Analysis of the Incircle predicate for the Euclidean Voronoi diagram of axes-aligned line segments*

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Abstract

In this paper we study the most-demanding predicate for computing the Euclidean Voronoi diagram of axes-aligned line segments, namely the Incircle predicate. In particular, we show that the Incircle predicate can be answered by evaluating the signs of algebraic expressions of degree at most 6; this is half the algebraic degree we get when we evaluate the Incircle predicate using the current state-of-the-art approach.

1 Introduction

The Euclidean Voronoi diagrams of a set of line segments is one of the most well studied structures in computational geometry. There are numerous algorithms for its computation [5, 14, 16, 21, 7, 1, 15]. There are implementations that assume that numerical computations are performed exactly [19, 13], i.e., they follow the Exact Geometric Computation (EGC) paradigm [22], as well as algorithms that use floating-point arithmetic [10, 20, 9]; the latter class of algorithms does not guarantee exactness, but rather topological correctness.

Efficient and exact predicate evaluation in geometric algorithms is of vital importance. It has to be fast for the algorithm to be efficient. It has to be complete in the sense that it has to cover all degenerate cases, which, despite that fact that they are "degenerate" from the theoretical/analysis point-of-view, they are commonplace in real world input. In the EGC paradigm context, exactness is the bare minimum that is required in order to guarantee the correctness of the algorithm. The efficiency of predicates is typically measured in terms of the algebraic degree of the expressions (in the input parameters) that are computed during the predicate evaluation, as well as the number (and possibly type) of arithmetic operations involved. Degree-driven approaches for either the evaluation of predicates, or the design of the algorithm as a whole, has become an important question in algorithm/predicate design over the past few years [3, 17, 2, 4, 6, 18].

*The full version of the paper may be found in [11].

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In this paper we are interested in the most demanding predicate of the Euclidean Voronoi diagram of axes-aligned line segments, namely the Incircle predicate. Axes-aligned segments are typical input instances in applications such as VLSI design [8]. Given three sites S_1, S_2, and S_3 we denote their Voronoi circle by $V(S_1, S_2, S_3)$ (if it exists). There are at most two Voronoi circles defined by the triplet (S_1, S_2, S_3); the notation $V(S_1, S_2, S_3)$ refers to the Voronoi circle that "discovers" the sites S_1, S_2 and S_3 in that (cyclic) order, when we walk on the circle's boundary in the counterclockwise sense. Given a fourth object O, which we call the *query object*, the Incircle predicate Incircle(S_1, S_2, S_3, O) determines the relative position O with respect to the disk D bounded by $V(S_1, S_2, S_3)$. The predicate is positive if O does not intersect D, zero if O touches the boundary but not the interior of D, and negative if the intersection of O with the interior of D is non-empty.

The Voronoi circle of three sites does not always exist. In this paper, however, we assume that the Incircle predicate is called during the execution of an incremental algorithm for computing the Euclidean Voronoi diagram of line segments, and thus the first three sites are always related to a Voronoi vertex in the diagram. Since we can circularly rotate the first three arguments of the Incircle predicate, there are only eight possible distinct configurations for the Incircle predicate: $PPPX$, $PPSX$, $PSSX$ and $SSSX$, where P stands for point, S stands for segment, and X stands for either P or S.

The predicates for the Euclidean Voronoi diagram of line segments, in the context of an incremental construction of the diagram, have already been studied by Burnikel [3]. Assuming that the input is either rational points, or segments described by their endpoints as rational points, Burnikel shows that the Incircle predicate can be evaluated using polynomial expressions of degree 40 in the input quantities (see the line dubbed "General [3]" in Table 1). Considering Burnikel's approach for the case of axes-aligned line segments, and performing the appropriate simplifications in his calculations, we arrive at a new set of algebraic degrees for the various configurations of the Incircle predicate (see line dubbed "Axes-aligned [3]" in Table 1); now the most demanding case the is $PPSX$ case, which gives algebraic degree 8 and 12, when the query object is a point and a segment, respectively.

	$PPPP$	$PPSP$	$PSSP$	$SSSP$
General [3]	4	12	16	32
Axes-aligned [3]	4	8	4	2
Axes-aligned [this paper]	4	6	4	2

	$PPPS$	$PPSS$	$PSSS$	$SSSS$
General [3]	8	24	32	40
Axes-aligned [3]	6	12	4	2
Axes-aligned [this paper]	6	6	4	2

Table 1: Maximum algebraic degrees for the eight types of the Incircle predicate according to: [3] for the general and the axes-aligned segments case, and this paper. Top/Bottom table: the query object is a point/segment.

In Section 3 we analyze the $PPSX$ configurations for the Incircle predicate, and show how we can reduce the algebraic degrees for this case from 8 and 12, to 6. This is done by means of three key ingredients: (1) we reduce the $PPSP$ case to the $PPPS$ case, (2) we express the Incircle predicate as a difference of distances, instead of as a difference of squares of distances, and (3) we formulate the Incircle predicate as an algebraic problem of the following form: we compute a linear polynomial $L(x) = l_1x + l_0$ and a quadratic polynomial $Q(x) = q_2x^2 + q_1x + q_0$, such that the result of the Incircle predicate is the sign of $L(x)$ evaluated at a specific root of $Q(x)$.

2 Evaluation of the sign of $L(x) = l_1x + l_0$ at a specific root of $Q(x) = q_2x^2 + q_1x + q_0$

Let $L(x) = l_1x + l_0$ and $Q(x) = q_2x^2 + q_1x + q_0$ be a linear and a quadratic polynomial, respectively, such that $Q(x)$ has non-negative discriminant. Let the algebraic degrees of l_1, l_2, q_2, q_1 and q_0 be δ_l, $\delta_l + 1$, δ_q, $\delta_q + 1$, and $\delta_q + 2$, respectively. We are interested in the sign of $L(r)$, where r is one of the two roots $x_1 \leq x_2$ of $Q(x)$. Below we assume, without loss of generality, that $l_1, q_2 > 0$.

The obvious approach is to solve for r and substitute into the equation of $L(x)$. Let $\Delta_Q = q_1^2 - 4q_2q_0$ be the discriminant of $Q(x)$. Then $r = (-q_1 \pm \sqrt{\Delta_Q})/(2q_2)$, which, in turn, yields $L(r) = (l_1q_1 + 2l_0q_2 \pm \sqrt{\Delta_Q})/(2q_2)$. Computing $\text{sign}(L(r))$ is dominated by the computation of $\text{sign}(l_1q_1 + 2l_0q_2 \pm l_1\sqrt{\Delta_Q})$. This ammounts to evaluating expressions of algebraic degree at most $2(\delta_l + \delta_q + 1)$.

Observe now that evaluating the sign of $L(r)$ is equivalent to evaluating $\text{sign}(Q(x^\star))$, and possibly $\text{sign}(Q'(x^\star))$, where $x^\star = -\frac{l_0}{l_1}$ stands for the unique root of $L(x)$. Since $Q(x^\star) = (l_1^2q_0 - l_1q_1l_0 + q_2l_0^2)/l_1^2$ and $Q'(x^\star) = (l_1q_1 - 2q_2l_0)/l_1$, we conclude that, in order to evaluate $\text{sign}(L(r))$, we need to consider expressions of algebraic degree at most $2\delta_l+\delta_q+2$, which is smaller than the algebraic degree of the approach described early in this section, when $\delta_q > 0$.

3 The $PPSX$ case

Let A and B be the two points and CD be the segment defining the Voronoi circle. Without loss of generality, we may assume that CD is x-axis parallel, since otherwise we can reduce $\text{Incircle}(A, B, CD, Q)$ to $\text{Incircle}(\mathcal{R}(B), \mathcal{R}(A), \mathcal{R}(CD), \mathcal{R}(Q))$, where $\mathcal{R} : \mathbb{E}^2 \to \mathbb{E}^2$ denotes the reflection transformation about the line $y = x$. Notice that $\mathcal{R}$ preserves circles and line segments, reverses orientations, and is inclusion preserving. Finally, $\mathcal{R}$ maps an x-axis parallel segment to a y-axis parallel segment, and vice versa. Hence, $\text{Incircle}(A, B, CD, QS) = \text{Incircle}(\mathcal{R}(B), \mathcal{R}(A), \mathcal{R}(CD), \mathcal{R}(QS))$.

The query object is a point. Let Q be the query point. For the Voronoi circle $V(A, B, CD)$ to be defined, both A and B must be on the same side with respect to ℓ_{CD}. Consider now Q: if Q does not lie on the side of ℓ_{CD} that A and B lie, we have $\text{Incircle}(A, B, CD, Q) > 0$. Testing the sideness of Q against ℓ_{CD} simply means testing the sign of $y_Q - y_C$, which is a quantity of algebraic degree 1.

Suppose now that Q lies on the same side of ℓ_{CD} as A and B, and let $\sigma = \text{Orientation}(B, A, Q)$. In the special case $\sigma = 0$ (i.e., Q lies on the line ℓ_{BA}), we observe that Q lies inside the Voronoi circle $V(A, B, CD)$ if and only if Q lies on ℓ_{BA} and between A and B. This can be determined by evaluating the signs of the differences $x_A - x_B$, $x_Q - x_A$ and $x_Q - x_B$, if $x_A \neq x_B$, or the signs of the differences $y_A - y_B$, $y_Q - y_A$ and $y_Q - y_B$, if $x_A = x_B$, which are all quantities of algebraic degree 1.

If $\sigma \neq 0$, we are going to reduce $\text{Incircle}(A, B, CD, Q)$ to $\text{Incircle}(A, B, Q, CD)$ (see also Fig. 1). Suppose first that $\sigma < 0$, i.e., Q lies to the right of the oriented line ℓ_{BA}. Since A, B and CD appear on $V(A, B, CD)$ in that order when we traverse it in the counterclockwise sense, we conclude that Q lies inside $V(A, B, CD)$ (resp., lies on $V(A, B, CD)$) if and only if the circle defined by A, B and Q, does not intersect with (resp., touches) the segment CD. Hence, $\text{Incircle}(A, B, CD, Q) = -\text{Incircle}(A, B, Q, CD)$. In a similar manner, if $\sigma > 0$, i.e., Q lies to the left of the oriented line ℓ_{BA}, Q lies inside $V(A, B, CD)$ (resp., lies on $V(A, B, CD)$) if and only if the circle defined by B, A and Q intersects the line segment CD. Hence, $\text{Incircle}(A, B, CD, Q) = \text{Incircle}(B, A, Q, CD)$.

Since the Incircle predicate, in the $PPPS$ case, is of degree 6 (cf. Table 1), while $\text{Orientation}(B, A, Q)$ is of degree 2, we deduce that $\text{Incircle}(A, B, CD, Q)$ can also be answered using quantities of algebraic degree at most 6.

The query object is a segment. Let K be the center of $V(A, B, CD)$. K is an intersection point of

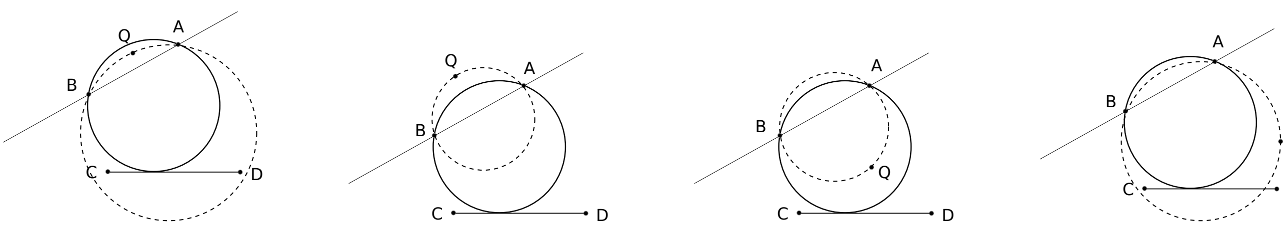

Figure 1: Reducing Incircle(A, B, CD, Q) to Incircle(A, B, Q, CD). Left/Right two: Q lies to the left/right of the oriented line ℓ_{BA}.

the bisector of A and B and the parabola with focal point A and directrix the supporting line ℓ_{CD} of CD. Solving the corresponding system of equations we deduce that, in the general case where A and B are not equidistant from ℓ_{CD} (i.e., if $y_A \neq y_B$), the x-coordinate of the Voronoi center x_K, is a root of a quadratic polynomial $P(x) = p_2x^2 + p_1x + p_0$, while the y-coordinate of the Voronoi center y_K, is a root of a quadratic polynomial $T(y) = t_2y^2 + t_1y + t_0$. Moreover, y_K and x_K are linearly dependent, i.e., $y_K = \frac{\alpha_1}{\beta}x_K + \frac{\alpha_0}{\beta}$. The algebraic degrees of p_2, p_1, p_0, t_2, t_1 and t_0 are 1, 2, 3, 2, 3 and 4, respectively. Furthermore, the degrees of α_1, α_0 and β are 1, 2 and 1, respectively. The roots $x_1 \leq x_2$ of the polynomial $P(x)$ (resp., $y_1 \leq y_2$ of $T(y)$) correspond to the centers of the two possible Voronoi circles $V(A, B, CD)$ and $V(B, A, CD)$. The roots of $P(x)$ and $T(y)$ of interest are shown in the following two tables.

Relative positions of A, B and CD	Root of $P(x)$ of interest
$y_C < y_A < y_B$ or $y_A < y_B < y_C$	x_1
$y_C < y_B < y_A$ or $y_B < y_A < y_C$	x_2

Relative positions of A, B	Root of $T(y)$ of interest
$x_A < x_B$	y_2
$x_A > x_B$	y_1

Let QS be the query segment. The first step is to compute Incircle(A, B, CD, Q) and, if needed, Incircle(A, B, CD, S). If at least one of Q and S lies inside $V(A, B, CD)$, we get Incircle$(A, B, CD, QS) < 0$. Otherwise, we need to determine if the line ℓ_{QS} intersects $V(A, B, CD)$. If ℓ_{QS} does not intersect the Voronoi circle, we have Incircle$(A, B, CD, QS) > 0$. If ℓ_{QS} intersects the Voronoi circle we have to check if Q and S lie on the same or opposite sides of the line $\ell^{\perp}_{QS}(K)$ that goes through the Voronoi center K and is perpendicular to ℓ_{QS}. Notice that since QS is axes-aligned, the line $\ell^{\perp}_{QS}(K)$ is either the line $x = x_K$ or the line $y = y_K$. Answering the Incircle predicate is equivalent to comparing the distance of K from the line ℓ_{QS} to the segment CD:

$$\mathsf{Incircle}(A, B, CD, \ell_{QS}) = d(K, \ell_{QS}) - d(K, CD). \quad (1)$$

Let us now examine and analyze the right-hand side difference of (1). Since the segment CD is x-axis parallel, $d(K, CD) = |y_K - y_C|$. Recall that y_K is a specific root of the quadratic polynomial $T(y)$. Therefore, determining the sign of $y_K - y_C$ reduces to evaluating the signs of $T(y_C)$ and $T'(y_C)$. Assume first that the segment QS is x-axis parallel. In this case, the equation of ℓ_{QS} is $y = y_Q$, and, hence, $d(K, \ell_{QS}) = |y_K - y_Q|$. As before, we can determine the sign of $y_K - y_Q$ by evaluating the signs of $T(y_Q)$ and $T'(y_Q)$. Hence, Incircle$(A, B, CD, \ell_{QS}) = |y_K - y_Q| - |y_K - y_C| = J_1y_K + J_0$, where J_1 and J_0 are given in the following table.

$y_K - y_Q$	$y_K - y_C$	J_1	J_0
≥ 0	≥ 0	0	$y_C - y_Q$
	< 0	2	$-y_Q - y_C$
< 0	≥ 0	-2	$y_Q + y_C$
	< 0	0	$-y_C + y_Q$

Clearly, if $J_1 = 0$ we have Incircle$(A, B, CD, \ell_{QS}) = \text{sign}(J_0)$. Otherwise, evaluating Incircle(A, B, CD, ℓ_{QS}) can be done as in Subsection 2. Since the algebraic degrees of J_1 and J_0 are 0 and 1, respectively, we can resolve the Incircle predicate using expressions of algebraic degree at most 4.

Consider now the case where QS is y-axis parallel. The equation of ℓ_{QS} is $x = x_Q$, and, thus, $d(K, \ell_{QS}) = |x_K - x_Q|$. As in the x-axis parallel case, x_K is a specific known root of the quadratic polynomial $P(x)$, i.e., determining the sign of $x_K - x_Q$ amounts to evaluating the signs of $P(x_Q)$ and $P'(x_Q)$. Using the fact that $y_K = \frac{\alpha_1}{\beta}x_K + \frac{\alpha_0}{\beta}$, we get Incircle$(S_1, S_2, S_3, \ell_{QS}) = |x_K - x_Q| - |y_K - y_C| = \frac{1}{\beta}(L_1x_K + L_0)$, where L_1 and L_0 are given in the following table.

$x_K - x_Q$	$y_K - y_C$	L_1	L_0
≥ 0	≥ 0	$-\alpha_1 + \beta$	$\beta(y_C - x_Q) - \alpha_0$
	< 0	$\alpha_1 + \beta$	$\beta(-y_C - x_Q) + \alpha_0$
< 0	≥ 0	$-\alpha_1 - \beta$	$\beta(y_C + x_Q) - \alpha_0$
	< 0	$\alpha_1 - \beta$	$\beta(-y_C + x_Q) + \alpha_0$

If $L_1 = 0$, Incircle$(S_1, S_2, S_3, \ell_{QS}) = \text{sign}(L_0)\text{sign}(\beta)$. Otherwise, given that x_K is a known root of $P(x)$, determining the sign of $L_1x_K + L_0$ can be done as in Subsection 2. Since the algebraic degrees of L_1 and L_0 are 1 and 2, respectively, evaluating the sign $L_1x_K + L_0$ reduces to computing the signs of expressions of algebraic degree at most 5.

As we mentioned at the beginning of this subsection, if Incircle$(A, B, CD, \ell_{QS}) \leq 0$, we need to check the position of Q and S with respect to the either line

$x = x_K$ (if QS is x-axis parallel), or the line $y = y_K$ (if QS is y-axis parallel). To check the position of I, $I \in \{Q, S\}$, against the line $x = x_K$, we simply have to compute the signs of $P(x_I)$ and $P'(x_I)$. The algebraic degrees of these quantities are 3 and 2, respectively. In a symmetric manner, to check the position of I, $I \in \{Q, S\}$, against the line $y = y_K$, we simply have to compute the signs of $T(y_I)$ and $T'(y_I)$; their algebraic degrees are 4 and 3, respectively.

For the special case $y_A = y_B$, we easily get $x_K = \frac{1}{2}(x_A + x_B)$ and $y_K = \frac{U_2}{U_1}$, where the algebraic degrees of U_2 and U_1 are 2 and 1, respectively. If QS is x-axis parallel, we need to determine the sign of the quantity $d(K, \ell_{QS}) - d(K, CD) = |y_K - y_Q| - |y_K - y_C|$, or, equivalently, the signs of U_1 and $|U_2 - U_1 y_Q| - |U_2 - U_1 y_C|$, which are of algebraic degree 1 and 2, respectively. If QS is y-axis parallel, we need to evaluate the sign of $d(K, \ell_{QS}) - d(K, CD) = |x_K - x_Q| - |y_K - y_C|$, or, equivalently, the signs of U_1 and $|U_1(x_A + x_B - 2x_Q)| - 2|U_2 - U_1 y_Q|$, which are also of algebraic degree 1 and 2, respectively.

Recalling that, in order to evaluate Incircle(A, B, CD, QS), the first step is to evaluate Incircle(A, B, CD, Q), and, if needed, Incircle(A, B, CD, S), we conclude that in order to evaluate the Incircle predicate in the $PPSS$ case, we need to compute the sign of expressions of algebraic degree at most 6.

4 Future work

Our analysis is so far theoretical. We would like to implement the approach presented in this paper and compare it against the generic implementation in CGAL [12]. Finally, we would like to study and implement the rest of the predicates involved in the computation of the Voronoi diagram, when the line segment are axes-aligned.

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