Guarding curvilinear art galleries with edge or mobile guards

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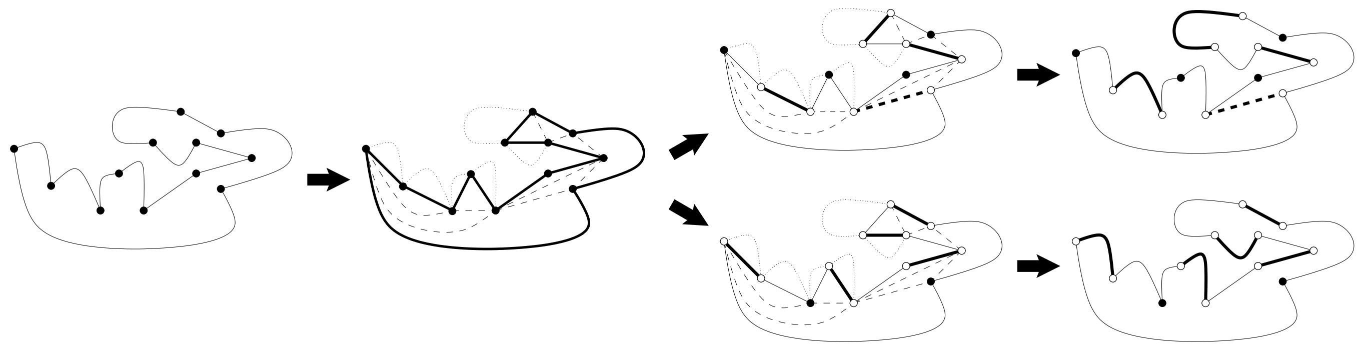

Abstract

In this paper we consider the problem of monitoring an art gallery modeled as a polygon, the edges of which are arcs of curves. We consider two types of guards: edge guards (these are edges of the polygon) and mobile guards (these are either edges or straight-line diagonals of the polygon). Our focus is on piecewise-convex polygons, i.e., polygons that are locally convex, except possibly at the vertices, and their edges are convex arcs. We reduce the problem of monitoring a piecewise-convex polygon to the problem of *2-dominating* a constrained triangulation graph with edges or diagonals, where 2-dominance requires that every triangle in the triangulation graph has at least two of its vertices in the 2-dominating set. We show that, given a triangulation graph T_P of a polygon P with $n \geq 3$ vertices: (1) $\lfloor \frac{n+1}{3} \rfloor$ diagonal guards are always sufficient and sometimes necessary, and (2) $\lfloor \frac{2n+1}{5} \rfloor$ edge guards are always sufficient and $\lfloor \frac{2n}{5} \rfloor$ edge guards are sometimes necessary, in order to 2-dominate T_P. We also show that a diagonal (resp., edge) 2-dominating set of size $\lfloor \frac{n+1}{3} \rfloor$ (resp., $\lfloor \frac{3n}{7} \rfloor$) can be computed in $O(n)$ time and space, whereas an edge 2-dominating set of size $\lfloor \frac{2n+1}{5} \rfloor$ can be computed in $O(n^2)$ time and $O(n)$ space. Based on these results we prove that, in order to monitor a piecewise-convex polygon P with $n \geq 2$ vertices: (1) $\lfloor \frac{n+1}{3} \rfloor$ mobile guards or $\lfloor \frac{2n+1}{5} \rfloor$ edge guards are always sufficient, and (2) $\lfloor \frac{n}{3} \rfloor$ mobile or edge guards are sometimes necessary. A mobile (resp., edge) guard set for P of size $\lfloor \frac{n+1}{3} \rfloor$ (resp., $\lfloor \frac{2n+1}{5} \rfloor$ or $\lfloor \frac{3n}{7} \rfloor$) can be computed in $O(n \log n + T(n))$ time and $O(n)$ space, where $T(n)$ denotes the time for computing a diagonal (resp., edge) 2-dominating set of size $\lfloor \frac{n+1}{3} \rfloor$ (resp., $\lfloor \frac{2n+1}{5} \rfloor$ or $\lfloor \frac{3n}{7} \rfloor$) for a triangulation graph with n vertices.

CR Categories: F.2.2 [Theory of Computation]: Analysis of Algorithms and Problem Complexity—Nonnumerical Algorithms and Problems; I.3.5 [Computing Methodologies]: Computer Graphics—Computational Geometry and Object Modeling

Keywords: art gallery, curvilinear polygons, triangulation graphs, 2-dominance, edge guards, diagonal guards, mobile guards, piecewise-convex polygons

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1 Introduction

In recent years computational geometry has made a shift towards curvilinear objects. Recent works have addressed both combinatorial properties and algorithmic aspects, as well as the necessary algebraic techniques for resolving predicates involving curvilinear objects. The pertinent literature is quite extensive; the interested reader may consult the recent book edited by Boissonnat and Teillaud [Boissonnat and Teillaud 2007] for a collection of recent results for various classical computational geometry problems involving curvilinear objects. Despite the apparent shift towards the curvilinear world, and despite the vast range of application areas for art gallery problems, including robotics [Kuc and Siegel 1987; Xie et al. 1986], motion planning [Lozano-Pérez and Wesley 1979; Mitchell 1989], computer vision [Stenstrom and Connolly 1986; Yachida 1986; Avis and ElGindy 1983; Toussaint 1980], graphics [McKenna 1987; Chazelle and Incerpi 1984], CAD/CAM [Bronsvoort 1988; Eo and Kyung 1989] and wireless networks [Eppstein et al. 2007], there are very few works addressing the well-known art gallery and illumination class of problems when the objects involved are curvilinear [Urrutia and Zaks 1989; Coullard et al. 1989; Czyzowicz et al. 1994; Czyzowicz et al. 1995; Karavelas and Tsigaridas 2008].

The original art gallery problem was posted by Klee to Chvátal: given a simple polygon P with n vertices, how many vertex guards are required in order to monitor the interior of P? Chvátal [Chvátal 1975] proved that $\lfloor \frac{n}{3} \rfloor$ vertex guards are always sufficient and sometimes necessary, while Fisk [Fisk 1978], a few years later, gave exactly the same result using a much simpler proof technique based on 3-coloring a triangulation of the polygon. In the context of curvilinear polygons, i.e., polygons the edges of which may be line segments or arcs of curves, Karavelas and Tsigaridas [Karavelas and Tsigaridas 2008] have shown that is always possible to monitor piecewise-convex polygons (i.e., polygons that are locally convex, except possibly at the vertices, and their edges are convex arcs) with $\lfloor \frac{2n}{3} \rfloor$ vertex guards, whereas there exist classes of such polygons that require a minimum of $\lfloor \frac{4n}{7} \rfloor - 1$ vertex or $\lfloor \frac{n}{2} \rfloor$ point guards. They also show that $2n-4$ point guards are always sufficient and sometimes necessary in order to monitor piecewise-concave polygons, i.e., polygons that are locally concave, except possibly at the vertices, and their edges are convex arcs.

Soon after the first results on monitoring polygons with vertex or point guards, other types of guarding models where considered. Toussaint introduced in 1981 the notion of *edge guards*. A point p in the interior of the polygon is considered to be monitored if it is visible from at least one point of an edge in the guard set. Edge guards were introduced as a guarding model in which guards were allowed to move along the edges of the polygon. Another variation, dating back to 1983, is due to O'Rourke: guards are allowed to move along the edges or diagonals of the polygon. This type of guards is called *mobile guards*. Toussaint conjectured that, except for a few polygons, $\lfloor \frac{n}{4} \rfloor$ edge guards are always sufficient. There are only two known counterexamples to this conjecture, with $n = 7, 11$, due to Paige and Shermer [Shermer 1992], requiring $\lfloor \frac{n+1}{4} \rfloor$ edge guards, whereas there exists a family of polygons that require $\lfloor \frac{n}{4} \rfloor$ edge guards. The first step towards Toussaint's conjecture was made by O'Rourke [O'Rourke 1983; O'Rourke 1987] who proved that $\lfloor \frac{n}{4} \rfloor$ mobile guards are always sufficient and occasionally necessary in order to monitor any polygon with n vertices. The technique of O'Rourke, for proving the upper bound, amounts to reducing the problem of monitoring a simple polygon to that of dominating a *triangulation graph* of the polygon. A triangulation graph is a maximal outerplanar graph, all internal faces of which are triangles. Dominance means that at least one of the vertices of each triangle in the triangulation graph is incident to a mobile guard. The reduction to the problem of triangulation graph dominance is applicable to the case of edge guards. Shermer [Shermer 1992] settled the problem of dominating triangulation graphs with edge guards by showing that $\lfloor \frac{3n}{10} \rfloor$ edge guards are always sufficient and sometimes necessary, except for $n = 3, 6$ or 13, in which case one extra edge guard may be necessary; this, also, constitutes the best known upper bound on the number of edge guards that are sufficient in order to monitor an n-vertex polygon.

In this paper we consider the problem of monitoring piecewise-convex polygons with edge or mobile guards; in our context, a mobile guard is either an edge or a straight-line diagonal of the polygon. Our proof technique capitalizes on the technique used by O'Rourke to prove tight bounds on the number of mobile guards that are sufficient for monitoring straight-line polygons [O'Rourke 1987]. Unlike O'Rourke's paradigm, where the solution for the dominance problem is trivially a solution for the geometric guarding problem, in our paradigm we first reduce the geometric problem to a combinatorial problem, and then map of the solution for the combinatorial problem to a solution for the geometric problem. More precisely, in order to monitor piecewise-convex polygons with mobile or edge guards, we first reduce the problem of monitoring a piecewise-convex polygon P to the problem of 2-dominating a constrained triangulation graph. Given a triangulation graph T_P of a polygon P, a set of edges and/or diagonals of T_P is a 2-dominating set of T_P if every triangle in T_P has at least two of its vertices incident to an edge or diagonal in the 2-dominating set. We prove that $\lfloor \frac{n+1}{3} \rfloor$ diagonal guards (i.e., edges or diagonals of T_P) are always sufficient and sometimes necessary in order to 2-dominate T_P, whereas $\lfloor \frac{2n+1}{5} \rfloor$ edge guards are always sufficient and $\lfloor \frac{2n}{5} \rfloor$ edge guards are sometimes necessary in order to 2-dominate T_P. The proofs of sufficiency are inductive on the number of vertices of P. In the case of diagonal 2-dominance, our proof yields a linear time and space algorithm. In the case of edge 2-dominance, the inductive step incorporates edge contraction operations, thus yielding an $O(n^2)$ time and $O(n)$ space algorithm, where n is the size of P. A linear time and space algorithm can be attained by slightly relaxing the size of the edge 2-dominating set. More precisely, we have shown that we can 2-dominate T_P with $\lfloor \frac{3n}{7} \rfloor$ edges; the proof does not make use of edge contractions and is analogous, though more complicated, to the proof, presented in this paper, for the case of diagonal 2-dominance.

Focusing back to the geometric guarding problem, the triangulation graph T_P of the piecewise-convex polygon P is a constrained triangulation graph: we require that certain diagonals of T_P are present. The remaining non-triangular subpolygons of T_P are straight-line polygons and may be triangulated arbitrarily. For the edge guarding problem, any edge 2-dominating set computed for T_P is also an edge guard set for P. A diagonal 2-dominating set D of T_P, however, may contain diagonals of T_P that are not embeddable as straight-line diagonals of P. To produce a mobile guard set for P, we keep all edges and straight-line diagonals of P in D and map non-straight-line diagonals in D to edges of P. In summary, we can compute: (1) a mobile guard set for P of size at most $\lfloor \frac{n+1}{3} \rfloor$ in $O(n \log n)$ time and $O(n)$ space; (2) an edge guard set for P of size at most $\lfloor \frac{2n+1}{5} \rfloor$ in $O(n^2)$ time and $O(n)$ space; (3) an edge guard set for P of size at most $\lfloor \frac{3n}{7} \rfloor$ in $O(n \log n)$ time and $O(n)$ space. Finally, we show that $\lfloor \frac{n}{3} \rfloor$ edge or mobile guards are sometimes necessary in order to monitor P.

The rest of the paper is structured as follows. Section 2 is devoted to 2-dominance of triangulation graphs using diagonal or edge guards. In Section 3 we discuss the problem of monitoring piecewise-convex polygons with mobile or edge guards. Finally, in Section 4 we conclude with a discussion of our results and open problems.

2 2-dominance of triangulation graphs

Given a polygon P with n vertices, its *triangulation graph* T_P is a maximal outerplanar graph, i.e., a Hamiltonian planar graph consisting of n vertices and $2n-3$ edges, all internal faces of which are triangles (cycles of size 3). The triangulation graph of a straight-line polygon, i.e., a polygon the edges of which are line segments, is the planar graph we get when the polygon has been triangulated.

A *dominating set* D of a triangulation graph T_P is a set of vertices, edges or diagonals of T_P such that at least one of the vertices of each triangle in T_P belongs to D. An *edge (resp., diagonal) dominating set* of T_P is a dominating set of T_P consisting of only edges (resp., edges or diagonals) of P. A *2-dominating set* D of T_P is a dominating set of T_P that has the property that every triangle in T_P has at least two of its vertices in D. In a similar manner, an *edge (resp., diagonal) 2-dominating set* of T_P is a 2-dominating set of T_P consisting only of edges (resp., edges or diagonals) of T_P.

Before proceeding with the main results of this section, we state the following lemma, which is a direct generalization of Lemmas 1.1 and 3.6 in [O'Rourke 1987].

Lemma 1 *Consider an integer $\lambda \geq 2$. Let P be a polygon of $n \geq 2\lambda$ vertices, and T_P a triangulation graph of P. There exists a diagonal d in T_P that partitions T_P into two pieces, one of which contains k arcs corresponding to edges of P, where $\lambda \leq k \leq 2(\lambda - 1)$.*

Proof. Choose d to be a diagonal of T_P that separates off a *minimum* number of polygon edges that is at least λ. Let $k \geq \lambda$ be this minimum number, and label the vertices of P with the labels $0, 1, \ldots, n-1$, such that d is $(0, k)$. The diagonal d supports a triangle whose apex is at vertex t, $0 \leq t \leq k$. Since k is minimal $t \leq \lambda - 1$ and $k - t \leq \lambda - 1$. Thus, $\lambda \leq k \leq 2(\lambda - 1)$. $\square$

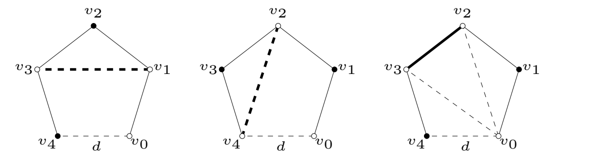

Figure 1: *The case $k = 4$. Left: $d_{13} \in T_1$. Middle: $d_{24} \in T_1$. Right: $d_{02}, d_{03} \in T_1$.*

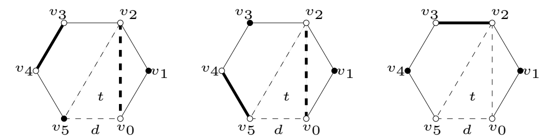

Figure 2: *The case $k = 5$. Left: $d_{02} \in D'$. Middle: $d_{02} \notin D'$ and $d_{25} \in D'$. Right: $d_{02}, d_{25} \notin D'$.*

Diagonal guards Using Lemma 1 for $\lambda = 4$, yields the following theorem concerning the worst-case number of diagonals that are sufficient and necessary in order to 2-dominate a triangulation graph. The inductive proof that follows is not the simplest possible. The interested reader may find a much simpler alternative proof in [Karavelas 2008]. The simpler proof, however, makes use of edge contractions, which make it unsuitable as a basis for a linear time and space algorithm. On the other hand, the proof presented below can be implemented in linear time and space, as will be discussed below. The proof that follows is a detailed, rather technical, case-by-case analysis; we present it, however, uncondensed, so as to illustrate the details that pertain to our linear time and space algorithm.

Theorem 2 *Every triangulation graph T_P of a polygon P with $n \geq 3$ vertices can be 2-dominated by $\lfloor \frac{n+1}{3} \rfloor$ diagonal guards. This bound is tight in the worst-case.*

Proof. The proof for $3 \leq n \leq 7$ is straightforward and is omitted. Let us now assume that $n \geq 8$ and that the theorem holds for all n' such that $3 \leq n' < n$. By means of Lemma 1 with $\lambda = 4$, there exists a diagonal d that partitions T_P into two triangulation graphs T_1 and T_2, where T_1 contains k boundary edges of T_P with $4 \leq k \leq 6$. Let v_i, $0 \leq i \leq k$, be the $k+1$ vertices of T_1, as we encounter them while traversing P counterclockwise, and let v_0v_k be the common edge of T_1 and T_2. For each value of k we are going to define a diagonal 2-dominating set D for T_P of size $\lfloor \frac{n+1}{3} \rfloor$. In what follows d_{ij} denotes the diagonal v_iv_j, whereas e_i denotes the edge v_iv_{i+1}[1]. Consider each value of k separately.

$k = 4$. In this case T_2 contains $n-3$ vertices. By our induction hypothesis we can 2-dominate T_2 with $f(n-3) = \lfloor \frac{n+1}{3} \rfloor - 1$ diagonal guards. Let D_2 be the diagonal 2-dominating set for T_2. At least one of v_0 and v_4 is in D_2. The cases are symmetric, so we can assume without loss of generality that $v_0 \in D_2$. Consider the following cases (see Fig. 1[2]):
- **$d_{13} \in T_1$.** Set $D = D_2 \cup \{d_{13}\}$.
- **$d_{24} \in T_1$.** Set $D = D_2 \cup \{d_{24}\}$.
- **$d_{02}, d_{03} \in T_1$.** Set $D = D_2 \cup \{e_2\}$.

$k = 5$. The presence of diagonals d_{04} and d_{15} would violate the minimality of k. Let t be the triangle supported by d in T_1. The apex v of this triangle can either be v_2 or v_3. The two cases are symmetric, so we assume, without loss of generality that the apex of t is v_2. Consider the triangulation graph $T' = T_2 \cup \{t\}$. It has $n-3$ vertices, hence, by our induction hypothesis, it can be 2-dominated with $f(n-3) = \lfloor \frac{n+1}{3} \rfloor - 1$ diagonal guards. Let D' be the 2-dominating set for T'. Consider the following cases (see Fig. 2):
- **$d_{02} \in D_2$.** Set $D = D' \cup \{e_3\}$.
- **$d_{02} \notin D_2$.** If $d_{25} \in D'$, set $D = (D' \setminus \{d_{25}\}) \cup \{d_{02}, e_4\}$. Otherwise, v_2 cannot belong to D' (both edges of T' incident to v_2 do not belong to D'). However, the triangle t is 2-dominated in T', which implies that both v_0 and v_5 belong to D'. Hence, set $D = D' \cup \{e_2\}$.

$k = 6$. The presence of diagonals d_{04}, d_{05}, d_{16} and d_{26} would violate the minimality of k. Let t be the triangle supported by d in T_1. The apex v of this triangle must be v_3. Let t' be the second triangle in T_1 beyond t supported by the diagonal d_{03}, and let v' be its vertex opposite to d_{03}. Symmetrically, let t'' be the second triangle in T_1 beyond t supported by the diagonal d_{36}, and let v'' be its vertex opposite to d_{36}. Consider the triangulation graph $T' = T_2 \cup \{t, t'\}$. It has $n-3$ vertices, hence, by our induction hypothesis, it can be 2-dominated with $f(n-3) = \lfloor \frac{n+1}{3} \rfloor - 1$ diagonal guards. Let D' be the 2-dominating set for T'.

Let us first consider the case $v' \equiv v_2$. Let d'' be the unique diagonal of the quadrilateral $v_3v_4v_5v_6$. Consider the following cases (see Fig. 3):
- **$d_{02} \in D'$.** Set $D = D' \cup \{d''\}$.
- **$d_{02} \notin D'$.** We further distinguish between the following two cases:
  - **$d_{36} \in D'$.** If $v_0 \in D'$, simply set $D = (D' \setminus \{d_{36}\}) \cup \{e_2, e_5\}$. If $v_0 \notin D'$, the diagonal d_{03} cannot belong to D'. Therefore, in order for the triangle t' to be 2-dominated by D', we must have that e_2 in D'. Thus, set $D = (D' \setminus \{d_{36}\}) \cup \{e_0, e_5\}$.
  - **$d_{36} \notin D'$.** In order for t' to be 2-dominated by D' we must have that either $d_{03} \in D'$ or $e_2 \in D'$. If $d_{03} \in D'$, set $D = (D' \setminus \{d_{03}\}) \cup \{d_{02}, d''\}$; otherwise, set $D = (D' \setminus \{e_2\}) \cup \{d_{02}, d''\}$.

The situation is entirely symmetric if $v'' \equiv v_4$. Hence, the only remaining case is the case where $v' \equiv v_1$ and $v'' \equiv v_5$. Consider the following cases (see Fig. 4):
- **$d_{13} \in D'$.** Set $D = D' \cup \{e_5\}$.
- **$d_{13} \notin D'$.** We further distinguish between the following two cases:
  - **$d_{03} \in D'$.** Set $D = (D' \setminus \{d_{03}\}) \cup \{e_0, d_{35}\}$.

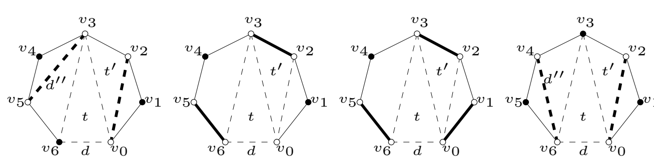

Figure 3: *The case $k = 6$ with $v' \equiv v_2$. Left: $d_{02} \in D'$; it is assumed in this subfigure that $d'' \equiv d_{35}$. Middle left: $d_{02} \notin D'$ and $d_{36} \in D'$ and $v_0 \in D'$. Middle right: $d_{02} \notin D'$ and $d_{36} \in D'$ and $v_0 \notin D'$. Right: $d_{02}, d_{36} \notin D'$; it is assumed in this subfigure that $d'' \equiv d_{46}$.*

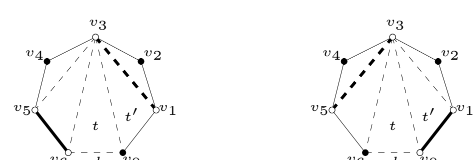

Figure 4: *The case $k = 6$ with $v' \equiv v_1$ and $v'' \equiv v_5$. Left: $d_{13} \in D'$; also $d_{13}, d_{03}, e_0 \notin D'$. Right: $d_{13} \notin D'$ and $d_{03} \in D'$; also $d_{13}, d_{03} \notin D'$ and $e_0 \in D'$.*

[1] Indices are considered to be evaluated modulo n.

[2] In all figures, edges/diagonals in a dominating/guard set are shown as thick solid/dashed lines, while vertices in a dominating/guard set are transparent.

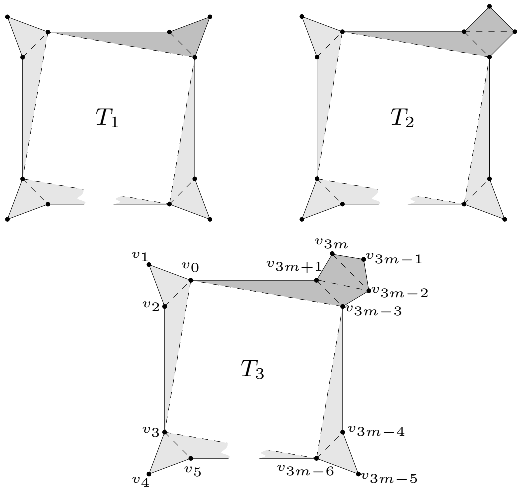

Figure 5: *Three triangulation graphs T_i, $i = 1, 2, 3$, with $n = 3m + i - 1$ vertices, respectively. All three triangulation graphs require at least $\lfloor \frac{n+1}{3} \rfloor$ diagonal guards in order to be 2-dominated.*

> $\boldsymbol{d_{03} \notin D'}$**.** If $e_0 \in D'$, set $D = D' \cup \{d_{35}\}$. Otherwise, i.e., if $e_0 \notin D'$, v_1 cannot be in D'. Since the triangle t' is 2-dominated in D', both v_0 and v_3 have to belong to D'. Since the diagonal d_{03} does not belong to D', the diagonal d_{36} has to belong to D' in order for v_3 to be in D'. Thus, set $D = (D' \setminus \{d_{36}\}) \cup \{d_{13}, e_5\}$.

Let us now turn our attention to establishing the lower bound. Consider the triangulation graphs T_i, $i = 1, 2, 3$, with $n = 3m + i - 1$ vertices, shown in Fig. 5, and let D_i be the diagonal 2-dominating set of T_i. The central part of T_i is triangulated arbitrarily. Notice that each subgraph of T_i, shown in either light or dark gray, requires at least one among its edges or diagonals to be in D_i in order to be 2-dominated. This observation immediately establishes a lower bound of $\lfloor \frac{n}{3} \rfloor$.

Let us now assume that $|D_3| = \lfloor \frac{n}{3} \rfloor$. Under this assumption, each shaded subgraph in T_3 must have *exactly one* among its edges/diagonals in D_3. Moreover, none of the diagonals in the central part of T_3 (not shown in Fig. 5(bottom)) can belong to D_3, since then we would have $|D_3| > \lfloor \frac{n}{3} \rfloor$. Consider the triangulated hexagon $H := v_0 v_{3m-3} v_{3m-2} v_{3m-1} v_{3m} v_{3m+1}$. In order for H to be 2-dominated with exactly one of its edges/diagonals, both v_0 and v_{3m-3} have to be in D_3 due to edges/diagonals in the neighboring shaded subgraphs, while the unique edge or diagonal of H in D_3 must be the diagonal $d_{3m-2,3m}$. Since we require that v_{3m-3} must belong to D_3 via an edge/diagonal of the quadrilateral $v_{3m-6} v_{3m-5} v_{3m-4} v_{em-3}$, and at the same time we require that exactly one of the edges/diagonals of $v_{3m-6} v_{3m-5} v_{3m-4} v_{em-3}$ to be in D_3, e_{3m-4} must belong to D_3 and v_{3m-6} must be in D_3 due to an edge/diagonal in the quadrilateral $v_{3m-9} v_{3m-8} v_{3m-7} v_{em-6}$. Cascading this argument, we conclude that, since v_3 must belong to D_3 due to an edge/diagonal of the quadrilateral $v_0 v_1 v_2 v_3$, and at the same time exactly one of the edges/diagonals of $v_0 v_1 v_2 v_3$ must be in D_3, e_2 must belong to D_3 and v_0 must belong to D_3 due to an edge/diagonal in H. This yields a contradiction, since the unique edge/diagonal of H in D_3 is $d_{3m-2,3m}$, which is, obviously, not incident to v_0. $\square$

The proof of Theorem 2 can almost immediately be transformed into a linear time and space algorithm. The triangulation graph T_P of P is assumed to be represented via a half-edge representation. Half-edges and vertices in our representation are assumed to have additional flags for indicating whether a half-edge is a boundary edge of the polygon, or whether a half-edge or a vertex of T_P is marked as being in the diagonal 2-dominating set of T_P. Under these assumptions, adding or removing a half-edge or a vertex from the sought-for 2-dominating set, querying a half-edge or a vertex for membership in the 2-dominating set, as well as forming the triangulation graph for the recursive calls, all take $O(1)$ time.

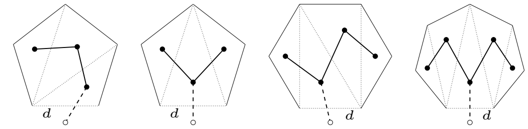

Figure 6: *The four possible configurations for the dual trees Δ_1 for $4 \leq k \leq 6$, shown as thick solid lines. The diagonal d separates T_1 from T_2.*

Consider a diagonal d that separates T_P into two triangulation graphs T_1 and T_2, where T_1 contains $k = 4, 5$ or 6 edges of P; recall from the proof of Lemma 1 (for $\lambda = 4$) that the value of k is minimal. Let Δ be the dual tree of T_P, Δ_1 (resp., Δ_2) the dual tree of T_1 (resp., T_2) and $\Delta_1' = \Delta_1 \cup \{d'\}$, where d' is the dual edge of d in Δ. Δ_1 consists of a subtree of Δ with 2, 3 or 4 edges of Δ, connected with the rest of Δ via a degree-2 or a degree-3 node (see Fig. 6). Moreover, for $n \geq 13$, the subtrees Δ_1' corresponding to different diagonals d of T_P must be edge disjoint (otherwise the number of vertices of P would be less than 13).

Having made these observations we can now describe the algorithm for computing the diagonal 2-dominating set D for T_P. We first describe the initialization steps: (1) initialize D to be empty; (2) create a queue Q, and initialize it to be empty. Q will consist of diagonals of T_P; (3) for each diagonal d of T_P determine whether it separates off 4, 5 or 6 edges of P in T_P and its size is minimal. In other words, determine if the dual edge d' of d in Δ is adjacent to subtrees of the form shown in Fig. 6. If so, put d in Q.

The recursive part of the algorithm is as follows:

1. If the number of vertices of T_P is less than 13, find a diagonal 2-dominating set D and return.
2. If Q is not empty:
   (a) Pop a diagonal d out of Q.
   (b) If T_2 has less than 13 vertices, empty the queue Q and find a 2-dominating set D_2 for T_2. Based on D_2, and according to the cases in the proof of Theorem 2, compute D and return.
   (c) Determine the case in the proof of Theorem 2, to which d corresponds. Let $\hat{T}$ be the triangulation graph for which we are supposed to find the 2-dominating set recursively, and let $\hat{\Delta}$ be the dual tree of $\hat{T}$. Let V be the set of vertices in $(\hat{\Delta} \cap \Delta_1')$. For any $v \in V$ determine if v is a leaf-node to a subtree of $\hat{\Delta}$ like the subtrees in Fig. 6. If so, add the corresponding diagonal to Q.
   (d) Recursively, find a diagonal 2-dominating $\hat{D}$ for $\hat{T}$, using Q as the queue.
   (e) Construct from $\hat{D}$ a diagonal 2-dominating set D for T_P and return.

It is straightforward to verify that the time $T(n)$ spent for the recursive part of our algorithm satisfies the recursion $T(n) = T(n-3) + O(1)$, which gives $T(n) = O(n)$. Since initialization takes linear time, and our space requirements are obviously linear in the size of P (we do not duplicate parts of T_P for the recursive calls, but rather set appropriately the boundary flags for some half-edges), we arrive at the following theorem.

Theorem 3 *Given the triangulation graph T_P of a polygon P with*

$n \geq 3$ *vertices, we can compute a diagonal 2-dominating set for* T_P *of size at most* $\lfloor \frac{n+1}{3} \rfloor$ *in* $O(n)$ *time and space.*

Edge guards Applying Lemma 1 for $\lambda = 5$ we can prove that $\lfloor \frac{2n+1}{5} \rfloor$ edge guards are sufficient in order to 2-dominate the triangulation of an n-vertex piecewise-convex polygon. The proof is similar to the proof of Theorem 2; however, exactly like the simple (omitted) proof of Theorem 2, it makes use of edge contractions, yielding an $O(n^2)$ time and $O(n)$ space algorithm. A linear time and space algorithm is feasible by relaxing the requirement on the size of the edge 2-dominating set. More precisely, applying Lemma 1 for $\lambda = 6$, we have shown that we can 2-dominate the triangulation graph of a piecewise-convex polygon with $\lfloor \frac{3n}{7} \rfloor$ edge guards. Although this result is weaker, it does not use edge contractions. We can, thus, devise a linear time and space algorithm for computing an edge 2-dominating set of size at most $\lfloor \frac{3n}{7} \rfloor$, in exactly the same manner as in the case of diagonal 2-dominance. The following theorem summarizes our results, including our worst-case lower bound on the number of edge guards required to 2-dominate the triangulation graph of a piecewise-convex polygon.

Theorem 4 ([Karavelas 2008]) *Given the triangulation graph* T_P *of a polygon* P *with* $n \geq 3$ *vertices, we can either compute: (1) an edge 2-dominating set for* T_P *of size at most* $\lfloor \frac{2n+1}{5} \rfloor$ *(except for* $n = 4$*, where one additional edge is required) in* $O(n^2)$ *time and* $O(n)$ *space, or (2) an edge 2-dominating set for* T_P *of size at most* $\lfloor \frac{3n}{7} \rfloor$ *(except for* $n = 4$*, where one additional edge is required) in* $O(n)$ *time and space. Finally, there exists a family of triangulation graphs with* $n \geq 5$ *vertices that require* $\lfloor \frac{2n}{5} \rfloor$ *edge guards in order to be 2-dominated.*

3 Piecewise-convex polygons

Let $v_1, \ldots, v_n$, $n \geq 2$, be a sequence of points and $a_1, \ldots, a_n$ a set of curvilinear arcs, such that a_i has as endpoints the points v_i and v_{i+1}. We will assume that the arcs a_i and a_j, $i \neq j$, do not intersect, except when $j = i - 1$ or $j = i + 1$, in which case they intersect only at the points v_i and v_{i+1}, respectively. We define a *curvilinear polygon* P to be the closed region of the plane delimited by the arcs a_i. The points v_i are called the vertices of P. An arc a_i is a *convex arc* if every line on the plane intersects a_i at at most two points or along a line segment. A polygon P is called a *locally convex polygon* if P is locally convex except possibly at its vertices (see Fig. 7(left)). A polygon P is called a *piecewise-convex polygon*, if it is locally convex and its edges are convex arcs (see Fig. 7(right)).

Let a_i be an edge of a piecewise-convex polygon P with endpoints v_i and v_{i+1}. We call the convex region r_i delimited by a_i and $\overline{v_i v_{i+1}}$ a *room*, where $\overline{xy}$ denotes the line segment from x to y. A room is called degenerate if the arc a_i is a line segment. For $p, q \in a_i$, $\overline{pq}$ is called a *chord* of a_i; the chord of r_i is $\overline{v_i v_{i+1}}$. An *empty room* is a non-degenerate room that does not contain any vertex of P in the interior of r_i or in the interior of $\overline{v_i v_{i+1}}$. A *non-empty room* is a non-degenerate room that contains at least one vertex of P in the interior of r_i or in the interior of $\overline{v_i v_{i+1}}$.

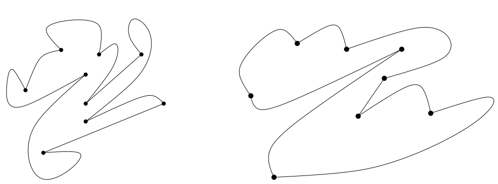

Figure 7: *Left: A locally convex polygon. Right: A piecewise-convex polygon.*

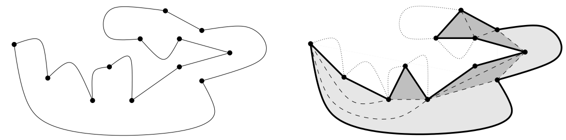

Figure 8: *Left: A piecewise-convex polygon* P*. Right: The triangulation graph* T_P *of* P*. The boundary edges of* T_P *are shown as thick solid lines. The crescents of* P *are shown in light gray, whereas the stars of* P *are shown in dark gray.*

We say that a point p in the interior of a piecewise-convex polygon P is visible from a point q if $\overline{pq}$ lies in the closure of P. We say that P is *monitored* by a *guard set* G if every point in P is visible from at least one point belonging to some guard in G. An *edge (resp., mobile) guard* is an edge (resp., edge or diagonal) of P belonging to a guard set G of P. An *edge (resp., mobile) guard set* is a guard set that consists of only edge (resp., mobile) guards.

Let P be a piecewise-convex polygon with $n \geq 3$ vertices. Consider a convex arc a_i of P, with endpoints v_i and v_{i+1}, and let r_i be the corresponding room. If r_i is a non-empty room, let X_i be the set of vertices of P that lie in the interior of $\overline{v_i v_{i+1}}$, and let R_i be the set of vertices of P in the interior of r_i or in X_i. If $R_i \neq X_i$, let C_i be the set of vertices in the convex hull of the vertex set $(R_i \setminus X_i) \cup \{v_i, v_{i+1}\}$; if $R_i = X_i$, let $C_i = X_i \cup \{v_i, v_{i+1}\}$. Finally, let $C_i^* = C_i \setminus \{v_i, v_{i+1}\}$. If r_i is an empty room, let $C_i = \{v_i, v_{i+1}\}$ and $C_i^* = \emptyset$. Let T_P be the sought-for triangulation graph of P. The vertex set of T_P is the set of vertices of P. The edges and diagonals of T_P, as well as their embedding, are defined as follows (see also Fig. 8):

- If a_i is a line segment or r_i is an empty room, the edge (v_i, v_{i+1}) is an edge in T_P, and is embedded as $\overline{v_i v_{i+1}}$.
- If r_i is a non-empty room, the following edges or diagonals belong to T_P:
  1. (v_i, v_{i+1}),
  2. $(c_{i,j}, c_{i,j+1})$, for $1 \leq j \leq K_i - 1$, where $K_i = |C_i|$, $c_{i,1} \equiv v_i$ and $c_{i,K_i} \equiv v_{i+1}$. The remaining $c_{i,j}$'s are the vertices of P in C_i^* as we encounter them when walking inside r_i and on the convex hull of the point set C_i from v_i to v_{i+1}, and
  3. $(v_i, c_{i,j})$, for $3 \leq j \leq K_i - 1$, provided that $K_i \geq 4$. We call these diagonals *weak diagonals*.

  The diagonals $(c_{i,j}, c_{i,j+1})$, $1 \leq j \leq K_i - 1$ are embedded as $\overline{c_{i,j}, c_{i,j+1}}$, whereas the diagonals $(v_i, c_{i,j})$, $3 \leq j \leq K_i - 1$, are embedded as curvilinear segments. Finally, the edges (v_i, v_{i+1}) are embedded as curvilinear segments, namely, the arcs a_i.

The edges (v_i, v_{i+1}), along with the diagonals $(c_{i,j}, c_{i,j+1})$, $1 \leq j \leq K_i - 1$, partition P into subpolygons of two types: (1) subpolygons that lie entirely inside a non-empty room, called *crescents*, and (2) subpolygons delimited by edges of the polygon P, as well as diagonals of the type $(c_{i,j}, c_{i,j+1})$, called *stars*. In general, a piecewise-convex polygon may only have crescents, or only stars, or both. The crescents are triangulated by means of the diagonals $(v_i, c_{i,j})$, $3 \leq j \leq K_i - 1$. To finish the definition of the triangulation graph T_P, we simply need to triangulate all stars inside P. Since stars are straight-line polygons, any polygon triangulation algorithm may be used to triangulate them.

In direct analogy to the types of subpolygons we can have inside P, we have two possible types of triangles in T_P: (1) triangles inside stars, called *star triangles*, and (2) triangles inside a crescent, called *crescent triangles*. Crescent triangles have at least one edge that is a weak diagonal, except when the number of vertices of P in the interior of the corresponding room r is exactly one, in which case none of the three edges of the unique crescent triangle in r is a weak diagonal. A crescent triangle that has at least one weak diagonal among its edges is called a *weak triangle*.

Mobile guards Let G_{T_P} be a diagonal 2-dominating set of T_P. Based on G_{T_P} we define a set G of edges or straight-line diagonals of P as follows (see also Fig. 9): (1) add to G every non-weak diagonal of G_{T_P}, and (2) for every weak diagonal in G_{T_P}, add to G the edge of P delimiting the crescent that contains the weak diagonal. Clearly, $|G| \leq |G_{T_P}|$.

Lemma 5 *Let P be a piecewise-convex polygon with $n \geq 3$ vertices, T_P its constrained triangulation graph, and G_{T_P} a diagonal 2-dominating set of T_P. The set G of mobile guards, defined by mapping every non-weak diagonal of G_{T_P} to itself, and every weak diagonal d of G_{T_P} to the corresponding convex arc of P delimiting the crescent that contains d, is a mobile guard set for P.*

Proof. Let q be a point in the interior of P. q is either inside: (1) an empty room r_i of P, (2) a star triangle t_s of T_P, (3) a non-weak crescent triangle t_{nw} of T_P, or (4) a weak crescent triangle t_w of T_P. In any of the four cases, q is visible from at least two vertices u_1 and u_2 of T_P that are connected via an edge or a diagonal in T_P. In the first case, q is visible from the two endpoints v_i and v_{i+1} of a_i. In the second case, q is visible from all three vertices of t_s. The third case arises when q is inside a non-empty room r_j with $|C_j^*| = 1$ (t_{nw} is the unique crescent triangle in r_j), in which case q is visible from at least two of the three vertices v_j, v_{j+1} and $c_{j,1}$. Finally, in the fourth case, q has to lie inside the crescent of a non-empty room r_j with $|C_j^*| \geq 2$, and is visible from at least two consecutive vertices $c_{j,k}$ and $c_{j,k+1}$ of C_j.

Since G is a diagonal 2-dominating set for T_P, and $(u_1, u_2) \in T_P$, at least one of u_1 and u_2 belongs to G_{T_P}. Without loss of generality, let us assume that $u_1 \in G_{T_P}$. If $u_1 \in G$, q is monitored by u_1. If $u_1 \notin G$, u_1 has to be an endpoint of a weak diagonal d_w in G_{T_P}. Let r_ℓ be the room, inside the crescent of which lies d_w. Since $d_w \in G_{T_P}$, we have that $a_\ell \in G$. If q lies inside the crescent of the room r_ℓ (this can only happen in case (4) above), q is visible from a_ℓ, and thus monitored by a_ℓ. Otherwise, u_1 cannot be an endpoint of a_ℓ ($a_\ell \in G$, whereas $u_1 \notin G$), which implies that $u_1 \in C_\ell^*$, i.e., $u_1 \equiv c_{\ell,m}$, with $2 \leq m \leq K_\ell - 1$. But then q lies inside the cone with apex $c_{\ell,m}$, delimited by the rays $c_{\ell,m}c_{\ell,m-1}$ and $c_{\ell,m}c_{\ell,m+1}$, and containing at least one of v_ℓ and $v_{\ell+1}$ in its interior. Since, q is visible from the intersection point of the line qu_1 with a_ℓ, q is monitored by a_ℓ. $\square$

Our approach for computing the mobile guard set G of P consists of three major steps: (1) Construct the constrained triangulation T_P of P; (2) Compute a diagonal 2-dominating set G_{T_P} for the triangulation graph T_P; (3) Map G_{T_P} to G. The sets C_i^*, needed in order to construct the constrained triangulation T_P of P can be computed in $O(n \log n)$ time and $O(n)$ space (cf. [Karavelas and Tsigaridas 2008]). Once we have the sets C_i^*, the constrained triangulation T_P of P can be constructed in linear time and space. By Theorem 3, computing G_{T_P} takes linear time; furthermore $|G_{T_P}| \leq \lfloor \frac{n+1}{3} \rfloor$, which implies that $|G| \leq \lfloor \frac{n+1}{3} \rfloor$. Finally, the construction of G from G_{T_P} takes $O(n)$ time and space: for every diagonal d in G_{T_P} we need to determine if it is a weak diagonal, in which case we need to add the edge of P delimiting the crescent in which d lies to G; by appropriate bookkeeping at the time of construction of T_P these operations can take $O(1)$ per diagonal. Summarizing, by Theorem 2, Lemma 5 and our analysis above, we arrive at the following theorem. The case $n = 2$ can be trivially established.

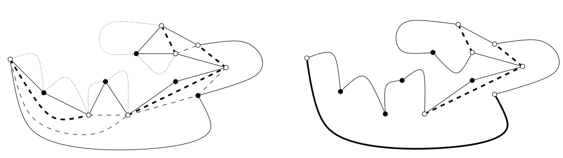

Figure 9: *Left: a diagonal 2-dominating set for the triangulation graph T_P from Fig. 8. Right: the corresponding mobile guard set.*

Theorem 6 *Let P be a piecewise-convex polygon with $n \geq 2$ vertices. We can compute a mobile guard set for P of size at most $\lfloor \frac{n+1}{3} \rfloor$ in $O(n \log n)$ time and $O(n)$ space.*

Edge guards Let G_{T_P} be and edge 2-dominating set of T_P (see Fig. 10). The set G of edge guards, defined by mapping every edge in G_{T_P} to the corresponding convex arc of P, is an edge guard set for P (cf. [Karavelas 2008]).

By Theorem 4, we can either compute an edge 2-dominating set G_{T_P} of size $\lfloor \frac{2n+1}{5} \rfloor$ in $O(n^2)$ time and $O(n)$ space, or an edge 2-dominating set G_{T_P} of size $\lfloor \frac{3n}{7} \rfloor$ (except for $n = 4$ where one additional edge is needed) in linear time and space. Since T_P can be computed in $O(n \log n)$ time and $O(n)$ space, and $|G| = |G_{T_P}|$, we arrive at the following theorem. The case $n = 2$ is trivial, since in this case any of the two edges of P is an edge guard set for P.

Theorem 7 *Let P be a piecewise-convex polygon with $n \geq 2$ vertices. We can either: (1) compute an edge guard set for P of size $\lfloor \frac{2n+1}{5} \rfloor$ (except for $n = 4$, where one additional edge guard is required) in $O(n^2)$ time and $O(n)$ space, or (2) compute an edge guard set for P of size $\lfloor \frac{3n}{7} \rfloor$ (except for $n = 2, 4$, where one additional edge guard is required) in $O(n \log n)$ time and $O(n)$ space.*

Lower bound construction Consider the piecewise-convex polygon P of Fig. 11. Each spike consists of three edges, namely, two line segments and a convex arc. In order for points in the non-empty room of the convex arc to be monitored, either one of the three edges of the spike, or a diagonal at least one endpoint of which is an endpoint of the convex arc, has to be in any guard set of P: the chosen edge or diagonal in a spike cannot monitor the non-empty room inside another spike of P. Since P consists of k spikes, yielding $n = 3k$ vertices, we need at least k edge or mobile guards to

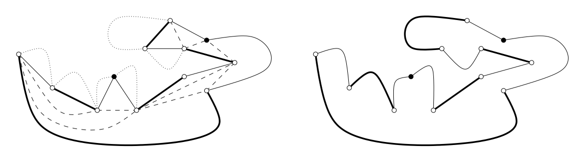

Figure 10: *Left: an edge 2-dominating set for the triangulation graph T_P from Fig. 8. Right: the corresponding edge guard set.*

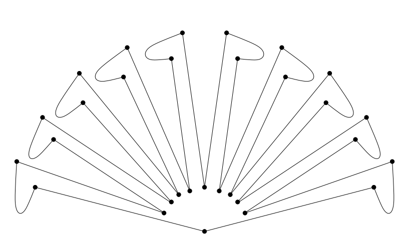

Figure 11: *The lower bound construction: the polygon shown contains $n = 3k$ vertices, and requires $k = \lfloor \frac{n}{3} \rfloor$ edge or mobile guards in order to be monitored.*

monitor P. We, thus, conclude that P requires at least $\lfloor \frac{n}{3} \rfloor$ edge or mobile guards in order to be monitored.

4 Discussion and open problems

As far as the problem of 2-dominance of triangulation graphs is concerned, we have not yet found a way to compute an edge 2-dominating set of size at most $\lfloor \frac{2n+1}{5} \rfloor$ in $o(n^2)$ time, whereas we have shown that is it is possible to compute an edge 2-dominating set of size at most $\lfloor \frac{3n}{7} \rfloor$ in linear time and space. It, thus, remains an open problem how to compute an edge 2-dominating set of size at most $\lfloor \frac{2n+1}{5} \rfloor$ in $o(n^2)$ time and linear space. Moreover, we conjecture that there exist triangulation graphs that require a minimum of $\lfloor \frac{2n+1}{5} \rfloor$ edge guards; thus far we have found such triangulation graphs for $n = 7, 12, 17, 22$.

Once a 2-dominating set D has been found for the constrained triangulation graph of a piecewise-convex polygon P, we either prove that D is also a guard set for P (this is the case for edge guards) or we map D to a mobile guard set for P. In the case of edge guards, the piecewise-convex polygon is actually monitored by the endpoints of the edges in the guard set. In the case of mobile guards, interior points of the edges may also be needed in order to monitor the interior of the polygon. The latter observation should be contrasted against the corresponding results for the class of straight-line polygons, where, for both edge and mobile guards, the polygon is essentially monitored by the endpoints of these guards (cf. [O'Rourke 1987]). Another important observation, due to the lower bound in Theorem 4, is that the proof technique of this paper cannot possibly yield better results for the edge guarding problem. If we are to close the gap between the upper and lower bounds, a fundamentally different technique will have to used.

Thus far we have limited our attention to the class of piecewise-convex polygons. It would be interesting to attain similar results for locally concave polygons (i.e., curvilinear polygons that are locally concave except possibly at the vertices), for piecewise-concave polygons (i.e., locally concave polygons the edges of which are convex arcs), or for curvilinear polygons with holes.

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