A geometric approach for the upper bound theorem for Minkowski sums of convex polytopes*

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Abstract

We derive tight expressions for the maximum number of k-faces, $0 \leq k \leq d-1$, of the Minkowski sum, $P_1 + \cdots + P_r$, of r convex d-polytopes $P_1, \ldots, P_r$ in $\mathbb{R}^d$, where $d \geq 2$ and $r < d$, as a (recursively defined) function on the number of vertices of the polytopes. Our results coincide with those recently proved by Adiprasito and Sanyal [1]. In contrast to Adiprasito and Sanyal's approach, which uses tools from Combinatorial Commutative Algebra, our approach is purely geometric and uses basic notions such as f- and h-vector calculus, stellar subdivisions and shellings, and generalizes the methodology used in [10] and [9] for proving upper bounds on the f-vector of the Minkowski sum of two and three convex polytopes, respectively. The key idea behind our approach is to express the Minkowski sum $P_1 + \cdots + P_r$ as a section of the Cayley polytope $\mathcal{C}$ of the summands; bounding the k-faces of $P_1 + \cdots + P_r$ reduces to bounding the subset of the $(k+r-1)$-faces of $\mathcal{C}$ that contain vertices from each of the r polytopes. We end our paper with a sketch of an explicit construction that establishes the tightness of the upper bounds.

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1 Introduction

Given two sets A and B in $\mathbb{R}^d$, $d \geq 2$, their Minkowski sum $A+B$ is the set $\{a+b \mid a \in A, b \in B\}$. The Minkowski sum definition can be extended naturally to any number of summands: $A_{[r]} := A_1 + A_2 + \cdots + A_r = \{a_1 + a_2 + \cdots + a_r \mid a_i \in A_i, 1 \leq i \leq r\}$. Minkowski sums have a wide range of applications, including algebraic geometry, computational commutative algebra, collision detection, computer-aided design, graphics, robot motion planning and game theory, just to name a few (see also [1], [9] and the references therein).

In this paper we focus on convex polytopes, and we are interested in computing the worst-case complexity of their Minkowski sum. More precisely, given r d-polytopes $P_1, \ldots, P_r$ in $\mathbb{R}^d$, we seek tight bounds on the number of k-faces $f_k(P_{[r]})$, $0 \leq k \leq d-1$, of their Minkowski sum $P_{[r]} := P_1 + P_2 + \cdots + P_r$. This problem, which can be seen as a generalization

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of the Upper Bound Theorem (UBT) for polytopes [14], has a history of more than 20 years. Gritzmann and Sturmfels [7] were the first to consider the problem, and gave a complete answer for it, for any number of d-polytopes in $\mathbb{R}^d$, in terms of the number of non-parallel edges of the r polytopes. More than 10 years later, Fukuda and Weibel [5] proved tight upper bounds on the number of k-faces of the Minkowski sum of two 3-polytopes, expressed either in terms of the number of vertices or number of facets of the summands. Fogel, Halperin, and Weibel [4] extended one of the results in [5], and expressed the number of facets of the Minkowski sum of r 3-polytopes in terms of the number of facets of the summands. Quite recently Weibel [16] provided a relation for the number of k-faces of the Minkowski sum of $r \geq d$ summands in terms of the k-faces of the Minkowski sums of subsets of size at most $d-1$ of these summands. This result should be viewed in conjunction with a result by Sanyal [15] stating that the number of vertices of the Minkowski sum of r d-polytopes, where $r \geq d$, is strictly less than the product of the vertices of the summands (whereas for $r < d$ this is indeed possible). About 3 years ago, the authors of this paper proved the first tight upper bound on the number of k-faces for the Minkowski sum of two d-polytopes in $\mathbb{R}^d$, for any $d \geq 2$ and for all $0 \leq k \leq d-1$ (cf. [10]), a result which was subsequently extended to three summands in collaboration with Konaxis (cf. [9]).

In a recent paper, Adiprasito and Sanyal [1] provide the complete resolution of the *Upper Bound Theorem for Minkowski sums (UBTM)*. In particular, they show that there exists, what they call, a *Minkowski-neighborly* family of r d-polytopes $N_1, \ldots, N_r$, with $f_0(N_i) = n_i$, $1 \leq i \leq r$, such that for any r d-polytopes $P_1, P_2, \ldots, P_r \subset \mathbb{R}^d$ with $f_0(P_i) = n_i$, $1 \leq i \leq r$, $f_k(P_{[r]})$ is bounded by above by $f_k(N_{[r]})$, for all $0 \leq k \leq d-1$. The majority of the arguments in the UBTM proof by Adiprasito and Sanyal make use of powerful tools from Combinatorial Commutative Algebra. The high-level layout of the proof is analogous to McMullen's proof of the UBT, as well as the proofs of the UBTM in [10] and [9] for two and three summands, respectively:

1. Consider the Cayley polytope $\mathcal{C} \subset \mathbb{R}^{d+r-1}$ of the r polytopes $P_1, P_2, \ldots, P_r$, and identify their Minkowski sum as a section of $\mathcal{C}$ with an appropriately defined d-flat $\overline{W}$. Let $\mathcal{F} \subset \mathbb{R}^{d+r-1}$ be the faces of $\mathcal{C}$ that intersect $\overline{W}$, and let $\mathcal{K}$ be the closure of $\mathcal{F}$ under subface inclusion ($\mathcal{K}$ is a $(d+r-1)$-polytopal complex). By the Cayley trick, there is a bijection between the faces of $\mathcal{F}$ and the faces of $P_{[r]}$; as a result, to bound the number of faces of $P_{[r]}$ it suffices to bound the number of faces of $\mathcal{F}$.
2. Define the h-vector $\boldsymbol{h}(\mathcal{F})$ of $\mathcal{F}$, and prove the Dehn-Sommerville equations for $\boldsymbol{h}(\mathcal{F})$, relating its elements to the elements of $\boldsymbol{h}(\mathcal{K})$.
3. Prove a recurrence relation for the elements of $\boldsymbol{h}(\mathcal{F})$.
4. Use the recurrence relation above to prove upper bounds for $h_k(\mathcal{F})$, for all $0 \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$.
5. Prove upper bounds for $h_k(\mathcal{K})$, for all $0 \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$.
6. Provide necessary and sufficient conditions under which the elements of both $\boldsymbol{h}(\mathcal{F})$ and $\boldsymbol{h}(\mathcal{K})$ are maximized for all k. These conditions are conditions on the *lower half* of the h-vector of $\mathcal{F}$. Due to the relation between the f- and h-vectors of $\mathcal{F}$, these are also conditions for the maximality of the elements of $\boldsymbol{f}(\mathcal{F})$.
7. Describe a family of polytopes for which the necessary and sufficient conditions hold; clearly, such a family establishes the tightness of the upper bounds.

In Adiprasito and Sanyal's proof steps 2, 3 and 4 are proved by introducing a powerful new theory that they call the *relative Stanley-Reisner theory* for simplicial complexes. The focus of this theory is on relative simplicial complexes, and is able to reveal properties of such complexes not only under topological restrictions, but also account for their combinatorial and geometric structure. To apply their theory, Adiprasito and Sanyal consider the simplicial complex $\mathcal{K}$ and then define $\mathcal{F}$ as a relative simplicial complex (they call them the Cayley and *relative Cayley* complex, respectively). They then apply their relative Stanley-Reisner theory to $\mathcal{F}$ to establish the Dehn-Sommerville equations of step 2, the recurrence relation of step 3 and finally the upper bounds for $\boldsymbol{h}(\mathcal{F})$ in 4. Steps 5 and 6 are done by clever algebraic manipulation of the h-vectors of $\mathcal{F}$ and $\mathcal{K}$, by exploiting the geometric properties of $\mathcal{K}$, and by making use of the recurrence relation in step 3. Step 7 is reduced to results by Matschke, Pfeifle, and Pilaud [13] and Weibel [16].

Our contribution. In what follows, we provide a completely geometric proof of the UBTM, that generalizes the technique we used in [10] and [9] for two and three summands to the case of r summands, when $r < d$. Instead of relying on algebraic tools, we use basic notions from combinatorial geometry, such as stellar subdivisions and shellings. Our proof, in essence, differs from that of Adiprasito and Sanyal in steps 2, 3, 4 and 5 of the layout above (the remaining steps do not use tools from Combinatorial Commutative Algebra anyway).

In more detail, to prove the various intermediate results, towards the UBTM, we consider the Cayley polytope $\mathcal{C}$ and we perform a series of stellar subdivisions to get a simplicial polytope $\mathcal{Q}$. From the analysis of the combinatorial structure of $\mathcal{Q}$, we derive the Dehn-Sommerville equations of step 2 (see Sections 3 and 4), as well as the recurrence relation of step 3 (see Section 5). This recurrence relation is then used for establishing the upper bounds for the elements of $\boldsymbol{h}(\mathcal{F})$ and $\boldsymbol{h}(\mathcal{K})$ (see Section 6). We end with a construction similar to the one presented in [13, Theorem 2.6], that establishes the tightness of the upper bounds (see Section 7). Due to space limitations, the majority of the proofs have been omitted; the interested reader may refer to the full version of the paper [11].

2 Preliminaries

Let P be a d-dimensional polytope, or d-polytope for short. Its dimension is the dimension of its affine span. The faces of P are $\varnothing, P$, and the intersections of P with its supporting hyperplanes. The $\varnothing$ and P faces are called *improper*, while the remaining faces are called *proper*. Each face of P is itself a polytope, and a face of dimension k is called a k-face. Faces of P of dimension $0, 1, d-2$ and $d-1$ are called vertices, edges, ridges, and facets, respectively.

A d-dimensional *polytopal complex* or, simply, *d-complex*, $\mathscr{C}$ is a finite collection of polytopes in $\mathbb{R}^d$ such that (i) $\varnothing \in \mathscr{C}$, (ii) if $P \in \mathscr{C}$ then all the faces of P are also in $\mathscr{C}$ and (iii) the intersection $P \cap Q$ for two polytopes P and Q in $\mathscr{C}$ is a face of both. The dimension $\dim(\mathscr{C})$ of $\mathscr{C}$ is the largest dimension of a polytope in $\mathscr{C}$. A polytopal complex is called *pure* if all its maximal (with respect to inclusion) faces have the same dimension. In this case the maximal faces are called the *facets* of $\mathscr{C}$. A polytopal complex is *simplicial* if all its faces are simplices. A polytopal complex $\mathscr{C}'$ is called a *subcomplex* of a polytopal complex $\mathscr{C}$ if all faces of $\mathscr{C}'$ are also faces of $\mathscr{C}$. For a polytopal complex $\mathscr{C}$, the *star* of v in $\mathscr{C}$, denoted by $\mathrm{star}(v, \mathscr{C})$, is the subcomplex of $\mathscr{C}$ consisting of all faces that contain v, and their faces. The *link* of v, denoted by $\mathscr{C}/v$, is the subcomplex of $\mathrm{star}(v, \mathscr{C})$ consisting of all the faces of $\mathrm{star}(v, \mathscr{C})$ that do not contain v.

A d-polytope P, together with all its faces, forms a d-complex, denoted by $\mathscr{C}(P)$. The polytope P itself is the only maximal face of $\mathscr{C}(P)$, i.e., the only facet of $\mathscr{C}(P)$, and is called the *trivial* face of $\mathscr{C}(P)$. Moreover, all proper faces of P form a pure $(d-1)$-complex, called the *boundary complex* $\mathscr{C}(\partial P)$, or simply ∂P, of P. The facets of ∂P are just the facets of P.

For a $(d-1)$-complex $\mathscr{C}$, its f-vector is defined as $\boldsymbol{f}(\mathscr{C}) = (f_{-1}, f_0, f_1, \dots, f_{d-1})$, where $f_k = f_k(\mathscr{C})$ denotes the number of k-faces of P and $f_{-1}(\mathscr{C}) := 1$ corresponds to the empty face of $\mathscr{C}$. From the f-vector of $\mathscr{C}$ we define its h-vector as the vector $\boldsymbol{h}(\mathscr{C}) = (h_0, h_1, \dots, h_d)$, where $h_k = h_k(\mathscr{C}) := \sum_{i=0}^{k}(-1)^{k-i}\binom{d-i}{d-k}f_{i-1}(\mathscr{C})$, $0 \le k \le d$.

Denote by $\mathcal{Y}$ a generic subset of faces of a polytopal complex $\mathscr{C}$, and define its dimension $\dim(\mathcal{Y})$ as the maximum of the dimensions of its faces. Let $\dim(\mathcal{Y}) = \delta - 1$; then we may define (if not already properly defined), the h-vector $\boldsymbol{h}(\mathcal{Y})$ of $\mathcal{Y}$ as:

$$h_k(\mathcal{Y}) = \sum_{i=0}^{\delta}(-1)^{k-i}\binom{\delta-i}{\delta-k}f_{i-1}(\mathcal{Y}). \tag{2.1}$$

We can further define the m-order g-vector of $\mathcal{Y}$ according to the following recursive formula:

$$g_k^{(m)}(\mathcal{Y}) = \begin{cases} h_k(\mathcal{Y}), & m = 0, \\ g_k^{(m-1)}(\mathcal{Y}) - g_{k-1}^{(m-1)}(\mathcal{Y}), & m > 0. \end{cases} \tag{2.2}$$

Clearly, $\boldsymbol{g}^{(m)}(\mathcal{Y})$ is nothing but the backward m-order finite difference of $\boldsymbol{h}(\mathcal{Y})$; therefore:

$$g_k^{(m)}(\mathcal{Y}) = \sum_{i=0}^{m}(-1)^i\binom{m}{i}h_{k-i}(\mathcal{Y}), \qquad k, m \ge 0. \tag{2.3}$$

Observe that for $m = 0$ we get the h-vector of $\mathcal{Y}$, while for $m = 1$ we get what is typically defined as the g-vector.

The relation between the f- and h-vector of $\mathcal{Y}$ is better manipulated using generating functions. We define the f-polynomial and h-polynomial of $\mathcal{Y}$ as follows:

$$\mathtt{f}(\mathcal{Y};t) = \sum_{i=0}^{\delta} f_{i-1}t^{\delta-i} = f_{\delta-1} + f_{\delta-2}t + \cdots + f_{-1}t^{\delta}, \quad \mathtt{h}(\mathcal{Y};t) = \sum_{i=0}^{\delta} h_i t^{\delta-i} = h_\delta + h_{\delta-1}t + \cdots + h_0 t^{\delta},$$

where, we simplified $f_i(\mathcal{Y})$ and $h_i(\mathcal{Y})$ to f_i and h_i. In this set-up, the relation between the f-vector and h-vector (cf. (2.1)) can be expressed as:

$$\mathtt{f}(\mathcal{Y};t) = \mathtt{h}(\mathcal{Y};t+1), \qquad \text{or, equivalently, as} \qquad \mathtt{h}(\mathcal{Y};t) = \mathtt{f}(\mathcal{Y};t-1). \tag{2.4}$$

2.1 The Cayley embedding, the Cayley polytope and the Cayley trick

Let $P_1, P_2, \ldots, P_r$ be r d-polytopes with vertex sets $\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r$, respectively. Let $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{r-1}$ be an affine basis of $\mathbb{R}^{r-1}$ and call $\mu_i : \mathbb{R}^d \to \mathbb{R}^{r-1} \times \mathbb{R}^d$ the affine inclusion given by $\mu_i(\boldsymbol{x}) = (\boldsymbol{e}_{i-1}, \boldsymbol{x})$, $1 \le i \le r$. The *Cayley embedding* $\mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r)$ of the point sets $\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r$ is defined as $\mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r) = \bigcup_{i=1}^{r} \mu_i(\mathscr{V}_i)$. The polytope corresponding to the convex hull $\operatorname{conv}\big(\mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r)\big)$ of the Cayley embedding $\mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r)$ of $\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r$ is typically referred to as the *Cayley polytope* of $P_1, P_2, \ldots, P_r$.

The following lemma, known as *the Cayley trick for Minkowski sums*, relates the Minkowski sum of the polytopes $P_1, P_2, \ldots, P_r$ with their Cayley polytope.

▶ **Lemma 2.1** ([8, Lemma 3.2]). *Let $P_1, P_2, \ldots, P_r$ be r d-polytopes with vertex sets $\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r \subset \mathbb{R}^d$. Moreover, let $\overline{W}$ be the d-flat defined as $\{\frac{1}{r}\boldsymbol{e}_0 + \cdots + \frac{1}{r}\boldsymbol{e}_{r-1}\} \times \mathbb{R}^d \subset \mathbb{R}^{r-1} \times \mathbb{R}^d$. Then, the Minkowski sum $P_{[r]}$ has the following representation as a section of the Cayley embedding $\mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r)$ in $\mathbb{R}^{r-1} \times \mathbb{R}^d$:*

$$\begin{aligned} P_{[r]} &\cong \mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r) \cap \overline{W} \\ &:= \Big\{\operatorname{conv}\{(\boldsymbol{e}_{i-1}, \boldsymbol{v}) \mid 1 \le i \le r\} \cap \overline{W} : (\boldsymbol{e}_{i-1}, \boldsymbol{v}) \in \mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r), 1 \le i \le r\Big\}. \end{aligned}$$

Moreover, F is a facet of $P_{[r]}$ if and only if it is of the form $F = F' \cap \overline{W}$ for a facet F' of $\mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r)$ containing at least one point $(\boldsymbol{e}_{i-1}, \boldsymbol{v})$ for all $1 \le i \le r$.

Let $\mathcal{C}_{[r]}$ be the Cayley polytope of $P_1, P_2, \ldots, P_r$, and call $\mathcal{F}_{[r]}$ the set of faces of $\mathcal{C}_{[r]}$ that have non-empty intersection with the d-flat $\overline{W}$. A direct consequence of Lemma 2.1 is a bijection between the $(k-1)$-faces of $\overline{W}$ and the $(k-r)$-faces of $\mathcal{F}_{[r]}$, for $r \le k \le d+r-1$. This further implies that:

$$f_{k-1}(\mathcal{F}_{[r]}) = f_{k-r}(P_{[r]}), \quad \text{for all } r \le k \le d+r-1. \tag{2.5}$$

In what follows, to keep the notation lean, we identify $V_i \coloneqq \mu_i(\mathscr{V}_i)$ with its pre-image $\mathscr{V}_i$. For any $\varnothing \subset R \subseteq [r]$, we denote by $\mathcal{C}_R$ the Cayley polytope of the polytopes P_i where $i \in R$. In particular, if $R = \{i\}$ for some $i \in [r]$, then $\mathcal{C}_{\{i\}} \equiv P_i$. We shall assume below that $\mathcal{C}_{[r]}$ is *"as simplicial as possible"*. This means that we consider all faces of $\mathcal{C}_{[r]}$ to be simplicial, except possibly for the trivial faces $\{\mathcal{C}_R\}$[1], $\varnothing \subset R \subseteq [r]$. Otherwise, we can employ the so called *bottom-vertex triangulation* [12, Section 6.5, pp. 160–161] to triangulate all proper faces of $\mathcal{C}_{[r]}$ except for the trivial ones, i.e., $\{\mathcal{C}_R\}$, $\varnothing \subset R \subseteq [r]$. The resulting complex is polytopal (cf. [2]) with all its faces being simplices, except possibly for the trivial ones. Moreover, it has the same number of vertices as $\mathcal{C}_{[r]}$, while the number of its k-faces is never less than the number of k-faces of $\mathcal{C}_{[r]}$.

For each $\varnothing \subset R \subseteq [r]$, we denote by $\mathcal{F}_R$ the set of faces of $\mathcal{C}_R$ having at least one vertex from each V_i, $i \in R$, and we call it the set of *mixed faces of* $\mathcal{C}_R$. We trivially have that $\mathcal{F}_{\{i\}} \equiv \partial P_i$. We define the dimension of $\mathcal{F}_R$ to be the maximum dimension of the faces in $\mathcal{F}_R$, i.e., $\dim(\mathcal{F}_R) = \max_{F \in \mathcal{F}_R} \dim(F) = d + |R| - 2$. Under the *"as simplicial as possible"* assumption above, the faces in $\mathcal{F}_R$ are simplices. We denote by $\mathcal{K}_R$ the *closure*, under subface inclusion, of $\mathcal{F}_R$. By construction, $\mathcal{K}_R$ contains: (1) all faces in $\mathcal{F}_R$, (2) all faces that are subfaces of faces in $\mathcal{F}_R$, and (3) the empty set. It is easy to see that $\mathcal{K}_R$ does not contain any of the trivial faces $\{\mathcal{C}_S\}$, $\varnothing \subset S \subseteq R$, and thus, $\mathcal{K}_R$ is a pure simplicial $(d+|R|-2)$-complex. It is also easy to verify that:

$$f_k(\mathcal{K}_R) = \sum_{\varnothing \subset S \subseteq R} f_k(\mathcal{F}_S), \qquad -1 \le k \le d+|R|-2, \tag{2.6}$$

where in order for the above equation to hold for $k = -1$, we set $f_{-1}(\mathcal{F}_S) = (-1)^{|S|-1}$ for all $\varnothing \subset S \subseteq R$. In what follows we use the convention that $f_k(\mathcal{F}_R) = 0$, for any $k < -1$ or $k > d + |R| - 2$.

A general form of the Inclusion-Exclusion Principle states that if f and g are two functions defined over the subsets of a finite set A, such that $f(A) = \sum_{\varnothing \subset B \subseteq A} g(B)$, then $g(A) = \sum_{\varnothing \subset B \subseteq A} (-1)^{|A|-|B|} f(B)$ [6, Theorem 12.1]. Applying this principle to (2.6), we deduce that:

$$f_k(\mathcal{F}_R) = \sum_{\varnothing \subset S \subseteq R} (-1)^{|R|-|S|} f_k(\mathcal{K}_S), \qquad -1 \le k \le d+|R|-2. \tag{2.7}$$

In the majority of our proofs that involve evaluation of f- and h-vectors, we use generating functions as they significantly simplify calculations. The starting point is to evaluate $\mathtt{f}(\mathcal{K}_R;t)$ (resp., $\mathtt{f}(\mathcal{F}_R;t)$) in terms of the generating functions $\mathtt{f}(\mathcal{F}_S;t)$ (resp., $\mathtt{f}(\mathcal{K}_S;t)$), $\varnothing \subset S \subseteq R$, for each fixed choice of $\varnothing \subset R \subseteq [r]$. Then, using (2.4) we derive the analogous relations between their h-vectors.

[1] We denote by $\{\mathcal{C}_R\}$ the polytope $\mathcal{C}_R$ as a trivial face itself (without its non-trivial faces).

Recalling that $\dim(\mathcal{K}_R) = d + |R| - 2$ and $\dim(\mathcal{F}_S) = d + |S| - 2$ we have:

$$\begin{aligned} \mathtt{f}(\mathcal{K}_R;t) &= \sum_{k=0}^{d+|R|-1} f_{k-1}(\mathcal{K}_R)t^{d+|R|-1-k} \overset{(2.6)}{=} \sum_{k=0}^{d+|R|-1} \sum_{\varnothing\subset S\subseteq R} f_{k-1}(\mathcal{F}_S)t^{d+|R|-1-k} \\ &= \sum_{\varnothing\subset S\subseteq R} t^{|R|-|S|} \sum_{k=0}^{d+|R|-1} f_{k-1}(\mathcal{F}_S)t^{d+|S|-1-k} = \sum_{\varnothing\subset S\subseteq R} t^{|R|-|S|}\mathtt{f}(\mathcal{F}_S;t). \end{aligned} \tag{2.8}$$

Rewriting the above relation as $t^{-|R|}\mathtt{f}(\mathcal{K}_R;t) = \sum_{\varnothing\subset S\subseteq R} t^{-|S|}\mathtt{f}(\mathcal{F}_S;t)$ and using Möbius inversion, we get:

$$\mathtt{f}(\mathcal{F}_R;t) = \sum_{\varnothing\subset S\subseteq R} (-1)^{|R|-|S|}t^{|R|-|S|}\mathtt{f}(\mathcal{K}_S;t). \tag{2.9}$$

Setting $t := t-1$ in (2.8) we have:

$$\begin{aligned} \mathtt{h}(\mathcal{K}_R;t) = \mathtt{f}(\mathcal{K}_R;t-1) &= \sum_{\varnothing\subset S\subseteq R} (t-1)^{|R|-|S|}\mathtt{f}(\mathcal{F}_S;t-1) \\ &= \sum_{\varnothing\subset S\subseteq R} (t-1)^{|R|-|S|}\mathtt{h}(\mathcal{F}_S;t) = \sum_{\varnothing\subset S\subseteq R} \mathtt{g}^{(|R|-|S|)}(\mathcal{F}_S;t). \end{aligned} \tag{2.10}$$

Similarly, from (2.9) we obtain:

$$\mathtt{h}(\mathcal{F}_R;t) = \sum_{\varnothing\subset S\subseteq R} (-1)^{|R|-|S|}\mathtt{g}^{(|R|-|S|)}(\mathcal{K}_S;t). \tag{2.11}$$

Comparing coefficients in the above generating functions, we deduce that:

$$h_k(\mathcal{K}_R) = \sum_{\varnothing\subset S\subseteq R} g_k^{(|R|-|S|)}(\mathcal{F}_S), \qquad \text{for all } 0 \le k \le d+|R|-1, \text{ and} \tag{2.12}$$

$$h_k(\mathcal{F}_R) = \sum_{\varnothing\subset S\subseteq R} (-1)^{|R|-|S|}g_k^{(|R|-|S|)}(\mathcal{K}_S), \qquad \text{for all } 0 \le k \le d+|R|-1. \tag{2.13}$$

3 The construction of the auxiliary simplicial polytope $\mathcal{Q}_{[r]}$

The proper faces of the Cayley polytope $\mathcal{C}_{[r]}$ of $P_1,\dots,P_r$ are the faces in each $\mathcal{F}_R$, $\varnothing\subset R\subseteq[r]$ as well as all trivial faces $\{\mathcal{C}_R\}$ with $\varnothing\subset R\subset[r]$. Since the latter are not necessarily simplices, the Cayley polytope $\mathcal{C}_{[r]}$ may not be simplicial. In order to exploit the combinatorial structure of $\mathcal{C}_{[r]}$, we add auxiliary points on $\mathcal{C}_{[r]}$ so that the resulting polytope, denoted by $\mathcal{Q}_{[r]}$, is simplicial.

The main tool for describing our construction is *stellar subdivisions*. Let $P\subset\mathbb{R}^d$ be a d-polytope, and consider a point y_F in the relative interior of a face F of ∂P. The *stellar subdivision* $\mathrm{st}(y_F,\partial P)$ of ∂P over F, replaces F by the set of faces $\{y_F, F'\}$ where F' is a non-trivial face of F. It is a well-known fact that stellar subdivisions preserve polytopality (cf. [3, pp. 70–73]), in the sense that the newly constructed complex is combinatorially equivalent to a polytope each facet of which lies on a distinct supporting hyperplane.

Our goal is to triangulate each face $\{\mathcal{C}_R\}$, $\varnothing\subset R\subset[r]$, of $\mathcal{C}_{[r]}$ so that the boundaries of the resulting complexes, denoted by $\mathcal{Q}_S$, $\varnothing\subset S\subseteq[r]$, are simplicial polytopes. We obtain this by performing a series of stellar subdivisions. First set $\mathcal{Q}_S := \mathcal{C}_S$, for all $\varnothing\subset S\subseteq[r]$. Then, we add auxiliary vertices as follows:

for s from 1 to $r-1$
  for all $S\subseteq[r]$ with $|S| = s$
    choose $y_S\in\mathrm{relint}(\mathcal{Q}_S)$ (3.1)
    for all T with $S\subset T\subseteq[r]$
      $\mathcal{Q}_T := \mathrm{st}(y_S,\mathcal{Q}_T)$

The recursive step of the previous definition is well defined due to the fact that for any fixed s, the order in which we add the auxiliary points y_S is independent of the S chosen, since the relative interiors of all $\mathcal{Q}_S$ with $|S| = s$ are pairwise disjoint. At the end of the s-th iteration, the faces of each $\mathcal{Q}_T$ of dimension less than $d+s-1$ are simplices. At the end of the iterative procedure above, and in view of the fact that stellar subdivisions preserve polytopality, the above construction results in simplicial $(d+|R|-1)$-polytopes $\mathcal{Q}_R$, for all $\varnothing \subset R \subseteq [r]$.

The next lemma shows how the iterated stellar subdivisions performed in (3.1) are captured in the enumerative structure of $\mathcal{Q}_R$.

▸ **Lemma 3.1.** *For all $\varnothing \subset R \subseteq [r]$ we have:*

$$\mathtt{f}(\partial\mathcal{Q}_R;t) = \mathtt{f}(\mathcal{F}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|} i!\, S^{i+1}_{|R|-|S|+1} t^{|R|-|S|-i}\mathtt{f}(\mathcal{F}_S;t), \tag{3.2}$$

$$\mathtt{f}(\partial\mathcal{Q}_R;t) = \mathtt{f}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|-2} (i+1)!\, S^{i+1}_{|R|-|S|} t^{|R|-|S|-i}\mathtt{f}(\mathcal{K}_S;t), \tag{3.3}$$

where $S^k_m = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^m$, $m \geq k \geq 0$, are the Stirling numbers of the second kind.

The h-vector relations stemming from the f-vector relations above are the subject of the following lemma.

▸ **Lemma 3.2.** *For all $\varnothing \subset R \subseteq [r]$ we have:*

$$\mathtt{h}(\partial\mathcal{Q}_R;t) = \mathtt{h}(\mathcal{F}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} E^{j}_{|R|-|S|}\, t^{j+1}\,\mathtt{h}(\mathcal{F}_S;t), \tag{3.4}$$

$$\mathtt{h}(\partial\mathcal{Q}_R;t) = \mathtt{h}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} E^{j}_{|R|-|S|}\, t^{j}\,\mathtt{h}(\mathcal{K}_S;t), \tag{3.5}$$

where $E^k_m = \sum_{i=0}^{k}(-1)^i\binom{m+1}{i}(k+1-i)^m$, $m \geq k+1 > 0$, are the Eulerian numbers.

4 The Dehn-Sommerville equations

A very important structural property of the Cayley polytope $\mathcal{C}_R$ is, what we call, the *Dehn-Sommerville equations*. For a single polytope they reduce to the well-known Dehn-Sommerville equations, whereas for two or more summands they relate the h-vectors of the sets $\mathcal{F}_R$ and $\mathcal{K}_R$. The Dehn-Sommerville equations for $\mathcal{C}_R$ are one of the major key ingredients for establishing our upper bounds, as they permit us to reason for the maximality of the elements of $\boldsymbol{h}(\mathcal{F}_R)$ and $\boldsymbol{h}(\mathcal{K}_R)$ by considering only the lower halves of these vectors.

▸ **Theorem 4.1** (Dehn-Sommerville equations)**.** *Let $\mathcal{C}_R$ be the Cayley polytope of the d-polytopes $P_i, i \in R$. Then, the following relations hold:*

$$t^{d+|R|-1}\mathtt{h}(\mathcal{F}_R;\tfrac{1}{t}) = \mathtt{h}(\mathcal{K}_R;t) \tag{4.1}$$

or, equivalently,

$$h_{d+|R|-1-k}(\mathcal{F}_R) = h_k(\mathcal{K}_R), \qquad 0 \leq k \leq d+|R|-1. \tag{4.2}$$

Proof. We prove our claim by induction on the size of R, the case $|R| = 1$ being the Dehn-Sommerville equations for a d-polytope. We next assume that our claim holds for all $\varnothing \subset S \subset R$

and prove it for R. The ordinary Dehn-Sommerville relations, written in generating function form, for the (simplicial) $(d+|R|-1)$-polytope $\mathcal{Q}_R$ imply that:

$$\mathtt{h}(\partial\mathcal{Q}_R;t)=t^{d+|R|-1}\mathtt{h}(\partial\mathcal{Q}_R;\tfrac{1}{t}). \tag{4.3}$$

In view of relation (3.4) of Lemma 3.2, the right-hand side of (4.3) becomes:

$$t^{d+|R|-1}\mathtt{h}(\mathcal{F}_R;\tfrac{1}{t})+t^{d+|R|-1}\sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}E^{j}_{|R|-|S|}t^{-j-1}\mathtt{h}(\mathcal{F}_S;\tfrac{1}{t}). \tag{4.4}$$

Using relation (3.5), along with the induction hypothesis, the left-hand side of (4.3) becomes:

$$\begin{aligned}
&\mathtt{h}(\mathcal{K}_R;t)+\sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}E^{j}_{|R|-|S|}t^{j}\mathtt{h}(\mathcal{K}_S;t) && (4.5)\\
&\qquad=\mathtt{h}(\mathcal{K}_R;t)+\sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}E^{j}_{|R|-|S|}t^{|R|-|S|-j-1}\mathtt{h}(\mathcal{K}_S;t) && (4.6)\\
&\qquad=\mathtt{h}(\mathcal{K}_R;t)+\sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}E^{j}_{|R|-|S|}t^{|R|-|S|-j-1}t^{d+|S|-1}\mathtt{h}(\mathcal{F}_S;\tfrac{1}{t})\\
&\qquad=\mathtt{h}(\mathcal{K}_R;t)+\sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}E^{j}_{|R|-|S|}t^{d+|R|-j-2}\mathtt{h}(\mathcal{F}_S;\tfrac{1}{t}), && (4.7)
\end{aligned}$$

where to go from (4.5) to (4.6) we changed variables and used the well-known symmetry of the Eulerian numbers, namely, $E^k_m=E^{m-k-1}_m$, for all $m\geq k+1>0$.

Now, substituting (4.4) and (4.7) in (4.3), we deduce that $t^{d+|R|-1}\mathtt{h}(\mathcal{F}_R;\frac{1}{t})=\mathtt{h}(\mathcal{K}_R;t)$, which is, coefficient-wise, equivalent to (4.2). ◂

5 The recurrence relation for $h(\mathcal{F}_R)$

The subject of this section is the generalization, for the h-vector of $\mathcal{F}_R$, $\varnothing\subset R\subseteq[r]$, of the recurrence relation

$$(k+1)h_{k+1}(\partial P)+(d-k)h_k(\partial P)\leq n\,h_k(\partial P),\quad 0\leq k\leq d-1, \tag{5.1}$$

that holds true for any simplicial d-polytope $P\subset\mathbb{R}^d$. This is the content of the next theorem.

▸ **Theorem 5.1** (Recurrence inequality). *For any $\varnothing\subset R\subseteq[r]$ we have:*

$$h_{k+1}(\mathcal{F}_R)\leq\frac{n_R-d-|R|+1+k}{k+1}h_k(\mathcal{F}_R)+\sum_{i\in R}\frac{n_i}{k+1}g_k(\mathcal{F}_{R\setminus\{i\}}),\qquad 0\leq k\leq d+|R|-2, \tag{5.2}$$

where: (1) $n_R=\sum_{i\in R}n_i$, and, (2) $g_k(\mathcal{F}_\varnothing)=g_k(\varnothing)=0$, for all k.

Sketch of proof. To prove the inequality in the statement of the theorem, we generalize McMullen's steps in the proof of his Upper Bound theorem [14].

Our starting point is relation (5.1) applied to the simplicial $(d+|R|-1)$-polytope $\mathcal{Q}_R$, expressed in terms of generating functions:

$$(d+|R|-1)\,\mathtt{h}(\partial\mathcal{Q}_R;t)+(1-t)\mathtt{h}'(\partial\mathcal{Q}_R;t)=\sum_{v\in\mathrm{vert}(\partial\mathcal{Q}_R)}\mathtt{h}(\partial\mathcal{Q}_R/v;t). \tag{5.3}$$

Exploiting the combinatorial structure of $\mathcal{Q}_R$ in order to express: (1) $\boldsymbol{h}(\partial\mathcal{Q}_R)$ in terms of $\boldsymbol{h}(\mathcal{F}_S)$, $\varnothing \subset S \subseteq R$, and (2) $\boldsymbol{h}(\partial\mathcal{Q}_R/v)$ in terms of $\boldsymbol{h}(\mathcal{F}_S/v)$, $\varnothing \subset S \subseteq R$, and $\boldsymbol{h}(\mathcal{F}_S)$, $\varnothing \subset S \subset R$, relation (5.3) yields:

$$(d+|R|-1)\mathtt{h}(\mathcal{F}_R;t) + (1-t)\mathtt{h}'(\mathcal{F}_R;t) = \sum_{v\in V_R} \mathtt{h}(\mathcal{F}_R/v;t),$$

the element-wise form of which is:

$$(k+1)h_{k+1}(\mathcal{F}_R) + (d+|R|-1-k)h_k(\mathcal{F}_R) = \sum_{v\in V_R} h_k(\mathcal{F}_R/v), \quad 0 \le k \le d+|R|-2.$$

Noticing that $h_k(\mathcal{F}_R/v)$ is equal to $\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{v\in V_S} g_k^{(|R|-|S|)}(\mathcal{K}_S/v)$ (by the Inclusion-Exclusion Principle; see also relations (2.12) and (2.13)), and using a particular shelling of $\partial\mathcal{Q}_R$, we show that:

$$\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{v\in V_S} g_k^{(|R|-|S|)}(\mathcal{K}_S/v) \le \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{v\in V_S} g_k^{(|R|-|S|)}(\mathcal{K}_S).$$

The right-hand side of the above relation simplifies to $n_R\, h_k(\mathcal{F}_R) + \sum_{i\in R} n_i\, g_k(\mathcal{F}_{R\setminus\{i\}})$, which in turn suggests the following inequality:

$$(k+1)h_{k+1}(\mathcal{F}_R) + (d+|R|-1-k)h_k(\mathcal{F}_R) \le n_R\, h_k(\mathcal{F}_R) + \sum_{i\in R} n_i\, g_k(\mathcal{F}_{R\setminus\{i\}}) \tag{5.4}$$

that holds true for all $0 \le k \le d+|R|-2$. Solving in terms of $h_{k+1}(\mathcal{F}_R)$ results in (5.2). ◂

6 Upper bounds

Let $S_1,\dots,S_r$ be a partition of a set S into r sets. We say that $A \subseteq \bigcup_{1\le i\le r} S_i$ is a *spanning subset of* S if $A \cap S_i \neq \varnothing$ for all $1 \le i \le r$.

▸ **Definition 6.1.** Let $P_i, i\in R$, be d-polytopes with vertex sets $V_i, i \in R$. We say that their Cayley polytope $\mathcal{C}_R$ is *R-neighborly* if every spanning subset of $\bigcup_{i\in R} V_i$ of size $|R| \le \ell \le \lfloor\frac{d+|R|-1}{2}\rfloor$ is a face of $\mathcal{C}_R$ (or, equivalently, a face of $\mathcal{F}_R$). We say that the Cayley polytope $\mathcal{C}_R$ is *Minkowski-neighborly* if, for every $\varnothing \subset S \subseteq R$, the Cayley polytope $\mathcal{C}_S$ is S-neighborly.

The following lemma characterizes R-neighborly Cayley polytopes in terms of the f- and h-vector of $\mathcal{F}_R$.

▸ **Lemma 6.2.** *The following are equivalent:*

(i) $\mathcal{C}_R$ *is R-neighborly,*
(ii) $f_{\ell-1}(\mathcal{F}_R) = \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S}{\ell}$, *for all* $0 \le \ell \le \lfloor\frac{d+|R|-1}{2}\rfloor$,
(iii) $h_\ell(\mathcal{F}_R) = \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|+\ell}{\ell}$, *for all* $0 \le \ell \le \lfloor\frac{d+|R|-1}{2}\rfloor$,
where n_i is the number of vertices of P_i and $n_S = \sum_{i\in S} n_i$.

From the recurrence relation in Theorem 5.1 we arrive at the following theorem. The proof is by induction on k.

▸ **Theorem 6.3.** *For any* $\varnothing \subset R \subseteq [r]$ *and* $0 \le k \le d+|R|-1$, *we have:*

$$g_k(\mathcal{F}_R) \le \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|-1+k}{k}, \quad \textit{and} \tag{6.1}$$

$$h_k(\mathcal{F}_R) \le \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|+k}{k}, \tag{6.2}$$

where $n_S = \sum_{i\in S} n_i$. *Equalities hold for all* $0 \le k \le \lfloor\frac{d+|R|-1}{2}\rfloor$ *if and only if the Cayley polytope* $\mathcal{C}_R$ *is R-neighborly.*

Before proceeding with proving upper bounds for the h-vectors of $\mathcal{F}_R$ and $\mathcal{K}_R$ we need to define the following functions.

▸ **Definition 6.4.** Let $d \geq 2$, $\varnothing \subset R \subseteq [r]$, $m \geq 0$, $0 \leq k \leq d+|R|-1$, and $n_i \in \mathbb{N}$, $i \in R$, with $n_i \geq d+1$. We define the functions $\Phi^{(m)}_{k,d}(\boldsymbol{n}_R)$ and $\Psi_{k,d}(\boldsymbol{n}_R)$ via the following conditions:

1. $\Phi^{(0)}_{k,d}(\boldsymbol{n}_R) = \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|+k}{k}$, $0 \leq k \leq \lfloor\frac{d+|R|-1}{2}\rfloor$,
2. $\Phi^{(m)}_{k,d}(\boldsymbol{n}_R) = \Phi^{(m-1)}_{k,d}(\boldsymbol{n}_R) - \Phi^{(m-1)}_{k-1,d}(\boldsymbol{n}_R)$, $m > 0$,
3. $\Psi_{k,d}(\boldsymbol{n}_R) = \sum_{\varnothing\subset S\subseteq R}\Phi^{(|R|-|S|)}_{k,d}(\boldsymbol{n}_S)$,
4. $\Phi^{(0)}_{k,d}(\boldsymbol{n}_R) = \Psi_{d+|R|-1-k,d}(\boldsymbol{n}_R)$,

where $\boldsymbol{n}_R$ stands for the $|R|$-dimensional vector whose elements are the values n_i, $i \in R$.

Notice that $\Phi^{(0)}_{k,d}(\boldsymbol{n}_R)$ and $\Psi_{k,d}(\boldsymbol{n}_R)$ are well defined, though in a recursive manner (in the size of R), since for any $k > \lfloor\frac{d+|R|-1}{2}\rfloor$, we have:

$$\begin{aligned}
\Phi^{(0)}_{k,d}(\boldsymbol{n}_R) &= \Psi_{d+|R|-1-k,d}(\boldsymbol{n}_R) = \sum_{\varnothing\subset S\subseteq R}\Phi^{(|R|-|S|)}_{d+|R|-1-k,d}(\boldsymbol{n}_S)\\
&= \Phi^{(0)}_{d+|R|-1-k,d}(\boldsymbol{n}_R) + \sum_{\varnothing\subset S\subset R}\Phi^{(|R|-|S|)}_{d+|R|-1-k,d}(\boldsymbol{n}_S)\\
&= \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-k-1}{d+|R|-1-k} + \sum_{\varnothing\subset S\subset R}\Phi^{(|R|-|S|)}_{d+|R|-1-k,d}(\boldsymbol{n}_S),
\end{aligned}\tag{6.3}$$

where the second sum in (6.3) is to be understood as 0 when $|R| = 1$. In other words, $\Phi^{(0)}_{k,d}(\boldsymbol{n}_R)$, and, thus, also $\Phi^{(m)}_{k,d}(\boldsymbol{n}_R)$ for any $m > 0$, is fully defined for some R and any k, once we know the values $\Phi^{(\ell)}_{k,d}(\boldsymbol{n}_S)$ for all $\varnothing \subset S \subset R$, for all $0 \leq k \leq d+|S|-1$, and for all $1 \leq \ell \leq |R|-1$. Moreover, it is easy to verify that $\Phi^{(0)}_{k,d}(\boldsymbol{n}_R)$ satisfies the following recurrence relation:

$$\Phi^{(0)}_{k+1,d}(\boldsymbol{n}_R) = \frac{n_R-d-|R|+k+1}{k+1}\Phi^{(0)}_{k,d}(\boldsymbol{n}_R)+\sum_{i\in R}\frac{n_i}{k+1}\Phi^{(1)}_{k,d}(\boldsymbol{n}_{R\setminus\{i\}}),\quad 0 \leq k < \lfloor\tfrac{d+|R|-1}{2}\rfloor.\tag{6.4}$$

The next theorem provides upper bounds for the h-vectors of $\mathcal{F}_R$ and $\mathcal{K}_R$, as well as necessary and sufficient conditions for these upper bounds to be attained.

▸ **Theorem 6.5.** *For all $0 \leq k \leq d+|R|-1$, we have:*

(i) $h_k(\mathcal{F}_R) \leq \Phi^{(0)}_{k,d}(\boldsymbol{n}_R)$,
(ii) $h_k(\mathcal{K}_R) \leq \Psi_{k,d}(\boldsymbol{n}_R)$.

Equalities hold for all k if and only if the Cayley polytope $\mathcal{C}_R$ is Minkowski-neighborly.

Proof. To prove the upper bounds we use recursion on the size of $|R|$. For $|R| = 1$, the result for both $h_k(\mathcal{F}_R)$ and $h_k(\mathcal{K}_R)$ comes from the UBT for d-polytopes. For $|R| > 1$, we assume that the bounds hold for all S with $\varnothing \subset S \subset R$, and for all k with $0 \leq k \leq d+|S|-1$. Furthermore, the upper bound for $h_k(\mathcal{F}_R)$ for $k \leq \lfloor\frac{d+|R|-1}{2}\rfloor$ is immediate from Theorem 6.3. To prove the upper bound for $h_k(\mathcal{K}_R)$, $0 \leq k \leq \lfloor\frac{d+|R|-1}{2}\rfloor$, we use the following expansion for $h_k(\mathcal{K}_R)$ (cf. [1, Lemma 5.14]):

$$\begin{aligned}
h_k(\mathcal{K}_R) &= \sum_{j=0}^{\lfloor\frac{|R|}{2}\rfloor}\sum_{s=c-2j-1}^{|R|-2j}\sum_{\substack{S\subseteq R\\|S|=s}}\binom{|R|-s}{2j}\left(h_{k-2j}(\mathcal{F}_S)-\frac{1}{2j+1}\sum_{i\in S}h_{k-2j-1}(\mathcal{F}_{S\setminus\{i\}})\right)\\
&\quad+\sum_{j=0}^{\lfloor\frac{|R|}{2}\rfloor}\sum_{\substack{S\subset R\\|S|=c-2j+1}}\binom{|R|-|S|}{2j}\left(h_{k-2j}(\mathcal{F}_S)-\frac{1}{2j+1}\sum_{i\in S}h_{k-2j-1}(\mathcal{F}_{S\setminus\{i\}})\right),
\end{aligned}\tag{6.5}$$

where c depends on k, d and $|R|$. Under the assumption that $r < d$, it is easy to show that:

$$h_{k-2j}(\mathcal{F}_S) - \frac{1}{2j+1}\sum_{i\in S} h_{k-2j-1}(\mathcal{F}_{S\setminus\{i\}}) \le \Phi^{(0)}_{k-2j,d}(\boldsymbol{n}_S) - \frac{1}{2j+1}\sum_{i\in S}\Phi^{(0)}_{k-2j-1,d}(\boldsymbol{n}_{S\setminus\{i\}}). \quad (6.6)$$

Substituting the upper bound from (6.6) in (6.5), and reversing the derivation logic for (6.5), we deduce that $h_k(\mathcal{K}_R) \le \Psi_{k,d}(\boldsymbol{n}_R)$.

For $k > \lfloor\frac{d+|R|-1}{2}\rfloor$ we have:

$$h_k(\mathcal{F}_R) = h_{d+|R|-1-k}(\mathcal{K}_R) \le \Psi_{d+|R|-1-k,d}(\boldsymbol{n}_R) = \Phi^{(0)}_{k,d}(\boldsymbol{n}_R), \quad \text{and,}$$
$$h_k(\mathcal{K}_R) = h_{d+|R|-1-k}(\mathcal{F}_R) \le \Phi^{(0)}_{d+|R|-1-k,d}(\boldsymbol{n}_R) = \Psi_{k,d}(\boldsymbol{n}_R).$$

The necessary and sufficient conditions are easy consequences of the equality claim in Theorem 6.3. ◂

For any $d \ge 2$, $\varnothing \subset R \subseteq [r]$, $0 \le k \le d+|R|-1$, and $n_i \in \mathbb{N}$, $i \in R$, with $n_i \ge d+1$, let

$$\Xi_{k,d}(\boldsymbol{n}_R) = \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|} f_k\big(C_{d+|R|-1}(n_S)\big) + \sum_{i=0}^{\lfloor\frac{d+|R|-2}{2}\rfloor}\binom{i}{k-d-|R|+1+i}\sum_{\varnothing\subset S\subset R}\Phi^{(|R|-|S|)}_{i,d}(\boldsymbol{n}_S),$$

where $C_\delta(n)$ stands for the cyclic δ-polytope with n vertices. It is straightforward to verify that for $0 \le k \le \lfloor\frac{d+|R|-1}{2}\rfloor$, $\Xi_{k,d}(\boldsymbol{n}_R)$ simplifies to $\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S}{k}$. We are finally ready to state and prove the main result of the paper.

▸ **Theorem 6.6.** *Let $P_1,\ldots,P_r$ be r d-polytopes, $r < d$, with $n_1,\ldots,n_r$ vertices respectively. Then, for all $1 \le k \le d$, we have:*

$$f_{k-1}(P_{[r]}) \le \Xi_{k+r,d}(\boldsymbol{n}_{[r]}).$$

Equality holds for all $0 \le k \le d$ if and only if the Cayley polytope $\mathcal{C}_{[r]}$ of $P_1,\ldots,P_r$ is Minkowski-neighborly.

Proof. We start by recalling that:

$$f_{k-1}(\mathcal{F}_{[r]}) = \sum_{i=0}^{d+r-1}\binom{d+r-1-i}{k-i}h_i(\mathcal{F}_{[r]}).$$

In view of Theorem 6.5, the above expression is bounded from above by:

$$\sum_{i=0}^{\lfloor\frac{d+r-1}{2}\rfloor}\binom{d+r-1-i}{k-i}\Phi^{(0)}_{i,d}(\boldsymbol{n}_{[r]}) + \sum_{i=\lfloor\frac{d+r-1}{2}\rfloor+1}^{d+r-1}\binom{d+r-1-i}{k-i}\Phi^{(0)}_{i,d}(\boldsymbol{n}_{[r]}) \quad (6.7)$$

$$= \sum_{i=0}^{\lfloor\frac{d+r-1}{2}\rfloor}\binom{d+r-1-i}{k-i}\Phi^{(0)}_{i,d}(\boldsymbol{n}_{[r]}) + \sum_{i=0}^{\lfloor\frac{d+r-2}{2}\rfloor}\binom{i}{k-d-r+1+i}\sum_{\varnothing\subset R\subseteq[r]}\Phi^{(r-|R|)}_{i,d}(\boldsymbol{n}_R) \quad (6.8)$$

$$= \sum_{i=0}^{\frac{d+r-1}{2}}{}^{*}\left(\binom{d+r-1-i}{k-i}+\binom{i}{k-d-r+1+i}\right)\sum_{\varnothing\subset R\subseteq[r]}(-1)^{r-|R|}\binom{n_R-d-r+i}{i}$$
$$+ \sum_{i=0}^{\lfloor\frac{d+r-2}{2}\rfloor}\binom{i}{k-d-r+1+i}\sum_{\varnothing\subset R\subset[r]}\Phi^{(r-|R|)}_{i,d}(\boldsymbol{n}_R) \quad (6.9)$$

$$= \sum_{\varnothing\subset R\subseteq[r]}(-1)^{r-|R|}f_k\big(C_{d+r-1}(n_R)\big) + \sum_{i=0}^{\lfloor\frac{d+r-2}{2}\rfloor}\binom{j}{k-d-r+1+i}\sum_{\varnothing\subset R\subset[r]}\Phi^{(r-|R|)}_{i,d}(\boldsymbol{n}_R), \quad (6.10)$$

where to go:

- from (6.7) to (6.8) we changed the variable of the second sum from i to $d+r-1-i$, and used conditions 3 and 4 of Definition 6.4,
- from (6.8) to (6.9) we wrote the explicit expression of $\Phi^{(0)}_{i,d}(\boldsymbol{n}_{[r]})$ from relation (6.3),
- from (6.9) to (6.10) we used that the number of $(k-1)$-faces of a cyclic δ-polytope with n vertices is $\sum^{*\frac{\delta}{2}}_{i=0}\left(\binom{\delta-i}{k-i}+\binom{i}{k-\delta+i}\right)\binom{n-\delta-1+i}{i}$, where $\sum^{\frac{\delta}{2}\,*}_{i=0} T_i$ denotes the sum of the elements $T_0, T_1, \dots, T_{\lfloor\frac{\delta}{2}\rfloor}$ where the last term is halved if δ is even.

Finally, observing that the expression in (6.10) is nothing but $\Xi_{k,d}(\boldsymbol{n}_{[r]})$, and recalling that $f_{k-1}(\mathcal{F}_{[r]}) = f_{k-r}(P_{[r]})$, we arrive at the upper bound in the statement of the theorem. The equality claim is immediate from Theorem 6.5. ◂

7 Tight bound construction

In this section we show that the bounds in Theorem 6.6 are tight. Before getting into the technical details, we outline our approach. We start by considering the $(d-r+1)$-dimensional moment curve, which we embed in r distinct subspaces of $\mathbb{R}^d$. We consider the r copies of the $(d-r+1)$-dimensional moment curve as different curves, and we perturb them appropriately, so that they become d-dimensional moment-like curves. The perturbation is controlled via a non-negative parameter ζ, which will be chosen appropriately. We then choose points on these r moment-like curves, all parameterized by a positive parameter τ, which will again be chosen appropriately. These points are the vertices of r d-polytopes $P_1, P_2, \dots, P_r$, and we show that, for all $\varnothing \subset R \subseteq [r]$, the number of $(k-1)$-faces of $\mathcal{F}_R$, where $|R| \le k \le \lfloor\frac{d+|R|-1}{2}\rfloor$, becomes equal to $\Xi_{k,d}(\boldsymbol{n}_R)$ for small enough positive values of ζ and τ. Our construction produces *projected prodsimplicial-neighborly* polytopes (cf. [13]). For $\zeta = 0$ our polytopes are essentially the same as those in [13, Theorem 2.6], while for $\zeta > 0$ we get *deformed* versions of those polytopes. The positivity of ζ allows us to ensure the tightness of the upper bound on $f_k(P_{[r]})$, not only for small, but also for large values of k.

At a more technical level, the proof that $f_{k-1}(\mathcal{F}_R) = \Xi_{k,d}(\boldsymbol{n}_R)$, for all $|R| \le k \le \lfloor\frac{d+|R|-1}{2}\rfloor$, is performed in two steps. We first consider the cyclic $(d-r+1)$-polytopes $\hat{P}_1, \dots, \hat{P}_r$, embedded in appropriate subspaces of $\mathbb{R}^d$. The $\hat{P}_i$'s are the *unperturbed*, with respect to ζ, versions of the d-polytopes $P_1, P_2, \dots, P_r$ (i.e., the polytope $\hat{P}_i$ is the polytope we get from P_i, when we set ζ equal to zero). For each $\varnothing \subset R \subseteq [r]$ we denote by $\hat{\mathcal{C}}_R$ the Cayley polytope of $\hat{P}_i, i \in R$, seen as a polytope in $\mathbb{R}^d$, and we focus on the set $\hat{\mathcal{F}}_R$ of its mixed faces. Recall that the polytopes $\hat{P}_i, i \in R$, are parameterized by the parameter τ; we show that there exists a sufficiently small positive value $\tau^\star$ for τ, for which the number of $(k-1)$-faces of $\hat{\mathcal{F}}_R$ is equal to $\Xi_{k,d}(\boldsymbol{n}_R)$ for all $|R| \le k \le \lfloor\frac{d+|R|-1}{2}\rfloor$. For τ equal to $\tau^\star$, we consider the polytopes $P_1, P_2, \dots, P_r$ (with τ set to $\tau^\star$), and show that for sufficiently small ζ (denoted by $\zeta^\Diamond$), $f_{k-1}(\mathcal{F}_R)$ is equal to $\Xi_{k,d}(\boldsymbol{n}_R)$.

In the remainder of this section we describe our construction in detail. For each $1 \le i \le r$, we define the d-dimensional moment-like curve[2]:

$$\gamma_i(t;\zeta) = (\zeta t^{d-r+2}, \dots, \zeta t^{d-r+i}, \overset{i\text{-}th\text{ coordinate}}{t}, \zeta t^{d-r+i+2}, \dots, \zeta t^{d+1}, t^2, \dots, t^{d-r+1}),$$

and the d-polytope

[2] The curve $\gamma_i(t;\zeta)$, $\zeta > 0$, is the image under an invertible linear transformation, of the curve $\hat{\gamma}_i(t) = (t, t^2, \dots, t^{d-r+i}, t^{d-r+i+2}, \dots, t^{d+1})$. Polytopes whose vertices are n distinct points on this curve are combinatorially equivalent to the cyclic d-polytope with n vertices.

$$P_i := \operatorname{conv}\left(\{\boldsymbol{\gamma}_i(y_{i,1};\zeta),\ldots,\boldsymbol{\gamma}_i(y_{i,n_i};\zeta)\}\right), \tag{7.1}$$

where the parameters $y_{i,j}$ belong to the sets $Y_i = \{y_{i,1},\ldots,y_{i,n_i}\}$, $1 \le i \le r$, whose elements are determined as follows. Choose

- $n_{[r]} + d + r$ arbitrary real numbers $x_{i,j}$ and M_s, such that:
  - $0 < x_{i,1} < x_{i,1} + \epsilon < x_{i,2} < x_{i,2} + \epsilon < \cdots < x_{i,n_i} + \epsilon$, for $1 \le i \le r-1$,
  - $0 < x_{r,1} < x_{r,1} + \epsilon < x_{r,2} < x_{r,2} + \epsilon < \cdots < x_{r,n_r} + \epsilon < M_1' < \cdots < M_{d+r}'$,

  where $\epsilon > 0$ is sufficiently small and $x_{i,n_i} < x_{i+1,1}$ for all i, and
- r non-negative integers $\beta_1, \beta_2, \ldots, \beta_r$, such that $\beta_1 > \beta_2 > \cdots > \beta_{r-1} > \beta_r \ge 0$.

We then set $y_{i,j} := x_{i,j}\tau^{\beta_i}$, $\tilde{y}_{i,j} := (x_{i,j}+\epsilon)\tau^{\beta_i}$ and $M_i := M_i'\tau^{\beta_r}$, where τ is a positive parameter. The $y_{i,j}$'s, $\tilde{y}_{i,j}$'s and M_i's are used to define determinants whose value is positive for a small enough value of τ. The positivity of these determinants is crucial in defining supporting hyperplanes for the Cayley polytopes $\hat{\mathcal{C}}_R$ and $\mathcal{C}_R$ in Lemmas 7.1 and 7.2 below.

Next, for each $1 \le i \le r$, we define $\hat{P}_i := \lim_{\zeta\to 0^+} P_i$. Clearly, each $\hat{P}_i$ is a cyclic $(d-r+1)$-polytope embedded in the $(d-r+1)$-flat F_i of $\mathbb{R}^d$, where $F_i = \{x_j = 0 \mid 1 \le j \le r \text{ and } j \ne i\}$. The following lemma establishes the first step towards our construction.

▸ **Lemma 7.1.** *There exists a sufficiently small positive value $\tau^\star$ for τ, such that, for any $\varnothing \subset R \subseteq [r]$, the set of mixed faces $\hat{\mathcal{F}}_R$ of the Cayley polytope of the polytopes $\hat{P}_1,\ldots,\hat{P}_r$ constructed above, has*

$$f_{k-1}(\hat{\mathcal{F}}_R) = \Xi_{k,d}(\boldsymbol{n}_R), \quad |R| \le k \le \lfloor \tfrac{d+|R|-1}{2} \rfloor.$$

Proof. Let $\mathscr{U}_i$ be the set of vertices of $\hat{P}_i$ for $1 \le i \le r$ and set $\mathscr{U} := \cup_{i=1}^r \mathscr{U}_i$. The objective in the proof is, for each $\varnothing \subset R \subseteq [r]$ and each spanning subset U of the partition $\mathsf{U} = \cup_{i\in R}\mathscr{U}_i$, to exhibit a supporting hyperplane of the $(d+|R|-1)$-dimensional Cayley polytope $\hat{\mathcal{C}}_R$, containing exactly the vertices in U. In that respect, our approach is similar in spirit to the proof showing, by defining supporting hyperplanes constructed from Vandermonde determinants, that the cyclic n-vertex d-polytope $C_d(n)$ is neighborly (see, e.g., [17, Corollary 0.8]).

In our proof we need to involve the parameter ζ before taking the limit $\zeta \to 0^+$. This is due to the fact that, when $\varnothing \subset R \subset [r]$, the information of the relative position of the polytopes $\hat{P}_i$, $i \in R$, is lost if we set $\zeta = 0$ from the very first step. To describe our construction, we write each spanning subset U of U as the disjoint union of non-empty sets U_i, $i \in R$, where $U_i = U \cap \mathscr{U}_i$ and $|U_i| = \kappa_i \le n_i$. For this particular U, we define the linear equation:

$$H_U(\boldsymbol{x}) = \lim_{\zeta\to 0^+} (-1)^{\frac{|R|(|R|-1)}{2}+\sigma(R)} \zeta^{|R|-r} \mathsf{D}_U(\boldsymbol{x};\zeta), \tag{7.2}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_{d+|R|-1})$, and $\mathsf{D}_U(\boldsymbol{x};\zeta)$ is the $(d+|R|)\times(d+|R|)$ determinant:

- whose first column is $(1,\boldsymbol{x})^\top$,
- the next κ_i, $i \in R$, pairs of columns are $(1, \boldsymbol{e}_{i-1}, \boldsymbol{\gamma}_i(y_{i,j};\zeta))^\top$ and $(1, \boldsymbol{e}_{i-1}, \boldsymbol{\gamma}_i(\tilde{y}_{i,j};\zeta))^\top$ where $\boldsymbol{e}_0, \ldots, \boldsymbol{e}_{|R|-1}$ is the standard affine basis of $\mathbb{R}^{|R|-1}$, $y_{i,j} \in \{y \in Y_i \mid \boldsymbol{\gamma}_i(y;0) \in U_i\}$, and
- the last $s := d + |R| - 1 - \sum_{i\in R}\kappa_i$ columns are $(1, \boldsymbol{e}_{|R|-1}, \boldsymbol{\gamma}_{|R|-1}(M_i;\zeta))^\top$, $1 \le i \le s$; these columns exist only if $s > 0$.

The quantity $\sigma(R)$ above is a non-negative integer counting the total number of row swaps required to shift, for all $j \in [r] \setminus R$, the $(|R|+j)$-th row of $\mathsf{D}_U(\boldsymbol{x};\zeta)$ to the bottom of the determinant, so that the powers of $y_{i,j}$ in each column are in increasing order (notice that if $R \equiv [r]$ no such row swaps are required). Moreover, $\sigma(R)$ depends only on R and not on the choice of the spanning subset U of U.

The equation $H_U(\boldsymbol{x}) = 0$ is the equation of a hyperplane in $\mathbb{R}^{d+|R|-1}$ that passes through the points in U. We claim that, for any choice of U, and for all vertices $\boldsymbol{u}$ in $\mathscr{U} \setminus U$, we have

$H_U(\boldsymbol{u}) > 0$. To prove our claim, notice first that, for each $j \in [r] \setminus R$, the $(|R|+j)$-th row of the determinant $\mathsf{D}_U(\boldsymbol{u};\zeta)$ will contain the parameters $y_{i,j}^{d-r+1+j}$, $\tilde{y}_{i,j}^{d-r+1+j}$ and $M_i^{d-r+1+j}$, multiplied by ζ. After extracting ζ from each of these rows and shifting them to their *proper* position (i.e., the position where the powers along each column increase), we will have a term $\zeta^{r-|R|}$ and a sign $(-1)^{\sigma(R)}$ (induced from the $\sigma(R)$ row swaps required altogether). These terms cancel out with the term $(-1)^{\sigma(R)}\zeta^{|R|-r}$ in (7.2). We can, therefore, transform $H_U(\boldsymbol{u})$ in the form of the determinant $D_N(\mathsf{Z};\alpha_1,\ldots,\alpha_m)$, $\mathsf{Z} = \{z_{i,j} \mid 1 \le i \le \rho, 1 \le j \le \nu_i\}$, $N = (\nu_1,\nu_2,\ldots,\nu_m)$, $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_m$, shown below:

$$D_N(\mathsf{Z};\alpha_1,\ldots,\alpha_m) := (-1)^{\frac{\rho(\rho-1)}{2}} \begin{vmatrix} z_{1,1}^{\alpha_1} & \cdots & z_{1,\nu_1}^{\alpha_1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & z_{2,1}^{\alpha_1} & \cdots & z_{2,\nu_2}^{\alpha_1} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & z_{\rho,1}^{\alpha_1} & \cdots & z_{\rho,\nu_1}^{\alpha_1} \\ z_{1,1}^{\alpha_2} & \cdots & z_{1,\nu_1}^{\alpha_2} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & z_{2,1}^{\alpha_2} & \cdots & z_{2,\nu_2}^{\alpha_2} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & z_{\rho,1}^{\alpha_2} & \cdots & z_{\rho,\nu_n}^{\alpha_2} \\ z_{1,1}^{\alpha_3} & \cdots & z_{1,\nu_1}^{\alpha_3} & z_{2,1}^{\alpha_3} & \cdots & z_{2,\nu_2}^{\alpha_3} & \cdots & z_{n,1}^{\alpha_3} & \cdots & z_{n,\nu_n}^{\alpha_3} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ z_{1,1}^{\alpha_m} & \cdots & z_{1,\nu_1}^{\alpha_m} & z_{2,1}^{\alpha_m} & \cdots & z_{2,\nu_2}^{\alpha_m} & \cdots & z_{\rho,1}^{\alpha_m} & \cdots & z_{\rho,\nu_n}^{\alpha_m} \end{vmatrix},$$

by means of the following determinant transformations:

(i) By subtracting rows 2 to $|R|$ of $H_U(\boldsymbol{u})$ from its first row.

(ii) By shifting the first column of $H_U(\boldsymbol{u})$ to the right, so that all columns of $H_U(\boldsymbol{u})$ are arranged in increasing order with respect to their parameters $z_{i,j}$. Clearly, this can be done with an *even* number of column swaps.

The determinant $D_N(\mathsf{Z};\alpha_1,\ldots,\alpha_m)$ is strictly positive for all τ between 0 and some value $\hat{\tau}(R,U,\boldsymbol{u})$, that, depends (only) on the choice of R, U and $\boldsymbol{u}$. Since there is a finite number of possible such determinants, the value $\hat{\tau}^\star := \min_{R,U,\boldsymbol{u}} \hat{\tau}(R,U,\boldsymbol{u})$ is necessarily positive. Choosing some $\tau^\star \in (0,\hat{\tau}^\star)$ makes all these determinants simultaneously positive; this completes our proof. ◂

The following lemma establishes the second (and last) step of our construction.

▸ **Lemma 7.2.** *There exists a sufficiently small positive value $\zeta^\Diamond$ for ζ, such that, for any $\varnothing \subset R \subseteq [r]$, the set $\mathcal{F}_R$ of mixed faces of the Cayley polytope $\mathcal{C}_R$ of the polytopes $P_1,\ldots,P_r$ in* (7.1) *has*

$$f_{k-1}(\mathcal{F}_R) = \Xi_{k,d}(\boldsymbol{n}_R), \qquad \textit{for all} \quad |R| \le k \le \lfloor \tfrac{d+|R|-1}{2} \rfloor.$$

Proof. Briefly speaking, the value $\zeta^\Diamond$ is determined by replacing the limit $\zeta \to 0^+$ in the previous proof, by a specific value of ζ for which the determinants we consider are positive.

More precisely, let $\mathscr{U}_i$ be the set of vertices of P_i, $1 \le i \le r$, and set $\mathscr{U} := \cup_{i=1}^r \mathscr{U}_i$. Our goal is, for each $\varnothing \subset R \subseteq [r]$ and each spanning subset U of the partition $\mathsf{U} = \cup_{i \in R} \mathscr{U}_i$, to exhibit a supporting hyperplane of the Cayley polytope $\mathcal{C}_R$, containing exactly the vertices in U. To this end, we define the hyperplane $\widetilde{H}_U(\boldsymbol{x};\zeta) = 0$, $\boldsymbol{x} = (x_1,x_2,\ldots,x_{d+|R|-1})$, with

$$\widetilde{H}_U(\boldsymbol{x};\zeta) = (-1)^{\frac{|R|(|R|-1)}{2}+\sigma(R)} \zeta^{|R|-r} \mathsf{D}_U(\boldsymbol{x};\zeta), \quad \zeta > 0, \tag{7.3}$$

where $\mathsf{D}_U(\boldsymbol{x};\zeta)$ is the determinant in the proof of Lemma 7.1, where we have set τ to $\tau^\star$. Clearly, for each $\boldsymbol{u} \in \mathscr{U} \setminus U$, we have $\lim_{\zeta \to 0^+} \widetilde{H}_U(\boldsymbol{u};\zeta) = H_U(\boldsymbol{u}) > 0$. This immediately implies that for each combination of R, U and $\boldsymbol{u}$ there exists a value $\hat{\zeta}(R,U,\boldsymbol{u})$ such that, for all $\zeta \in (0,\hat{\zeta}(R,U,\boldsymbol{u}))$, $\widetilde{H}_U(\boldsymbol{u};\zeta) > 0$. Since the number of possible combinations for R, U and $\boldsymbol{u}$ is finite, the minimum $\hat{\zeta}^{\Diamond} := \min_{R,U,\boldsymbol{u}}\{\hat{\zeta}(R,U,\boldsymbol{u})\}$ is well defined and positive. Taking $\zeta^{\Diamond}$ to be any value in $(0,\hat{\zeta}^{\Diamond})$, satisfies our demands. ◂

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