K and L are two origin symmetric convex bodies in $\mathbb{R}^n$ such that for every $\xi \in S^{n-1}$,

$$\mathrm{Vol}_{n-1}(K \cap \xi^{\perp}) \leq \mathrm{Vol}_{n-1}(L \cap \xi^{\perp}).$$

Does it follow that

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L) \quad ?$$

The problem was solved in the late 90's as a result of the work of many mathematicians, ([Ha], [Gie], [LR], [Ba], [Gi], [Bo], [Bu], [Lu], [Pa], [Ga1], [Ga2], [K2], [K3], [Zh1], [Zh2], [GKS]; see [K9, p.3] for the history of the solution). The answer to the Busemann-Petty problem is affirmative for $n \leq 4$ and negative for $n \geq 5$.

Since the answer to the original real Busemann-Petty problem is negative in most dimensions, it is natural to ask what condition on the $(n-1)$-dimensional volumes of central sections allow to compare the n-dimensional volumes. Such conditions were found by Koldobsky, Yaskin and Yaskina ([KYY]). The result is as follows: For an origin symmetric convex body D in $\mathbb{R}^n$ define the section function

$$S_D(\xi) = \mathrm{Vol}_{n-1}(D \cap \xi^{\perp}),\ \xi \in S^{n-1}.$$

Suppose K and L are origin symmetric convex smooth bodies in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$ with $\alpha \geq n-4$. Then, the inequality

$$\left(-\Delta\right)^{\alpha/2} S_K(\xi) \leq \left(-\Delta\right)^{\alpha/2} S_L(\xi),\ \xi \in S^{n-1}$$

implies that $\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L)$. If $\alpha < n-4$ this is not necessarily true. Here, Δ is the Laplace operator in $\mathbb{R}^n$.

A few years after the complete solution of the Busemann-Petty problem a generalization of the problem was found by Zvavitch [Zv], who showed that one can replace the volume by essentially any measure on $\mathbb{R}^n$. Namely, if we consider any even continuous positive function f on $\mathbb{R}^n$ and denote by μ the measure with density f, we can define

$$\mu(D) = \int_D f(x)dx \qquad \text{and} \qquad \mu(D \cap \xi^{\perp}) = \int_{D\cap\xi^{\perp}} f(x)dx,$$

for every closed bounded set D in $\mathbb{R}^n$ and every direction $\xi \in S^{n-1}$. Then, we have the following problem: Suppose that K and L are two origin symmetric convex bodies in $\mathbb{R}^n$ such that, for every $\xi \in S^{n-1}$,

$$\mu(K \cap \xi^{\perp}) \leq \mu(L \cap \xi^{\perp}).$$

Does it follow that

$$\mu(K) \leq \mu(L) \ ?$$

Surprisingly, the answer remains the same as in the original problem, namely, it is affirmative $n \leq 4$ and negative for $n \geq 5$.

Zvavitch's ideas for general measures were applied and further developed in [Rb1], [Y1] and [Y2], for hyperbolic and spherical spaces and for sections of lower dimensions.

The study of extremal sections of l_p-balls has long history and many mathematicians have contribute to it. The extremal hyperplane sections of the cube are known in both real and complex cases. Hadwiger [Ha] proved that the minimal volume of hyperplane sections of the real unit cube is equal to 1 and corresponds to the sections parallel to the faces. Different proofs of this fact were later given by Vaaler [V], who generalized the result to sections of arbitrary dimensions, Hensley [He] and Ball [B1]. It was shown by Barthe and Koldobsky, [BK], that this property of the cube is in some sense stable, i.e. for every $0 < t < 3/4$ the slab parallel to the face has minimal volume among all central slabs of the cube with fixed width t. The exact upper bound $\sqrt{2}$ for the volume of hyperplane sections of the real unit cube was found by Ball [B1] and corresponds to the hyperplane orthogonal to the vector $(1, 1, 0, \ldots, 0)$. The case of the complex cube was studied by Oleszkiewicz and Pelczynski [OP], who proved that the minimal sections are the ones orthogonal to vectors with only one non-zero coordinate, and the maximal sections are orthogonal to vectors of the form $e_j + \sigma e_k$, where $j \neq k$, e_j and e_k are standard basic vectors, and $\sigma \in \mathbb{C}$, $|\sigma| = 1$. Note that the "minimal" part also follows from an earlier result of Meyer and Pajor [MP], Corollary 2.5.

The critical sections of l_p-balls, $0 < p < \infty$ are different for $p > 2$ and $p < 2$. Meyer and Pajor [MP] proved that the section orthogonal to the vector $(1, 0, \ldots, 0)$ is minimal for $p > 2$ and maximal for $1 \leq p < 2$. The latter result also holds for $0 < p < 1$, as proved by Caetano [Ca]. In the same paper, Meyer and Pajor proved that the minimal hyperplane section of $B_1(\mathbb{R}^n)$ is the one perpendicular to the vector $(1, 1, \ldots, 1)$ and conjectured that this is also true for every $p \in [1, 2]$. This conjecture was proved in [K2] for $0 < p \leq 2$. It is still an open question what are the maximal sections of $B_p(\mathbb{R}^n)$ when $2 < p < \infty$. Oleszkiewicz [O] showed that the answer must depend on p and the dimension.

1.2 Basic concepts and notation

In this thesis a *star body* K is a closed bounded set K in $\mathbb{R}^n$, with $0 \in int(K)$, where every straight line passing through the origin, crosses the boundary at exactly two points different from the origin. Moreover, the boundary of K is continuous in the sense that the *Minkowski functional* of K, defined by

$$\|x\|_K = min\,\{a \geq 0 : x \in aK\}\,,$$

is a continuous function on $\mathbb{R}^n$. A star body K is called origin symmetric, if $\|x\|_K = \|-x\|_K$ for every $x \in \mathbb{R}^n$. If, in addition, K is a convex subset of $\mathbb{R}^n$, then it is an origin symmetric *convex body.* Note that then the Minkowski functional $\|x\|_K$ becomes a norm as a homogeneous function of degree 1 on $\mathbb{R}^n$ and the body K is the unit ball of the normed space $\big(\mathbb{R}^n, \|\cdot\|_K\big)$.

We define the *support function* of a body K in $\mathbb{R}^n$ as

$$h_K(x) = \max\{< x, u >, u \in K\},\ x \in \mathbb{R}^n.$$

If K is origin symmetric, then h_K is a norm on $\mathbb{R}^n$. The convex body K^* which is the unit ball of this norm, is called the *polar body* of K:

$$h_K(x) = \|x\|_{K^*},\ x \in \mathbb{R}^n.$$

The *radial function* of K is defined by

$$\rho_K(u) = \sup\{\lambda > 0 : \lambda u \in K\},\ u \in \mathbb{R}^n \setminus \{0\}.$$

For bodies that contain 0 in their interior the radial function is positive and homogeneous of degree -1. From the definition we can easily see that $\rho_K(x) = \|x\|_K^{-1},\ x \in \mathbb{R}^n \setminus \{0\}$. If $u \in S^{n-1}$, then $\rho_K(u)$ is the "radius" of K in the direction of u, i.e. the distance from the origin to the boundary of K in the direction of u.

We denote by B_2^n and S^{n-1}, the unit ball and the unit sphere of $\mathbb{R}^n$, with volumes

$$|B_2^n| = \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})}, \qquad |S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})},$$

respectively. For any star body D, we write $vol_m(D)$ for the m-dimensional volume of D and use the notation $|\cdot|_2$ for the Euclidean norm in the proper space. If χ is the indicator function of the interval $[-1,1]$, then $\chi\big(\|\cdot\|_K\big)$ is the indicator function of the body K.

Suppose K is a star body in $\mathbb{R}^n$, then we can easily obtain a *polar formula for the volume of* K,

$$\begin{aligned} \mathrm{Vol}_n(K) &= \int_{\mathbb{R}^n} \chi\big(\|x\|_K\big)dx = \int_{S^{n-1}} \int_0^{\infty} r^{n-1}\chi\big(\|r\theta\|_K\big)drd\theta \\ &= \int_{S^{n-1}} \int_0^{\|\theta\|_K^{-1}} r^{n-1}drd\theta = \frac{1}{n}\int_{S^{n-1}} \|\theta\|_K^{-n}d\theta. \qquad (1.2.1) \end{aligned}$$

An important tool for the approximation of a body is the *radial metric* which is defined on the set of all origin symmetric star bodies in $\mathbb{R}^n$ by

$$\rho\big(K,L\big) = \max_{u\in S^{n-1}} |\rho_K(u) - \rho_L(u)|.$$

We denote by $C^k\big(S^{n-1}\big),\ \big(C^\infty\big(S^{n-1}\big)\big)$, the space of k-times (infinitely) differentiable functions on the unit sphere S^{n-1}, equipped with the topology of the uniform convergence of all the derivatives.

A star body K is m-smooth (infinitely) smooth, if the restriction of $\|x\|_K$ to the sphere belongs to the class $C^k\big(S^{n-1}\big),\ \big(C^\infty\big(S^{n-1}\big)\big)$.

It is well known that a convex body in $\mathbb{R}^n$ can be approximated, in the radial metric, by a sequence of infinitely smooth convex bodies, preserving certain useful properties of the original body. Since the Minkowski functional of a convex body is the support function of the polar body, the following theorem describes this approximation.

Theorem 1.2.1. ([Sch Theorem 3.3.1]) *Let $\varepsilon > 0$ and suppose $\phi : [0, \infty) \to 0, \infty)$ is a C^∞ function with support in $[\frac{\varepsilon}{2}, \varepsilon]$ so that $\int_{\mathbb{R}^n} \phi(|z|_2)dz = 1$. If $h_D : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the support function of a convex body D then the function $\tilde{h}_D$ defined by*

$$\tilde{h}_D(x) := \int_{\mathbb{R}^n} h_D(x + |x|z)\phi(|z|)dz, \quad x \in \mathbb{R}^n$$

is infinitely differentiable on $\mathbb{R}^n \setminus \{0\}$ and it is the support function of a convex body D_ε, i.e. $\tilde{h}_D = h_{D_\varepsilon}$. Moreover, $\tilde{h}_D$ has the following properties:

(a) $\rho(\tilde{h}_D) = \tilde{h}_{\rho D}$, for every rotation ρ of $\mathbb{R}^n$,

(b) $|h_{D_\varepsilon}(u) - h_D(u)| < r\varepsilon$.

Proof. Since h_D is the support function of D, the homogeneity of $\tilde{h}$ immediately follows from the definition. For simplicity we will use h for h_D. To prove the subadditivity we define a function g_z as follows: Let $z \in \mathbb{R}^n$ and

$$g_z(x) := h(x + |x|z) + h(x - |x|z), \quad x \in \mathbb{R}^n.$$

For $x, y \in \mathbb{R}^n$ and $\alpha \in [0,1]$ we have that

$$\begin{aligned} g_z(x+y) &\leq h(x+\alpha|x+y|z) + h(y+(1-\alpha)|x+y|z) \\ &+ h(x-\alpha|x+y|z) + h(y-(1-\alpha)|x+y|z), \end{aligned} \quad (1.2.2)$$

since h is subadditive. Without loss of generality we may assume that x and y are linearly independent. We put $\alpha = \frac{|x|}{|x|+|y|}, \beta = \frac{|x+y|}{|x|}, \gamma = \frac{|x+y|}{|y|}$, then $1-\alpha\beta > 0$ and $1-(1-\alpha)\gamma > 0$. Since,

$$2(x+\alpha|x+y|z) = (1+\alpha\beta)(x+|x|z) + (1-\alpha\beta)(x-|x|z),$$

$$2(x-\alpha|x+y|z) = (1-\alpha\beta)(x+|x|z) + (1+\alpha\beta)(x-|x|z),$$

we have that

$$h(x+\alpha|x+y|z)+h(x-\alpha|x+y|z) \leq h(x+|x|z)+h(x-|x|z) = g_z(x). \quad (1.2.3)$$

Similarly, we obtain that

$$h(y+(1-\alpha)|x+y|z) + h(y-(1-\alpha)|x+y|z) \leq g_z(y). \quad (1.2.4)$$

Now, by equations (1.2.2), (1.2.3) and (1.2.4), we have that

$$g_z(x+y) \leq g_z(x) + g_z(y).$$

By the definition of g_z and the fact that $\phi \geq 0$ the subadditivity is proved. So, $\tilde{h}$ is a support function.

To prove that $\tilde{h}$ is a C^∞ function on $\mathbb{R}^n \setminus \{0\}$, we write

$$\tilde{h}(u) = \int_{\mathbb{R}^n} h(u+z)\phi(|z|)dz = \int_{\mathbb{R}^n} h(y)\phi(|y-u|)dy, \ u \in S^{n-1}.$$

Then, for $x \in \mathbb{R}^n \setminus \{0\}$,

$$\tilde{h}(x) = |x| \int_{\mathbb{R}^n} h(y)\phi\Big(\big|y - \frac{x}{|x|}\big|\Big)dy.$$

Let ρ be a rotation of $\mathbb{R}^n$, proper or improper. We define $(\rho h)(x) := h(\rho^{-1}x), \ x \in \mathbb{R}^n$ and have that

$$\begin{aligned}(\rho\tilde{h})(x) &= \int_{\mathbb{R}^n} h(\rho^{-1}x + |\rho^{-1}x|z)\phi(|z|)dz \\ &= \int_{\mathbb{R}^n} h(\rho^{-1}(x + |x|\rho z)\phi(\rho|z|)dz \\ &= \int_{\mathbb{R}^n} (\rho h)(x + |x|z)\phi(|z|)dz.\end{aligned}$$

This proves that $\widetilde{(\rho h)} = \rho\tilde{h}$, which means that $\rho(\tilde{h}_D) = \tilde{h}_{\rho D}$. This completes the proof of part (a).

To prove part (b) we need the following fact, proved in [Sch, Lemma 1.8.10]: Suppose that K and L are two convex bodies contained in the Euclidean ball of radius r, $B_2^n(r)$. Then for $x, y \in \mathbb{R}^n$

$$|h_K(x) - h_L(y)| \leq r|x-y| + \max\{|x|, |y|\}\delta(K, L), \tag{1.2.5}$$

where $\delta(K,L)$ is the Hausdorf metric, defined as $\delta(K,L) = \min\{\lambda \geq 0 : K \subset L + \lambda B_2^n, L \subset K + \lambda B_2^n\}$.

Now, we assume that there exists $r > 0$, so that the body $D \subset B_2^n(r)$. Using the triangle inequality, the fact that $\int_{\mathbb{R}^n} \phi(|z|_2)dz = 1$ and equation (1.2.5), we have that for $u \in S^{n-1}$,

$$\begin{aligned}
|h_{D_\varepsilon}(u) - h_D(u)| &= \Big| \int_{R^n} h_D(u+z)\phi(|z|)dz - h_D(u) \Big| \\
&\leq \int_{R^n} \big| h_D(u+z) - h_D(u) \big| \phi(|z|)dz \\
&\leq \int_{\mathbb{R}^n} r|z|\phi(|z|)dz < r\varepsilon,
\end{aligned}$$

since $\phi(|z|) = 0$ for $|z| > \varepsilon$. Part (b) is proved since $u \in S^{n-1}$ was arbitrary.

□

For $z \in \mathbb{C}$, with $Rez > 0$, the *Gamma function* is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt. \tag{1.2.6}$$

It is not difficult to see that, if we apply a change of variables, we have that, for any $p, q > 0$,

$$\int_0^\infty r^{q-1}e^{-r^p}dr = \frac{\Gamma(\frac{q}{p})}{p}. \tag{1.2.7}$$

The *Beta function* is defined as follows: For every $\alpha, \beta > 0$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \tag{1.2.8}$$

Lastly, we often use different generalization of Brunn's theorem, (see for example [K9, Theorem 2.3]). The theorem states that, for an origin symmetric convex body and a fixed direction, the central hyperplane section has the maximal volume among all hyperplane sections perpendicular to the given direction.

1.3 Complex convex bodies and hyperplanes in $\mathbb{C}^n$

Let $\xi \in \mathbb{C}^n, \ |\xi| = 1$ we denote by

$$H_\xi = \{z \in \mathbb{C}^n \ : \ (z, \xi) = \sum_{k=1}^n z_k \overline{\xi}_k = 0\}$$

the complex hyperplane perpendicular to ξ.

Origin symmetric convex bodies in $\mathbb{C}^n$ are the unit balls of norms on $\mathbb{C}^n$. We denote by $\| \cdot \|_K$ the norm corresponding to the body K

$$K = \{z \in \mathbb{C}^n \ : \|z\|_K \leq 1\}.$$

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the mapping

$$\xi = (\xi_1, \dots, \xi_n) = (\xi_{11} + i\xi_{12}, \dots, \xi_{n1} + i\xi_{n2}) \longmapsto (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}) \tag{1.3.1}$$

and observe that under this mapping the complex hyperplane H_ξ turns into a two-codimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors

$$\xi = (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}) \text{ and } \xi^\perp = (-\xi_{12}, \xi_{11}, \ldots, -\xi_{n2}, \xi_{n1}).$$

We denote by $B_p(\mathbb{C}^n)$ and use the notation $\|\cdot\|_p$ for the unit ball and the norm, respectively, of the complex n-dimensional l_p space. Then $B_p(\mathbb{C}^n)$ becomes a $2n$-dimensional subset of $\mathbb{R}^{2n}$ under the mapping (1.3.1):

$$\begin{aligned} B_p(\mathbb{C}^n) &= \{(x_{11} + ix_{12}, \ldots x_{n1} + ix_{n2}) \in \mathbb{C}^n : \sum_{j=1}^n (x_{j1}^2 + x_{j2}^2)^{\frac{p}{2}} \leq 1\} \\ &= \{(x_{11}, \ldots x_{n1}, x_{n2}) \in \mathbb{R}^{2n} : \sum_{j=1}^n (x_{j1}^2 + x_{j2}^2)^{\frac{p}{2}} \leq 1\}. \end{aligned}$$

If $p \geq 1$ then $B_p(\mathbb{C}^n)$ is an origin symmetric convex body in $\mathbb{R}^{2n}$.

Since norms on $\mathbb{C}^n$ satisfy the equality

$$\|\lambda z\| = |\lambda| \|z\|, \quad \forall z \in \mathbb{C}^n, \ \forall \lambda \in \mathbb{C},$$

origin symmetric complex convex bodies correspond to those origin symmetric convex bodies K in $\mathbb{R}^{2n}$ that are invariant with respect to any coordinate-wise two-dimensional rotation. Namely, for each $\theta \in [0, 2\pi]$ and for each $x \in \mathbb{R}^{2n}$,

$$\|x\|_K = \|R_\theta(x_{11}, x_{12}), \ldots, R_\theta(x_{n1}, x_{n2})\|_K, \tag{1.3.2}$$

where R_θ stands for the counterclockwise rotation of $\mathbb{R}^2$ by the angle θ with respect to the origin. If a convex body satisfies (1.3.2) we will say that *it is invariant with respect to all* R_θ. The unit ball $B_p(\mathbb{C}^n)$ is an example of an origin symmetric invariant with respect to all R_θ body in $\mathbb{R}^{2n}$.

It is easy for one to see that the central section of a complex convex body K by a hyperplane H in $\mathbb{C}^n$, when viewed as a subset of $\mathbb{R}^{2n-2}$, is a convex body that is also invariant with respect to all R_θ. In other words, if ρ_θ is an R_θ-rotation, $\theta \in [0, 2\pi]$, of $\mathbb{R}^{2n}$, then $\|x\|_{\rho_\theta(K\cap H)} = \|x\|_{K\cap H}$.

As a natural consequence, our problems on sections of complex convex bodies will be reformulated as problems on sections of invariant with respect to all R_θ convex bodies in $\mathbb{R}^{2n}$, under the theory described in this section.

1.4 The Fourier analytic approach

Throughout this study, we use some important facts from the theory of distributions, as well as from the theory of the Fourier transform of distributions. Here, we present the Fourier analytic approach mostly using the definitions from the real space $\mathbb{R}^n$, (see [GS], [K9] and [Ru] for more details). We will apply this analytic approach on $\mathbb{R}^{2n}$.

We denote by $\mathcal{S}$ the set of *test functions* which is defined to be the class of all the C^∞ functions ϕ on $\mathbb{R}^n$ with values in $\mathbb{C}$ that are rapidly decreasing at infinity, and with $\mathcal{S}'$ we denote the space of distributions on $\mathcal{S}$.

The space $\mathcal{S}$, known as the *Schwartz class*, consists of all the functions with all their partial derivatives continuous and so that

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \big(D^\beta \phi\big)(x)| < \infty,$$

for all n-tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ of non-negative integers. The n-tuples α, β are called multi-indices and are defined for every $x \in \mathbb{R}^n$ to be so that $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and $D^\beta = \partial^{\beta_1+\beta_2+\cdots+\beta_n}/\partial x_1^{\beta_1}\partial x_2^{\beta_2}\cdots\partial x_n^{\beta_n}$.

The set $\mathcal{S}'$, of all the continuous linear on $\mathcal{S}$, forms the space of *tempered distributions* when $\mathcal{S}$ is induced with the topology described below.

For every ordered pair (α, β) of non negative n-tuples we define the semi-norm $\rho_{\alpha\beta}(\phi) = \sup_{x\in\mathbb{R}^n} |x^\alpha (D^\beta \phi)(x)|$ in $\mathcal{S}$ and the semi-metric $d'_{\alpha\beta}(\phi, \psi) = \rho_{\alpha\beta}(\phi - \psi)$. Let $d'_1, d'_2, \dots$ be all the semi-metrics of this form, we define $d_n := \frac{d'_n}{1+d'_n}, \quad n = 1, 2, \dots$. It is not difficult to see that d_n is also a semi-norm, equivalent to d'_n and that $d_n \leq 1$. The metric that defines the topology of $\mathcal{S}$, is

$$d = \sum_{n=1}^{\infty} 2^{-n} d_n.$$

We can easily see that $\phi_k \xrightarrow{d} \phi$ if and only if $\phi_k \to \phi$ with respect to every d_n, as $k \to +\infty$. Hence, we have that the vector operations

$$\begin{aligned} (\phi, \psi) &\longrightarrow \phi + \psi \\ (\alpha, \phi) &\longrightarrow \alpha\phi, \quad \alpha \in \mathbb{C} \end{aligned}$$

are continuous. This implies that the space $\big(\mathcal{S}, d\big)$ is a topological vector space.

A linear functional L of $\mathcal{S}$ is a *tempered distribution* if and only if there exists a constant $c > 0$ and $m, l \in \mathbb{Z}$ such that

$$|L(\phi)| \leq c \sum_{\substack{|\alpha| \leq l \\ |\beta| \leq m}} \rho_{\alpha\beta}(\phi),$$

for every $\phi \in \mathcal{S}$ ([SW p.22]).

If $f \in \mathcal{S}'$, $\phi \in \mathcal{S}$, then

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx.$$

As mentioned before, the main tool of this study is the Fourier transform of distributions. See [GS], [Ru] and [K9] for more details on this subject. Our definition is slightly different from those in [GS] or [Ru]. Let $f \in L^1(\mathbb{R}^n)$, we define the Fourier transform of f by

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{-i\xi \cdot x} d\xi.$$

If $\phi \in \mathcal{S}$, then $\hat{\phi} \in \mathcal{S}$. Moreover, since the Fourier transform is invertible on $\mathcal{S}$, we have that for every $\phi \in \mathcal{S}$,

$$(\hat{\phi})^\wedge(x) = (2\pi)^n \phi(-x).$$

Suppose $f \in \mathcal{S}'$, then the Fourier transform $\hat{f}$ of a distribution is defined to be the distribution given by

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle, \quad \phi \in \mathcal{S}.$$

From the above we have that, if a test function ϕ is even, then

$$(\hat{\phi})^\wedge = (2\pi)^n \phi \quad \text{and} \quad \langle \hat{f}, \hat{\phi} \rangle = (2\pi)^n \langle f, \phi \rangle,$$

for every $f \in \mathcal{S}'$. Here, we only consider real-valued even test functions ϕ, for which the Fourier transform $\hat{\phi}$ is also an even real-valued test function.

A distribution f is called even homogeneous of degree $p \in \mathbb{R}$, if

$$\langle f, \phi\big(\cdot/t\big) \rangle = |t|^{n+p} \langle f, \phi \rangle,$$

for every $\phi \in \mathcal{S}$, $t \in \mathbb{R}$ me $t \neq 0$. The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$, ([K9, Lemma 2.21]).

A distribution f is called *positive definite* if, for every test function ϕ,

$$\langle f, \phi * \overline{\phi(-x)} \rangle \geq 0.$$

By Schwartz's generalization of Bochner's theorem, this is equivalent to $\hat{f}$ being a positive distribution in the sense that $\langle \hat{f}, \phi \rangle \geq 0$, for every non-negative test function ϕ.

We denote by $\Delta = \sum_{i=1}^{k} \partial^2 / \partial u_i^2$ the k-dimensional Laplace operator in $\mathbb{R}^k$. Then the fractional powers of the Laplacian in $\mathbb{R}^n$ are defined by

$$\left((-\Delta)^{\alpha/2} f \right)^{\wedge} = \frac{1}{(2\pi)^n} |x|_2^{\alpha} \hat{f}(x), \tag{1.4.1}$$

where the Fourier transform is considered in the sense of distributions.

Let q be non integer, then the Fourier transform of the function $|z|^q,\ z \in \mathbb{R}$, is equal to (see [GV vol.1, p.173] and [K9, Lemma 2.23] for more details)

$$(|z|^q)^{\wedge}(t) = -2\Gamma(1+q) \sin \frac{q\pi}{2} |t|^{-q-1}, \qquad t \in \mathbb{R} \tag{1.4.2}$$

A very important tool in the analytic theory, is the Minkowski functional raised to certain powers and treated as a distribution. The following proposition, see [K9, Lemma 2.1], allows us to consider the functional as a locally integrable function on $\mathbb{R}^n$.

Proposition 1.4.1. *Let D be an origin symmetric star body in $\mathbb{R}^n$. Then, for $0 < p < n$, the function $\| \cdot \|_D^{-p}$ is locally integrable on $\mathbb{R}^n$. Also, if f is a* *bounded integrable function on* $\mathbb{R}^n$, *then the function* $\|\cdot\|_D^{-p} f(\cdot)$ *is integrable on* $\mathbb{R}^n$.

Another important fact, that is often used, is the following proposition, proved in [K9, Lemma 3.16]. It shows that the Fourier transform of a smooth homogeneous function is also smooth.

Proposition 1.4.2. ([K9, Lemma 3.16]) *Let* $k \in \mathbb{N} \cup \{0\}$ *and let* $f \in C^{2k}(S^{n-1})$ *be an even function. Suppose that* $q \leq 2k$, *where* q *is not an odd integer. Then, the following hold:*

(i) The Fourier transform of the distribution $f(\theta) r^{-n+q+1}$ *is a homogeneous of degree* $-1-q$, *continuous function on* $\mathbb{R}^n \setminus \{0\}$. *If* $q < 2k$, *then, for every* $x \in \mathbb{R}^n$,

$$|x|_2^{2k} \big(f(\theta) r^{-n+q+1}\big)^\wedge (x)$$

$$= \frac{(-1)^k \pi}{-2\Gamma(2k-q) \sin(\pi(2k-q-1)/2)} \int_{S^{n-1}} |(x,\xi)|^{2k-q-1} \Delta^k \big(f(\theta) r^{-n+q+1}\big)(\xi) d\xi.$$

If $q = 2k$, *then*

$$|x|_2^{2k} \big(f(\theta) r^{-n+q+1}\big)^\wedge (x)$$

$$= (-1)^k \pi |x|_2^{-1} \int_{S^{n-1} \cap (x/|x|_2)^\perp} \Delta^k \big(f(\theta) r^{-n+q+1}\big)(\xi) d\xi.$$

(ii) If $f \in C^\infty(S^{n-1})$*, then, there exists an even function* $g \in C^\infty(S^{n-1})$ *so that for every* $x = t\xi \in \mathbb{R}^n,\ t \neq 0,\ t \in S^{n-1}$,

$$\big(f(\theta)r^{-n+q+1}\big)^\wedge(x) = g(\xi)t^{-1-q},$$

so the Fourier transform of $f(\theta)r^{-n+q+1}$ *is an infinitely smooth function on* $\mathbb{R}^n \setminus \{0\}$.

The following result is a Parseval type identity on the unit sphere S^{n-1}, established in [K5], (see also [K9, Lemma 3.22]). Note that, the key to this formula is that the total homogeneity of the integrands has to be of the order of $-n$.

Proposition 1.4.3. *Let* D *be an infinitely smooth origin symmetric star body in* $\mathbb{R}^n$ *and* $g \in C^{k-1}(\mathbb{R}^n)$ *even homogeneous of degree* $-n+k$ *function. Then* $\hat{g}$ *and* $\big(\|\cdot\|_D^{-k}\big)^\wedge$ *are continuous functions on* S^{n-1}*, extended to homogeneous functions on the whole* $\mathbb{R}^n$ *and*

$$\int_{S^{n-1}} g(\theta)\|\theta\|_D^{-k} d\theta = (2\pi)^n \int_{S^{n-1}} \hat{g}(\xi)\big(\|\theta\|_D^{-k}\big)^\wedge(\xi) d\xi.$$

The classes of k-intersection bodies were introduced in [K5], [K8] as follows: Let $1 \leq k < n$, and let D and L be origin symmetric star bodies in $\mathbb{R}^n$. We say that D is a *k-intersection body of L* if for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$

$$\mathrm{Vol}_k(D \cap H^{\perp}) = \mathrm{Vol}_{n-k}(L \cap H).$$

More generally, we say that an origin symmetric star body D in $\mathbb{R}^n$ is a *k-intersection body* if there exists a finite Borel measure μ on S^{n-1} so that for every even test function $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \|x\|_D^{-k} \phi(x)\, dx = \int_{S^{n-1}} \left(\int_0^{\infty} t^{k-1} \hat{\phi}(t\xi)\, dt \right) d\mu(\xi).$$

Note that k-intersection bodies of star bodies are those k-intersection bodies for which the measure μ has a continuous strictly positive density; see [K8] or [K9, p. 77]. When $k = 1$ we get the class of intersection bodies introduced by Lutwak in [Lu].

A more general concept of embedding in L_{-p} was introduced in [K7]. Let D be an origin symmetric star body in $\mathbb{R}^n$, and $X = (\mathbb{R}^n, \|\cdot\|_D)$. For $0 < p < n$, we say that X embeds in L_{-p} if there exists a finite Borel measure μ on S^{n-1} so that, for every even $\phi \in \mathcal{S}$

$$\int_{\mathbb{R}^n} \|x\|_D^{-p} \phi(x)\, dx = \int_{S^{n-1}} \left(\int_{\mathbb{R}} |z|^{p-1} \hat{\phi}(z\theta)\, dz \right) d\mu(\theta).$$

Obviously, an origin symmetric star body D in $\mathbb{R}^n$ is a k-intersection body if and only if the space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-k}. In this article we use embeddings in L_{-p} only to state some results in continuous form; for more applications of this concept, see [K9, Ch.6].

Embeddings in L_{-p} and k-intersection bodies admit a Fourier analytic characterization that we are going to use throughout this text:

Proposition 1.4.4. ([K8], [K9, Th. 6.16]) *Let D be an origin symmetric star body in $\mathbb{R}^n$, $0 < p < n$. The space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-p} if and only if the function $\|x\|_D^{-p}$ represents a positive definite distribution on $\mathbb{R}^n$. In particular, D is a k-intersection body if and only if $\|x\|_D^{-k}$ is a positive definite distribution on $\mathbb{R}^n$.*

Most of the problems that are presented here require an analytic representation of the $(2n-k)$-dimensional volume, $1 \leq k < 2n$, of sections of convex bodies by subspaces of $\mathbb{R}^{2n}$. A very important tool, in order to obtain such representation, is the distribution $|u|_2^{-q-k}/\Gamma(-q/2)$. This idea was first introduced in [K8] and it is a generalization of the connection between the distribution t_+^q, and the fractional derivatives of the section function, (see [K9, Section 2.6, Theorem 3.18] for details). Here we introduce the theory on $\mathbb{R}^n$.

Let $1 < k < n$. We consider the action of the distributions $|u|_2^{-q-k}/\Gamma(-q/2)$,

$u \in \mathbb{R}^k$ on an infinitely differentiable function A with compact support on $\mathbb{R}^k$. Note that, for our purposes A will only need to be infinitely differentiable or even differentiable up to a certain order, at the origin. For simplicity here we assume that it is a test function.

For $Req < 0$, the distribution is an analytic function of q and

$$\left\langle \frac{|u|_2^{-q-k}}{\Gamma(-\frac{q}{2})}, A(u) \right\rangle = \frac{1}{\Gamma(-q/2)} \int_{\mathbb{R}^k} |u|_2^{-q-k} A(u) du. \tag{1.4.3}$$

A standard normalization argument allows us to have the following proposition, (see [GS, p.71-74]).

Proposition 1.4.5. *Let A be an infinitely differentiable function with compact support, on $\mathbb{R}^k$. Then, the function*

$$q \longmapsto \left\langle \frac{|u|_2^{-q-k}}{\Gamma(-\frac{q}{2})}, A(u) \right\rangle \tag{1.4.4}$$

is an entire function of $q \in \mathbb{C}$. Moreover, if $q = 2m,\ m \in \mathbb{N} \cup \{0\}$, then

$$\left\langle \frac{|u|_2^{-q-k}}{\Gamma(-q/2)}\Big|_{q=2m}, A_{D,H}(u) \right\rangle$$

$$= \frac{(-1)^m |S^{k-1}|}{2^{m+1} k(k+2)...(k+2m-2)} \Delta^m A_{D,H}(0), \tag{1.4.5}$$

Proof : Let $u \in \mathbb{R}^k$. For the first part, we only need to show that the poles of $\langle |u|_2^{-q-k}, A(u) \rangle$ and $\Gamma(-\frac{q}{2})$ coincide.

We use polar coordinates to get that

$$\begin{aligned}\langle |u|_2^{-q-k}, A(u)\rangle &= \int_{\mathbb{R}^k} |u|_2^{-q-k} A(u)du \\ &= \int_0^\infty r^{-q-1}\Big(\int_{S^{k-1}} A(r\theta)d\theta\Big)dr. \end{aligned} \tag{1.4.6}$$

Observe that the inner integral in (1.4.6) is an infinitely differentiable function of r, with compact support, since $A(r\theta)$ vanishes for r large enough. Moreover, all of its odd derivatives vanish at $r = 0$. So, if we expand it in a Taylor series we can easily see that is also an even function of r.

Let $\alpha(r) := \int_{S^{k-1}} A(r\theta)d\theta$. We consider (1.4.6) as the action of r_+^{-q-1}, $t \in \mathbb{R}$ on $\alpha(r)$, where $r_+ = \max\{r, 0\}$. It is known, that r_+^{-q-1} is analytic for $Req > 0$, (see [GS, p.48] and [K9, p.36] for more details). In addition, one may apply analytic continuation to extend r_+^{-q-1} to an entire function of q on $\mathbb{C} \setminus \{0, 1, 2, \ldots\}$. But, since all the odd derivatives of $\alpha(r)$ vanish at zero, we have that the function

$$q \longmapsto \left\langle r_+^{-q-1}, \alpha(r) \right\rangle$$

has simple poles only at $q = 0, 2, 4, \ldots$

On the other hand, as it is known, the Gamma function has simple poles at the points $-m$, $m \in \mathbb{N}_0$. This proves that the function in (1.4.4) is an entire function of $q \in \mathbb{C}$.

To prove the second part of the proposition, first, we need to compute the residue of $\langle |u|_2^{-q-k}, A(u)\rangle$ at $q = 2m,\ m \in \mathbb{N}_0$:

As mentioned above, the distribution r_+^{-q-1} has simple poles for $q = 0, 1, 2, \ldots,$ and the residue of $\langle r_+^{-q-1}, \alpha(r)\rangle$ at $q = m,\ m \in \mathbb{N}_0$, is $\frac{\alpha^{(m)}(0)}{m!}$.

Accordingly, we have that

$$\mathrm{Res}(\langle |u|_2^{-q-k}, A(u)\rangle)\ |_{q=2m} = \frac{\alpha^{(2m)}(0)}{(2m)!}. \tag{1.4.7}$$

On the other hand, if we use the fact that

$$\Delta(|u|_2^{-q-k+2}) = q(q+k-2)|u|_2^{-q-k} \tag{1.4.8}$$

we can find a different representation of the residue of $\langle |u|_2^{-q-k}, A(u)\rangle$ at $q = 2m$. Note that, the equality in (1.4.8) can be proved with a straightforward computation if $Re q < k$, and by analytic continuation for the other values of q. By iteration we find that

$$|u|_2^{-q-k} = \frac{\Delta^m(|u|_2^{-q-k+2m})}{(q+k-2)(q+k-4)\cdots(q+k-2m)q(q-2)\cdots(q-2m+2)} \tag{1.4.9}$$

Now, by the definition of differentiation of distributions, it is true that

$$\langle \Delta^m(|u|_2^{-q-k+2m}), A(u)\rangle = \langle \Delta(|u|_2^{-q-k+2m}), \Delta^m A(u)\rangle. \tag{1.4.10}$$

Hence, by equations (1.4.9) and (1.4.10), we have that the residue of $\langle |u|_2^{-q-k}, A(u)\rangle$ at $q = 2m$ is equal to

$$\frac{\Delta^m A(u)}{2^m m! k(k+2)\cdots(k+2m-2)}, \tag{1.4.11}$$

since at $q = k$ the function has a simple pole with residue $A(0)$.

To complete the proof, we need to calculate the residue of $\Gamma(-\frac{q}{2})$ at $q = 2m$. Since,

$$\Gamma(-\frac{q}{2}) = \frac{2}{|S^{k-1}|}\int_{\mathbb{R}^k} |u|_2^{-q-k} e^{-|u|_2^2} du,$$

equation (1.4.7) gives that the residue is equal to

$$\frac{2}{|S^{k-1}|}\mathrm{Res}(\langle |u|_2^{-q-k}, e^{-|u|_2^2}\rangle) \mid_{q=2m} = \frac{2}{|S^{k-1}|}\frac{(e^{-r^2})^{(2m)}(0)}{(2m)!}. \tag{1.4.12}$$

But $(e^{-r^2})^{(2m)}(0) = (-1)^m \frac{(2m)!}{m!}$. Hence, for $q = 2m$,

$$\left\langle |u|_2^{-q-k}, A(u)\right\rangle = \frac{\mathrm{Res}(\langle |u|_2^{-q-k}, A(u)\rangle) \mid_{q=2m}}{\mathrm{Res}\Gamma(-\frac{q}{2}) \mid_{q=2m}},$$

and by (1.4.11) and (1.4.12) we obtain equation (1.4.5). □

If the function A is even, for $0 < q < 2$, we have (see also [K9, p.39])

$$\left\langle \frac{|u|_2^{-q-k}}{\Gamma(-q/2)}, A(u)\right\rangle$$

$$= \frac{1}{\Gamma(-q/2)} \int_{S^{n-1}} \left(\int_0^\infty \frac{A(t\theta) - A(0)}{t^{1+q}} \, dt \right) \, d\theta. \tag{1.4.13}$$

Note that the function (1.4.4) is equal (up to a constant) to the fractional power of the Laplacian $\Delta^{q/2} A_{D,H}$.

Chapter 2

The complex Busemann-Petty problem

2.1 Introductory

The complex Busemann-Petty problem can be formulated as follows: suppose K and L are origin symmetric invariant with respect to all R_θ convex bodies in $\mathbb{R}^{2n}$ such that

$$\mathrm{Vol}_{2n-2}(K \cap H_\xi) \leq \mathrm{Vol}_{2n-2}(L \cap H_\xi)$$

for each $\xi \in S^{2n-1}$. Does it follow that

$$\mathrm{Vol}_{2n}(K) \leq \mathrm{Vol}_{2n}(L)?$$

This formulation is reminiscent of the lower-dimensional Busemann-Petty problem, where one tries to deduce the inequality for $2n$-dimensional volumes of arbitrary origin-symmetric convex bodies from the inequalities for volumes of all $(2n-2)$-dimensional sections. In the case where $n = 2$ this amounts to considering two-dimensional sections of four-dimensional bodies, where the answer to the lower dimensional problem is affirmative by the solution to the original Busemann-Petty problem - we first get inequalities for the volumes of all three-dimensional sections and then the inequality for the four-dimensional volumes. However, if $n = 3$ we get four-dimensional sections of six-dimensional bodies, where the answer to the lower-dimensional problem is negative by a result of Bourgain and Zhang [BZ]. Our problem is different from the lower-dimensional Busemann-Petty problem in two aspects. First, we do not have all $(2n-2)$-dimensional sections, only sections by subspaces coming from complex hyperplanes, which makes the situation worse than for the lower-dimensional problem. Secondly, we consider only those convex bodies in $\mathbb{R}^{2n}$ that are invariant with respect to all R_θ,

It appears that the complex Busemann-Petty problem is closely related to the class of 2-intersection bodies introduced in [K5], [K8]. Namely, the answer to the problem is affirmative if and only if every origin symmetric invariant with respect to all R_θ convex body in $\mathbb{R}^{2n}$ is a 2-intersection body. This extends to the complex case the connection between the real Busemann-Petty problem and intersection bodies, established by Lutwak, ([L]). This connection played the crucial role in the solution of the real Busemann-Petty problem. We shall prove this connection in Theorem 2.3.2. After that we prove that every origin symmetric invariant with respect to all R_θ convex body in $\mathbb{R}^{2n}$ is a $(2n-4)$-intersection body, but not every such body is a $(2n-6)$-intersection body. Putting $n=3$ and then $n=4$, one can see how these results imply the solution of the complex Busemann-Petty problem.

2.2 The analytic aspect

Let $1 \le k < 2n$ and let H be an $(2n-k)$-dimensional subspace of $\mathbb{R}^{2n}$. Fix any orthonormal basis $e_1, ..., e_k$ in the orthogonal subspace $H^\perp$. For a convex body D in $\mathbb{R}^{2n}$, define the $(2n-k)$-dimensional parallel section function $A_{D,H}$ as a function on $\mathbb{R}^k$ such that

$$A_{D,H}(u) = \mathrm{Vol}_{2n-k}(D \cap \{H + u_1e_1 + ... + u_ke_k\})$$

$$= \int_{\{x\in\mathbb{R}^{2n}:(x,e_1)=u_1,...,(x,e_k)=u_k\}} \chi(\|x\|_D)\, dx, \quad u \in \mathbb{R}^k. \tag{2.2.1}$$

If the body D is infinitely smooth, the function $A_{D,H}$ is infinitely differentiable at the origin (see [K9, Lemma 2.4]). So, we can consider the action of the distribution $|u|_2^{-q-k}/\Gamma(-q/2)$, $u \in \mathbb{R}^k$ on $A_{D,H}$ and apply Proposition 1.4.5 for $A = A_{D,H}$. Also, equations (1.4.3) and (1.4.13) hold for $A_{D,H}$. Note that, if the body D is origin symmetric, the function $A_{D,H}$ is even.

The following proposition was proved in [K8, Th. 2], (see also [KKZ]). We reproduce the proof here, because we formulate the proposition in a slightly different form. We use a well-known formula (see for example [GS p.76]): for any $v \in \mathbb{R}^k$ and $q < -k+1$,

$$(v_1^2 + ... + v_k^2)^{(-q-k)/2}$$

$$= \frac{\Gamma(-q/2)}{2\Gamma((-q-k+1)/2)\pi^{(k-1)/2}} \int_{S^{k-1}} |(v,u)|^{-q-k} \, du. \qquad (2.2.2)$$

Proposition 2.2.1. *Let D be an infinitely smooth origin symmetric convex body in $\mathbb{R}^{2n}$ and $1 \le k < 2n$. Then for every $(2n-k)$-dimensional subspace H of $\mathbb{R}^{2n}$ and any $q \in \mathbb{R}$, $-k < q < 2n-k$,*

$$\left\langle \frac{|u|_2^{-q-k}}{\Gamma(-q/2)}, A_{D,H}(u) \right\rangle$$

$$= \frac{2^{-q-k}\pi^{-k/2}}{\Gamma((q+k)/2)(2n-q-k)} \int_{S^{2n-1}\cap H^\perp} \left(\|x\|_D^{-2n+q+k}\right)^\wedge(\theta) \, d\theta. \qquad (2.2.3)$$

Also for every $m \in \mathbb{N} \cup \{0\}$, $m < (2n-k)/2$,

$$\Delta^m A_{D,H}(0) = \frac{(-1)^m}{(2\pi)^k(2n-2m-k)} \int_{S^{2n-1}\cap H^\perp} (\|x\|_D^{-2n+2m+k})^\wedge(\eta) \, d\eta,$$

$$(2.2.4)$$

Proof : First let $q \in (-k, -k+1)$. Then,

$$\left\langle \frac{|u|_2^{-q-k}}{\Gamma(-q/2)}, A_{D,H}(u) \right\rangle = \frac{1}{\Gamma(-q/2)} \int_{\mathbb{R}^k} |u|_2^{-q-k} A_{D,H}(u) \, du.$$

Using the expression (2.2.1) for the function $A_{D,H}$, writing the integral in polar coordinates and then using (3.2.4), we see that the right-hand side of the latter equation is equal to

$$\frac{1}{\Gamma(\frac{-q}{2})}\int_{\mathbb{R}^{2n}} \left((x,e_1)^2+...+(x,e_k)^2\right)^{(-q-k)/2}\chi(\|x\|_D)\ dx =$$

$$\frac{1}{\Gamma(\frac{-q}{2})(2n-q-k)}\int_{S^{2n-1}} \left((\theta,e_1)^2+...+(\theta,e_k)^2\right)^{(-q-k)/2}\|\theta\|_D^{-2n+q+k}\ d\theta =$$

$$\frac{1}{2\Gamma(\frac{-q-k+1}{2})\pi^{\frac{k-1}{2}}(2n-q-k)}\times$$

$$\int_{S^{2n-1}}\|\theta\|_D^{-2n+q+k}\left(\int_{S^{k-1}}\left|(\sum_{i=1}^k u_ie_i,\theta)\right|^{-q-k}\ du\right)\ d\theta =$$

$$\frac{1}{2\Gamma(\frac{-q-k+1}{2})\pi^{\frac{k-1}{2}}(2n-q-k)}\times$$

$$\int_{S^{k-1}}\left(\int_{S^{n-1}}\|\theta\|_D^{-2n+q+k}\left|(\sum_{i=1}^k u_ie_i,\theta)\right|^{-q-k}\ d\theta\right)du. \tag{2.2.5}$$

Let us show that the function under the integral over S^{k-1} is the Fourier transform of $\|x\|_D^{-2n+q+k}$ at the point $\sum u_ie_i$. For any even test function $\phi \in \mathcal{S}(\mathbb{R}^{2n})$, using the well-known connection between the Fourier and Radon transforms (see [K9 p. 27]) and the expression for the Fourier transform of the distribution $|z|^{q+k-1}$ (see [K9 p. 38]), we get

$$\langle(\|x\|_D^{-2n+q+k})^\wedge,\phi\rangle = \langle\|x\|_D^{-2n+q+k},\hat{\phi}\rangle = \int_{\mathbb{R}^n}\|x\|_D^{-2n+q+k}\hat{\phi}(x)\ dx =$$

$$\int_{S^{2n-1}}\|\theta\|_D^{-2n+q+k}\left(\int_0^\infty z^{q+k-1}\hat{\phi}(z\theta)\ dz\right)d\theta =$$

$$\frac{1}{2}\int_{S^{2n-1}} \|\theta\|_D^{-2n+q+k}\Big\langle |z|^{q+k-1}, \hat{\phi}(z\theta)\Big\rangle\, d\theta =$$

$$\frac{2^{q+k}\sqrt{\pi}\ \Gamma((q+k)/2)}{2\Gamma((-q-k+1)/2)}\int_{S^{2n-1}} \|\theta\|_D^{-2n+q+k}\Big\langle |t|^{-q-k}, \int_{(y,\theta)=t}\phi(y)\ dy\Big\rangle\, d\theta =$$

$$\frac{2^{q+k}\sqrt{\pi}\Gamma((q+k)/2)}{2\Gamma((-q-k+1)/2)}\int_{\mathbb{R}^{2n}}\Big(\int_{S^{2n-1}} |(\theta,y)|^{-q-k}\|\theta\|_D^{-2n+q+k}\ d\theta\Big)\phi(y)\ dy.$$

Since ϕ is an arbitrary test function, this proves that, for every $y \in \mathbb{R}^{2n}\backslash\{0\}$,

$$\big(\|x\|_D^{-2n+q+k}\big)^\wedge(y) = \frac{2^{q+k}\sqrt{\pi}\Gamma((q+k)/2)}{2\Gamma((-q-k+1)/2)}\int_{S^{2n-1}} |(\theta,y)|^{-q-k}\|\theta\|_D^{-2n+q+k}\ d\theta.$$

Together with (3.2.7), the latter equality shows that

$$\Big\langle \frac{|u|_2^{-q-k}}{\Gamma(-q/2)}, A_{D,H}(u)\Big\rangle$$

$$= \frac{2^{-q-k}\pi^{-k/2}}{\Gamma((q+k)/2)(2n-q-k)}\int_{S^{2n-1}\cap H^\perp}\big(\|x\|_D^{-2n+q+k}\big)^\wedge(\theta)\ d\theta, \qquad (2.2.6)$$

because in our notation $S^{k-1} = S^{2n-1}\cap H^\perp$.

We have proved (3.2.8) under the assumption that $q \in (-k, -k+1)$. However, both sides of (3.2.8) are analytic functions of $q \in \mathbb{C}$ in the domain where $-k < Re(q) < 2n-k$. This implies that the equality (3.2.8) holds for every q from this domain (see [K9 p.61] for the details of a similar argument).

Putting $q = 2m,\ m \in \mathbb{N}\cup\{0\},\ m < (2n-k)/2$ in (3.2.8) and applying (1.4.5) and the fact that $\Gamma(x+1) = x\Gamma(x)$, we get the second formula. $\square$

As mentioned in the Introduction (Section 1.2) we use a generalization of the Brunn's theorem.

Lemma 2.2.1. *If D is a 2-smooth origin symmetric convex body in $\mathbb{R}^n$, then the function $A_{D,H}$ is twice differentiable at the origin and*

$$\Delta A_{D,H}(0) \leq 0.$$

Besides that for any $q \in (0,2)$,

$$\left\langle \frac{|u|_2^{-q-k}}{\Gamma(-q/2)}, A_{D,H}(u) \right\rangle \geq 0.$$

Proof : Differentiability follows from [K9 Lemma 2.4]. Applying the Brunn's theorem to the bodies $D \cap (H + t\theta)$, $\theta \in S^{n-1} \cap H^{\perp}$, we see that the function $t \mapsto A_{D,H}(t\theta)$ has maximum at zero. Therefore, the interior integral in (1.4.13) for the function $A_{D,H}$ instead of A, is negative, but $\Gamma(-q/2) < 0$ for $q \in (0,2)$, which implies the second statement. The first inequality also follows from the fact that each of the functions $t \mapsto A_{D,H}(te_j)$, $j = 1, ..., k$ has maximum at the origin. □

We often use Lemma 4.10 from [K9] for the purpose of approximation by infinitely smooth bodies. For convenience, let us formulate this lemma:

Lemma 2.2.2. ([K9, Lemma 4.10]) *Let $1 \leq k < n$. Suppose that D is an origin symmetric convex body in $\mathbb{R}^n$ that is not a k-intersection body. Then there exists a sequence D_m of origin symmetric convex bodies so that D_m converges to D in the radial metric, each D_m is infinitely smooth, has strictly* *positive curvature and each D_m is not a k-intersection body.*

If, in addition, D is invariant with respect to R_θ, one can choose D_m with the same property.

2.3 Connection with intersection bodies

We now return to the complex case. The following simple observation is crucial for applications of the Fourier methods to convex bodies in the complex case:

Lemma 2.3.1. *Suppose that K is an origin symmetric infinitely smooth invariant with respect to all R_θ star body in $\mathbb{R}^{2n}$. Then for every $0 < p < 2n$ and $\xi \in S^{2n-1}$ the Fourier transform of the distribution $\|x\|_K^{-p}$ is a constant function on $S^{2n-1} \cap H_\xi^\perp$.*

Proof : By Proposition 1.4.2 ([K9, Lemma 3.16]), the Fourier transform of $\|x\|_K^{-p}$ is a continuous function outside of the origin in $\mathbb{R}^{2n}$. The function $\|x\|_K$ is invariant with respect to all R_θ, so by the connection between the Fourier transform of distributions and linear transformations, the Fourier transform of $\|x\|_K^{-p}$ is also invariant with respect to all R_θ. Recall that the two-dimensional space $H_\xi^\perp$ is spanned by vectors ξ and $\xi^\perp$ (see Section 1.3).

Every vector in $S^{2n-1} \cap H_\xi^\perp$ is the image of ξ under one of the coordinate-wise rotations R_θ, so the Fourier transform of $\|x\|_K^{-p}$ is a constant function on $S^{2n-1} \cap H_\xi^\perp$. $\square$

Of course, this argument also applies to the Fourier transform of any distribution of the form $h(\|x\|_K)$.

Similarly to the real case (see [K1], [K9 Theorem 3.8]), one can express the volume of hyperplane sections in terms of the Fourier transform.

Theorem 2.3.1. *Let K be an infinitely smooth origin symmetric invariant with respect to R_θ convex body in $\mathbb{R}^{2n}, n \geq 2$. For every $\xi \in S^{2n-1}$, we have*

$$Vol_{2n-2}(K \cap H_\xi) = \frac{1}{4\pi(n-1)} \left(\|x\|_K^{-2n+2} \right)^\wedge (\xi).$$

Proof : Let us fix $\xi \in S^{2n-1}$. We apply formula (2.2.4) with $H = H_\xi,\ k = 2,\ m = 0$. We get

$$\mathrm{Vol}_{2n-2}(K \cap H_\xi) = A_{K,H_\xi}(0) = \frac{1}{8\pi^2(n-1)} \int_{S^{2n-1} \cap H_\xi^\perp} \left(\|x\|_K^{-2n+2} \right)^\wedge (\eta)\, d\eta.$$

By Lemma 2.3.1, the function under the integral in the right hand side is constant on the circle $S^{2n-1} \cap H_\xi^\perp$. Since $\xi \in H_\xi^\perp$, the integral is equal to $2\pi \left(\|x\|_K^{-2n+2} \right)^\wedge (\xi)$. $\square$

The connection between the complex Busemann-Petty problem and intersection bodies is as follows:

Theorem 2.3.2. *The answer to the complex Busemann-Petty problem in* $\mathbb{C}^n$ *is affirmative if and only if every origin symmetric invariant with respect to all* R_θ *convex body in* $\mathbb{R}^{2n}$ *is a* 2*-intersection body.*

This theorem will follow from the next two lemmas. Note that, since we can approximate the body K in the radial metric from inside by infinitely smooth convex bodies invariant with respect to all R_θ, and also approximate L from outside in the same way, we can argue that if the answer to the complex Busemann-Petty problem is affirmative for infinitely smooth bodies K and L then it is affirmative in general.

Lemma 2.3.2. *Let* K *and* L *be infinitely smooth origin symmetric invariant with respect to* R_θ *convex bodies in* $\mathbb{R}^{2n}$ *so that* K *is a* 2*-intersection body and, for every* $\xi \in S^{2n-1}$,

$$Vol_{2n-2}(K \cap H_\xi) \leq Vol_{2n-2}(L \cap H_\xi).$$

Then

$$Vol_{2n}(K) \leq Vol_{2n}(L).$$

Proof : By Proposition 1.4.2, the Fourier transforms of the distributions $\|x\|_K^{-2n+2}$, $\|x\|_L^{-2n+2}$ and $\|x\|_K^{-2}$ are continuous functions outside of the origin in $\mathbb{R}^{2n}$. By Theorem 2.3.1 and Proposition 1.4.4, the conditions of the lemma imply that for every $\xi \in S^{2n-1}$,

$$\left(\|x\|_K^{-2n+2}\right)^\wedge(\xi) \le \left(\|x\|_L^{-2n+2}\right)^\wedge(\xi)$$

and

$$\left(\|x\|_K^{-2}\right)^\wedge(\xi) \ge 0.$$

Therefore,

$$\int_{S^{2n-1}} \left(\|x\|_K^{-2n+2}\right)^\wedge(\xi) \left(\|x\|_K^{-2}\right)^\wedge(\xi)\, d\xi$$
$$\le \int_{S^{2n-1}} \left(\|x\|_L^{-2n+2}\right)^\wedge(\xi) \left(\|x\|_K^{-2}\right)^\wedge(\xi)\, d\xi.$$

Now we apply Parseval's formula on the sphere, Proposition 1.4.3, to remove the Fourier transforms in the latter inequality and then use the polar formula for the volume and Hölder's inequality:

$$2n\ \mathrm{Vol}_{2n}(K) = \int_{S^{2n-1}} \|x\|_K^{-2n}\, dx \le \int_{S^{2n-1}} \|x\|_L^{-2n+2}\|x\|_K^{-2}\, dx$$
$$\le \left(\int_{S^{2n-1}} \|x\|_L^{-2n}\, dx\right)^{\frac{n-1}{n}} \left(\int_{S^{2n-1}} \|x\|_K^{-2n}\, dx\right)^{\frac{1}{n}}$$
$$= \left(2n\ \mathrm{Vol}_{2n}(L)\right)^{\frac{n-1}{n}} \left(2n\ \mathrm{Vol}_{2n}(K)\right)^{\frac{1}{n}},$$

which gives the result. □

Lemma 2.3.3. *Suppose that there exists an origin symmetric invariant with respect to all R_θ convex body L in $\mathbb{R}^{2n}$ which is not a 2-intersection body. Then one can perturb L twice to construct other origin symmetric invariant* *with respect to R_θ convex bodies L' and K in $\mathbb{R}^{2n}$ such that for every $\xi \in S^{2n-1}$,*

$$Vol_{2n-2}(K \cap H_\xi) \leq Vol_{2n-2}(L' \cap H_\xi),$$

but

$$Vol_{2n}(K) > Vol_{2n}(L').$$

Proof : We can assume that the body L is infinitely smooth and has strictly positive curvature. In fact, approximating L in the radial metric by infinitely smooth invariant with respect to all R_θ convex bodies with strictly positive curvature, we get by Lemma 2.2.2 that the approximating bodies cannot all be 2-intersection bodies. So there exists an infinitely smooth invariant with respect to all R_θ convex body L' with strictly positive curvature that is not a 2-intersection body. In the following we write L instead of L'.

Now as L is infinitely smooth, by Proposition 1.4.2, the Fourier transform of $\|x\|_L^{-2}$ is a continuous function outside of the origin in $\mathbb{R}^{2n}$. The body L is not a 2-intersection body, so by Proposition 1.4.4, the Fourier transform $\left(\|x\|_L^{-2}\right)^\wedge$ is negative on some open subset Ω of the sphere S^{2n-1}.

Since L is invariant with respect to rotations R_θ, we can assume that the set Ω is also invariant with respect to rotations R_θ. This allows us to choose an even non-negative invariant with respect to rotations R_θ function

$f \in C^\infty(S^{2n-1})$ which is supported in Ω. Extend f to an even homogeneous function $f(x/|x|_2)|x|_2^{-2}$ of degree -2 on $\mathbb{R}^{2n}$. By Proposition 1.4.2, the Fourier transform of this extension is an even homogeneous function of degree -2n+2 on $\mathbb{R}^{2n}$, whose restriction to the sphere is infinitely smooth:

$$\left(f(x/|x|_2)|x|_2^{-2}\right)^\wedge (y) = g(y/|y|_2)|y|_2^{-2n+2},$$

where $g \in C^\infty(S^{2n-1})$. By the connection between the Fourier transform and linear transformations, the function g is also invariant with respect to rotations R_θ.

Define a body K in $\mathbb{R}^{2n}$ by

$$\|x\|_K^{-2n+2} = \|x\|_L^{-2n+2} - \epsilon g(x/|x|_2)|x|_2^{-2n+2}. \tag{2.3.1}$$

For small enough $\epsilon > 0$ the body K is convex. This essentially follows from a simple two-dimensional argument: if h is a strictly concave function on an interval $[a, b]$ and u is a twice differentiable function on $[a, b]$, then for small ϵ the function $h + \epsilon u$ is also concave. Note that here we use the condition that L has strictly positive curvature. Besides that, the body K is invariant with respect to rotations R_θ because so are the body L and the function g. We can now choose ϵ so that K is an origin symmetric invariant with respect to all R_θ convex body in $\mathbb{R}^{2n}$.

Let us prove that the bodies K and L provide the necessary counterexample. We apply the Fourier transform to both sides of (2.3.1). By definition of the function g and since f is non-negative, we get that for every $\xi \in S^{2n-1}$

$$\left(\|x\|_K^{-2n+2}\right)^\wedge(\xi) = \left(\|x\|_L^{-2n+2}\right)^\wedge(\xi) - (2\pi)^{2n}\epsilon f(\xi) \leq \left(\|x\|_L^{-2n+2}\right)^\wedge(\xi).$$

By Theorem 2.3.1, this means that for every $\xi \in S^{2n-1}$

$$\mathrm{Vol}_{2n-2}(K \cap H_\xi) \leq \mathrm{Vol}_{2n-2}(L \cap H_\xi).$$

On the other hand, the function f is positive only where $\left(\|x\|_L^{-2}\right)^\wedge$ is negative, so

$$\begin{aligned}
&\int_{S^{2n-1}} \left(\|x\|_K^{-2n+2}\right)^\wedge(\xi)\left(\|x\|_L^{-2}\right)^\wedge(\xi)\, d\xi \\
&= \int_{S^{2n-1}} \left(\|x\|_L^{-2n+2}\right)^\wedge(\xi)\left(\|x\|_L^{-2}\right)^\wedge(\xi)\, d\xi \\
&\quad -(2\pi)^{2n}\epsilon \int_{S^{2n-1}} \left(\|x\|_L^{-2}\right)^\wedge(\xi) f(\xi)\, d\xi \\
&> \int_{S^{2n-1}} \left(\|x\|_L^{-2n+2}\right)^\wedge(\xi)\left(\|x\|_L^{-2}\right)^\wedge(\xi)\, d\xi.
\end{aligned}$$

The end of the proof is similar to that of the previous lemma - we apply Parseval's formula to remove Fourier transforms and then use Hölder's inequality and the polar formula for the volume to get $\mathrm{Vol}_{2n}(K) > \mathrm{Vol}_{2n}(L)$. $\square$

2.4 The solution of the problem

It is known (see [K6] or [K9 Corollary 4.9] plus Proposition 1.4.4) that for every origin symmetric convex body K in $\mathbb{R}^{2n}, n \geq 2$ the space $(\mathbb{R}^{2n}, \|\cdot\|_K)$ embeds in L_{-p} for each $p \in [2n-3, 2n)$, or, in other words, every origin symmetric convex body in $\mathbb{R}^{2n}$ is a $(2n-3)$-, $(2n-2)$- and $(2n-1)$-intersection body. On the other hand, for $q > 2$ the unit ball of the real space ℓ_q^{2n} is not a $(2n-4)$-intersection body , and, moreover, $\mathbb{R}^{2n}$ provided with the norm of this space does not embed in L_{-p} with $p < 2n-3$ (see [K3] or [K9 Th. 4.13]).

Now we have to find out what happens if we consider convex bodies invariant with respect to all R_θ. As it is proved in Chapter 6, for $q > 2$ the complex space ℓ_q^n does not embed in L_{-p} with $p < 2n-4$, which means that the unit ball $B_q^n(\mathbb{C}^n)$ of this space (which is invariant with respect to all R_θ) is not a k-intersection body with $k < 2n-4$.

Recall that we denote $B_q(\mathbb{C}^n)$ the unit ball of the complex space ℓ_q^n considered as a subset of $\mathbb{R}^{2n}$, (see Section 1.3 for definition).

Theorem 2.4.1. *If $q > 2$ then the space $(\mathbb{R}^{2n}, \|\cdot\|_q)$ does not embed in L_{-p} with $0 < p < 2n-4$. In particular, the body $B_q(\mathbb{C}^n)$ is not a k-intersection body for any $1 \leq k < 2n-4$.*

We postpone the proof for Chapter 6.

The only question that remains open is what happens in the interval $p \in [2n-4, 2n-3)$. The following result answers this question.

Theorem 2.4.2. *Let $n \geq 3$. Every origin symmetric invariant with respect to R_θ convex body K in $\mathbb{R}^{2n}$ is a $(2n-4)$-intersection body. Moreover, the space $(\mathbb{R}^{2n}, \|\cdot\|_K)$ embeds in L_{-p} for every $p \in [2n-4, 2n)$.*

If $n = 2$ the space $(\mathbb{R}^{2n}, \|\cdot\|_K)$ embeds in L_{-p} for every $p \in (0, 4)$.

Proof : By Lemma 2.2.2, it is enough to prove the result in the case where K is infinitely smooth. Fix $\xi \in S^{2n-1}$.

Let $n \geq 3$. Applying formula (2.2.4) and then Lemma 2.3.1 with $H = H_\xi,\ m = 1$ and $k = 2$, we get

$$\begin{aligned}\Delta A_{K,H_\xi}(0) &= \frac{-1}{8\pi^2(n-2)} \int_{S^{n-1}\cap H_\xi^\perp} (\|x\|_K^{-2n+4})^\wedge(\eta)\, d\eta \\ &= \frac{-2\pi}{8\pi^2(n-2)} \left(\|x\|_K^{-2n+4}\right)^\wedge(\xi).\end{aligned}$$

By Brunn's theorem (see Lemma 2.2.1), $\left(\|x\|_K^{-2n+4}\right)^\wedge(\xi) \geq 0$ for every $\xi \in S^{2n-1}$, so $\|x\|_K^{-2n+4}$ is a positive definite distribution on $\mathbb{R}^{2n}$. By Proposition 1.4.4, K is a $(2n-4)$-intersection body.

Now let $n \geq 2$. For $0 < q < 2$, formula (2.2.3) and Lemma 2.2.1 imply that $\left(\|x\|_K^{-2n+q+2}\right)^\wedge(\xi) \geq 0$. By Proposition 1.4.4, the space $(\mathbb{R}^{2n}, \|\cdot\|_K)$

embeds in $L_{-2n+q+2}$, and, using the range of q, every such space embeds in $L_{-p},\ p \in (2n-4, 2n-2)$. As mentioned before, these spaces also embed in $L_{-p},\ p \in [2n-3, 2n)$, because so does any $2n$-dimensional normed space. $\square$

We are now ready to prove the main result of this article:

Theorem 2.4.3. *The solution to the complex Busemann-Petty problem in $\mathbb{C}^n$ is affirmative if $n \leq 3$ and it is negative if $n \geq 4$.*

Proof : By Theorem 2.4.2, every origin symmetric invariant with respect to R_θ convex body in $\mathbb{R}^6$ (where $n = 3$) is a $2n-4 = 2$-intersection body, and in $\mathbb{R}^4$ (where $n = 2$) it is a $2n-2 = 2$-intersection body. The affirmative answers for $n = 3$ and $n = 2$ follow now from Theorem 2.3.2.

If $n \geq 4$ then $2n-4 > 2$, so by Theorem 2.4.1 the body B_q^n is not a 2-intersection body. The negative answer follows from Theorem 2.3.2. $\square$

Remark 1. The transition between the dimensions $n = 3$ and $n = 4$ is due to the fact that convexity controls only derivatives of the second order. To see this let us look again at formula (2.2.4), which we apply with $k = 2$. We want to get information about the Fourier transform of $\|x\|_D^{-2}$, so we need to choose m so that $-2n+2m+2 = -2$. If $n = 3$ then $m = 1$, but when $n = 4$ we need $m = 2$. This means that for $n = 3$ we consider $\Delta A_{K,H}(0)$, which is always negative by convexity, but when $n = 4$ we look at $\Delta^2 A_{K,H}(0)$, which is not controlled by convexity and can be sign-changing. One can construct a counterexample in dimension $n = 4$ using this argument, similarly to how it was done for the "real" Busemann-Petty problem; see [K9 Corollary 4.4].

Remark 2. Applying Theorem 2.4.2 to $n = 2$ we get that every two-dimensional complex normed space (which is a 4-dimensional real normed space) embeds in L_{-p} for every $p \in (-4, 0)$. By [KKYY Th. 6.4], this implies that every such space embeds isometrically in L_0. The concept of embedding in L_0 was introduced in [KKYY]: a normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_0 if there exist a probability measure μ on S^{n-1} and a constant C so that for every $x \in \mathbb{R}^n,\ x \neq 0$

$$\log \|x\| = \int_{S^{n-1}} \log |(x, \xi)| \, d\mu(\xi) + C.$$

We have

Theorem 2.4.4. *Every two-dimensional complex normed space embeds in L_0. On the other hand, there exist two-dimensional complex normed spaces that do not embed isometrically in any $L_p,\ p > 0$.*

An example supporting the second claim is the complex space ℓ_q^2 with $q > 2$. This follows from a version of the second derivative test proved in [KL] (see also [K9 Theorem 6.11]). Recall that every two-dimensional real normed space embeds isometrically in L_1 (see [Fe], [He], [Li] or [K9 p.120]), but the real space ℓ_q^2 does not embed isometrically in any $L_p,\ 1 < p \leq 2$, as proved by Dor [Do]; see also [K9 p.124].

Chapter 3

The modified complex Busemann-Petty problem

3.1 Introductory

As it is proved in [KKZ] the answer to the complex Busemann-Petty problem is affirmative if $n \leq 3$ and negative if $n \geq 4$. In this article our aim is to extend [KYY] to the complex case.

Let D be an origin symmetric convex body in $\mathbb{C}^n$. For every $\xi \in \mathbb{C}^n,\ |\xi| = 1$ we define the section function by

$$S_{CD}(\xi) = \mathrm{Vol}_{2n-2}(D \cap H_\xi),\ \forall \xi \in S^{2n-1}. \tag{3.1.1}$$

Extending S_{CD} to the whole $\mathbb{R}^{2n}$ as a homogeneous function of degree -2 we prove the following:

Main Theorem. *Suppose K and L are two origin symmetric invariant with respect to all R_θ convex bodies in $\mathbb{R}^{2n}$. Suppose that $\alpha \in [2n-6, 2n-2),\ n \geq$*

3. *If*

$$\left(-\Delta\right)^{\alpha/2} S_{CK}(\xi) \leq \left(-\Delta\right)^{\alpha/2} S_{CL}(\xi), \tag{3.1.2}$$

for every $\xi \in S^{2n-1}$. *Then*

$$Vol_{2n}(K) \leq Vol_{2n}(L).$$

If $\alpha \in (2n-7, 2n-6)$ *then one can construct two convex bodies* K *and* L *that satisfy (3.1.2), but* $Vol_{2n}(K) > Vol_{2n}(L)$.

This means that one needs to differentiate the section functions at least $2n-6$ times and compare them in order to obtain the same inequality for the volume of the original bodies. Note that if $\alpha = 0$ the problem coincides with the original complex Busemann-Petty problem.

3.2 The Fourier analytic approach

Let H be an $(2n-2)-$dimensional subspace of $\mathbb{R}^{2n}$ and $p \leq 2n-2$. We fix an orthonormal basis , $\{e_1, e_2\}$, in the orthogonal subspace $H^\perp$. For any convex body D in $\mathbb{R}^{2n}$ we define the function $A_{D,H,p}$ as a function on $\mathbb{R}^2$ such that

$$A_{D,H,p}(u) = \int_{D \cap H_u} |x|_2^{-p} dx, \ u \in \mathbb{R}^2, \tag{3.2.1}$$

where $H_u = \{x \in \mathbb{R}^{2n} : (x, e_1) = u_1, (x, e_2) = u_2\}$.

If the body D is infinitely smooth and $0 \le p < 2n - m - 2$, then $A_{D,H,p}$ is m-times continuously differentiable near the origin. To see this we need to observe that the function can be written in the form

$$A_{D,H,p}(u) = \int_{S_u^{2n-3}} \Big(\int_0^{\|\theta\|_{D\cap H_u}^{-1}} r^{2n-3}(r^2 + |u|_2^2)^{-p/2} dr \Big) d\theta, \tag{3.2.2}$$

where S_u^{2n-3} is the unit sphere of the subspace H_u and then follow similar steps as in [K9, Lemma 2.4]. Note that $A_{D,H,p} \in C^m$ near the origin since differentiating the inner integral in (3.2.2) more than m times, it is no longer convergent for $t = 0$.

In addition, we can consider the action of the distribution $|u|_2^{-q-k}/\Gamma(-q/2)$, $u \in \mathbb{R}^k$ on $A_{D,H,p}$ and apply Proposition 1.4.5 for $A = A_{D,H,p}$.

For $q \in \mathbb{C}$ with $\Re q \le 2n - p - 3$ the function is an analytic function of q. Also, equations (1.4.3) and (1.4.13) hold for $A_{D,H,p}$.

If the body D is origin symmetric the function $A_{D,H,p}$ is even and for $0 < q < 2$ we use equation (1.4.13) for $A_{D,H,p}$ to get

$$\Big\langle \frac{|u|_2^{-q-2}}{\Gamma(-\frac{q}{2})}, A_{D,H,p}(u) \Big\rangle$$

$$= \frac{1}{\Gamma(-\frac{q}{2})} \int_0^{2\pi} \Big(\int_0^\infty \frac{A_{D,H,p}(t\theta) - A_{D,H,p}(0)}{t^{1+q}} dt \Big) d\theta. \tag{3.2.3}$$

The following proposition is a generalization of Proposition 2.2.1 with $k = 2$. We prove it using a well-known formula (see for example [GS, p.76]):

for any $v \in \mathbb{R}^2$ and $q < -1$,

$$(v_1^2 + v_2^2)^{\frac{-q-2}{2}} = \frac{\Gamma(-q/2)}{2\Gamma((-q-1)/2)\pi^{1/2}} \int_0^{2\pi} |(v,u)|^{-q-2}\, du. \qquad (3.2.4)$$

Proposition 3.2.1. *Let D be an infinitely smooth origin symmetric convex body in $\mathbb{R}^{2n}$. If $-2 < q < 2n-2,\ 0 \le p \le 2n-q-3$. Then for every $(2n-2)$-dimensional subspace H of $\mathbb{R}^{2n}$*

$$\left\langle \frac{|u|_2^{-q-2}}{\Gamma(-\frac{q}{2})}, A_{D,H,p}(u) \right\rangle$$

$$= \frac{2^{-q-2}}{\pi\Gamma\left(\frac{q+2}{2}\right)(2n-q-p-2)} \int\limits_{S^{2n-1}\cap H^\perp} \left(\|x\|_D^{-2n+q+p+2}|x|_2^{-p}\right)^\wedge(\theta)d\theta. \quad (3.2.5)$$

Also, for every $d \in \mathbb{N} \cup 0,\ d < n-1$

$$\Delta^d A_{D,H,p}(0) = \frac{(-1)^d}{8\pi^2(n-d-1)} \int\limits_{S^{2n-1}\cap H^\perp} \left(\|x\|_D^{-2n+2d+p+2}|x|_2^{-p}\right)^\wedge(\eta)d\eta. \qquad (3.2.6)$$

Proof. First we assume that $q \in (-2,-1)$. Then

$$\left\langle \frac{|u|_2^{-q-2}}{\Gamma(-\frac{q}{2})}, A_{D,H,p}(u) \right\rangle = \frac{1}{\Gamma(-q/2)} \int_{\mathbb{R}^2} |u|_2^{-q-2} A_{D,H,p}(u)du$$

Using the expression (3.2.1) for the function $A_{D,H,p}$, writing the integral in polar coordinates and then using (3.2.4), we see that the right-hand side of the latter equation is equal to

$$\frac{1}{\Gamma(\frac{-q}{2})} \int_{\mathbb{R}^n} \left((x,e_1)^2 + (x,e_2)^2\right)^{\frac{-q-2}{2}} |x|_2^{-p} \chi(\|x\|_D)\, dx$$

$$= \frac{1}{\Gamma(\frac{-q}{2})(n-q-p-2)} \int_{S^{n-1}} \left((\theta,e_1)^2+(\theta,e_2)^2\right)^{\frac{-q-2}{2}} \|\theta\|_D^{-n+q+2}\ d\theta$$

$$= \frac{1}{2\Gamma(\frac{-q-1}{2})\pi^{\frac{1}{2}}(n-q-p-2)} \times$$

$$\int_{S^{n-1}} \|\theta\|_D^{-n+q+p+2} \left(\int_0^{2\pi} \left|(u_1e_1+u_1e_2,\theta)\right|^{-q-2}\ du\right)\ d\theta$$

$$= \frac{1}{2\Gamma(\frac{-q-1}{2})\pi^{\frac{1}{2}}(n-q-p-2)} \times$$

$$\int_0^{2\pi} \left(\int_{S^{n-1}} \|\theta\|_D^{-n+q+p+2} \left|(u_1e_1+u_2e_2,\theta)\right|^{-q-2}\ d\theta\right) du. \qquad (3.2.7)$$

Let us show that the function under the integral over $[0,2\pi]$ is the Fourier transform of $\|x\|_D^{-n+q+p+2}|x|_2^{-p}$ at the point $u_1e_1+u_2e_2$. For any even test function $\phi \in \mathcal{S}(\mathbb{R}^n)$, using the well-known connection between the Fourier and Radon transforms (see [K9, p.27]) and the expression for the Fourier transform of the distribution $|z|_2^{q-1}$ (see [K9, p.38]), we get

$$\langle (\|x\|_D^{-n+q+p+2}|x|_2^{-p})^\wedge, \phi\rangle = \int_{\mathbb{R}^n} \|x\|_D^{-n+q+p+2}|x|_2^{-p}\hat{\phi}(x)\ dx$$

$$= \int_{S^{n-1}} \|\theta\|_D^{-n+q+p+2} \left(\int_0^\infty r^{q+1}\hat{\phi}(r\theta)\ dr\right) d\theta$$

$$= \frac{1}{2}\int_{S^{n-1}} \|\theta\|_D^{-n+q+p+2} \left\langle |r|^{q+1}, \hat{\phi}(r\theta)\right\rangle\ d\theta$$

$$= \frac{2^{q+2}\sqrt{\pi}\ \Gamma((q+2)/2)}{2\Gamma((-q-1)/2)} \int_{S^{n-1}} \|\theta\|_D^{-n+q+p+2} \left\langle |t|^{-q-2}, \int_{(y,\theta)=t} \phi(y)\ dy\right\rangle\ d\theta$$

$$= \frac{2^{q+1}\sqrt{\pi}\Gamma((q+2)/2)}{2\Gamma((-q-1)/2)} \int_{\mathbb{R}^n} \left(\int_{S^{n-1}} |(\theta,y)|^{-q-2}\|\theta\|_D^{-n+q+p+2}\ d\theta\right)\phi(y)\ dy.$$

Since ϕ is an arbitrary test function, this proves that, for every $y \in \mathbb{R}^n \setminus \{0\}$,

$$\left(\|x\|_D^{-n+q+p+2}|x|_2^{-p}\right)^\wedge(y)$$

$$= \frac{2^{q+2}\sqrt{\pi}\Gamma((q+2)/2)}{2\Gamma((-q-1)/2)} \int_{S^{n-1}} |(\theta, y)|^{-q-2}\|\theta\|_D^{-n+q+p+2}\, d\theta.$$

Together with (3.2.7), the latter equality shows that

$$\left\langle \frac{|u|_2^{-q-2}}{\Gamma(-q/2)}, A_{D,H,p}(u) \right\rangle$$

$$= \frac{2^{-q-2}\pi^{-1}}{\Gamma((q+2)/2)(n-q-p-2)} \int_{S^{n-1}\cap H^\perp} \left(\|x\|_D^{-n+q+p+2}|x|_2^{-p}\right)^\wedge(\theta)\, d\theta, \tag{3.2.8}$$

because in our notation $S^{n-1} \cap H^\perp = [0, 2\pi]$.

We have proved (3.2.8) under the assumption that $q \in (-2, -1)$. However, both sides of (3.2.8) are analytic functions of $q \in \mathbb{C}$ in the domain where $-2 < \Re q < 2n - 2$. This implies that the equality (3.2.8) holds for every q from this domain (see [K9, p.61] for the details of a similar argument).

Putting $q = 2m,\ m \in \mathbb{N} \cup \{0\},\ m < n - 1$ in (3.2.8) and applying (**??**) and the fact that $\Gamma(x + 1) = x\Gamma(x)$, we get the second formula. □

The following proposition is a generalization of Brunn's theorem (see Section 1.2), proved in Chapter 2, Proposition 2.2.1 for $p = 0$.

Proposition 3.2.2. *Suppose D is a 2-smooth origin symmetric convex body* *in* $\mathbb{R}^{2n}$, *then the function* $A_{D,H,p}$ *is twice differentiable at the origin and*

$$\Delta A_{D,H,p}(0) \le 0.$$

Moreover, for any $q \in (0,2)$,

$$\left\langle \frac{|u|_2^{-q-2}}{\Gamma(-\frac{q}{2})}, A_{D,H,p}(u) \right\rangle \ge 0.$$

Proof. Since D is 2-smooth, it is not difficult to see that the function

$$\int_0^{\|\theta\|_{D\cap H_u}^{-1}} r^{2n-3}(r^2 + |u|_2^2)^{-p/2} dr$$

is twice differentiable in a neighborhood of 0. By equation (3.2.2) this proves the differentiability of $A_{D,H,p}$ at the origin.

The body D is origin symmetric and convex, so to prove the first inequality we need to observe that the function $u \longmapsto A_{D,H,p}(u)$, $u \in \mathbb{R}^2$, attains its maximum at the origin:

If $p = 0$ then it follows immediately from Brunn's theorem (see Lemma 2.2.1, or [K9, Theorem 2.3]). Let $p > 0$. Since $|x|_2^{-p} = p\int_0^\infty \chi(z|x|_2) z^{q-1} dz$, we have that for any $u \in \mathbb{R}^2$

$$\begin{aligned} A_{D,H,p}(u) &= \int_{D\cap H_u} |x|_2^{-p} dx = p \int_{D\cap H_u} \int_0^\infty \chi(z|x|_2) z^{q-1} dz dx \\ &= p \int_0^\infty z^{q-1} \int_{D\cap H_u} \chi(z|x|_2) dx dz \\ &= p \int_0^\infty z^{q-1} \int_{B(1/z)\cap H_u} \chi(\|x\|_D) dx dz, \end{aligned}$$

where $B(1/z)$ is the unit ball of radius $1/z,$. Applying Brunn's theorem to the body $B(1/z) \cap D$, we have that the latter integral is

$$\leq p \int_0^\infty z^{q-1} \int_H \chi(\|x\|_{B(1/z)\cap D}) dx dz = A_{D,H,p}(0).$$

If $q \in (0,2)$ then $\Gamma(-q/2) < 0$. Hence, for the second inequality we use (3.2.3) to get that

$$\left\langle \frac{|u|_2^{-q-2}}{\Gamma(-\frac{q}{2})}, A_{D,H,p}(u) \right\rangle$$

$$= \frac{1}{\Gamma(-\frac{q}{2})} \int_0^{2\pi} \Big(\int_0^\infty \frac{A_{D,H,p}(t\theta) - A_{D,H,p}(0)}{t^{1+q}} dt \Big) d\theta \geq 0,$$

since $A_{D,H,p}(u) \leq A_{D,H,p}(0)$, for every $u \in \mathbb{R}^2$.

3.3 Distributions of the form $|x|_2^{-\beta}\|x\|^{-\gamma}$

As in the modified real Busemann-Petty problem the solution is closely related to distributions of the form $|x|_2^{-\beta}\|x\|^{-\gamma}$.

First, we need a simple observation. The following lemma is crucial for the solution of the problem.

Lemma 3.3.1. *For every infinitely smooth origin symmetric invariant with respect to all R_θ convex body D in $\mathbb{R}^{2n}$ and every $\xi \in S^{2n-1}$, the Fourier transform of the distribution $|x|_2^{-\beta}\|x\|_D^{-\gamma}$, $0 < \beta, \gamma < 2n$ is a constant function on $S^{2n-1} \cap H_\xi^\perp$.*

Proof : The proof (see also Lemma 2.3.1 in Chapter 2, when $\beta = 0$) is based on the following observation:

The body D is invariant with respect to all R_θ. So, because of the connection between the Fourier transform and linear transformations, the Fourier transform of $|x|_2^{-\beta}\|x\|_D^{-\gamma}$ is also invariant with respect to all R_θ. This implies that it is a constant function on $S^{2n-1} \cap H_\xi^\perp$ because this circle can be represented as the set of all the rotations $R_\theta, \theta \in [0, 2\pi]$, of the vector $\xi \in S^{2n-1}$.

As a consequence of the above we have that

$$\int_{S^{2n-1}\cap H_\xi^\perp} \left(|x|_2^{-\beta}\|x\|_D^{-\gamma}\right)^\wedge(\theta)d\theta = 2\pi\left(|x|_2^{-\beta}\|x\|_D^{-\alpha}\right)^\wedge(\xi). \tag{3.3.1}$$

$\square$

Lemma 3.3.2. *Let D be an origin symmetric invariant with respect to all R_θ convex body in $\mathbb{R}^{2n}, n \geq 3$. If $q \in (-2, 2]$ and $0 \leq p < 2n - q - 3$ then*

$$|x|_2^{-p}\|x\|_D^{-2n+p+q+2}$$

is a positive definite distribution.

Proof.

If $p = 0$ then by Theorem 2.4.2, $\left(\|x\|_D^{-2n+q+2}\right)^\wedge \geq 0$, since $2n - q - 2 \in [2n - 4, 2n)$.

Let $p > 0$. If $q \in (-2, 0)$ then by equation (1.4.3) for $A_{D,H,p}$ and Proposition 3.2.1 (formula (3.2.5)) we have that

$$\frac{2^{-q-2}}{\pi\Gamma\left(\frac{q+2}{2}\right)(2n-q-p-2)} \int\limits_{S^{2n-1}\cap H^{\perp}} \left(\|x\|_D^{-2n+q+p+2}|x|_2^{-p}\right)^{\wedge}(\theta)d\theta$$

$$= \frac{1}{\Gamma(-q/2)} \int_{\mathbb{R}^2} |u|_2^{-q-2} A_{D,H,p}(u)du \geq 0.$$

By Lemma 3.3.1, the Fourier transform of the distribution $|x|_2^{-p}\|x\|_D^{-2n+p+q+2}$ is a constant function on $S^{2n-1} \cap H_\xi^{\perp}$ (equation (3.3.1)). So,

$$\left(|x|_2^{-p}\|x\|_D^{-2n+p+q+2}\right)^{\wedge} \geq 0,$$

since $\Gamma(\frac{q+2}{2}) > 0,\ \Gamma(-\frac{q}{2}) > 0$ and $q < 2n-p-2$.

Now, if $q = 0$, (3.2.6) and (3.3.1) give that

$$A_{D,H,p}(0) = \frac{1}{4\pi(n-1)}\left(|x|_2^{-p}\|x\|_D^{-2n+p+q+2}\right)^{\wedge}(\xi) \geq 0.$$

For the case where $q \in (0, 2)$ we use Proposition 3.2.1 and Lemma 3.3.1 to get that

$$\left\langle \frac{|u|_2^{-q-2}}{\Gamma(-\frac{q}{2})}, A_{D,H,p}(u) \right\rangle$$

$$= \frac{2^{-q-1}}{\Gamma\left(\frac{q+2}{2}\right)(2n-q-p-2)}\left(\|x\|_D^{-2n+q+p+2}|x|_2^{-p}\right)^{\wedge}(\xi).$$

Then, by the generalization of Brunn's theorem, Proposition 3.2.2, the desired follows.

Lastly, if $q = 2$, (3.2.6) and (3.3.1) imply that

$$\Delta A_{D,H,p}(0) = \frac{-1}{4\pi(n-2)} \left(\|x\|_D^{-2n+p+4} |x|_2^{-p} \right)^{\wedge} (\xi).$$

Combining this with Brunn's generalization, since the Laplacian of the function $A_{D,H,p}$ at 0 is non-positive, we have that

$$\left(|x|_2^{-p} \|x\|_D^{-2n+p+4} \right)^{\wedge} (\xi) \geq 0.$$

□

Before we prove the main result of this article, we also need the following:

Lemma 3.3.3. *Let D be an infinitely smooth origin symmetric invariant with respect to all R_θ convex body in $\mathbb{R}^{2n}$ and $\alpha \in \mathbb{R}$. Then*

$$(-\Delta)^{\alpha/2} S_{CD}(\xi) = \frac{1}{4\pi(n-1)} \left(|x|_2^{\alpha} \|x\|_D^{-2n+2} \right)^{\wedge} (\xi) \tag{3.3.2}$$

Proof. Let $\xi \in S^{2n-1}$. As proved in Theorem 2.3.1, using the same idea as in Lemma 3.3.1 (with $r = 0$)

$$\mathrm{Vol}_{2n-2}(D \cap H_\xi) = \frac{1}{4\pi(n-1)} \left(\|x\|_D^{-2n+2} \right)^{\wedge} (\xi). \tag{3.3.3}$$

By the definition of the section function of D, and equation (3.3.3) we obtain the following formula:

$$S_{CD}(\xi) = \frac{1}{4\pi(n-1)} \left(\|x\|_D^{-2n+2} \right)^{\wedge} (\xi). \tag{3.3.4}$$

We extend S_{CD} to the whole $\mathbb{R}^{2n}$ as a homogeneous function of degree -2 and apply the definition of the fractional powers of the Laplacian. Then, since $\|x\|_D^{-2n+2}$ is an even distribution, equation (3.3.2) immediately follows.

□

3.4 The solution of the problem.

We consider the affirmative and negative part of the main result separately. The proof follows by the next two theorems.

Theorem 3.4.1. *(AFFIRMATIVE PART) Let K and L be two infinitely smooth origin symmetric invariant with respect to all R_θ convex bodies in $\mathbb{R}^{2n}$. Suppose that $\alpha \in [2n-6, 2n-2), n \geq 3$. Then for every $\xi \in S^{2n-1}$*

$$(-\Delta)^{\alpha/2} S_{CK}(\xi) \leq (-\Delta)^{\alpha/2} S_{CL}(\xi) \tag{3.4.1}$$

implies that

$$Vol_n(K) \leq Vol_n(L).$$

Proof. The bodies K and L are infinitely smooth and invariant with respect to all R_θ convex bodies. So by equation (3.3.2) the condition in (3.4.1) can be written as

$$\left(|x|_2^\alpha \|x\|_K^{-2n+2}\right)^\wedge \leq \left(|x|_2^\alpha \|x\|_L^{-2n+2}\right)^\wedge. \tag{3.4.2}$$

We apply Lemma 3.3.2 with $p = \alpha$ and $q = 2n - \alpha - 4$ so that the distribution $|x|_2^{\alpha}\|x\|_K^{-2}$ is positive definite. By Bochner's theorem this implies that its Fourier transform is a non-negative function on $\mathbb{R}^{2n} \setminus \{0\}$. By Proposition 1.4.2, it is also continuous, since K is infinitely smooth. Multiply both sides in (3.4.2) by $\left(|x|_2^{-\alpha}\|x\|_K^{-2}\right)^{\wedge}$ and integrate over the unit sphere S^{2n-1}. Then we can apply Parseval's spherical version, Proposition 1.4.3, to get that

$$\int_{S^{2n-1}} \|x\|_K^{-2n} dx \leq \int_{S^{2n-1}} \|x\|_K^{-2}\|x\|_L^{-2n+2}. \tag{3.4.3}$$

Then, by a simple application of Hölder's inequality to formula (3.4.3) and the polar formula of the bodies (see equation (1.2.1), we obtain the affirmative answer to the problem, since

$$2n \text{ Vol}_{2n}(K) \leq \Big(2n \text{ Vol}_{2n}(K)\Big)^{1/n}\Big(2n \text{ Vol}_{2n}(L)\Big)^{(n-1)/n}.$$

□

To prove the negative part we need the following lemma.

Lemma 3.4.1. *Let $\alpha \in (2n-7, 2n-6)$. There exists an infinitely smooth origin symmetric convex body L with positive curvature, so that*

$$|x|_2^{-\alpha}\|x\|_L^{-2}$$

is not a positive definite distribution.

We postpone the proof of Lemma 3.4.1 until the end of this section to show that the existence of such a body gives a negative answer to the problem.

Theorem 3.4.2. *(NEGATIVE PART) Suppose there exists an infinitely smooth origin symmetric convex body L for which $|x|_2^{-\alpha}\|x\|_L^{-2}$ is not a positive definite distribution. Then one can construct an origin symmetric convex body K in $\mathbb{R}^{2n}, n \geq 3$, so that together with L they satisfy (3.4.1), for every $\xi \in S^{2n-1}$ but*

$$Vol_{2n}(K) > Vol_{2n}(L).$$

Proof. The body L is infinitely smooth, so by Proposition 1.4.2, the Fourier transform of the distribution $|x|_2^{-\alpha}\|x\|_L^{-2}$ is a continuous function on the unit sphere S^{2n-1}. Moreover there exists an open subset Ω of S^{2n-1} in which $\left(|x|_2^{-\alpha}\|x\|_L^{-2}\right)^\wedge < 0$. Since L is invariant with respect to all R_θ we may assume that Ω is also invariant we respect to rotations R_θ.

We use a standard perturbation procedure for convex bodies, see for example [K9, p.96] (similar argument was used in Section 2, Lemma 2.3.3). Consider a non-negative infinitely differentiable even function g supported on Ω that is also invariant with respect to rotations R_θ. We extend it to a homogeneous function of degree $-\alpha-2$ on $\mathbb{R}^{2n}$. By Proposition 1.4.2 its Fourier transform is an even homogeneous function of degree $-2n+\alpha+2$ on $\mathbb{R}^{2n}$, whose restriction to the sphere is infinitely smooth: $\big(g(x/|x|_2)|x|_2^{-\alpha-2}\big)^\wedge(y) = h(y/|y|_2)|y|_2^{-2n+\alpha+2}$, where $h \in C^\infty(S^{2n-1})$.

We define a body K so that

$$\|x\|_K^{-2n+2} = \|x\|_L^{-2n+2} + \varepsilon|x|_2^{-2n+2}h\big(\frac{x}{|x|_2}\big),$$

for small enough $\varepsilon > 0$ so that the body K is strictly convex. Note that K is also invariant with respect to all R_θ. We multiply both sides by $\frac{1}{4\pi(n-1)}|x|_2^\alpha$ and apply Fourier transform. Then

$$\begin{aligned}(-\Delta)^{\alpha/2}S_{CK}(\xi) &= (-\Delta)^{\alpha/2}S_{CL}(\xi) + \frac{\varepsilon(2\pi)^{2n}}{4\pi(n-1)}|x|_2^{-\alpha-2}g\big(\frac{x}{|x|_2}\big) \qquad (3.4.4)\\ &\leq (-\Delta)^{\alpha/2}S_{CL}(\xi),\end{aligned}$$

since g is non-negative.

On the other hand, we multiply both sides of (3.4.4) by $\Big(|x|_2^{-\alpha}\|x\|_L^{-2}\Big)^\wedge$ and integrate over the sphere,

$$\int_{S^{2n-1}} \Big(|x|_2^{-\alpha}\|x\|_L^{-2}\Big)^\wedge(\theta)\big(-\Delta\big)^{\alpha/2}S_{CK}(\theta)d\theta$$

$$\begin{aligned} &= \int_{S^{2n-1}} \Big(|x|_2^{-\alpha}\|x\|_L^{-2}\Big)^\wedge(\theta)\big(-\Delta\big)^{\alpha/2} S_{CL}(\theta)d\theta \\ &+ \varepsilon\frac{(2\pi)^{2n}}{4\pi(n-1)}\int_{S^{2n-1}} \Big(|x|_2^{-\alpha}\|x\|_L^{-2}\Big)^\wedge(\theta)g(\theta)d\theta \\ &> \int_{S^{2n-1}} \Big(|x|_2^{-\alpha}\|x\|_L^{-2}\Big)^\wedge(\theta)\big(-\Delta\big)^{\alpha/2} S_{CL}(\theta)d\theta, \end{aligned}$$

since $\Big(|x|_2^{-\alpha}\|x\|_L^{-2}\Big)^\wedge < 0$ on the support of g. Using equation (3.3.2) and the spherical version of Parseval's identity, the latter becomes

$$\int_{S^{2n-1}} \|x\|_L^{-2}\|x\|_K^{-2n+2} > \int_{S^{2n-1}} \|x\|_L^{-2n}dx.$$

As in Theorem 3.4.1, we apply Hölder's inequality and the polar representation of the volume to obtain the desired inequality for the volumes of the bodies.

□

Proof of Lemma 3.4.1. The construction of the body follows similar steps as in [KYY]. We put $q = 2n - \alpha - 4$, so $q \in (2, 3)$. From the definition of the fractional derivatives, Proposition 3.2.1 and Lemma 3.3.1, we see that for a $\xi \in S^{2n-1}$ we need to construct a convex body D so that

$$\int_0^{2\pi}\int_0^\infty t^{-q-1}\Big(A_{D,H_\xi,\alpha}(t\theta) - A_{D,H_\xi,\alpha}(0) - \Delta A_{D,H_\xi,\alpha}(0)\frac{t^2}{2}\Big)dtd\theta < 0$$

since $\Gamma(-\frac{q}{2}) > 0$ for $q \in (2, 3)$.

We define the function

$$f(|u|) = (1 - |u|_2^2 - N|u|_2^4)^{\frac{1}{2n-\alpha-2}},\ u \in \mathbb{R}^2$$

and consider the body D in $\mathbb{R}^{2n}$ as

$$D = \Big\{(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in \mathbb{R}^{2n} : |\bar{x}|_2 = |(x_{n1}, x_{n2})|_2 \in [-\alpha_N, \alpha_N],$$
$$\big(\sum_{\substack{i=1 \\ j=1,2}}^{n-1} x_{ij}^2\big)^{1/2} \leq f(|\bar{x}|_2)\Big\},$$

where a_N is the first positive root of the equation $f(t) = 0$. From its definition, the body D is strictly convex with an infinitely smooth boundary. We choose $\xi \in S^{2n-1}$ in the direction of $\bar{x}$. Then for $u \in \mathbb{R}^2$ with $|u|_2 \in [0, a_N]$, equation (3.2.2) gives that

$$\begin{aligned} A_{D,H_\xi,\alpha}(u) &= \int_{S_u^{2n-3}} \int_0^{f(|u|_2)} (r^2 + |u|_2^2)^{-\frac{\alpha}{2}} r^{2n-3} dr d\theta \\ &= |S_u^{2n-3}| \int_0^{f(|u|_2)} (r^2 + |u|_2^2)^{-\frac{\alpha}{2}} r^{2n-3} dr, \end{aligned}$$

where $|S_u^{2n-3}|$ is the volume of the $(2n-3)$-dimensional unit sphere. Note that if $|u|_2 > \alpha_N$ then $A_{D,H_\xi,\alpha}(u) = 0$. Moreover, if $u = t\theta,\ t \in [0, \infty)$ and $\theta \in S^1$, the parallel section function $A_{D,H_\xi,\alpha}(t\theta)$ is independent of θ since

$$A_{D,H_\xi,\alpha}(t\theta) = |S_t^{2n-3}| \int_0^{f(t)} (r^2 + t^2)^{-\frac{\alpha}{2}} r^{2n-3} dr. \tag{3.4.5}$$

Hence, we need to prove that the above construction of the body D gives that

$$\int_0^\infty t^{-q-1}\Big(A_{D,H_\xi,\alpha}(t\theta) - A_{D,H_\xi,\alpha}(0) - \Delta A_{D,H_\xi,\alpha}(0)\frac{t^2}{2}\Big)dt < 0. \qquad (3.4.6)$$

Note that the condition $|u|_2 \in [0, \alpha_N]$ is now equivalent to $t \in [0, \alpha_N]$. In order to prove the above we first compute

$$A_{D,H_\xi,\alpha}(0) = \frac{|S^{2n-3}|}{2n-\alpha-2}$$

and

$$\Delta A_{D,H_\xi,\alpha}(0) = -|S^{2n-3}|\Big[\frac{1}{2n-\alpha-2} + \frac{\alpha}{2n-\alpha-4}\Big].$$

Let β_N be the positive root of the equation $1 - t^2 - Nt^4 = t^{q+1}$. We split the integral in (3.4.6) in three parts: $[0, \beta_N]$, $[\beta_N, \alpha_N]$ and $[\alpha_N, \infty)$ and work separately. It is not difficult to see that for large N, $\alpha_N, \beta_N \simeq N^{-\frac{1}{4}}$. Also, for every $t \in [0, \alpha_N]$, $f(t) > 0$ and $f(t) \geq t$ if and only if $t \in [0, \beta_N]$.

For the first part, the interval $[0, \beta_N]$, since $f(t) \geq t$, the 2-dimensional parallel section function $A_{D,H_\xi,\alpha}$ can be easily estimated if we split it into two integrals. For the second we use the inequality $(1+x)^{-\gamma} \leq 1 - \gamma x + \frac{\gamma(\gamma+1)}{2}x^2$, for $\gamma > 0$ and $0 < x < 1$. Then

$$\int_0^t (r^2+t^2)^{-\frac{\alpha}{2}} r^{2n-3} dr \leq \int_0^t r^{-\alpha+2n-3} dr = \frac{t}{2n-\alpha-2}$$

and

$$\int_t^{f(t)} (r^2+t^2)^{-\frac{\alpha}{2}} r^{2n-3} dr \le \int_t^{f(t)} \Big[1 - \frac{\alpha}{2}\frac{t^2}{r^2} + \frac{\alpha(\alpha+2)}{4}\frac{t^4}{r^4}\Big] r^{2n-\alpha-3} dr$$

$$= \frac{r^{2n-\alpha-2}}{2n-\alpha-2} - \frac{\alpha}{2}t^2 \frac{r^{2n-\alpha-4}}{2n-\alpha-4} + \frac{\alpha(\alpha+2)}{4} t^4 \frac{r^{2n-\alpha-6}}{2n-\alpha-6}\Bigg|_t^{f(t)}$$

$$= \frac{f^{2n-\alpha-2}(t)}{2n-\alpha-2} - \frac{\alpha}{2}t^2 \frac{f^{2n-\alpha-4}(t)}{2n-\alpha-4} + \frac{\alpha(\alpha+2)}{4} t^4 \frac{f^{2n-\alpha-6}(t)}{2n-\alpha-6} - Ct^{2n-\alpha-2},$$

where $C = \frac{1}{2n-\alpha-2} - \frac{p}{2(2n-\alpha-4)} + \frac{\alpha(\alpha+2)}{4(2n-\alpha-6)} > 0$, since $n \ge 3$ and $\alpha \in (2n-7, 2n-6)$.

We now use the definition of the function f and the inequality $(1-x)^\gamma \ge 1 - \gamma x(1-x)^{\gamma-1}$, for $0 < \gamma < 1$ and $0 < x < 1$. We then write

$$\begin{aligned} &= \frac{1-t^2-Nt^4}{2n-\alpha-2} - \frac{\alpha}{2}\frac{t^2(1-t^2-Nt^4)^{\frac{2n-\alpha-4}{2n-\alpha-2}}}{2n-\alpha-4} \\ &+ \frac{\alpha(\alpha+2)}{4}\frac{t^4(1-t^2-Nt^4)^{\frac{2n-\alpha-6}{2n-\alpha-2}}}{2n-\alpha-6} - Ct^{2n-\alpha-2} \\ &\le \frac{1-t^2-Nt^4}{2n-\alpha-2} - \frac{\alpha t^2}{2(2n-\alpha-4)} + \frac{\alpha t^2(t^2+Nt^4)}{2(2n-\alpha-2)(1-t^2-Nt^4)^{\frac{2}{2n-\alpha-2}}} \\ &+ \frac{\alpha(\alpha+2)t^4}{4(2n-\alpha-6)} - \frac{\alpha(\alpha+2)}{4(2n-\alpha-2)}\frac{t^4(t^2+Nt^4)}{(1-t^2-Nt^4)^{\frac{4}{2n-\alpha-2}}} - Ct^{2n-\alpha-2}. \end{aligned}$$

Hence, we have that

$$\int_0^{\beta_N} t^{-q-1}\Big(A_{D,H_\xi,\alpha}(t\theta) - A_{D,H_\xi,\alpha}(0) - \Delta A_{D,H_\xi,\alpha}(0)\frac{t^2}{2}\Big) dt$$

$$= \int_0^{\beta_N} t^{-q-1}\Big(Ct^{2n-\alpha-2} - Dt^4 + E\frac{t^2(t^2+Nt^4)}{(1-t^2-Nt^4)^{\frac{2}{2n-\alpha-2}}}$$

$$-F\frac{t^4(t^2+Nt^4)}{(1-t^2-Nt^4)^{\frac{4}{2n-\alpha-2}}}\Big)dt, \qquad (3.4.7)$$

where $E = \frac{\alpha}{2(2n-\alpha-2)} > 0$, $F = \frac{\alpha(\alpha+2)}{2n-\alpha-2} > 0$ and $D = \frac{N}{2n-\alpha-2} - \frac{\alpha(\alpha+2)}{4(2n-\alpha-6)} > 0$, for N large enough.

Now, in order to obtain an upper bound for (3.4.7) we need to estimate four different integrals. The first one simply gives $\frac{C}{2}\beta_N^2 \simeq C_1 N^{-\frac{1}{2}}$, and the second $\frac{D}{4-q}\beta_N^{-q+4} \simeq D_1 N^{\frac{q-4}{4}}$, for large N. For the third one, we make a change of variables, $u = N^{\frac{1}{4}}t$ and get

$$E\int_0^{\beta_N} \frac{t^{-q+1}(t^2+Nt^4)}{(1-t^2-Nt^4)^{\frac{2}{2n-\alpha-2}}}dt = EN^{\frac{q-2}{4}} \int_0^{\beta_N N^{\frac{1}{4}}} \frac{u^{-q+3}(N^{-\frac{1}{2}}+u^2)}{(1-u^2N^{-\frac{1}{2}}-u^4)^{\frac{2}{2n-\alpha-2}}}du$$

$$\leq E_1 N^{\frac{q-2}{4}},$$

since $\beta_N N^{\frac{1}{4}} \longrightarrow 1$ as $N \to \infty$ and the integral $\int_0^1 \frac{u^{-q+5}}{(1-u^4)^{\frac{2}{2n-\alpha-2}}}du$ converges.

We apply the same change of variables, $u = N^{\frac{1}{4}}t$, for the last integral and find that it is comparable to $N^{\frac{q}{4}-1}$.

$$F\int_0^{\beta_N} \frac{t^{-q+3}(t^2+Nt^4)}{(1-t^2-Nt^4)^{\frac{4}{2n-\alpha-2}}}dt = FN^{\frac{q}{4}-1} \int_0^{\beta_N N^{\frac{1}{4}}} \frac{u^{-q+3}(u^2N^{-\frac{1}{2}}+u^4)}{(1-u^2N^{-\frac{1}{2}}-u^4)^{\frac{4}{2n-\alpha-2}}}du. \qquad (3.4.8)$$

The integrand in the latter is a positive increasing function of u and $\beta_N N^{\frac{1}{4}} \longrightarrow 1$ as $N \to \infty$. So, we can roughly bound the integral from below by a positive constant and have that equation (3.4.8) is greater than $F_1 N^{\frac{q}{4}-1}$, where $F_1 > 0$.

In the second interval, we use the fact that $A_{D,H,\alpha}(t\theta) \leq A_{D,H,\alpha}(0)$ since central sections have maximum volume. Then, since $t << 1$, we have that

$$\int_{\beta_N}^{\alpha_N} t^{-q-1}\Big(A_{D,H_\xi,\alpha}(t\theta) - A_{D,H_\xi,\alpha}(0) - \Delta A_{D,H_\xi,\alpha}(0)\frac{t^2}{2}\Big)dt$$

$$\leq \int_{\beta_N}^{\alpha_N} t^{-q-1}\Big(\frac{1}{2n-\alpha-2} + \frac{\alpha}{2(2n-\alpha-4)}\Big)t^2 dt < A\int_{\beta_N}^{\alpha_N} t^{-q-1}dt.$$

Recall that α_N and β_N are the positive solutions of the equations $f(t) = 0$ and $1 - t^2 - Nt^4 = t^{q+1}$ respectively, and that for large N, $\alpha_N \simeq N^{-\frac{1}{4}}$. Then, it is not difficult to see that

$$A\int_{\beta_N}^{\alpha_N} t^{-q-1}dt \leq \frac{A}{(\alpha_N + \beta_N)(1 + N(\alpha_N^2 + \beta_N^2))} \simeq AN^{-\frac{1}{4}},$$

see [KYY, p.204] for details.

Lastly, for the interval $[\alpha_N, \infty)$, we use the fact that $A_{D,H_\xi,\alpha}(t\theta) = 0$. Then, we have that

$$\int_{\alpha_N}^{\infty} t^{-q-1}\Big(-A_{D,H_\xi,\alpha}(0) - \Delta A_{D,H_\xi,\alpha}(0)\frac{t^2}{2}\Big)dt$$

$$= \int_{\alpha_N}^{\infty}\Big[-\frac{t^{-q-1}}{2n-\alpha-2} + \Big(\frac{2}{2n-\alpha-2} + \frac{\alpha}{2n-\alpha-4}\Big)\frac{t^{-q+1}}{2}\Big]dt$$

$$= -A_1\alpha_N^{-q} + A_2\alpha^{-q+2} \simeq -A_1 N^{\frac{q}{4}} + A_2 N^{\frac{q-2}{4}},$$

where $A_1, A_2 > 0$.

Combining all the above estimations, for N large enough, we obtain the following upper bound for the integral in (3.4.6),

$$\int_0^\infty t^{-q-1}\Big(A_{D,H_\xi,\alpha}(t\theta) - A_{D,H_\xi,\alpha}(0) - \Delta A_{D,H_\xi,\alpha}(0)\frac{t^2}{2}\Big)dtd\theta$$

$$< C_1 N^{-\frac{1}{2}} + D_1 N^{\frac{q-4}{4}} + E_1 N^{\frac{q-2}{4}} - F_1 N^{\frac{q}{4}-1} + AN^{-\frac{1}{4}} - A_1 N^{\frac{q}{4}} + A_2 N^{\frac{q-2}{4}},$$

which clearly shows that it is negative since all the constants are positive and $q \in (2, 3)$.

□

Chapter 4

The complex Busemann-Petty problem for arbitrary measures

4.1 Introductory

In this chapter we present a generalization of the complex Busemann-Petty problem where the volume is replaced by an "almost" arbitrary measure with positive continuous density. Surprisingly, the answer remains the same. The result can be considered as the complex analogue to Zvavitch's generalization, the Busemann-Petty problem for arbitrary measures [Zv].

Let f be an even positive and continuous function on $\mathbb{R}^{2n}$. We define a measure μ with density f, so that

$$\mu(D) = \int_D f(x)dx \quad \text{and} \quad \mu(D \cap H) = \int_{D\cap H} f(x)dx$$

for every closed bounded invariant with respect to all R_θ set D in $\mathbb{R}^{2n}$ and every $(2n-2)$-dimensional subspace H of $\mathbb{R}^{2n}$.

As it is proved in Section 4.3, (Lemma 4.3.1), one may assume that the density is also invariant with respect to all rotations R_θ. We shall call this function *R_θ-invariant.* Then, the complex Busemann-Petty problem for arbitrary measures is stated as follows: Suppose K and L are origin symmetric invariant with respect to all R_θ convex bodies in $\mathbb{R}^{2n}$ so that for every $\xi \in S^{2n-1}$

$$\mu(K \cap H_\xi) \leq \mu(L \cap H_\xi),$$

does it follow that

$$\mu(K) \leq \mu(L) \ ?$$

Note that the problem is stated for any measure with positive continuous density. The positivity assumption on f is necessary, because otherwise one may assume that the density is identically zero where the affirmative answer to the problem holds trivially in all dimensions.

4.2 The Fourier analytic connection to the problem

Let $1 \leq k < 2n$ and let H be an $(2n-k)-$dimensional subspace of $\mathbb{R}^{2n}$. As it is done for the complex Busemann-Petty problem and the modified

Busemann-Petty problem, we define a lower-dimensional section function. We fix an orthonormal basis $e_1, \dots, e_k$ in the orthogonal subspace $H^\perp$. For any convex body D in $\mathbb{R}^{2n}$ and any even positive continuous function f on $\mathbb{R}^{2n}$ we define the $(2n-k)-$dimensional parallel section function $A_{f,D,H}$ as a function on $\mathbb{R}^k$ such that

$$A_{f,D,H}(u) = \int\limits_{\{x\in\mathbb{R}^{2n}:(x,e_1)=u_1,\dots,(x,e_k)=u_k\}} \chi(\|x\|_D)f(x)dx,\ u \in \mathbb{R}^k. \quad (4.2.1)$$

The original lower-dimensional parallel section function that corresponds to the $(n-k)$-dimensional volume of the section of D with a subspace H (put n instead of $2n$ and $f = 1$), was defined in [K8]. Note that at 0 the function $A_{f,D,H}$ measures the central section of the body D by the subspace H. Passing to polar coordinates on H we have that

$$\begin{aligned} A_{f,D,H}(0) &= \mu(D\cap H) = \int_H \chi(\|x\|_D)f(x)dx \\ &= \int\limits_{S^{2n-1}\cap H} \Big(\int_0^{\|\theta\|_D^{-1}} r^{2n-3}f(r\theta)dr\Big)d\theta. \end{aligned} \quad (4.2.2)$$

If D is infinitely smooth and $f \in \mathbb{C}^\infty(\mathbb{R}^{2n})$, the function $A_{f,D,H}$ is infinitely differentiable at the origin (see [K9, Lemma 2.4]). So, we consider the distribution $|u|_2^{-q-k}/\Gamma(-q/2)$ and replace A by $A_{f,D,H}$ in the regularization argument, described in Section 1.4. Then the function

$$q \longmapsto \left\langle \frac{|u|_2^{-q-k}}{\Gamma(-\frac{q}{2})}, A_{f,D,H}(u) \right\rangle \tag{4.2.3}$$

is an entire function of $q \in \mathbb{C}$. If $q = 2m,\ m \in \mathbb{N} \cup \{0\}$, then

$$\left\langle \frac{|u|_2^{-q-k}}{\Gamma(-\frac{q}{2})}\Big|_{q=2m}, A_{f,D,H}(u) \right\rangle$$

$$= \frac{(-1)^m |S^{k-1}|}{2^{m+1} k(k+2)\cdots(k+2m-2)} \Delta^m A_{f,D,H}(0),$$

Remark. If a body D is m-smooth (or infinitely) and $f \in C^m(\mathbb{R}^{2n})$, or $C^\infty(\mathbb{R}^{2n})$) it is easy for one to see that the function $x \mapsto |x|_2^{-m} \int_0^{\frac{|x|_2}{\|x\|_K}} r^{2n-3} f\big(r\frac{x}{|x|}\big) dr$ is also m-times, (infinitely) continuously differentiable on $\mathbb{R}^{2n} \setminus \{0\}$.

The proof of following proposition is similar to that of Proposition 2.2.1 in Chapter 2.

Proposition 4.2.1. *Let D be an infinitely smooth origin symmetric convex body in $\mathbb{R}^{2n}$, $f \in C^\infty(\mathbb{R}^{2n})$, and $1 \le k < 2n$. Then for every $(2n-k)-$dimensional subspace H of $\mathbb{R}^{2n}$ and any $q \in \mathbb{R},\ -k < q < 2n-k$,*

$$\left\langle \frac{|u|_2^{-q-k}}{\Gamma(-\frac{q}{2})}, A_{f,D,H}(u) \right\rangle$$

$$= \frac{2^{-q-k}\pi^{-\frac{k}{2}}}{\Gamma\big(\frac{q+k}{2}\big)} \int_{S^{2n-1}\cap H^\perp} \Big(|x|_2^{-2n+k+q} \int_0^{\frac{|x|_2}{\|x\|_D}} r^{2n-k-1-q} f\big(r\frac{x}{|x|_2}\big) dr\Big)^\wedge(\theta) d\theta. \tag{4.2.4}$$

Now, if $m \in \mathbb{N} \cup \{0\}$,

$$\Delta^m A_{f,D,H}(0)$$

$$= \frac{(-1)^m}{(2\pi)^k} \int\limits_{S^{2n-1}\cap H^\perp} \Big(|x|_2^{-2n+k+2m} \int_0^{\frac{|x|_2}{\|x\|_D}} r^{2n-k-1-2m} f\big(r\frac{x}{|x|_2}\big) dr \Big)^\wedge (\theta) d\theta \tag{4.2.5}$$

The following (elementary) inequality is similar to Lemma 1 in [Zv].

Lemma 4.2.1. *Let* $a, b > 0$ *and let* α *be a non-negative function on* $(0, \max\{a, b\}]$ *so that the integrals below converge. Then*

$$\int_0^a t^{2n-1}\alpha(t)dt - a^2 \int_0^a t^{2n-3}\alpha(t)dt \leq \int_0^b t^{2n-1}\alpha(t)dt - a^2 \int_0^b t^{2n-3}\alpha(t)dt. \tag{4.2.6}$$

4.3 Measure of sections and k-intersection bodies

As mentioned in Section 4.1, we can assume that the density function is R_θ-invariant. This simple observation plays an important role to the solution of the problem.

Lemma 4.3.1. *Suppose* f *is an even non-negative continuous function on* $\mathbb{R}^{2n}$ *and* μ *is a measure with density* f*. Then there exists an even non-*

negative continuous function $\tilde{f}$ that is invariant with respect to all rotations R_θ such that

$$\mu(D) = \int_D \tilde{f}(x)dx \quad \textit{and} \quad \mu(D \cap H_\xi) = \int_{D \cap H_\xi} \tilde{f}(x)dx,$$

for every closed bounded invariant with respect to all R_θ set D in $\mathbb{R}^{2n}$ and $\xi \in S^{2n-1}$.

Proof. We define $\tilde{f}(x) = \frac{1}{2\pi}\int_0^{2\pi} f(R_\theta x)d\theta$, for every $x \in \mathbb{R}^{2n}$. Then for every compact invariant with respect to all R_θ set D in $\mathbb{R}^{2n}$,

$$\begin{aligned} \int_D \tilde{f}(x)dx &= \frac{1}{2\pi}\int_D \int_0^{2\pi} f(R_\theta x)d\theta dx \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_{R_\theta^{-1}D} f(y)dyd\theta = \mu(D), \end{aligned}$$

since $R_\theta^{-1}D = D$, for all $\theta \in [0, 2\pi]$.

Moreover, since central sections of complex convex bodies by complex hyperplanes correspond to convex bodies in $\mathbb{R}^{2n-2}$ that are also invariant with respect to the R_θ rotations, we similarly get that for every $\xi \in S^{2n-1}$,

$$\mu(D \cap H_\xi) = \int_{D \cap H_\xi} \tilde{f}(x)dx.$$

□

Now, we are ready to express the measure of the central sections in terms of the Fourier transform.

Theorem 4.3.1. *Suppose K is an infinitely smooth origin symmetric invariant with respect to all R_θ convex body in $\mathbb{R}^{2n}$, $n \geq 2$, and f is an infinitely differentiable even positive and R_θ-invariant function on $\mathbb{R}^{2n}$. Then for every $\xi \in S^{2n-1}$*

$$\mu(K \cap H_\xi) = \frac{1}{2\pi}\Big(|x|_2^{-2n+2}\int_0^{\frac{|x|_2}{\|x\|_K}} r^{2n-3} f\big(r\frac{x}{|x|_2}\big)dr\Big)^\wedge(\xi) \qquad (4.3.1)$$

In order to prove Theorem 4.3.1 we need the following:

Lemma 4.3.2. *Let K and f as in Theorem 4.3.1. Then for every $\xi \in S^{2n-1}$ the Fourier transform of the distribution $|x|_2^{-2n+2}\int_0^{\frac{|x|_2}{\|x\|_K}} r^{2n-3} f(r\frac{x}{|x|_2})dr$ is a constant function on $S^{2n-1} \cap H_\xi^\perp$.*

Proof. The function $\|x\|_K^{-1}$ is invariant with respect to all R_θ (see Section 1.3), so, since f is R_θ-invariant it is easy to see that the function

$$|x|_2^{-2n+2}\int_0^{\frac{|x|_2}{\|x\|_K}} r^{2n-3} f(r\frac{x}{|x|_2})dr$$

is a continuous function which is also invariant with respect to all rotations R_θ. By the connection between the Fourier transform of distributions and linear transformations, its Fourier transform is also invariant with respect to all R_θ. As mentioned in the Introduction, the space $H_\xi^\perp$ is spanned by the vectors ξ and $\xi^{\perp}$. So every vector in $S^{2n-1}\cap H_\xi^{\perp}$ is a rotation R_θ, for some $\theta\in[0,2\pi]$, of ξ and hence the Fourier transform of

$$|x|_2^{-2n+2}\int_0^{\frac{|x|_2}{\|x\|_K}} r^{2n-3}f(r\frac{x}{|x|_2})dr$$

is a constant function on $S^{2n-1}\cap H_\xi^{\perp}$. □

Proof of Theorem 4.3.1. Let $\xi\in S^{2n-1}$. In formula (4.2.5) we put $H_\xi = H$, $k=2$ and $m=0$. Then, by the definition of the lower-dimensional section function $A_{f,D,H}(0)$, equation (4.2.2), we have that

$$\mu(K\cap H_\xi)=\frac{1}{(2\pi)^2}\int_{S^{2n-1}\cap H_\xi^{\perp}}\Big(|x|_2^{-2n+2}\int_0^{\frac{|x|_2}{\|x\|_K}} r^{2n-3}f\big(r\frac{x}{|x|_2}\big)dr\Big)^{\wedge}(\eta)d\eta.$$

By Lemma 4.3.2, the function under the integral is constant on the circle $S^{2n-1}\cap H_\xi^{\perp}$. Since $\xi\in H_\xi^{\perp}$ we have that

$$\mu(K\cap H_\xi)=\frac{1}{(2\pi)^2}2\pi\Big(|x|_2^{-2n+2}\int_0^{\frac{|x|_2}{\|x\|_K}} r^{2n-3}f\big(r\frac{x}{|x|_2}\big)dr\Big)^{\wedge}(\xi)$$

which proves the theorem. □

As in the case of the complex Busemann-Petty problem, the property of a body to be a 2-intersection body is closely related to the solution of the complex Busemann-Petty problem for arbitrary measures.

Theorem 4.3.2. *The solution of the complex Busemann-Petty problem for arbitrary measures in* $\mathbb{C}^n$ *has an affirmative answer if and only if every origin symmetric invariant with respect to all* R_θ *convex body in* $\mathbb{R}^{2n}$ *is a* 2-*intersection body.*

The proof of Theorem 4.3.2 will follow from the Remarks and the next lemmas.

Remark 1. To prove the affirmative part of the problem it is enough to consider infinitely smooth origin symmetric invariant with respect to all R_θ bodies. This is true because one can approximate, in the radial metric, from inside the body K and from outside the body L by infinitely smooth convex invariant with respect to all R_θ bodies. Then if the affirmative answer holds for infinitely smooth bodies it also holds in the general case.

Remark 2. Let D be an origin symmetric convex body which is not a k-intersection body. Then, there exists a sequence of infinitely smooth convex bodies with strictly positive curvature which are not k-intersection bodies that converges in the radial metric to D, (see [K9, Lemma 4.10]). If, in addition, D is invariant with respect to all R_θ, one can choose a sequence of bodies with the same property.

Remark 3. A simple approximation argument allows us to prove Theorem 4.3.2 only for measures whose density is an infinitely differentiable even positive and R_θ-invariant function on $\mathbb{R}^{2n}$. Let f be the even positive continuous R_θ-invariant density function of a measure μ, as it is defined in the Introduction. Then there exists an increasing sequence g_n of even positive functions in $C^\infty(\mathbb{R}^{2n})$ such that $g_n(x)\chi(\|x\|_D) \to f(x)\chi(\|x\|_D)$, a.e., for every compact set D. Then by the Monotone Convergence Theorem we have that

$$\int_{\mathbb{R}^{2n}} g_n(x)\chi(\|x\|_D)dx \to \mu(D) \quad \text{and} \quad \int_H g_n(x)\chi(\|x\|_D)dx \to \mu(H \cap D),$$

as $n \to \infty$, for every subspace H of $\mathbb{R}^{2n}$. In addition, by Lemma 4.3.1, we may assume that every g_n is also R_θ-invariant.

Now we are ready to prove the affirmative part of the complex Busemann-Petty problem for arbitrary measures.

Lemma 4.3.3. *Suppose K and L are infinitely smooth origin symmetric invariant with respect to all R_θ convex bodies in $\mathbb{R}^{2n}$ so that K is a 2-intersection body and let f be an infinitely differentiable even positive R_θ-invariant function on $\mathbb{R}^{2n}$. Then, if for every $\xi \in S^{2n-1}$*

$$\mu(K \cap H_\xi) \le \mu(L \cap H_\xi) \tag{4.3.2}$$

then

$$\mu(K) \le \mu(L).$$

Proof. By the Remark before Proposition 4.2.1 and Proposition 1.4.2, the Fourier transform of the distributions

$$|x|_2^{-2n+2} \int_0^{\frac{|x|_2}{\|x\|_K}} r^{2n-3} f\big(r\tfrac{x}{|x|_2}\big)\,dr \quad \text{and} \quad |x|_2^{-2n+2} \int_0^{\frac{|x|_2}{\|x\|_L}} r^{2n-3} f\big(r\tfrac{x}{|x|_2}\big)\,dr$$

are homogeneous of degree -2 and continuous functions on $\mathbb{R}^{2n} \setminus \{0\}$. So, by Theorem 4.3.1, the inequality (4.3.2) becomes

$$\Big(|x|_2^{-2n+2} \int_0^{\frac{|x|_2}{\|x\|_K}} r^{2n-3} f\big(r\frac{x}{|x|_2}\big)\,dr\Big)^\wedge(\xi)$$
$$\le \Big(|x|_2^{-2n+2} \int_0^{\frac{|x|_2}{\|x\|_L}} r^{2n-3} f\big(r\frac{x}{|x|_2}\big)\,dr\Big)^\wedge(\xi).$$

Since K is an infinitely smooth 2-intersection body, by Proposition 1.4.4 and Proposition 1.4.2, the Fourier transform of the distribution $\|x\|_K^{-2}$ is a non-negative continuous, outside the origin, function on $\mathbb{R}^{2n}$. Multiplying both sides of the latter inequality by $\big(\|x\|_K^{-2}\big)^\wedge$ and applying the spherical version of Parseval, we have that

$$\int_{S^{2n-1}} \big(\|x\|_K^{-2}\big)^\wedge(\xi) \Big(|x|_2^{-2n+2} \int_0^{\frac{|x|_2}{\|x\|_K}} r^{2n-3} f\big(r\frac{x}{|x|_2}\big)\,dr\Big)^\wedge(\xi)\,d\xi$$
$$\le \int_{S^{2n-1}} \big(\|x\|_K^{-2}\big)^\wedge(\xi) \Big(|x|_2^{-2n+2} \int_0^{\frac{|x|_2}{\|x\|_L}} r^{2n-3} f\big(r\frac{x}{|x|_2}\big)\,dr\Big)^\wedge(\xi)\,d\xi,$$

which gives

$$\int_{S^{2n-1}} \|x\|_K^{-2} \int_0^{\|x\|_K^{-1}} r^{2n-3} f(rx) dr dx$$
$$\leq \int_{S^{2n-1}} \|x\|_K^{-2} \int_0^{\|x\|_L^{-1}} r^{2n-3} f(rx) dr dx. \tag{4.3.3}$$

We use the elementary inequality, equation (4.2.6), with $a = \|x\|_K^{-1}, b = \|x\|_L^{-1}$ and $\alpha(r) = f(rx)$ and integrate over S^{2n-1}. Then

$$\int_{S^{2n-1}} \Big(\int_0^{\|x\|_K^{-1}} r^{2n-1} f(rx) dr\Big) dx - \int_{S^{2n-1}} \|x\|_K^{-2} \Big(\int_0^{\|x\|_K^{-1}} r^{2n-3} f(rx) dr\Big) dx$$
$$\leq \int_{S^{2n-1}} \Big(\int_0^{\|x\|_L^{-1}} r^{2n-1} f(rx) dr\Big) dx - \int_{S^{2n-1}} \|x\|_K^{-2} \Big(\int_0^{\|x\|_L^{-1}} r^{2n-3} f(rx) dr\Big) dx \tag{4.3.4}$$

We add (4.3.3) and (4.3.4) and have that

$$\int_{S^{2n-1}} \Big(\int_0^{\|x\|_K^{-1}} r^{2n-1} f(rx) dr\Big) dx \leq \int_{S^{2n-1}} \Big(\int_0^{\|x\|_K^{-1}} r^{2n-1} f(rx) dr\Big) dx$$

which immediately implies that

$$\mu(K) \leq \mu(L).$$

□

For the negative part we need a perturbation argument to construct a body that will give a counter-example to the problem. The following lemma (without the assumption of invariance with respect to R_θ rotations) was proved in [Zv, Proposition 2], (see also [K9, Lemma 5.16]). The new body immediately inherits the additional property of invariance with respect to all R_θ of the original convex body.

Lemma 4.3.4. *Let L be an infinitely smooth origin symmetric convex body with positive curvature and let $f, g \in C^2(\mathbb{R}^{2n})$, such that f is strictly positive on $\mathbb{R}^{2n}$. For $\varepsilon > 0$ we define a star body K so that*

$$\int_0^{\|x\|_K^{-1}} t^{2n-3} f(tx)dt = \int_0^{\|x\|_L^{-1}} t^{2n-3} f(tx)dt - \varepsilon g(x), \ \forall x \in S^{2n-1}.$$

Then, if ε is small enough the body K is convex. Moreover, if L is invariant with respect to all R_θ, and f, g are R_θ-invariant then K is also invariant with respect to all R_θ.

Lemma 4.3.5. *Let $f \in C^\infty(\mathbb{R}^{2n})$ is an even positive R_θ-invariant function. Suppose L is an infinitely smooth origin symmetric invariant with respect to all R_θ convex body in $\mathbb{R}^{2n}$ with positive curvature which is not a 2-intersection body. Then there exists an origin symmetric invariant with respect to all R_θ convex body K in $\mathbb{R}^{2n}$ so that for every $\xi \in S^{2n-1}$*

$$\mu(K \cap H_\xi) \le \mu(L \cap H_\xi)$$

but

$$\mu(K) > \mu(L).$$

Proof. The body L is infinitely smooth, so, by Proposition 1.4.2, the Fourier transform of $\|x\|_L^{-2}$ is a continuous function on $\mathbb{R}^{2n}$. Since L is not a 2-intersection body, by Proposition 1.4.4 there exists an open set $\Omega \subset S^{2n-1}$ where the Fourier transform of $\|x\|_L^{-2}$ is negative. We can assume that Ω is invariant with respect to rotations R_θ since L is.

We define an even non-negative invariant with respect to all R_θ function $h \in C^\infty(S^{2n-1})$ whose support is in Ω. We extend h to an even homogeneous function $h(\frac{x}{|x|_2})|x|_2^{-2}$ of degree -2 on $\mathbb{R}^{2n}$. Then, by Proposition 1.4.2, the Fourier transform of $h(\frac{x}{|x|_2})|x|_2^{-2}$ is an even homogeneous function $g(\frac{x}{|x|_2})|x|_2^{-2n+2}$ of degree $-2n+2$ on $\mathbb{R}^{2n}$, with $g \in C^\infty(S^{2n-1})$. Moreover, g is also invariant with respect to rotations R_θ, since the Fourier transform preserves linear transformations.

The assumptions for the body L allow us to apply Lemma 4.3.4 and take $\varepsilon > 0$ small enough to define a convex body K by

$$|x|_2^{-2n+2} \int_0^{\frac{|x|_2}{\|x\|_K}} t^{2n-3} f\big(t\frac{x}{|x|_2}\big)dt$$
$$= |x|_2^{-2n+2} \int_0^{\frac{|x|_2}{\|x\|_L}} t^{2n-3} f\big(t\frac{x}{|x|_2}\big)dt - \varepsilon g\big(\frac{x}{|x|_2}\big)|x|_2^{-2n+2}.$$

We apply Fourier transform to both sides of the latter inequality. Then, by Theorem 4.3.1, since $h \geq 0$, we obtain the following inequality for the measures of the central sections of K and L by the subspace H_ξ,

$$\mu(K\cap H_\xi)=\frac{1}{2\pi}\Big(|x|_2^{-2n+2}\int_0^{\frac{|x|_2}{\|x\|_K}} r^{2n-3}f\big(r\frac{x}{|x|_2}\big)dr\Big)^\wedge(\xi)$$

$$=\frac{1}{2\pi}\Big(|x|_2^{-2n+2}\int_0^{\frac{|x|_2}{\|x\|_L}} r^{2n-3}f\big(r\frac{x}{|x|_2}\big)dr\Big)^\wedge(\xi)-(2\pi)^{2n-1}\varepsilon h(\xi)$$

$$\leq \mu(L\cap H_\xi)$$

On the other hand, the function h is positive only where $\big(\|\cdot\|_L^{-2}\big)^\wedge$ is negative. So, for every $\xi\in S^{2n-1}$,

$$\big(\|\cdot\|_L^{-2}\big)^\wedge(\xi)\Big(|x|_2^{-2n+2}\int_0^{\frac{|x|_2}{\|x\|_K}} r^{2n-3}f\big(r\frac{x}{|x|_2}\big)dr\Big)^\wedge(\xi)$$

$$=\big(\|\cdot\|_L^{-2}\big)^\wedge(\xi)\Big(|x|_2^{-2n+2}\int_0^{\frac{|x|_2}{\|x\|_L}} r^{2n-3}f\big(r\frac{x}{|x|_2}\big)dr\Big)^\wedge(\xi)$$

$$-(2\pi)^{2n}\big(\|\cdot\|_L^{-2}\big)^\wedge(\xi)\varepsilon h(\xi)$$

$$>\big(\|\cdot\|_L^{-2}\big)^\wedge(\xi)\Big(|x|_2^{-2n+2}\int_0^{\frac{|x|_2}{\|x\|_L}} r^{2n-3}f\big(r\frac{x}{|x|_2}\big)dr\Big)^\wedge(\xi),$$

Now, we integrate the latter inequality over S^{2n-1} and apply the spherical version of Parseval's identity. Then similarly to Lemma 4.3.3, we apply the elementary inequality for integrals, Lemma 4.2.1, and conclude that

$$\mu(K)>\mu(L).$$

□

4.4 The solution of the problem

To prove the main result of this paper we need to determine the dimensions in which an origin symmetric invariant with respect to all R_θ convex body in $\mathbb{R}^{2n}$ is a 2-intersection body.

Theorem 4.4.1. *The solution to the complex Busemann-Petty problem for arbitrary measures is affirmative if $n \le 3$ and negative if $n \ge 4$.*

Proof : It is known that an origin symmetric invariant with respect to R_θ, convex body in $\mathbb{R}^{2n}, n \ge 3$, is a k-intersection body if $k \ge 2n-4$ (see Theorem 2.4.2). Hence, we obtain an affirmative answer to the complex Busemann-Petty problem for arbitrary measures if $n \le 3$.

Now, suppose that $n \ge 4$. The unit ball $B_q^n(\mathbb{C}^n)$ of the complex space $\ell_q^n,\ q > 2$, considered as a subset of $\mathbb{R}^{2n}$:

$$B_q^n(\mathbb{C}^n) = \{x \in \mathbb{R}^{2n} : \|x\|_q = \left((x_{11}^2 + x_{12}^2)^{q/2} + \cdots + (x_{n1}^2 + x_{n2}^2)^{q/2}\right)^{1/q} \le 1\}$$

provides a counter-example for the Lebegue measure ($f = 1$), of a body that is not a k-intersection body for $k < 2n-4$ (see Chapter 6). By Proposition 1.4.4 this implies that for $n \ge 4$ the distribution $\|x\|_q^{-2}$ is not positive definite. Then the result follows by Theorem 4.3.2. $\square$

Chapter 5

Extremal sections of complex ℓ_p-balls, $0 < p \leq 2$

5.1 The complex volume of sections

In this chapter we present a continuation to the study of extremal sections of ℓ_p-balls. We characterize the extremal sections of complex ℓ_p-balls $B_p(\mathbb{C}^n)$, for $0 < p \leq 2$, (see Section 1.1 for the history).

Theorem 5.1.1. *Let* $0 < p \leq 2$. *For* $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n,\ \xi \neq 0$. *The* $(n-1)-$*dimensional complex volume of* $B_p(\mathbb{C}^n) \cap H_\xi$ *is minimal if* $|\xi_1| = \ldots = |\xi_n|$, *and it is maximal if* ξ *has only one non-zero coordinate.*

The part of this theorem related to the maximal sections was established earlier by Meyer and Pajor [MP], Corollary 2.5 for $1 \leq p \leq 2$, and by Barthe [Bar] for $0 < p < 1$. In fact, these papers cover a more general case of the unit balls of the real spaces $\ell_p^n(l_2^m)$ and show that, for every integer k, the "standard" sections of these balls of dimension km are minimal for $p \geq 2$ and maximal for $0 < p \leq 2$.

We prove Theorem 5.1.1 by generalizing the method of [K2] to the complex case. Like in the real case in [K2], the minimal and maximal sections are identified simultaneously.

Recall that we identify $\ell_p(\mathbb{C}^n)$ with the real $2n$-dimensional space equipped with the norm

$$\|x\|_p = \Big[(x_{11}^2 + x_{12}^2)^{\frac{p}{2}} + \ldots + (x_{n1}^2 + x_{n2}^2)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \tag{5.1.1}$$

where

$$\mathbb{C}^n \ni x = (x_{11} + ix_{12}, \ldots, , x_{n1} + ix_{n2}) \mapsto (x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \in \mathbb{R}^{2n}.$$

As mentioned in the Section 1.3, for every $\xi \in \mathbb{C}^n$, this mapping identifies the complex hyperplane H_ξ perpendicular to $\xi \in \mathbb{C}^n$ with the $(2n-2)$-dimensional subspace orthogonal to the vectors

$$\xi = (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}) \quad \text{and} \quad \xi^\perp = (-\xi_{12}, \xi_{11}, \ldots, -\xi_{n2}, \xi_{n1}).$$

We use the same notation H_ξ for this subspace and for the unit ball of the complex ℓ_p space when viewed as a subset of $\mathbb{R}^{2n}$. Then

$$\mathrm{Volc}_{n-1}\big(B_p(\mathbb{C}^n) \cap H_\xi\big) = \mathrm{Vol}_{2n-2}\big(B_p(\mathbb{C}^n) \cap H_\xi\big),$$

with

$$\begin{aligned} B_p(\mathbb{C}^n) &= \{(x_{11}+ix_{12},\dots x_{n1}+ix_{n2}) \in \mathbb{C}^n : \sum_{j=1}^{n}(x_{j1}^2+x_{j2}^2)^{\frac{p}{2}} \le 1\} \\ &= \{(x_{11},\dots x_{n1},x_{n2}) \in \mathbb{R}^{2n} : \sum_{j=1}^{n}(x_{j1}^2+x_{j2}^2)^{\frac{p}{2}} \le 1\}. \end{aligned} \tag{5.1.2}$$

5.2 The Fourier transform formula for sections of $B_p(\mathbb{C}^n)$.

For every origin symmetric star body K and every $(2n-2)$-dimensional subspace H, the polar formula for the volume of the central section of K by H, (see equation (1.2.1)), is given by

$$\mathrm{Vol}_{2n-2}(K\cap H) = \frac{1}{(2n-2)} \int\limits_{S^{n-1}\cap H} \|x\|^{-2n+2} dx. \tag{5.2.1}$$

On the other hand, if K is infinitely smooth, then using Theorem 2.2.1, with $k=2$ and $m=0$, we have that

$$\mathrm{Vol}_{2n-2}(K\cap H) = \frac{1}{(2\pi)^2}\frac{1}{(2n-2)} \int\limits_{S^{n-1}\cap H^{\perp}} \left(\|x\|_K^{-2n+2}\right)^{\wedge}(\theta)d\theta. \tag{5.2.2}$$

Although the bodies $B_p(\mathbb{C}^n)$ are not always smooth, we assume that formula (5.2.2) holds for the norm $\|\cdot\|_p$ introduced in (5.1.1). In Section 5.4 we present a simple approximation argument proving this assumption.

Lemma 5.2.1. *Let* $0<p<\infty$, $y=(y_{11},y_{12}\dots y_{n1},y_{n2}) \in \mathbb{R}^{2n}$. *Then the Fourier transform of* $\|\cdot\|_p^{-2n+2}$, *in the sense of distributions in* $\mathbb{R}^{2n}$, *is equal* *to a locally integrable function on* $\mathbb{R}^{2n}$

$$\big(\|\cdot\|_p^{-2n+2}\big)^\wedge(y)$$

$$= \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_0^\infty t\Big(\prod_{j=1}^n \int_{\mathbb{R}^2} e^{-it(y_{j1}x_{j1}+y_{j2}x_{j2})} e^{-(x_{j1}^2+x_{j2}^2)^{p/2}} dx_{j1}\ dx_{j2}\Big)\ dt.$$

Proof : From the definition of the Gamma function, equation (1.2.7), we have that

$$\|x\|_p^{-2n+2} = \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_0^\infty t^{2n-3} e^{-t^p\|x\|_p^p} dt. \tag{5.2.3}$$

We first fix $t > 0$ and compute the Fourier transform of the function $x \longmapsto e^{-t^p\|x\|_p^p}$: for any $y \in \mathbb{R}^{2n}$, making a change of variables $tx = z$ we get

$$\begin{aligned} \big(e^{-t^p\|x\|_p^p}\big)^\wedge(y) &= \int_{\mathbb{R}^{2n}} e^{-i(y,x)} e^{-t^p\|x\|_p^p} dx = \int_{\mathbb{R}^{2n}} e^{-i(y,z/t)} e^{-\|z\|_p^p} t^{-2n} dz \\ &= t^{-2n} \prod_{j=1}^n \int_{\mathbb{R}^2} e^{-i(\frac{y_{j1}}{t}z_{j1}+\frac{y_{j2}}{t}z_{j2})} e^{-(z_{j1}^2+z_{j2}^2)^{\frac{p}{2}}} dz_{j1}dz_{j2}. \end{aligned} \tag{5.2.4}$$

The function $\|x\|_p^{-2n+2}$ is locally integrable on $\mathbb{R}^{2n}$. Using (5.2.3), (5.2.4), Fubini and the change of variables $\frac{1}{t} = s$, we get that for any even test function ϕ,

$$\Big\langle \big(\|\cdot\|_p^{-2n+2}\big)^\wedge, \phi \Big\rangle = \Big\langle \|x\|_p^{-2n+2}, \widehat{\phi} \Big\rangle$$

$$= \int_{\mathbb{R}^{2n}} \|x\|_p^{-2n+2} \widehat{\phi}(x) dx = \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_{\mathbb{R}^{2n}} \Big(\int_0^\infty t^{2n-3} e^{-t^p\|x\|_p^p} dt\Big) \widehat{\phi}(x) dx$$

$$= \frac{p}{\Gamma\big(\frac{2n-2}{p}\big)} \int_0^\infty t^{2n-3} \int_{\mathbb{R}^{2n}} e^{-t^p\|x\|_p^p} \widehat{\phi}(x) dx\ dt$$

$$= \frac{p}{\Gamma\left(\frac{2n-2}{p}\right)} \int_0^\infty t^{2n-3} \int\limits_{\mathbb{R}^{2n}} \left(e^{-t^p\|x\|_p^p}\right)^\wedge(y)\phi(y)dy\ dt$$

$$= \frac{p}{\Gamma\left(\frac{2n-2}{p}\right)} \int_0^\infty t^{-3}\Big(\int\limits_{\mathbb{R}^{2n}} \phi(y)\times$$

$$\prod_{j=1}^{n} \int\limits_{\mathbb{R}^2} e^{-i(\frac{y_{j1}}{t}z_{j1}+\frac{y_{j2}}{t}z_{j2})} e^{-(z_{j1}^2+z_{j2}^2)^{\frac{p}{2}}} dz_{j1}dz_{j2}\ dy\Big)\ dt$$

$$= \frac{p}{\Gamma\left(\frac{2n-2}{p}\right)} \int\limits_{\mathbb{R}^{2n}} \phi(y)\times$$

$$\int_0^\infty s\Big(\prod_{j=1}^{n}\int_{\mathbb{R}^2} e^{-is(y_{j1}z_{j1}+y_{j2}z_{j2})} e^{-(z_{j1}^2+z_{j2}^2)^{p/2}} dz_{j1}dz_{j2}\Big)\ dsdy.$$

Since ϕ is an arbitrary even test function, the result follows. □

Remark. We define a function g on $\mathbb{R}^2$ by

$$g\big(y_{j1}, y_{j2}\big) := \int\limits_{\mathbb{R}^2} e^{-i(y_{j1}z_{j2}+y_{j2}z_{j2})} e^{-(z_{j1}^2+z_{j2}^2)^{\frac{p}{2}}} dz_{j1}dz_{j2}.$$

The function $\big(z_{j1}, z_{j2}\big) \longmapsto e^{-(z_{j1}^2+z_{j2}^2)^{\frac{p}{2}}}$ is a radial function on $\mathbb{R}^2$, so is its Fourier transform. Therefore,

$$\begin{aligned} g\big(ty_{j1}, ty_{j2}\big) &= g(t\sqrt{y_{j1}^2+y_{j2}^2}, 0) \\ f\big(t(y_{j1}^2+y_{j2}^2)^{1/2}\big) &= \int\limits_{\mathbb{R}^2} e^{-it(y_{j1}^2+y_{j2}^2)^{\frac{1}{2}}z_{j1}} e^{-(z_{j1}^2+z_{j2}^2)^{\frac{p}{2}}} dz_{j1}dz_{j2}. \end{aligned}$$

The latter formula defines a function f on $\mathbb{R}$ that we are going to use throughout this chapter:

$$f(u) = \int\limits_{\mathbb{R}^2} e^{-iuz_{j1}} e^{-(z_{j1}^2+z_{j2}^2)^{\frac{p}{2}}} dz_{j1}dz_{j2}, \quad u \in \mathbb{R}. \tag{5.2.5}$$

The following simple observation has played an important role in the current study. Here we only state it as a lemma, (see for example Chapter 2, Lemma 2.3.1).

Lemma 5.2.2. *Let f defined in (5.2.5) and $\xi \in S^{2n-1}$. Then, the function $\xi \longmapsto \prod_{j=1}^{n} f\big(t(\xi_{j1}^2 + \xi_{j2}^2)^{1/2}\big)$ is constant on $S^{2n-1} \cap H_\xi^\perp$.*

Now, we are ready to obtain the $(2n-2)$-dimensional volume of the sections in terms of the Fourier transform.

Theorem 5.2.1. *Let $0 < p \leq 2$. For every $\xi \in S^{2n-1}$*

$$Vol_{2n-2}\big(B_p(\mathbb{C}^n) \cap H_\xi\big) = \frac{1}{2\pi}\frac{1}{(2n-2)}\frac{p}{\Gamma\big(\frac{2n-2}{p}\big)}\int_0^\infty t\prod_{j=1}^{n} f\big(t(\xi_{j1}^2 + \xi_{j2}^2)^{1/2}\big)dt \tag{5.2.6}$$

Proof: Fix $\xi \in S^{2n-1}$. We apply formula (5.2.2) with $K = B_p(\mathbb{C}^n)$ and $H = H_\xi$, (via the approximation argument of Section 5.4). Then

$$\text{Vol}_{2n-2}\big(B_p(\mathbb{C}^n) \cap H_\xi\big) = \frac{1}{(2\pi)^2}\frac{1}{(2n-2)}\int\limits_{S^{2n-1}\cap H_\xi^\perp} \big(\|x\|_p^{-2n+2}\big)^\wedge(y)dy$$

By Lemma 5.2.1, the Remark and Fubini's theorem, the latter quantity is equal to

$$\frac{1}{(2\pi)^2}\frac{1}{(2n-2)}\frac{p}{\Gamma\big(\frac{2n-2}{p}\big)}\int\limits_{S^{2n-1}\cap H_\xi^\perp}\int_0^\infty t\prod_{j=1}^{n} f\big(t(y_{j1}^2 + y_{j2}^2)^{1/2}\big)dt\ dy$$

$$= \frac{1}{(2\pi)^2} \frac{1}{(2n-2)} \frac{p}{\Gamma\left(\frac{2n-2}{p}\right)} \int_0^\infty t \int_{S^{2n-1}\cap H_\xi^\perp} \prod_{j=1}^n f\big(t(y_{j1}^2 + y_{j2}^2)^{1/2}\big) dy\ dt. \tag{5.2.7}$$

Now, by Lemma 5.2.2, the function under the inner integral is constant on the circle $S^{2n-1} \cap H_\xi^\perp$.

Hence, the inner integral in (5.2.7) is equal to

$$\int_{S^{2n-1}\cap H_\xi^\perp} \prod_{j=1}^n f\big(t(y_{j1}^2 + y_{j2}^2)^{1/2}\big) dy = \int_{S^{2n-1}\cap E_\xi^\perp} \prod_{j=1}^n f\big(t(\xi_{j1}^2 + \xi_{j2}^2)^{1/2}\big) dy$$

$$= 2\pi \prod_{j=1}^n f\big(t(\xi_{j1}^2 + \xi_{j2}^2)^{1/2}\big).$$

The latter equality and (5.2.7) imply (5.2.6).

5.3 Extremal sections

The proof of the main result of this chapter, Theorem 5.1.1, immediately follows from Theorem 5.2.1 and the next lemma.

Lemma 5.3.1. *If $0 < p \leq 2$ then the function $f(\sqrt{\cdot})$ is log-convex on $[0, +\infty)$.*

Proof: For every $0 < p \leq 2$, the function $\exp(-|\cdot|^{p/2})$ is completely monotone, so by Bernstein's Theorem (see [W]), there exists a measure μ_p on $[0, +\infty)$ so that, for every $t \in \mathbb{R}$,

$$e^{-|t|^{\frac{p}{2}}} = \int_0^\infty e^{-ut} d\mu_p(u) \Rightarrow e^{-|t|^p} = \int_0^\infty e^{-ut^2} d\mu_p(u)$$

$$\Rightarrow e^{-(x_1^2+x_2^2)^{\frac{p}{2}}} = \int_0^\infty e^{-u(x_1^2+x_2^2)} d\mu_p(u), \quad x_1, x_2 \in \mathbb{R}.$$

Therefore, by Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}} e^{-isx_1} e^{-(x_1^2+x_2^2)^{\frac{p}{2}}} dx_1 &= \int_{\mathbb{R}} e^{-isx_1} \int_0^\infty e^{-u(x_1^2+x_2^2)} d\mu_p(u) dx_1 \\
&= \int_0^\infty e^{-ux_2^2} \int_{\mathbb{R}} e^{-isx_1} e^{-ux_1^2} dx_1 d\mu_p(u) \\
&= \sqrt{\pi} \int_0^\infty e^{-ux_2^2} \frac{1}{\sqrt{u}} e^{-\frac{s^2}{4u}} d\mu_p(u).
\end{aligned}$$

Now integrate the latter by x_2 over $\mathbb{R}$. Using Fubini and the well-known identity $\int_{\mathbb{R}} e^{-t^2} dt = \sqrt{\pi}$, we get the following expression for the function f :

$$\begin{aligned}
f(s) &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-isx_1} e^{-(x_1^2+x_2^2)^{\frac{p}{2}}} dx_1 dx_2 \\
&= \sqrt{\pi} \int_{\mathbb{R}} \int_0^\infty e^{-ux_2^2} \frac{1}{\sqrt{u}} e^{-s^2/4u} d\mu_p(u) dx_2 \\
&= \sqrt{\pi} \int_0^\infty \frac{1}{\sqrt{u}} e^{-s^2/4u} \big(\int_{\mathbb{R}} e^{-ux_2^2} dx_2 \big) d\mu_p(u) \\
&\Rightarrow \quad f(s) = \pi \int_0^\infty \frac{1}{u} e^{-s^2/4u} d\mu_p(u).
\end{aligned}$$

For any $\alpha_1, \alpha_2 > 0$, using the latter formula and the Cauchy-Schwartz inequality, we get

$$\left(f\Big(\sqrt{\frac{a_1+a_2}{2}}\Big)\right)^2 = \pi^2\Big[\int_0^\infty \frac{1}{\sqrt{u}}e^{-a_1/8u}e^{-a_2/8u}\frac{1}{\sqrt{u}}d\mu_p(u)\Big]^2$$
$$\leq \pi^2\Big(\int_0^\infty \frac{1}{u}e^{-a_1/4u}d\mu_p(u)\Big)\Big(\int_0^\infty \frac{1}{u}e^{-a_2/4u}d\mu_p(u)\Big)$$
$$= f(\sqrt{a_1})f(\sqrt{a_2})$$

which implies that $f(\sqrt{\cdot})$ is log-convex.

□

Now, to prove Theorem 5.1.1, note that the log-convexity of f immediately implies that for any $0 < \alpha_1 < \beta_1 < \beta_2 < \alpha_2,\ \alpha_1^2 + \alpha_2^2 = \beta_1^2 + \beta_2^2 = 1$, we have

$$f(t\beta_1)f(t\beta_2) \leq f(t\alpha_1)f(t\alpha_2), \quad \forall t > 0.$$

Therefore, the integrand in the formula of Theorem 5.2.1 decreases pointwise when we change the vector $\xi = (\xi^1, ..., \xi^n) \in \mathbb{C}^n$ so that it remains a unit vector but the absolute values of any two coordinates become closer to each other. In particular, the integrand is maximal when only one of the coordinates is non-zero, and minimal when the absolute values of the coordinates are equal. The latter property immediately implies the result of Theorem 5.2.1.

5.4 An approximation argument for $B_p(\mathbb{C}^n)$

In this section we prove that formula (5.2.2) can be applied to the bodies $B_p(\mathbb{C}^n)$ and subspaces H_ξ in spite of the fact that these bodies are not always smooth. This will give a formally correct proof of the formula of Theorem 5.2.1.

For $\varepsilon > 0$, we introduce a star body $B_{p,\varepsilon}(\mathbb{C}^n)$ defined as the unit ball of the norm

$$\|x\|_{p,\varepsilon} = \Big[\big((x_{11}^2 + x_{12}^2) + \varepsilon(x_{21}^2 + \cdots + x_{n2}^2)\big)^{\frac{p}{2}} + \cdots$$

$$\cdots + \big((x_{n1}^2 + x_{n2}^2) + \varepsilon(x_{11}^2 + \cdots + x_{(n-1)2}^2)\big)^{\frac{p}{2}}\Big]^{\frac{1}{p}}.$$

Clearly, $\|x\|_{p,\varepsilon}$ is a continuous function of ε, and $\|\cdot\|_{p,\varepsilon} \in C^\infty(S^{2n-1})$. Moreover, $\|x\|_{p,\varepsilon} \longrightarrow \|x\|_p$, as $\varepsilon \to 0^+$, uniformly with respect to $x \in S^{2n-1}$.

Combining (5.2.1) and (5.2.2), with $K = B_{p,\varepsilon}(\mathbb{C}^n)$ and $H = H_\xi$, we get that

$$\int_{S^{2n-1}\cap H_\xi} \|x\|_{p,\varepsilon}^{-2n+2}\, dx = \frac{1}{(2\pi)^2} \int\limits_{S^{2n-1}\cap H_\xi^\perp} \big(\|x\|_{p,\varepsilon}^{-2n+2}\big)^\wedge(\theta)d\theta. \qquad (5.4.1)$$

Obviously, the left-hand side of the latter equality converges to the same integral with $\|x\|_p$ in place of $\|x\|_{p,\varepsilon}$, as $\varepsilon \to 0$. Therefore, it suffices to prove that the same happens in the right-hand side.

Recall that the measure $\mu_p,\ 0 < p \le 2$ introduced in Section 5.3, has the property that, for any $x_1, x_2, \dots x_{2n} \in \mathbb{R}$ and $\varepsilon > 0$,

$$e^{-\left(x_1^2+x_2^2+\varepsilon(x_3^2+\cdots+x_{2n}^2)\right)^{\frac{p}{2}}} = \int_0^\infty e^{-v\left(x_1^2+x_2^2+\varepsilon(x_3^2+\cdots+x_{2n}^2)\right)} d\mu_p(v) \qquad (5.4.2)$$

where $\varepsilon > 0$. Let $u = (u_1, \dots, u_n) \in \mathbb{R}_+^n = [0, \infty) \times \cdots \times [0, \infty)$. We shall use the same notation μ_p to denote the product measure on $\mathbb{R}_+^n$, $\mu_p(u) = \mu_p(u_1) \cdots \mu_p(u_n)$.

Following the steps of Lemma 5.2.1 and using formula (5.4.2), one can easily show that the Fourier transform of $\| \cdot \|_{p,\varepsilon}^{-2n+2}$ (in the sense of distributions) is given by the formula

$$\left(\| \cdot \|_{p,\varepsilon}^{-2n+2}\right)^{\wedge}(y)$$

$$= \frac{p}{\Gamma\left(\frac{2n-2}{p}\right)} \int_0^\infty t^{-3} \int_{\mathbb{R}_+^n} \prod_{j=1}^n \frac{\pi}{U_j(u)} e^{-\frac{1}{4U_j(u)t^2}(y_{j1}^2+y_{j2}^2)} d\mu_p(u)\ dt, \qquad (5.4.3)$$

where $U_j(u) = u_j + \varepsilon \sum_{\substack{i=1 \\ i \ne j}}^n u_i$. Therefore, the right-hand side of (5.4.2) is equal to

$$\frac{1}{(2\pi)^2} \frac{p}{\Gamma\left(\frac{2n-2}{p}\right)} \times$$

$$\int_0^\infty t^{-3} \int_{\mathbb{R}_+^n} \int_{S^{2n-1} \cap H_\xi^\perp} \prod_{j=1}^n \frac{\pi}{U_j(u)} e^{-\frac{1}{4U_j(u)t^2}(y_{j1}^2+y_{j2}^2)} dy\ d\mu_p(u)\ dt. \qquad (5.4.4)$$

In the same way, as in the proof of Theorem 5.2.6, one can show that for every $y \in S^{2n-1} \cap H_\xi^\perp$

$$\prod_{j=1}^n \frac{\pi}{U_j(u)} e^{-\frac{1}{4U_j(u)t^2}(y_{j1}^2+y_{j2}^2)} = \prod_{j=1}^n \frac{\pi}{U_j(u)} e^{-\frac{1}{4U_j(u)t^2}(\xi_{j1}^2+\xi_{j2}^2)},$$

so the inner integral in (5.4.4) equals to

$$2\pi \prod_{j=1}^n \frac{\pi}{U_j(u)} e^{-\frac{1}{4U_j(u)t^2}(\xi_{j1}^2+\xi_{j2}^2)},$$

and the expression in (5.4.4) equals to

$$\frac{1}{2\pi} \frac{p}{\Gamma\left(\frac{2n-2}{p}\right)} \int_0^\infty t^{-3} \int_{\mathbb{R}_+^n} \prod_{j=1}^n \frac{\pi}{U_j(u)} e^{-\frac{1}{4U_j(u)t^2}(\xi_{j1}^2+\xi_{j2}^2)} d\mu_p(u)\, dt. \tag{5.4.5}$$

It remains to prove that the latter quantity converges to

$$\frac{1}{2\pi} \frac{p}{\Gamma\left(\frac{2n-2}{p}\right)} \int_0^\infty t^{-3} \int_{\mathbb{R}_+^n} \prod_{j=1}^n \frac{\pi}{u_j} e^{-\frac{1}{4u_jt^2}(\xi_{j1}^2+\xi_{j2}^2)} d\mu_p(u)\, dt, \tag{5.4.6}$$

as $\varepsilon \to 0$, because (5.4.6) is equal to

$$\frac{1}{(2\pi)^2} \int\limits_{S^{n-1}\cap E_\xi^\perp} \left(\|x\|_p^{-2n+2}\right)^\wedge(\theta) d\theta,$$

which follows from Lemma 5.2.1 and (5.4.2), in the same way as it was done for the norm $\|\cdot\|_{p,\varepsilon}$.

The pointwise convergence of functions under the integral in (5.4.4) is obvious, so we can apply the dominated convergence theorem to finish our argument. To do that, recall the properties of the measure μ_p on $\mathbb{R}$ (see for example [Z]). The measure μ_p has density that decreases at infinity like $|v|^{-1-p/2}$. Besides, $\int_0^\infty \frac{1}{v}\ d\mu_p(v) < \infty$. Now, break the integral over dt in (10) into two integrals: from 1 to ∞ and from 0 to 1. To find a dominating function in the integral from 1 to ∞, just estimate the exponential by 1. In the integral from 0 to 1, use the fact that $\exp(-1/x^2) \leq kx^{1+p/8}$ for every $x \in [0, \infty)$ and some fixed $k > 0$. The integrability of the dominating function follows from the order of decay of the density of the measure μ_p. $\square$

Chapter 6

$B_p(\mathbb{C}^n)$, $p > 2$, is not a $k-$intersection body for $k \in (0, 2n-4)$.

6.1 Introductory

In this chapter, we prove that the complex ℓ_p-ball with $p > 2$ is not a k-intersection body, if $1 \leq k < 2n-4$. In a more general concept, if we consider the body $B_p(\mathbb{C}^n)$ as a subset of $\mathbb{R}^{2n}$ with norm $\|\cdot\|_p$, (introduced in Section 1.3), we prove that the normed space $\left(\mathbb{R}^{2n}, \|\cdot\|_p\right)$ does not embed isometrically to any L_{-q}, $q > 0$, if $q < 2n-4$. This proves Theorem 2.4.1.

The proof is based on investigating the moments of the Fourier transform of the function $e^{-|\cdot|^p}$. This method was used in [K1], (see also [K3] or [K9, Lemma 4.12]), proving the real version of this problem.

For $x = (x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in \mathbb{R}^{2n}$, the norm of the complex ℓ_p-ball is defined as

$$\|x\|_p = \left[(x_{11}^2 + x_{12}^2)^{\frac{p}{2}} + \cdots + (x_{n1}^2 + x_{n2}^2)^{\frac{p}{2}}\right]^{\frac{1}{p}}.$$

6.2 Moments

For any $p > 0$ we define the function δ_p as the Fourier transform of the function $\mathbb{R}^2 \ni (x_1, x_2) \longmapsto \frac{1}{4}e^{-(x_1^2+x_2^2)^{\frac{p}{2}}}$. This is clearly a radial function, so, by the well-known connection between the Fourier transform and linear transformations, δ_p will also be a radial function.

Lemma 6.2.1. *Let $p > 0$. Then, for every $(\xi_1, \xi_2) \in \mathbb{R}^2$, the Fourier transform of the function $\frac{1}{4}e^{-|\cdot|_2^p}$ is given by*

$$\delta_p(\xi) = \int_0^\infty re^{-r^p} J_0(\xi r)dr,$$

where $\xi = (\xi_1^2 + \xi^2)^{\frac{1}{2}}$ and J_0 is the Bessel function of the first kind of order 0.

Proof : For every $(\xi_1, \xi_2) \in \mathbb{R}^2$, we compute the Fourier transform of $\frac{1}{4}e^{-(x_1^2+x_2^2)^{\frac{p}{2}}}$.

$$\begin{aligned}\Big(\frac{1}{4}e^{-(x_1^2+x_2^2)^{\frac{p}{2}}}\Big)^{\wedge}(\xi_1,\xi_2) &= \frac{1}{4}\int_{\mathbb{R}^2} e^{-i(x_1\xi_1+x_2\xi_2)}e^{-(x_1^2+x_2^2)^{\frac{p}{2}}}dx_1dx_2 \\ &= \frac{1}{4}\int_{\mathbb{R}^2} e^{-ix_1(\xi_1^2+\xi_2^2)^{1/2}}e^{-(x_1^2+x_2^2)^{\frac{p}{2}}}dx_1dx_2 \\ &= \int_0^\infty\int_0^\infty \cos(x_1(\xi_1^2+\xi_2^2)^{1/2})e^{-(x_1^2+x_2^2)^{\frac{p}{2}}}dx_1dx_2.\end{aligned}$$

We write $\xi = (\xi_1^2+\xi_2^2)^{1/2}$, and make a change of variables into polar coordinates. Then, the latter is equal to

$$\int_0^\infty\int_0^{\frac{\pi}{2}}\cos(\xi r\cos\theta)re^{-r^p}d\theta dr = \int_0^\infty re^{-r^p}\Big(\int_0^{\frac{\pi}{2}}\cos(\xi r\cos\theta)d\theta\Big)dr$$

$$= \int_0^\infty re^{-r^p}J_0(\xi r)dr = \delta_p(\xi), \qquad (6.2.1)$$

where J_0 is the Bessel function of the first kind of order 0. □

As mentioned in Section 6.1, we want to compute the moments of δ_p. If p is not an even integer, $\delta_p(t)$ behaves like t^{-p-2} at infinity. In particular,

$$\lim_{t\to\infty} t^{p+2}\delta_p(t) = 2^{p+2}\Big[\Gamma\Big(\frac{p+2}{2}\Big)\Big]^2\sin\frac{\pi p}{2},$$

see [Rb2, Lemma 7.2] for details. If p is an even integer, then the function decreases exponentially at infinity.

We consider the integral

$$M_p(\alpha) = \int_{\mathbb{R}}|t|^\alpha\delta_p(t)dt. \qquad (6.2.2)$$

By the properties of the Bessel functions of the first kind, $J_0 \sim 1$ near 0 and $J_0 \sim \frac{1}{\sqrt{t}}$, for t large enough. Therefore, the integral in (6.2.2) converges absolutely for every $\alpha \in (-1, p+1)$.

Lemma 6.2.2. *Let $p > 2$. Then, for every $\alpha \in (-1, p+1)$, with $\alpha \notin \{0, 1, 3, \dots, 2[\frac{p}{2}] + 1\}$.*

$$M_p(\alpha) = \frac{4\pi^{3/2}}{p} \frac{\Gamma(\alpha)\Gamma(\frac{-\alpha+1}{p})}{\Gamma(\frac{-\alpha+1}{2})\Gamma(\frac{\alpha}{2})}. \tag{6.2.3}$$

In particular, $M_p(\alpha) > 0$, if $\alpha \in (-1, 0) \cup (0, 3)$ and $M_p(\alpha) < 0$, if $\alpha \in (3, \min\{5, p+1\})$.

Proof : First we assume that $\alpha \in (-1, -\frac{1}{2})$. We apply Fubini's theorem and a change of variables $tr = s$. Then formula (6.2.2) becomes

$$M_p(\alpha) = \int_0^\infty r^{-\alpha} e^{-r^p} dr \int_{\mathbb{R}} |s|^\alpha J_0(s) ds.$$

We use Parseval's identity and formula (1.4.2). Then, by the definition of the Bessel function of order 0 and the Beta function, the latter is equal to

$$\begin{aligned} &-2\Gamma(\alpha+1) \sin \frac{\pi\alpha}{2} \int_0^\infty r^{-\alpha} e^{-r^p} dr \int_{-1}^1 |y|^{-1-\alpha} (1-y^2)^{-1/2} dy \\ &\quad = -2\Gamma(\alpha+1) \sin \frac{\pi\alpha}{2} B\Big(\frac{-\alpha}{2}, \frac{1}{2}\Big) \int_0^\infty r^{-\alpha} e^{-r^p} dr. \end{aligned} \tag{6.2.4}$$

Applying a change of variables, $r^p = s$, we conclude that (6.2.4) is equal to

$$\frac{4\pi^{3/2}}{p}\frac{\Gamma(\alpha)\Gamma(\frac{-\alpha+1}{p})}{\Gamma(\frac{-\alpha+1}{2})\Gamma(\frac{\alpha}{2})}, \tag{6.2.5}$$

since $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$, for $x \in \mathbb{C} \setminus \mathbb{Z}$.

Let $\alpha \in \mathbb{C}$. The functions in (6.2.5) are analytic in the domain $\{-1 < Re\alpha < p+1, \alpha \neq 0, 1, 3, \ldots, 2[\frac{p}{2}]+1\}$. We apply analytic continuation from the interval $(-1, -\frac{1}{2})$. Then, (6.2.5) remains valid for all $\alpha \in (-1, p+1)$, with $\alpha \notin \{0, 1, 3, \ldots, 2[\frac{p}{2}]+1\}$.

To complete the proof, we only need to observe that $\Gamma(x) > 0$, if $x > 0$ or if $x \in (-2k, -2k+1), k \in \mathbb{N}$. □

Now, let us compute the Fourier transform of the distribution $\|\cdot\|_p^{-q}$.

Lemma 6.2.3. *Let $p > 2$ and $\xi = (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$. For $-np < q < 2n$, the Fourier transform of the function $\|\cdot\|_p^{-q}$, in the sense of distributions, is given by*

$$\left(\|\cdot\|_p^{-q}\right)^\wedge(\xi) = \frac{4^n p}{\Gamma\left(\frac{q}{p}\right)} \int_0^\infty s^{2n-q-1} \prod_{j=1}^n \delta_p(s\xi_j) ds, \tag{6.2.6}$$

where $\xi_j = (\xi_{ji}^2 + \xi_{j2}^2)^{\frac{1}{2}}$.

Proof : First we assume that $0 < q < 1$. Let $x = (x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \in$

$\mathbb{R}^{2n}$, then by the definition of the Γ-function (see section 1.2), we have that

$$\begin{aligned} \|x\|_p^{-q} &= \frac{p}{\Gamma\left(\frac{q}{p}\right)} \int_0^\infty t^{q-1} e^{t^p \|x\|_p^p} dt \\ &= \frac{p}{\Gamma\left(\frac{q}{p}\right)} \int_0^\infty t^{q-1} \prod_{j=1}^n e^{t^p (x_{j1}^2 + x_{j2}^2)^{\frac{p}{2}}} dt. \end{aligned} \tag{6.2.7}$$

Now, we compute the Fourier transform. For every $\xi = (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$, we can apply Fubini's theorem to (6.2.7) and have that

$$\begin{aligned} \left(\| \cdot \|_p^{-q}\right)^\wedge(\xi) &= \frac{p}{\Gamma\left(\frac{q}{p}\right)} \int_{\mathbb{R}^{2n}} e^{-ix\cdot\xi} \Big(\int_0^\infty t^{q-1} e^{t^p \|x\|_p^p} dt \Big) dx \\ &= \frac{p}{\Gamma\left(\frac{q}{p}\right)} \int_0^\infty t^{q-1} \Big(\int_{\mathbb{R}^{2n}} \left(e^{t^p \|x\|_p^p}\right)^\wedge(\xi) d\xi \Big) dt. \end{aligned} \tag{6.2.8}$$

We use the linearity of the Fourier transform and apply the change of variables $y = tx$. Then by Lemma 6.2.1, equation (6.2.8) is equal to

$$\frac{p}{\Gamma\left(\frac{q}{p}\right)} \int_0^\infty t^{q-1} \Big(\int_{\mathbb{R}^{2n}} \prod_{j=1}^n \left(e^{-t^p (x_{1j}^2 + x_{2j}^2)^{\frac{p}{2}}}\right)^\wedge(\xi_{1j}, \xi_{2j}) d\xi \Big) dt$$

$$= \frac{4^n p}{\Gamma\left(\frac{q}{p}\right)} \int_0^\infty t^{-2n+q-1} \prod_{j=1}^n \delta_p(\xi_j/t) dt.$$

Now, we make another change of variables, $t^{-1} = s$, and use Fubini's theorem, to have that the latter integral is equal to

$$\frac{4^n p}{\Gamma\left(\frac{q}{p}\right)} \int_0^\infty s^{2n-q-1} \prod_{j=1}^n \delta_p(\xi_j s) ds,$$

where $\xi_j = (\xi_{1j}^2 + \xi_{2j}^2)^{1/2}$. The latter integral converges since $-np < q < 2n$. This is because the function $t \mapsto \prod_{j=1}^n \delta_p(\xi_j s)$ behaves like t^{-np-2n} at infinity, (see comments before Lemma 6.2.2). For $q \in \mathbb{C}$ with $-np < q < 2n$, both sides in (6.2.6) are analytic functions in the domain $\{-np < Req < 2n\}$. We can apply analytic continuation from the interval $(-1, -\frac{1}{2})$. Then (6.2.6) remains valid for all $q \in (-np, 2n)$, (see [GS] for details on analytic continuation). □

Now, we are ready to prove the main result of this section.

Theorem 6.2.1. *Let $p > 2, n \geq 4$. Then, if $q \in (0, 2n-4)$, then the distribution $\|x\|_p^{-q}$ is not positive definite.*

Proof : Let $p > 2, n \in \mathbb{N}$ and $q \in (-np, 2n)$. For $\xi = (\xi_1, \xi_2, \ldots, \xi_{n-1}, 1) \in \mathbb{R}_+^{n-1}$ with $\xi_j = (\xi_{j1}^2 + \xi_{j2}^2)^{1/2}$, we consider the integral

$$I(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = \int_{\mathbb{R}^{n-1}} |\xi_1|^{\alpha_1} \cdots |\xi_{n-1}|^{\alpha_{n-1}} \left(\|x\|_p^{-q}\right)^\wedge (\xi_1, \ldots, \xi_{n-1}, 1) d\xi.$$

We substitute equation (6.2.6), apply Fubini's theorem and make a change of variables $s\xi_i = \eta_i$. Then the latter integral is equal to

$$\begin{aligned} & C \int_{\mathbb{R}^{n-1}} |\xi_1|^{\alpha_1} \cdots |\xi_{n-1}|^{\alpha_{n-1}} \int_0^\infty s^{2n-q-1} \prod_{j=1}^{n-1} \delta_p(s\xi_j) \delta_p(s) ds d\xi \\ = \; & C \int_0^\infty s^{n-q-\alpha_1 \cdots - \alpha_{n-1}} M_p(\alpha_1) \cdots M_p(\alpha_{n-1}) \delta_p(s) ds \\ = \; & C M_p(n - q - \alpha_1 \cdots - \alpha_{n-1}) M_p(\alpha_1) \cdots M_p(\alpha_{n-1}). \qquad (6.2.9) \end{aligned}$$

where $C = \left[\frac{4^n p}{\Gamma\left(\frac{q}{p}\right)}\right]^n$.

The integrals in (6.2.9) converge absolutely, if all the numbers $\alpha_1, \dots, a_{n-1}$, $n - q - \alpha_1 \cdots - \alpha_{n-1} \in (-1, p+1)$. We choose $\alpha_k \in (-1, -\frac{1}{2})$, for $k = 1, \dots, n-1$, so that all the moments $M_p(\alpha_k),\ k = 1, \dots, n-1$ are positive. Then, $n - q - \alpha_1 \cdots - \alpha_{n-1} \in (\frac{3}{2}n - q - \frac{1}{2}, 2n - q - 1) \cap (-1, p+1)$. Since $0 < q < 2n - 4$, these intervals contain a neighborhood of 3. Recall that the moments $M_p(\alpha)$ change sign at 3. This means that for different choices of the numbers $\alpha_k,\ k = 1, \dots, n-1$, the integral $I(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$ can become both positive or negative. In other words, the Fourier transform of the distribution $\|\cdot\|_p^{-q}$ is sign-changing, and thus, by Bohner's theorem it is not positive definite. $\square$

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