The maximum number of faces of the Minkowski sum of two convex polytopes*

Menelaos I. Karavelas†‡ Eleni Tzanaki†‡

Abstract

We derive tight bounds for the maximum number of k-faces, $0 \leq k \leq d-1$, of the Minkowski sum, $P_1 \oplus P_2$, of two d-dimensional convex polytopes P_1 and P_2, as a function of the number of vertices of the polytopes.

For even dimensions $d \geq 2$, the maximum values are attained when P_1 and P_2 are cyclic d-polytopes with disjoint vertex sets. For odd dimensions $d \geq 3$, the maximum values are attained when P_1 and P_2 are $\lfloor \frac{d}{2} \rfloor$-neighborly d-polytopes, whose vertex sets are chosen appropriately from two distinct d-dimensional moment-like curves.

1 Introduction

Given two d-dimensional polytopes, or simply d-polytopes, P and Q, their Minkowski sum, $P \oplus Q$, is defined as the set $\{p+q \mid p \in P, q \in Q\}$. Minkowski sums are fundamental structures in both Mathematics and Computer Science. They appear in a variety of different subjects, including Combinatorial Geometry, Computational Geometry, Computer Algebra, Computer-Aided Design & Solid Modeling, Motion Planning, Assembly Planning, Robotics (see [15, 4] and the references therein), and, more recently, Game Theory [12], Computational Biology [11] and Operations Research [16].

Despite their apparent importance, little is known about the worst-case complexity of Minkowski sums in dimensions four and higher. In two dimensions, the worst-case complexity of Minkowski sums is well understood. Given two convex polygons P and Q with n and m vertices, respectively, the maximum number of vertices and edges of $P \oplus Q$ is $n+m$ [2]. This result can be generalized to any number of summands. If P is convex and Q is non-convex (or vice versa), the worst-case complexity of $P \oplus Q$ is $\Theta(nm)$, while if both P and Q are non-convex, the complexity of their Minkowski sum can be as high as $\Theta(n^2m^2)$ [2]. When P and Q are convex 3-polytopes (embedded in the 3-dimensional Euclidean space), the worst-case complexity of $P \oplus Q$ is $\Theta(nm)$, if both P and Q are convex, and $\Theta(n^3m^3)$, if both P and Q are non-convex (e.g., see [3]). For the intermediate cases, i.e., if only one of P and Q is convex, see [14].

Given two convex d-polytopes P_1 and P_2 in $\mathbb{E}^d$, $d \geq 2$, with n_1 and n_2 vertices, respectively, embed P_1 and P_2 in the hyperplanes $\{x_{d+1}=0\}$ and $\{x_{d+1}=1\}$ of $\mathbb{E}^{d+1}$, respectively. Then the weighted Minkowski sum $(1-\lambda)P_1 \oplus \lambda P_2 = \{(1-\lambda)p_1 + \lambda p_2 \mid p_1 \in P_1, p_2 \in P_2\}$, $\lambda \in (0,1)$, of P_1 and P_2 is the intersection of the convex hull $CH_{d+1}(\{P_1, P_2\})$ with the hyperplane $\{x_{d+1}=\lambda\}$. The embedding and reduction described above are essentially what are known as the *Cayley embedding* and *Cayley trick*, respectively [7]. From this reduction we immediately get that the worst-case complexity of $(1-\lambda)P_1 \oplus \lambda P_2$ is bounded from above by the complexity of $CH_{d+1}(\{P_1, P_2\})$, which is $O((n_1+n_2)^{\lfloor \frac{d+1}{2} \rfloor})$. Furthermore, the complexity of $(1-\lambda)P_1 \oplus \lambda P_2$ is independent of λ, which implies that the complexity of the weighted Minkowski sum of two convex polytopes is the same as the complexity of their unweighted sum. Very recently (cf. [8]), the authors of this paper have considered the problem of computing the asymptotic worst-case complexity of the convex hull of a fixed number r of convex d-polytopes lying on r parallel hyperplanes of $\mathbb{E}^{d+1}$. A direct corollary of our results is a tight bound on the worst-case complexity of the Minkowski sum of two convex d-polytopes for all odd dimensions $d \geq 3$. More precisely, we have shown that for $d \geq 3$ odd, the worst-case complexity of $P_1 \oplus P_2$ is in $\Theta(n_1 n_2^{\lfloor \frac{d}{2} \rfloor} + n_2 n_1^{\lfloor \frac{d}{2} \rfloor})$, which is a refinement of the obvious upper bound when n_1 and n_2 asymptotically differ.

In terms of exact bounds on the number of faces of the Minkowski sum of two polytopes, results are known only when the two summands are convex. Besides the trivial bound for convex polygons (2-polytopes), mentioned above, the first result of this nature was shown by Gritzmann and Sturmfels [6]: given r polytopes $P_1, P_2, \ldots, P_r$ in $\mathbb{E}^d$, with a total of n non-parallel edges, the number of l-faces, $f_l(P_1 \oplus P_2 \oplus \cdots \oplus P_r)$, of $P_1 \oplus P_2 \oplus \cdots \oplus P_r$ is bounded from above by $2\binom{n}{l}\sum_{j=0}^{d-1-l}\binom{n-l-1}{j}$. This bound is attained when the polytopes P_i are *zonotopes*, and their generating edges are in general position. Regarding bounds as a function

*The full version of the paper may be found in [9].

†Department of Applied Mathematics, University of Crete, P.O. Box 2208, GR-714 09 Heraklion, Greece; emails: {mkaravel,etzanaki}@tem.uoc.gr

‡Institute of Applied and Computational Mathematics, Foundation for Research and Technology - Hellas, P.O. Box 1385, GR-711 10 Heraklion, Greece.

of the number of vertices or facets of the summands, Fukuda and Weibel [5] have shown that, given two 3-polytopes P_1 and P_2 in $\mathbb{E}^3$, the number, $f_k(P_1 \oplus P_2)$, of k-faces of $P_1 \oplus P_2$, $0 \leq k \leq 2$, is bounded from above as follows:

$$\begin{aligned}
&f_0(P_1 \oplus P_2) \leq n_1 n_2,\\
&f_1(P_1 \oplus P_2) \leq 2n_1 n_2 + n_1 + n_2 - 8, \quad \text{and}\\
&f_2(P_1 \oplus P_2) \leq n_1 n_2 + n_1 + n_2 - 6,
\end{aligned}$$

where n_j is the number of vertices of P_j, $j = 1, 2$. Weibel [15] has also derived similar expressions in terms of the number of facets m_j of P_j, $j = 1, 2$, namely:

$$\begin{aligned}
&f_0(P_1 \oplus P_2) \leq 4m_1 m_2 - 8m_1 - 8m_2 + 16,\\
&f_1(P_1 \oplus P_2) \leq 8m_1 m_2 - 17m_1 - 17m_2 + 40, \quad \text{and}\\
&f_2(P_1 \oplus P_2) \leq 4m_1 m_2 - 9m_1 - 9m_2 + 26.
\end{aligned}$$

All these bounds are tight. Fogel, Halperin and Weibel [3] have extended the bound on the number of facets of the Minkowski sum in the case of r summands. More precisely, they have shown that given r 3-polytopes $P_1, P_2, \ldots, P_r$ in $\mathbb{E}^3$, where P_j has $m_j \geq d+1$ facets, the number of facets of the Minkowski sum $P_1 \oplus P_2 \oplus \cdots \oplus P_r$ is bounded from above by

$$\sum_{1 \leq i < j \leq r} (2m_i - 5)(2m_j - 5) + \sum_{i=1}^{r} m_i + \binom{r}{2},$$

and this bound is tight.

For dimensions four and higher, the only known results are worst-case bounds on the number of k-faces of the Minkowski sum of convex polytopes, as a function of the number of vertices of the summands. Fukuda and Weibel [5] have shown that the number of vertices of the Minkowski sum of r d-polytopes $P_1, \ldots, P_r$, where $r \leq d-1$ and $d \geq 2$, is bounded from above by $\prod_{i=1}^{r} n_i$, where n_i is the number of vertices of P_i, and this bound is tight. On the other hand, for $r \geq d$ this bound cannot be attained [13]. For higher-dimensional faces, i.e., for $k \geq 1$, Fukuda and Weibel [5] have shown that $f_k(P_1 \oplus P_2 \oplus \cdots \oplus P_r)$ is bounded by:

$$\sum_{\substack{1 \leq s_i \leq n_i \\ s_1 + \ldots + s_r = k + r}} \prod_{i=1}^{r} \binom{n_i}{s_i}, \qquad 0 \leq k \leq d-1,$$

where n_i is the number of vertices of P_i. These bounds are tight for $d \geq 4$, $r \leq \lfloor \frac{d}{2} \rfloor$, and for all k with $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - r$.

In this paper, we extend previous results on the exact maximum number of faces of the Minkowski sum of two convex d-polytopes[1]. We show that given two d-polytopes P_1 and P_2 in $\mathbb{E}^d$ with $n_1 \geq d+1$ and $n_2 \geq d+1$ vertices, respectively, the maximum number of k-faces of $P_1 \oplus P_2$ is bounded as follows:

$$f_{k-1}(P_1 \oplus P_2) \leq f_k(C_{d+1}(n_1 + n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k+1-i} \left(\binom{n_1 - d - 2 + i}{i} + \binom{n_2 - d - 2 + i}{i} \right),$$

where $1 \leq k \leq d$, and $C_d(n)$ stands for the cyclic d-polytope with n vertices. These expressions are shown to be tight for any $d \geq 2$ and for all $1 \leq k \leq d$, and, clearly, match all relevant previous bounds (cf. [2, 5]).

To prove the upper bounds we use the embedding in one dimension higher already described above: We consider the convex hull $P = CH_{d+1}(\{P_1, P_2\})$, where P_1 and P_2 are embedded in $\{x_{d+1} = 0\}$ and $\{x_{d+1} = 1\}$, respectively. We first argue that, for the purposes of the worst-case upper bounds, it suffices to consider the case where P is simplicial, except possibly for its two facets P_1 and P_2. We concentrate on the set $\mathcal{F}$ of faces of P that are neither faces of P_1 nor faces of P_2. The reason that we focus on $\mathcal{F}$ is that there is a bijection between the k-faces of $\mathcal{F}$ and the $(k-1)$-faces of $P_1 \oplus P_2$, $1 \leq k \leq d$, and, thus, deriving upper bounds of the number of $(k-1)$-faces of $P_1 \oplus P_2$ reduces to deriving upper bounds for the number of k-faces of $\mathcal{F}$. We then proceed in a manner analogous to that used by McMullen [10] to prove the Upper Bound Theorem for polytopes. We consider the f-vector $\boldsymbol{f}(\mathcal{F})$ of $\mathcal{F}$, from this we define the h-vector $\boldsymbol{h}(\mathcal{F})$ of $\mathcal{F}$, and continue by:

(i) deriving Dehn-Sommerville-like equations for $\mathcal{F}$, expressed in terms of the elements of $\boldsymbol{h}(\mathcal{F})$ and the g-vectors of the boundary complexes of P_1 and P_2, and,

(ii) establishing a recurrence relation for the elements of $\boldsymbol{h}(\mathcal{F})$.

From the latter, we inductively compute upper bounds on the elements of $\boldsymbol{h}(\mathcal{F})$, which we combine with the Dehn-Sommerville-like equations for $\mathcal{F}$, to get refined upper bounds for the "left-most half" of the elements of $\boldsymbol{h}(\mathcal{F})$, i.e., for the values $h_k(\mathcal{F})$ with $k > \lfloor \frac{d+1}{2} \rfloor$. We then establish our upper bounds by computing $\boldsymbol{f}(\mathcal{F})$ from $\boldsymbol{h}(\mathcal{F})$.

To prove the lower bounds we distinguish between even and odd dimensions. In even dimensions $d \geq 2$, we show that the k-faces of the Minkowski sum of any two cyclic d-polytopes with n_1 and n_2 vertices, respectively, whose vertex sets are distinct, attain the upper bounds we have proved. In odd dimensions $d \geq 3$, the construction that establishes the tightness of our bounds is more intricate. We consider the $(d-1)$-dimensional moment curve $\boldsymbol{\gamma}(t) = (t, t^2, t^3, \ldots, t^{d-1})$, $t > 0$, and define two vertex sets V_1 and V_2 with n_1 and n_2 vertices on $\boldsymbol{\gamma}(t)$, respectively. We then embed

[1]In the sequel, all polytopes are considered to be convex.

V_1 (resp., V_2) on the hyperplane $\{x_2 = 0\}$ (resp., $\{x_1 = 0\}$) of $\mathbb{E}^d$ and perturb the x_2-coordinates (resp., x_1-coordinates) of the vertices in V_1 (resp., V_2), so that the polytope P_1 (resp., P_2) defined as the convex hull of the vertices in V_1 (resp., V_2) is full-dimensional. We then argue that by *appropriately choosing* the vertex sets V_1 and V_2, the number of k-faces of the Minkowski sum $P_1 \oplus P_2$ attains its maximum possible value. At a very high/qualitative level, the appropriate choice we refer to above amounts to choosing V_1 and V_2 so that the parameter values on $\boldsymbol{\gamma}(t)$ of the vertices in V_1 and V_2, lie within two disjoint intervals of $\mathbb{R}$ that are far away from each other.

The structure of the rest of the paper is as follows. In Section 2 we formally give various definitions, and define what we call *bineighborly* polytopal complexes and discuss some properties associated with them. In Section 3 we prove our upper bounds on the number of faces of the Minkowski sum of two polytopes. In Section 4 we describe our lower bound constructions and show that these constructions attain the upper bounds proved in Section 3. We conclude the paper with Section 5, where we summarize our results and state open problems and directions for future work.

2 Definitions and preliminaries

A *convex polytope*, or simply *polytope*, P in $\mathbb{E}^d$ is the convex hull of a finite set of points V in $\mathbb{E}^d$, called the *vertex set* of P. A *face* of P is the intersection of P with a hyperplane for which the polytope is contained in one of the two closed halfspaces delimited by the hyperplane. The dimension of a face of P is the dimension of its affine hull. A k-face of P is a k-dimensional face of P. We consider the polytope itself as a trivial d-dimensional face; all the other faces are called *proper* faces. We use the term *d-polytope* to refer to a polytope the trivial face of which is d-dimensional. For a d-polytope P, the 0-faces of P are its *vertices*, while the $(d-1)$-faces are called *facets*. For $0 \le k \le d$ we denote by $f_k(P)$ the number of k-faces of P. Note that every k-face F of P is also a k-polytope whose faces are all the faces of P contained in F. A k-simplex in $\mathbb{E}^d$, $k \le d$, is the convex hull of any $k+1$ affinely independent points in $\mathbb{E}^d$. A polytope is called *simplicial* if all its proper faces are simplices. Equivalently, P is simplicial if for every vertex v of P and every face $F \in P$, v does not belong to the affine hull of the vertices in $F \setminus \{v\}$.

A *polytopal complex* $\mathcal{C}$ is a finite collection of polytopes in $\mathbb{E}^d$ such that (i) $\emptyset \in \mathcal{C}$, (ii) if $P \in \mathcal{C}$ then all the faces of P are also in $\mathcal{C}$ and (iii) the intersection $P \cap Q$ for two polytopes P and Q in $\mathcal{C}$ is a face of both P and Q. The dimension $\dim(\mathcal{C})$ of $\mathcal{C}$ is the largest dimension of a polytope in $\mathcal{C}$. A polytopal complex is called *pure* if all its maximal (with respect to inclusion) faces have the same dimension. In this case the maximal faces are called the *facets* of $\mathcal{C}$. We use the term *d-complex* to refer to a polytopal complex whose maximal faces are d-dimensional (i.e., the dimension of $\mathcal{C}$ is d). A polytopal complex is simplicial if all its faces are simplices. Finally, a polytopal complex $\mathcal{C}'$ is called a *subcomplex* of a polytopal complex $\mathcal{C}$ if all faces of $\mathcal{C}'$ are also faces of $\mathcal{C}$. An important class of polytopal complexes arise from polytopes. More precisely, a d-polytope P, together with all its faces and the empty set, form a d-complex, denoted by $\mathcal{C}(P)$. The only maximal face of $\mathcal{C}(P)$, which is clearly the only facet of $\mathcal{C}(P)$, is the polytope P itself. Moreover, all proper faces of P form a pure $(d-1)$-complex, called the *boundary complex* $\mathcal{C}(\partial P)$, or simply ∂P, of P. The facets of ∂P are just the facets of P, and its dimension is, clearly, $\dim(\partial P) = \dim(P) - 1 = d - 1$. For a vertex v of P, the *star* of v, denoted by $\mathrm{star}(v, P)$, is the polytopal complex of all faces of P that contain v, and their faces. The *link* of v, denoted by $\mathrm{link}(v, P)$, is the subcomplex of $\mathrm{star}(v, P)$ consisting of all the faces of $\mathrm{star}(v, P)$ that do not contain v.

Definition 2.1. ([17, Remark 8.3]) *Let $\mathcal{C}$ be a pure simplicial polytopal d-complex. A shelling $\mathbb{S}(\mathcal{C})$ of $\mathcal{C}$ is a linear ordering $F_1, F_2, \ldots, F_s$ of the facets of $\mathcal{C}$ such that for all $1 < j \le s$ the intersection, $F_j \cap \left(\bigcup_{i=1}^{j-1} F_i\right)$, of the facet F_j with the previous facets is non-empty and pure $(d-1)$-dimensional.*
In other words, for every $i < j$ there exists some $\ell < j$ such that the intersection $F_i \cap F_j$ is contained in $F_\ell \cap F_j$, and such that $F_\ell \cap F_j$ is a facet of F_j.

Every polytopal complex that has a shelling is called *shellable*. In particular, the boundary complex of a polytope of always shellable (cf. [1]). Consider a pure shellable simplicial d-complex $\mathcal{C}$ and let $\mathbb{S}(\mathcal{C}) = \{F_1, \ldots, F_s\}$ be a shelling order of its facets. The *restriction* $R(F_j)$ of a facet F_j is the set of all vertices $v \in F_j$ such that $F_j \setminus \{v\}$ is contained in one of the earlier facets[2]. The main observation here is that when we construct $\mathcal{C}$ according to the shelling $\mathbb{S}(\mathcal{C})$, the new faces at the j-th step of the shelling are exactly the vertex sets G with $R(F_j) \subseteq G \subseteq F_j$ (cf. [17, Section 8.3]). Moreover, notice that $R(F_1) = \emptyset$ and $R(F_i) \neq R(F_j)$ for all $i \neq j$.

The f-vector $\boldsymbol{f}(P) = (f_{-1}(P), f_0(P), \ldots, f_{d-1}(P))$ of a d-polytope P (or its boundary complex ∂P) is defined as the $(d+1)$-dimensional vector consisting of the number, $f_k(P)$, of k-faces of P, $-1 \le k \le d-1$,

[2] For simplicial faces, we identify the face with its vertex set.

where $f_{-1}(P) = 1$ refers to the empty set. The h-vector $\boldsymbol{h}(P) = (h_0(P), h_1(P), \ldots, h_d(P))$ of a d-polytope P (or its boundary complex ∂P) is defined as the $(d+1)$-dimensional vector, where

$$h_k(P) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}(P), \tag{2.1}$$

$0 \leq k \leq d$. It is easy to verify from equations (2.1) that the elements of $\boldsymbol{f}(P)$ determine the elements of $\boldsymbol{h}(P)$ and vice versa.

For simplicial polytopes, the number $h_k(P)$ counts the number of facets of P in a shelling of ∂P, whose restriction has size k; this number is independent of the particular shelling chosen (cf. [17, Theorem 8.19]). Moreover, the elements of $\boldsymbol{f}(P)$ (or, equivalently, $\boldsymbol{h}(P)$) are not linearly independent; they satisfy the so called *Dehn-Sommerville equations*, which can be written in a very concise form as: $h_k(P) = h_{d-k}(P)$, $0 \leq k \leq d$. An important implication of the existence of the Dehn-Sommerville equations is that if we know the face numbers $f_k(P)$ for all $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - 1$, we can determine the remaining face numbers $f_k(P)$ for all $\lfloor \frac{d}{2} \rfloor \leq k \leq d-1$. Both the f-vector and h-vector of a simplicial d-polytope are related to the so called g-vector. For a simplicial d-polytope P its g-vector is the $(\lfloor \frac{d}{2} \rfloor + 1)$-dimensional vector $\boldsymbol{g}(P) = (g_0(P), g_1(P), \ldots, g_{\lfloor \frac{d}{2} \rfloor}(P))$, where $g_0(P) = 1$, and $g_k(P) = h_k(P) - h_{k-1}(P)$, $1 \leq k \leq \lfloor \frac{d}{2} \rfloor$ (see also [17, Section 8.6]). Using the convention that $h_{d+1}(P) = 0$, we can actually extend the definition of $g_k(P)$ for all $0 \leq k \leq d+1$, while using the Dehn-Sommerville equations for P yields: $g_{d+1-k}(P) = -g_k(P)$, $0 \leq k \leq d+1$. The Upper Bound Theorem for polytopes can equivalently be expressed in terms of the g-vector:

Corollary 2.1. ([17, Corollary 8.38]) *We consider simplicial d-polytopes P of fixed dimension d and fixed number of vertices $n = g_1(P) + d + 1$. $\boldsymbol{f}(P)$ has its componentwise maximum if and only if all the components of $\boldsymbol{g}(P)$ are maximal, with*

$$g_k(P) = \binom{g_1(P)+k-1}{k} = \binom{n-d-2+k}{k}. \tag{2.2}$$

Also, $f_{k-1}(P)$ is maximal if an only if $g_i(P)$ is maximal for all i with $i \leq \min\{k, \lfloor \frac{d}{2} \rfloor\}$.

2.1 Bineighborly polytopal complexes. Let $\mathcal{C}$ be a d-complex, and let V be the vertex set of $\mathcal{C}$. Let $\{V_1, V_2\}$ be a partition of V and define $\mathcal{C}_1$ (resp., $\mathcal{C}_2$) to be the subcomplex of $\mathcal{C}$ containing the faces of $\mathcal{C}$ whose vertices are vertices in V_1 (resp., V_2).

Definition 2.2. *Let $\mathcal{C}$ be a d-complex. We say that $\mathcal{C}$ is (k, V_1)-bineighborly if we can partition the vertex set V of $\mathcal{C}$ into two non-empty subsets V_1 and $V_2 = V \setminus V_1$ such that for every $\emptyset \subset S_j \subseteq V_j$, $j = 1, 2$, with $|S_1| + |S_2| \leq k$, the vertices of $S_1 \cup S_2$ define a face of $\mathcal{C}$ (of dimension $|S_1| + |S_2| - 1$).*

We introduce the notion of bineighborly polytopal complexes because they play an important role when considering the maximum complexity of the Minkowski sum of two d-polytopes P_1 and P_2. As we will see in the upcoming section, the number of $(k-1)$-faces of $P_1 \oplus P_2$ is maximal for all $1 \leq k \leq l$, $l \leq \lfloor \frac{d-1}{2} \rfloor$, if and only if the convex hull P of P_1 and P_2, when embedded in the hyperplanes $\{x_{d+1} = 0\}$ and $\{x_{d+1} = 1\}$ of $\mathbb{E}^{d+1}$, respectively, is $(l+1, V_1)$-bineighborly, where V_1 stands for the vertex set of P_1. Even more interestingly, in any odd dimension $d \geq 3$, the number of k-faces of $P_1 \oplus P_2$ is maximized for all $0 \leq k \leq d-1$, if and only if P is $(\lfloor \frac{d+1}{2} \rfloor, V_1)$-bineighborly.

A direct consequence of our definition is the following: suppose that $\mathcal{C}$ is a (l, V_1)-bineighborly polytopal complex, and let F be a k-face F of $\mathcal{C}$, $1 \leq k < l$, such that at least one vertex of F is in V_1 and at least one vertex of F is in V_2; then F is simplicial (i.e., F is a k-simplex). Another immediate consequence of Definition 2.2 is that a k-neighborly d-complex $\mathcal{C}$ is also (k, V')-bineighborly for every non-empty subset V' of its vertex set. It is also easy to see that if a d-complex $\mathcal{C}$ is (k, V_1)-bineighborly, $k \geq 2$, then $\mathcal{C}$ is $(k-1)$-neighborly. On the other hand, if $\mathcal{C}$ is (k, V_1)-bineighborly, while $\mathcal{C}_1$ and $\mathcal{C}_2$ are k-neighborly, then $\mathcal{C}$ is also k-neighborly. Let $\mathcal{B}$ be the set of faces of $\mathcal{C}$ that are not faces of either $\mathcal{C}_1$ or $\mathcal{C}_2$, and denote by n_j the cardinality of V_j, $j = 1, 2$. Then, for all $1 \leq k \leq d$,

$$f_{k-1}(\mathcal{B}) \leq \sum_{j=1}^{k-1} \binom{n_1}{j} \binom{n_2}{k-j} = \binom{n_1+n_2}{k} - \binom{n_1}{k} - \binom{n_2}{k},$$

where equality holds if and only if $\mathcal{C}$ is (k, V_1)-bineighborly (cf. [9]). As a final remark, notice that, if $f_{l-1}(\mathcal{B})$ is equal to its maximal value for some l, then $f_{k-1}(\mathcal{B})$ is equal to its maximal value for all k with $1 \leq k \leq l-1$.

3 Upper bounds

Let P_1 and P_2 be two d-polytopes in $\mathbb{E}^d$, with n_1 and n_2 vertices, respectively. Let us embed P_1 (resp., P_2) in the hyperplane Π_1 (resp., Π_2) of $\mathbb{E}^{d+1}$ with equation $\{x_{d+1} = 0\}$ (resp., $\{x_{d+1} = 1\}$), and let $\tilde{\Pi}$ be a hyperplane in $\mathbb{E}^{d+1}$ parallel and in-between Π_1 and Π_2. (see Fig. 1). Call P the convex hull $CH_{d+1}(\{P_1, P_2\})$, and let $\mathcal{F}$ be the set of proper faces of P having non-

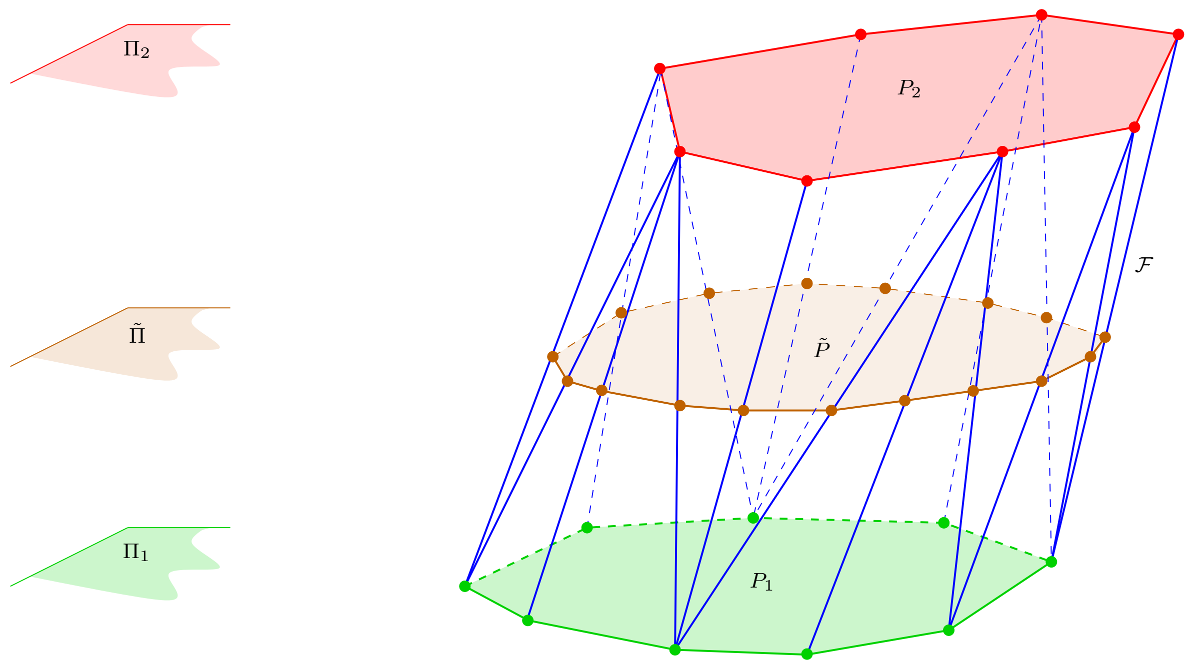

Figure 1: The d-polytopes P_1 and P_2 are embedded in the hyperplanes $\Pi_1 = \{x_{d+1} = 0\}$ and $\Pi_2 = \{x_{d+1} = 0\}$ of $\mathbb{E}^{d+1}$. The polytope $\tilde{P}$ is the intersection of $CH_{d+1}(\{P_1, P_2\})$ with the hyperplane $\tilde{\Pi} = \{x_{d+1} = \lambda\}$.

empty intersection with $\tilde{\Pi}$. Note that $\tilde{P} = P \cap \tilde{\Pi}$ is a d-polytope, which is, in general, non-simplicial, and whose proper non-trivial faces are intersections of the form $F \cap \tilde{\Pi}$ where $F \in \mathcal{F}$. As we have already observed in the introductory section, $\tilde{P}$ is combinatorially equivalent to the Minkowski sum $P_1 \oplus P_2$. Furthermore, $f_{k-1}(P_1 \oplus P_2) = f_{k-1}(\tilde{P}) = f_k(\mathcal{F})$, $1 \leq k \leq d$. The rest of this section is devoted to deriving upper bounds for $f_k(\mathcal{F})$, which also become upper bounds for $f_{k-1}(P_1 \oplus P_2)$.

Karavelas and Tzanaki [8, Lemma 2] have shown that the vertices of P_1 and P_2 can be perturbed in such a way that:

(i) the vertices of P_1' and P_2' remain in Π_1 and Π_2, respectively, and both P_1' and P_2' are simplicial;

(ii) $P' = CH_{d+1}(\{P_1', P_2'\})$ is also simplicial, except possibly the facets P_1' and P_2';

(iii) the number of vertices of P_1' and P_2' is the same as the number of vertices of P_1 and P_2, respectively, whereas $f_k(P) \leq f_k(P')$ for all $k \geq 1$,

where P_1' and P_2' are the polytopes in Π_1 and Π_2 we get after perturbing the vertices of P_1 and P_2, respectively. It, thus, suffices to consider the case where all three P_1, P_2, and P are simplicial, except possibly the facets P_1 and P_2 of P.

Let $\mathcal{K}$ be the polytopal complex whose faces are all the faces of $\mathcal{F}$, as well as the faces of P that are subfaces of faces in $\mathcal{F}$. It is easy to see that the d-faces of $\mathcal{K}$ are exactly the d-faces of $\mathcal{F}$, and, thus, $\mathcal{K}$ is a pure simplicial d-complex, with the d-faces of $\mathcal{F}$ being the facets of $\mathcal{K}$. Moreover, the set of k-faces of $\mathcal{K}$ is the disjoint union of the sets of k-faces of $\mathcal{F}$, ∂P_1 and ∂P_2. This implies, for $-1 \leq k \leq d$:

$$f_k(\mathcal{K}) = f_k(\mathcal{F}) + f_k(\partial P_1) + f_k(\partial P_2),$$

where $f_d(\partial P_j) = 0$, $j = 1, 2$, and conventionally we set $f_{-1}(\mathcal{F}) = -1$.

Let y_1 (resp., y_2) be a point below Π_1 (resp., above Π_2), such that the vertices of P_1 (resp., P_2) are the only vertices of P visible from y_1 (resp., y_2) (see Fig. 2). Let Q be the $(d+1)$-polytope that is the convex hull of the vertices of P_1, P_2, y_1 and y_2. Observe that the faces of ∂P (and thus all faces of $\mathcal{F}$), except for the facets P_1 and P_2 of ∂P, are all faces of the boundary complex ∂Q. The faces of ∂Q that are not faces of $\mathcal{F}$ are the faces in the star $\mathcal{S}_1$ of y_1 (resp., the star $\mathcal{S}_2$ of y_2), while the boundary complex ∂P_1 of P_1 (resp., ∂P_2 of P_2) is nothing but $\text{link}(y_1, \partial Q)$ (resp., $\text{link}(y_2, \partial Q)$).

It is easy to realize that the set of k-faces of ∂Q is the disjoint union of the k-faces of $\mathcal{F}$, $\mathcal{S}_1$ and $\mathcal{S}_2$. This implies that, for $-1 \leq k \leq d$:

$$f_k(\partial Q) = f_k(\mathcal{F}) + f_k(\mathcal{S}_1) + f_k(\mathcal{S}_2). \tag{3.3}$$

The k-faces of ∂Q in $\mathcal{S}_j$ are either k-faces of ∂P_j or k-faces defined by y_j and a $(k-1)$-face of ∂P_j. Hence, we

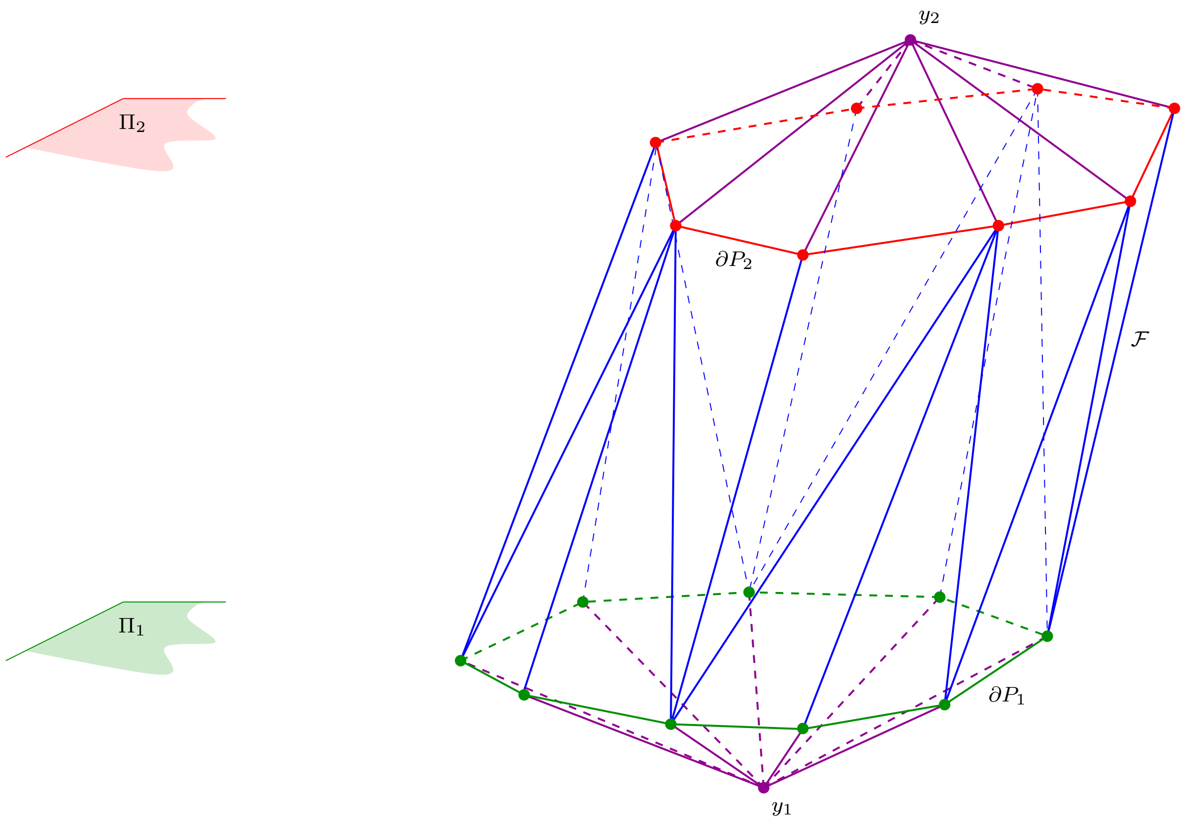

Figure 2: The polytope Q is created by adding two vertices y_1 and y_2. The vertex y_1 (resp., y_2) is below P_1 (resp., above P_2), and is visible by the vertices of P_1 (resp., P_2) only.

have, for $j = 1, 2$, and $0 \leq k \leq d$:

$$f_k(\mathcal{S}_j) = f_k(\partial P_j) + f_{k-1}(\partial P_j), \tag{3.4}$$

where $f_{-1}(\partial P_j) = 1$ and $f_d(\partial P_j) = 0$. Combining relations (3.3) and (3.4), we get, for $0 \leq k \leq d$:

$$\begin{aligned} f_k(\partial Q) = f_k(\mathcal{F}) + f_k(\partial P_1) + f_{k-1}(\partial P_1) \\ + f_k(\partial P_2) + f_{k-1}(\partial P_2). \end{aligned} \tag{3.5}$$

We call $\mathcal{K}_j$, $j = 1, 2$, the subcomplex of ∂Q consisting of either faces of $\mathcal{K}$ or faces of $\mathcal{S}_j$. $\mathcal{K}_j$ is a pure simplicial d-complex the facets of which are either facets in the star $\mathcal{S}_j$ of y_j or facets of $\mathcal{K}$. Furthermore, $\mathcal{K}_j$ is shellable. To see this first notice that ∂Q is shellable (Q is a polytope). Consider a line shelling $F_1, F_2, \ldots, F_s$ of ∂Q that shells $\text{star}(y_2, \partial Q)$ last, and let $F_{\lambda+1}, F_{\lambda+2}, \ldots, F_s$ be the facets of ∂Q that correspond to $\mathcal{S}_2$. Trivially, the subcomplex of ∂Q, the facets of which are $F_1, F_2, \ldots, F_\lambda$, is shellable; however, this subcomplex is nothing but $\mathcal{K}_1$. The argument for $\mathcal{K}_2$ is analogous.

Notice that Q is a simplicial $(d+1)$-polytope, while $\mathcal{K}_1$ and $\mathcal{K}_2$ are simplicial d-complexes; hence, for $0 \leq k \leq d+1$:

$$h_k(\mathcal{Y}) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}(\mathcal{Y}),$$

where $\mathcal{Y}$ stands for either ∂Q, $\mathcal{K}_1$ or $\mathcal{K}_2$. We define the f-vector of $\mathcal{F}$ to be the $(d+2)$-vector $\boldsymbol{f}(\mathcal{F}) = (f_{-1}(\mathcal{F}), f_0(\mathcal{F}), \ldots, f_d(\mathcal{F}))$ (recall that $f_{-1}(\mathcal{F}) = -1$); from this we can also define the $(d+2)$-vector $\boldsymbol{h}(\mathcal{F}) = (h_0(\mathcal{F}), h_1(\mathcal{F}), \ldots, h_{d+1}(\mathcal{F}))$, where

$$h_k(\mathcal{F}) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}(\mathcal{F}), \tag{3.6}$$

$0 \leq k \leq d+1$. We call this vector the h-vector of $\mathcal{F}$. As for polytopal complexes and polytopes, the f-vector of $\mathcal{F}$ defines the h-vector of $\mathcal{F}$ and vice versa. In particular, solving the defining equations of the elements of $\boldsymbol{h}(\mathcal{F})$ in terms of the elements of $\boldsymbol{f}(\mathcal{F})$ we get, for $0 \leq k \leq d+1$:

$$f_{k-1}(\mathcal{F}) = \sum_{i=0}^{d+1} \binom{d+1-i}{k-i} h_i(\mathcal{F}). \tag{3.7}$$

The next lemma associates the elements of $\boldsymbol{h}(\partial Q)$, $\boldsymbol{h}(\mathcal{K}_1)$, $\boldsymbol{h}(\mathcal{K}_2)$, $\boldsymbol{h}(\mathcal{F})$, $\boldsymbol{h}(\partial P_1)$ and $\boldsymbol{h}(\partial P_2)$. The last relation in the lemma can be thought of as the analogue of the Dehn-Sommerville equations for $\mathcal{F}$.

LEMMA 3.1. *For all $0 \le k \le d+1$, and $j = 1, 2$, we have:*

$$h_k(\partial Q) = h_k(\mathcal{F}) + h_k(\partial P_1) + h_k(\partial P_2), \tag{3.8}$$

$$h_k(\mathcal{K}_j) = h_k(\mathcal{F}) + h_k(\partial P_j) + g_k(\partial P_{3-j}), \tag{3.9}$$

$$h_{d+1-k}(\mathcal{F}) = h_k(\mathcal{F}) + g_k(\partial P_1) + g_k(\partial P_2). \tag{3.10}$$

Proof. Let $\mathcal{Y}$ denote either $\mathcal{F}$ or a pure simplicial subcomplex of ∂Q. We define the operator $\mathcal{S}_k(\cdot;\delta,\nu)$ whose action on $\mathcal{Y}$ is as follows:

$$\mathcal{S}_k(\mathcal{Y};\delta,\nu) = \sum_{i=1}^{\delta}(-1)^{k-i}\tbinom{\delta-i}{\delta-k}f_{i-\nu}(\mathcal{Y}).$$

It is easy to verify (cf. [9]) that if $\mathcal{Y}$ is δ-dimensional (this includes the case $\mathcal{Y} \equiv \mathcal{F}$), then

$$\mathcal{S}_k(\mathcal{Y};\delta,1) = h_k(\mathcal{Y}) - (-1)^k\tbinom{\delta}{\delta-k}f_{-1}(\mathcal{Y}),$$

while if $\mathcal{Y}$ is $(\delta-1)$-dimensional, then

$$\mathcal{S}_k(\mathcal{Y};\delta,1) = h_k(\mathcal{Y}) - h_{k-1}(\mathcal{Y}) - (-1)^k\tbinom{\delta}{\delta-k}f_{-1}(\mathcal{Y}),$$

and

$$\mathcal{S}_k(\mathcal{Y};\delta,2) = h_{k-1}(\mathcal{Y}).$$

Applying the operator $\mathcal{S}_k(\cdot;d+1,1)$ to ∂Q and using relation (3.5) we get:

$$\begin{aligned}\mathcal{S}_k(\partial Q;d+1,1) = \mathcal{S}_k(\mathcal{F};d+1,1)\\ + \mathcal{S}_k(\partial P_1;d+1,1)\\ + \mathcal{S}_k(\partial P_1;d+1,2)\\ + \mathcal{S}_k(\partial P_2;d+1,1)\\ + \mathcal{S}_k(\partial P_2;d+1,2),\end{aligned}$$

which, given that $f_{-1}(\partial Q) = f_{-1}(\partial P_1) = f_{-1}(\partial P_2) = 1$ and $f_{-1}(\mathcal{F}) = -1$, simplifies to relation (3.8).

Recall that the set of k-faces of $\mathcal{K}$ is the disjoint union of the k-faces of $\mathcal{F}$, the k-faces of ∂P_1, and the k-faces of ∂P_2. Moreover, the k-faces of $\mathcal{K}_j$, $j = 1, 2$, are either k-faces of $\mathcal{K}$ or k-faces of the star $\mathcal{S}_j$ of y_j that contain y_j. The latter faces are in one-to-one correspondence with the $(k-1)$-faces of ∂P_j, i.e., we get, for $0 \le k \le d$ and $j = 1, 2$:

$$f_k(\mathcal{K}_j) = f_k(\mathcal{K}) + f_{k-1}(\partial P_j),$$

which finaly gives, for $0 \le k \le d$ and $j = 1, 2$:

$$f_k(\mathcal{K}_j) = f_k(\mathcal{F}) + f_k(\partial P_1) + f_k(\partial P_2) + f_{k-1}(\partial P_j).$$

Once again, applying the operator $\mathcal{S}_k(\cdot;d+1,1)$ to the expression for $f_k(\mathcal{K}_j)$, we get relation (3.9).

We end the proof of this lemma by considering relations (3.10). Using (3.8), the Dehn-Sommerville equations for ∂Q can be rewritten as

$$\begin{aligned}h_{d+1-k}(\mathcal{F}) + h_{d+1-k}(\partial P_1) + h_{d+1-k}(\partial P_2)\\ = h_k(\mathcal{F}) + h_k(\partial P_1) + h_k(\partial P_2).\end{aligned}$$

Using the Dehn-Sommerville equations for P_j: $h_{d-k}(\partial P_j) = h_k(\partial P_j)$, $0 \le k \le d$, $j = 1, 2$, in the above relations, we get, for $0 \le k \le d+1$:

$$\begin{aligned}h_{d+1-k}(\mathcal{F}) + h_{k-1}(\partial P_1) + h_{k-1}(\partial P_2)\\ = h_k(\mathcal{F}) + h_k(\partial P_1) + h_k(\partial P_2),\end{aligned}$$

which finally give relations (3.10). □

Recall that the main goal in this section is to derive upper bounds for the elements of $\boldsymbol{h}(\mathcal{F})$. The most critical step toward this goal is a recurrence inequality for the elements of $\boldsymbol{h}(\mathcal{F})$ described in the following lemma.

LEMMA 3.2. *For all $0 \le k \le d$,*

$$\begin{aligned}h_{k+1}(\mathcal{F}) \le \tfrac{n_1+n_2-d-1+k}{k+1}\,h_k(\mathcal{F})\\ + \tfrac{n_1}{k+1}\,g_k(\partial P_2) + \tfrac{n_2}{k+1}\,g_k(\partial P_1).\end{aligned} \tag{3.11}$$

Proof. Let us denote by V the vertex set of ∂Q, and by V_j the vertex set of ∂P_j, $j = 1, 2$. Let $\mathcal{Y}/v$ be a shorthand for $\text{link}(v, \mathcal{Y})$, where v is a vertex of $\mathcal{Y}$, and $\mathcal{Y}$ stands for either ∂Q, $\mathcal{K}_1$, $\mathcal{K}_2$, ∂P_1 or ∂P_2. Then (cf. [10]), for $0 \le k \le d$, we have:

$$(k+1)h_{k+1}(\partial Q) + (d+1-k)h_k(\partial Q) = \sum_{v \in V} h_k(\partial Q/v), \tag{3.12}$$

while, for $0 \le k \le d-1$ and $j = 1, 2$, we have:

$$(k+1)h_{k+1}(\partial P_j) + (d-k)h_k(\partial P_j) = \sum_{v \in V_j} h_k(\partial P_j/v). \tag{3.13}$$

Recall that the link of y_j in ∂Q is ∂P_j, and observe that the link of $v \in V_j$ in ∂Q coincides with the link of v in $\mathcal{K}_j$. Expressing $h_k(\partial Q)$ in terms of $h_k(\mathcal{F})$ and $h_k(\partial P_j)$, $j = 1, 2$, in conjuction with relations (3.12) and (3.13), and noting that:

$$h_k(\partial Q/y_j) = h_k(\partial P_j),$$

and

$$\sum_{v \in V_1 \cup V_2} h_k(\partial Q/v) = \sum_{v \in V_1} h_k(\mathcal{K}_1/v) + \sum_{v \in V_2} h_k(\mathcal{K}_2/v),$$

we arrive at the following equality:

$$(k+1)h_{k+1}(\mathcal{F}) + (d+1-k)h_k(\mathcal{F}) = \sum_{j=1}^{2}\sum_{v\in V_j}[h_k(\mathcal{K}_j/v) - h_k(\partial P_j/v)].$$

Let us now consider a vertex $v \in V_1$, and a shelling $\mathbb{S}(\partial Q)$ of ∂Q that shells $\text{star}(v, \partial Q)$ first and $\text{star}(y_2, \partial Q)$ last (such a shelling does exit since v and y_2 are not visible to each other). Notice that $\mathbb{S}(\partial Q)$ induces a shelling $\mathbb{S}(\mathcal{K}_1)$ for $\mathcal{K}_1$ that shells $\text{star}(v, \mathcal{K}_1)$ first. On the other hand, $\mathbb{S}(\mathcal{K}_1)$ also induces (cf. [17, Lemma 8.7]):

(i) a shelling $\mathbb{S}(\mathcal{K}_1/v)$ for $\mathcal{K}_1/v$, and
(ii) a shelling $\mathbb{S}(\partial P_1)$ for ∂P_1 that shells $\text{star}(v, \partial P_1)$ first (recall that $\partial P_1 \equiv \partial Q/y_1 \equiv \mathcal{K}_1/y_1$).

Finally, $\mathbb{S}(\partial P_1)$ induces a shelling $\mathbb{S}(\partial P_1/v)$ for $\partial P_1/v$. The interested reader may refer to Figs. 5–10 at the end of this paper, where we show a shelling $\mathbb{S}(\mathcal{K}_1)$ of $\mathcal{K}_1$ that shells $\text{star}(v, \mathcal{K}_1)$ first, along with the induced shellings $\mathbb{S}(\mathcal{K}_1/v)$ and $\mathbb{S}(\partial P_1)$. In particular, Figs. 5–7 show the step-by-step construction of $\mathcal{K}_1$ from $\mathbb{S}(\mathcal{K}_1)$. Fig. 8 shows the step-by-step construction of $\text{star}(v, \mathcal{K}_1)$ from $\mathbb{S}(\mathcal{K}_1)$, as well as the corresponding induced construction of $\mathcal{K}_1/v$ from the induced shelling $\mathbb{S}(\mathcal{K}_1/v)$. Finally, Figs. 9 and 10 show the step-by-step construction of ∂P_1 from the shelling $\mathbb{S}(\partial P_1)$ induced by $\mathbb{S}(\mathcal{K}_1)$, along with the corresponding steps of the construction of $\mathcal{K}_1$ from $\mathbb{S}(\mathcal{K}_1)$, i.e., we only depict the steps of $\mathbb{S}(\mathcal{K}_1)$ that induce facets of $\mathbb{S}(\partial P_1)$.

Let F be a facet in $\mathbb{S}(\mathcal{K}_1)$. If F induces a facet for $\mathbb{S}(\mathcal{K}_1/v)$, denote by F/v this facet of $\mathcal{K}_1/v$. Similarly, if F induces a facet for $\mathbb{S}(\partial P_1)$, call F_1 this facet of ∂P_1. Finally, if F_1 induces a facet for $\mathbb{S}(\partial P_1/v)$, let F_1/v be this facet of $\partial P_1/v$. Let $G \subseteq F$, $G/v \subseteq F/v$, $G_1 \subseteq F_1$ and $G_1/v \subseteq F_1/v$ be the minimal new faces associated with F, F/v, F_1 and F_1/v in the corresponding shellings, let λ be the cardinality of G, and observe that $F_1 = F \cap \partial P_1$, $F_1/v = (F/v) \cap \partial P_1$, $G_1 = G \cap \partial P_1$ and $G_1/v = (G/v) \cap \partial P_1$. As long as we shell $\text{star}(v, \mathcal{K}_1)$, G induces G/v, and, in fact, the faces G and G/v coincide (see also Fig. 8). Similarly, as long as we shell $\text{star}(v, \partial P_1)$, G_1 induces G_1/v, and, in fact, the faces G_1 and G_1/v coincide. Hence, as long as we shell $\text{star}(v, \mathcal{K}_1)$ (i.e., as long as $v \in F$), we have $h_k(\mathcal{K}_1/v) = h_k(\mathcal{K}_1)$ and $h_k(\partial P_1/v) = h_k(\partial P_1)$, for all $k \geq 0$, and, thus, $h_k(\mathcal{K}_1/v) - h_k(\partial P_1/v) = h_k(\mathcal{K}_1) - h_k(\partial P_1)$, for all $k \geq 0$. After the shelling $\mathbb{S}(\mathcal{K}_1)$ has left $\text{star}(v, \mathcal{K}_1)$, there are no more facets in $\mathbb{S}(\mathcal{K}_1/v)$ and $\mathbb{S}(\partial P_1/v)$. This implies that, after $\mathbb{S}(\mathcal{K}_1)$ has left $\text{star}(v, \mathcal{K}_1)$ (i.e., v is not a vertex of F anymore), the values of $h_k(\mathcal{K}_1/v)$ and $h_k(\partial P_1/v)$ remain unchanged for all $k \geq 0$. However, the values of $h_k(\mathcal{K}_1)$ and $h_k(\partial P_1)$ may increase for some k. More precisely, if F does not induce a facet for $\mathbb{S}(\partial P_1)$, then $h_\lambda(\mathcal{K}_1)$ is increased by one, $h_k(\mathcal{K}_1)$ does not change for $k \neq \lambda$, while $h_k(\partial P_1)$ remains unchanged for all $k \geq 0$. Thus, $h_\lambda(\mathcal{K}_1/v) - h_\lambda(\partial P_1/v) < h_\lambda(\mathcal{K}_1) - h_\lambda(\partial P_1)$, while $h_k(\mathcal{K}_1/v) - h_k(\partial P_1/v) \leq h_k(\mathcal{K}_1) - h_k(\partial P_1)$, for all $k \neq \lambda$. If, however, F induces F_1, then the minimal new face G_1 in $\mathbb{S}(\partial P_1)$ due to F_1 coincides with G (see also Figs. 9 and 10). Therefore, in this case, both $h_\lambda(\mathcal{K}_1)$ and $h_\lambda(\partial P_1)$ are increased by one, while $h_k(\mathcal{K}_1)$ and $h_k(\partial P_1)$ remain unchanged for $k \neq \lambda$. Summarizing, for all $v \in V_1$, and for all $0 \leq k \leq d$, we have:

$$h_k(\mathcal{K}_1/v) - h_k(\partial P_1/v) \leq h_k(\mathcal{K}_1) - h_k(\partial P_1).$$

The argument for $v \in V_2$ is analogous, which means that for all $v \in V_2$, and for all $0 \leq k \leq d$:

$$h_k(\mathcal{K}_2/v) - h_k(\partial P_2/v) \leq h_k(\mathcal{K}_2) - h_k(\partial P_2).$$

Using relations (3.9), we, thus, get for every vertex $v \in V_j$, $j = 1, 2$, and for all $0 \leq k \leq d$:

$$\sum_{v\in V_j}[h_k(\mathcal{K}_j/v) - h_k(\partial P_j/v)] \leq \sum_{v\in V_j}[h_k(\mathcal{K}_j) - h_k(\partial P_j)] = n_j[h_k(\mathcal{F}) + g_k(\partial P_{3-j})].$$

We thus arrive at the following inequality, for $0 \leq k \leq d$:

$$(k+1)h_{k+1}(\mathcal{F}) + (d+1-k)h_k(\mathcal{F}) \leq (n_1+n_2)h_k(\mathcal{F}) + n_1 g_k(\partial P_2) + n_2 g_k(\partial P_1),$$

which gives the recurrence inequality in the statement of the lemma. □

Using the recurrence relation from Lemma 3.2 we get the following bounds on the elements of $\boldsymbol{h}(\mathcal{F})$ (a detailed proof of the lemma may be found in [9]):

LEMMA 3.3. *For all* $0 \leq k \leq d+1$,

$$h_k(\mathcal{F}) \leq \binom{n_1+n_2-d-2+k}{k} - \binom{n_1-d-2+k}{k} - \binom{n_2-d-2+k}{k}.$$

Equality holds for all k *with* $0 \leq k \leq l$ *if and only if* $l \leq \lfloor\frac{d+1}{2}\rfloor$ *and* P *is* (l, V_1)*-bineighborly.*

Sketch of proof. The upper bound holds (as equality) for $k = 0$. For $k \geq 1$, we use induction on k in conjunction with the upper bounds for the elements of the g-vector of a polytope (cf. Corollary 2.1).

Regarding the equality claim, the claim for $l = 0$ is obvious. Consider now that $l \geq 1$. Suppose first that P is (l, V_1)-bineighborly. Then, for all i with $0 \leq i \leq l$ we have:

$$f_{i-1}(\mathcal{F}) = \binom{n_1+n_2}{i} - \binom{n_1}{i} - \binom{n_2}{i}.$$

Substituting $f_{i-1}(\mathcal{F})$ in the defining equations (3.6) for $\boldsymbol{h}(\mathcal{F})$, and after some calculations, we conclude that

$$h_k(\mathcal{F}) = \binom{n_1+n_2-d-2+k}{k} - \binom{n_1-d-2+k}{k} - \binom{n_2-d-2+k}{k},$$

for all $0 \le k \le l$.

Suppose now that the inequality for $h_k(\mathcal{F})$ holds as equality for all $0 \le k \le l$. Substituting $h_i(\mathcal{F})$, $0 \le i \le l$, in (3.7), and after some calculations, we get:

$$f_{l-1}(\mathcal{F}) = \binom{n_1+n_2}{l} - \binom{n_1}{l} - \binom{n_2}{l}.$$

Hence, P is (l, V_1)-bineighborly. □

Using the Dehn-Sommerville-like relations (3.10), along with the bounds from the previous lemma, we derive alternative bounds for $h_k(\mathcal{F})$, which are of interest, since they refine the bounds for $h_k(\mathcal{F})$ from Lemma 3.3 for large values of k, namely for $k > \lfloor \frac{d+1}{2} \rfloor$.

LEMMA 3.4. *For all* $0 \le k \le d+1$,

$$h_{d+1-k}(\mathcal{F}) \le \binom{n_1+n_2-d-2+k}{k}.$$

Equality holds for all k *with* $0 \le k \le l$ *if and only if* $l \le \lfloor \frac{d}{2} \rfloor$ *and* P *is* l*-neighborly.*

Proof. The upper bound claim is a direct consequence of the Dehn-Sommerville-like relations (3.10) for $\boldsymbol{h}(\mathcal{F})$, the upper bounds from Lemma 3.3, and the Upper Bound Theorem for polytopes as stated in Corollary 2.1. Furthermore, the inequality in the statement of the lemma holds as equality for all $0 \le k \le l$, where $l \le \lfloor \frac{d}{2} \rfloor$, if and only if the following two conditions hold:

(i) Inequalities in Lemma 3.3 hold as equalities for all $0 \le k \le l \le \lfloor \frac{d}{2} \rfloor$.

(ii) For $j = 1, 2$, and for all $0 \le k \le l \le \lfloor \frac{d}{2} \rfloor$, we have $g_k(\partial P_j) = \binom{n_j-d-2+k}{k}$.

The first condition holds true if and only if P is (l, V_1)-bineighborly, while the second condition holds true if and only if P_j, $j = 1, 2$, is l-neighborly, i.e., conditions (i) and (ii) hold true if and only if P is l-neighborly. □

We are now ready to compute upper bounds for the face numbers of $\mathcal{F}$. Writing $\boldsymbol{f}(\mathcal{F})$ in terms of $\boldsymbol{h}(\mathcal{F})$ (cf. eq. (3.7)), in conjunction with the bounds on the elements of $\boldsymbol{h}(\mathcal{F})$ from Lemmas 3.3 and 3.4, we get, for $0 \le k \le d+1$:

$$\begin{aligned}
f_{k-1}(\mathcal{F}) &= \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} h_i(\mathcal{F}) + \sum_{i=\lfloor \frac{d+1}{2} \rfloor+1}^{d+1} \binom{d+1-i}{k-i} h_i(\mathcal{F}) \\
&= \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} h_i(\mathcal{F}) + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{i}{k-d-1+i} h_{d+1-i}(\mathcal{F}) \\
&\le \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \binom{n_1+n_2-d-2+i}{i} \\
&\quad - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \sum_{j=1}^{2} \binom{n_j-d-2+i}{i} \\
&\quad + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{i}{k-d-1+i} \binom{n_1+n_2-d-2+i}{i} \\
&= \sum_{i=0}^{\frac{d+1}{2}}{}^{*} \left(\binom{d+1-i}{k-i} + \binom{i}{k-d-1+i} \right) \binom{n_1+n_2-d-2+i}{i} \\
&\quad - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \sum_{j=1}^{2} \binom{n_j-d-2+i}{i} \\
&= f_{k-1}(C_{d+1}(n_1+n_2)) \\
&\quad - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k-i} \sum_{j=1}^{2} \binom{n_j-d-2+i}{i},
\end{aligned}$$

where $C_d(n)$ stands for the cyclic d-polytope with n vertices, and $\sum_{i=0}^{\frac{\delta}{2}}{}^{*} T_i$ denotes the sum of the elements $T_0, T_1, \ldots, T_{\lfloor \frac{\delta}{2} \rfloor}$ where the last term is halved if δ is even. Since for all $1 \le k \le d$, $f_{k-1}(P_1 \oplus P_2) = f_k(\mathcal{F})$, we arrive at the central theorem of this paper, stating upper bounds for the face numbers of the Minkowski sum of two d-polytopes (the equality claims follow from the equality claims in Lemmas 3.3 and 3.4).

THEOREM 3.1. *Let* P_1 *and* P_2 *be two* d*-polytopes in* $\mathbb{E}^d$, $d \ge 2$, *with* $n_1 \ge d+1$ *and* $n_2 \ge d+1$ *vertices, respectively. Let also* P *be the convex hull in* $\mathbb{E}^{d+1}$ *of* P_1 *and* P_2 *embedded in the hyperplanes* $\{x_{d+1} = 0\}$ *and* $\{x_{d+1} = 1\}$ *of* $\mathbb{E}^{d+1}$, *respectively. Then, for* $1 \le k \le d$, *we have:*

$$\begin{aligned}
f_{k-1}(P_1 \oplus P_2) &\le f_k(C_{d+1}(n_1+n_2)) \\
&\quad - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k+1-i} \left(\binom{n_1-d-2+i}{i} + \binom{n_2-d-2+i}{i} \right).
\end{aligned}$$

Furthermore:

(i) Equality holds for all $1 \le k \le l$ *if an only if* $l \le \lfloor \frac{d-1}{2} \rfloor$ *and* P *is* $(l+1, V_1)$*-bineighborly.*

(ii) For $d \ge 2$ *even, equality holds for all* $1 \le k \le d$ *if an only if* P *is* $\lfloor \frac{d}{2} \rfloor$*-neighborly.*

(iii) For $d \ge 3$ *odd, equality holds for all* $1 \le k \le d$ *if an only if* P *is* $(\lfloor \frac{d+1}{2} \rfloor, V_1)$*-bineighborly.*

4 Lower bounds

In this section we show that the upper bounds given in Theorem 3.1 are tight.

Fukuda and Weibel [5] have proved tight bounds for $f_k(P_1 \oplus P_2 \oplus \ldots \oplus P_r)$ for $d \geq 4$, $r \leq \lfloor \frac{d}{2} \rfloor$, and for all k with $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - r$. These upper bounds are attained, for $r = 2$, when considering two cyclic d-polytopes P_1 and P_2, with n_1 and n_2 vertices, respectively, with disjoint vertex sets. This construction gives, in fact, tight bounds on the number of k-faces of the Minkowski sum for all $0 \leq k \leq d-1$, when d is even. As in Section 3, embed P_1 and P_2 in the hyperplanes $\{x_{d+1} = 0\}$ and $\{x_{d+1} = 1\}$ of $\mathbb{E}^{d+1}$, let $P = CH_{d+1}(\{P_1, P_2\})$ and, call $\mathcal{F}$ the set of proper faces of P that are neither faces of P_1 nor faces of P_2. We then have

$$\begin{aligned} f_{\lfloor \frac{d}{2} \rfloor - 1}(\mathcal{F}) &= f_{\lfloor \frac{d}{2} \rfloor - 2}(P_1 \oplus P_2) \\ &= \sum_{j=1}^{\lfloor \frac{d}{2} \rfloor - 1} \tbinom{n_1}{j}\tbinom{n_2}{\lfloor \frac{d}{2} \rfloor - j} \\ &= \tbinom{n_1+n_2}{\lfloor \frac{d}{2} \rfloor} - \tbinom{n_1}{\lfloor \frac{d}{2} \rfloor} - \tbinom{n_2}{\lfloor \frac{d}{2} \rfloor}, \end{aligned}$$

which implies that P is $(\lfloor \frac{d}{2} \rfloor, V_1)$-bineighborly. Since P_1 and P_2 are $\lfloor \frac{d}{2} \rfloor$-neighborly, we further conclude that P is $\lfloor \frac{d}{2} \rfloor$-neighborly, which, by Theorem 3.1, suggests that $f_k(P_1 \oplus P_2)$ is equal to its maximum value for all $0 \leq k \leq d-1$.

If $d \geq 5$ and d is odd, however, the construction in [5] gives tight bounds for $f_k(P_1 \oplus P_2)$ for all $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - 2$, which, according to Theorem 3.1, are not sufficient to establish that the bounds are tight for the face numbers of all dimensions. To establish the tightness of the bounds in Theorem 3.1 for all k, we need to construct two d-polytopes P_1 and P_2, with n_1 and n_2 vertices, respectively, such that $f_{\lfloor \frac{d}{2} \rfloor}(\mathcal{F}) = f_{\lfloor \frac{d}{2} \rfloor - 1}(P_1 \oplus P_2) = \binom{n_1+n_2}{\lfloor \frac{d+1}{2} \rfloor} - \binom{n_1}{\lfloor \frac{d+1}{2} \rfloor} - \binom{n_2}{\lfloor \frac{d+1}{2} \rfloor}$, i.e., we need to construct P_1 and P_2 such that P is $(\lfloor \frac{d+1}{2} \rfloor, V_1)$-bineighborly. We start off with a technical lemma (its proof may be found in [9]).

Lemma 4.1. *Let $k \geq 2$ and $\ell \geq 2$, such that $k + \ell$ is odd, and let $D_{k,\ell}(\tau)$ be the $(k+\ell) \times (k+\ell)$ determinant:*

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ x_1\tau & x_2\tau & \cdots & x_k\tau & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & y_1 & y_2 & \cdots & y_\ell \\ x_1^2\tau^2 & x_2^2\tau^2 & \cdots & x_k^2\tau^2 & y_1^2 & y_2^2 & \cdots & y_\ell^2 \\ x_1^3\tau^3 & x_2^3\tau^3 & \cdots & x_k^3\tau^3 & y_1^3 & y_2^3 & \cdots & y_\ell^3 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_1^m\tau^m & x_2^m\tau^m & \cdots & x_k^m\tau^m & y_1^m & y_2^m & \cdots & y_\ell^m \end{vmatrix},$$

where $m = k + \ell - 3$, $0 < x_1 < x_2 < \ldots < x_k$, $0 < y_1 < y_2 < \ldots < y_\ell$, and $\tau > 0$. Then, there exists some $\tau_0 > 0$ such that for all $\tau \in (0, \tau_0)$, the determinant $D_{k,\ell}(\tau)$ is strictly positive.

In what follows $d \geq 3$ and d is odd. We denote by $\boldsymbol{\gamma}(t)$, $t > 0$, the $(d-1)$-dimensional moment curve $\boldsymbol{\gamma}(t) = (t, t^2, \ldots, t^{d-1})$, and we define two additional d-dimensional moment-like curves in $\mathbb{E}^{d+1}$:

$$\begin{aligned} \boldsymbol{\gamma}_1(t;\zeta) &= (t, \zeta t^d, t^2, t^3, \ldots, t^{d-1}, 0), \quad \text{and} \\ \boldsymbol{\gamma}_2(t;\zeta) &= (\zeta t^d, t, t^2, t^3, \ldots, t^{d-1}, 1), \end{aligned}$$

where $t > 0$ and $\zeta \geq 0$. Choose $n_1 + n_2$ real numbers α_i, $i = 1, \ldots, n_1$, and β_i, $i = 1, \ldots, n_2$, such that $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{n_1}$ and $0 < \beta_1 < \beta_2 < \ldots < \beta_{n_2}$. Let τ be a strictly positive parameter determined below, and let U_1 and U_2 be the $(d-1)$-dimensional point sets

$$\begin{aligned} U_1 &= \{\boldsymbol{\gamma}_1(\alpha_1\tau), \ldots, \boldsymbol{\gamma}_1(\alpha_{n_1}\tau)\}, \quad \text{and} \\ U_2 &= \{\boldsymbol{\gamma}_2(\beta_1), \ldots, \boldsymbol{\gamma}_2(\beta_{n_2})\}, \end{aligned}$$

where $\boldsymbol{\gamma}_j(\cdot)$ is used to denote $\boldsymbol{\gamma}_j(\cdot;0)$, for simplicity. Call Q_j the cyclic $(d-1)$-polytope defined as the convex hull of the points in U_j, $j = 1, 2$. Let $Q = CH_{d+1}(\{Q_1, Q_2\})$, and let $\mathcal{F}_Q$ be the set of proper faces of Q that are neither faces of Q_1 nor faces of Q_2. Then:

Lemma 4.2. *There exists a sufficiently small positive value $\tau^\star$ for τ, such that the $(d+1)$-polytope Q is $(\lfloor \frac{d+1}{2} \rfloor, U_1)$-bineighborly.*

Proof. Let $t_i = \alpha_i\tau$, $t_i^\epsilon = (\alpha_i + \epsilon)\tau$, $1 \leq i \leq n_1$, and $s_i = \beta_i$, $s_i^\epsilon = \beta_i + \epsilon$, $1 \leq i \leq n_2$, where $\epsilon > 0$ is chosen such that $\alpha_i + \epsilon < \alpha_{i+1}$, for all $1 \leq i < n_1$, and $\beta_i + \epsilon < \beta_{i+1}$, for all $1 \leq i < n_2$. Choose a subset U of $U_1 \cup U_2$ of size $\lfloor \frac{d+1}{2} \rfloor$, such that $U \cap U_j \neq \emptyset$, $j = 1, 2$. We denote by μ (resp., ν) the cardinality of $U \cap U_1$ (resp., $U \cap U_2$), and, clearly, $\mu + \nu = \lfloor \frac{d+1}{2} \rfloor$. Let $\boldsymbol{\gamma}_1(t_{i_1}), \boldsymbol{\gamma}_1(t_{i_2}), \ldots, \boldsymbol{\gamma}_1(t_{i_\mu})$ be the vertices in $U \cap U_1$, where $i_1 < i_2 < \ldots < i_\mu$, and analogously, let $\boldsymbol{\gamma}_2(s_{j_1}), \boldsymbol{\gamma}_2(s_{j_2}), \ldots, \boldsymbol{\gamma}_2(s_{j_\nu})$ be the vertices in $U \cap U_2$, where $j_1 < j_2 < \ldots < j_\nu$. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_{d+1})$ and define the $(d+2) \times (d+2)$ determinant $H_U(\boldsymbol{x})$ as shown in Fig. 3. The equation $H_U(\boldsymbol{x}) = 0$ is the equation of a hyperplane in $\mathbb{E}^{d+1}$ that passes through the points in U.

Consider the case $\boldsymbol{u} \in U_1 \setminus U$. Then, $\boldsymbol{u} = \boldsymbol{\gamma}_1(t) = (t, 0, t^2, t^3, \ldots, t^{d-1}, 0)$, $t = \alpha\tau$, for some $\alpha \notin \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_\mu}\}$. In this case, we can transform $H_U(\boldsymbol{u})$ in the form of the determinant $D_{k,\ell}(\tau)$ of Lemma 4.1, where $k = 2\mu+1$ and $\ell = 2\nu$, by subtracting the last row of $H_U(\boldsymbol{u})$ from the first, and by performing an *even* number of row and column swaps. The case $\boldsymbol{u} \in U_2 \setminus U$ is entirely analogous.

$$H_U(\boldsymbol{x}) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ \boldsymbol{x} & \boldsymbol{\gamma}_1(t_{i_1}) & \boldsymbol{\gamma}_1(t^\epsilon_{i_1}) & \cdots & \boldsymbol{\gamma}_1(t_{i_\mu}) & \boldsymbol{\gamma}_1(t^\epsilon_{i_\mu}) & \boldsymbol{\gamma}_2(s_{j_1}) & \boldsymbol{\gamma}_2(s^\epsilon_{j_1}) & \cdots & \boldsymbol{\gamma}_2(s_{j_\nu}) & \boldsymbol{\gamma}_2(s^\epsilon_{j_\nu}) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ x_1 & t_{i_1} & t^\epsilon_{i_1} & \cdots & t_{i_\mu} & t^\epsilon_{i_\mu} & 0 & 0 & \cdots & 0 & 0 \\ x_2 & 0 & 0 & \cdots & 0 & 0 & s_{j_1} & s^\epsilon_{j_1} & \cdots & s_{j_\nu} & s^\epsilon_{j_\nu} \\ x_3 & t^2_{i_1} & (t^\epsilon_{i_1})^2 & \cdots & t^2_{i_\mu} & (t^\epsilon_{i_\mu})^2 & s^2_{j_1} & (s^\epsilon_{j_1})^2 & \cdots & s^2_{j_\nu} & (s^\epsilon_{j_\nu})^2 \\ x_4 & t^3_{i_1} & (t^\epsilon_{i_1})^3 & \cdots & t^3_{i_\mu} & (t^\epsilon_{i_\mu})^3 & s^3_{j_1} & (s^\epsilon_{j_1})^3 & \cdots & s^3_{j_\nu} & (s^\epsilon_{j_\nu})^3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_d & t^{d-1}_{i_1} & (t^\epsilon_{i_1})^{d-1} & \cdots & t^{d-1}_{i_\mu} & (t^\epsilon_{i_\mu})^{d-1} & s^{d-1}_{j_1} & (s^\epsilon_{j_1})^{d-1} & \cdots & s^{d-1}_{j_\nu} & (s^\epsilon_{j_\nu})^{d-1} \\ x_{d+1} & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

Figure 3: The determinant $H_U(\boldsymbol{x})$.

$$F_V(\boldsymbol{x};\zeta) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \boldsymbol{x} & \boldsymbol{\gamma}_1(t_{i_1};\zeta) & \boldsymbol{\gamma}_1(t^\epsilon_{i_1};\zeta) & \cdots & \boldsymbol{\gamma}_1(t^\epsilon_{i_\mu};\zeta) & \boldsymbol{\gamma}_2(s_{j_1};\zeta) & \boldsymbol{\gamma}_2(s^\epsilon_{j_1};\zeta) & \cdots & \boldsymbol{\gamma}_2(s^\epsilon_{j_\nu};\zeta) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ x_1 & t_{i_1} & t^\epsilon_{i_1} & \cdots & t_{i_\mu} & t^\epsilon_{i_\mu} & \zeta s^d_{j_1} & \zeta(s^\epsilon_{j_1})^d & \cdots & \zeta s^d_{j_\nu} & \zeta(s^\epsilon_{j_\nu})^d \\ x_2 & \zeta t^d_{i_1} & \zeta(t^\epsilon_{i_1})^d & \cdots & \zeta t^d_{i_\mu} & \zeta(t^\epsilon_{i_\mu})^d & s_{j_1} & s^\epsilon_{j_1} & \cdots & s_{j_\nu} & s^\epsilon_{j_\nu} \\ x_3 & t^2_{i_1} & (t^\epsilon_{i_1})^2 & \cdots & t^2_{i_\mu} & (t^\epsilon_{i_\mu})^2 & s^2_{j_1} & (s^\epsilon_{j_1})^2 & \cdots & s^2_{j_\nu} & (s^\epsilon_{j_\nu})^2 \\ x_3 & t^3_{i_1} & (t^\epsilon_{i_1})^3 & \cdots & t^3_{i_\mu} & (t^\epsilon_{i_\mu})^3 & s^3_{j_1} & (s^\epsilon_{j_1})^3 & \cdots & s^3_{j_\nu} & (s^\epsilon_{j_\nu})^3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_d & t^{d-1}_{i_1} & (t^\epsilon_{i_1})^{d-1} & \cdots & t^{d-1}_{i_\mu} & (t^\epsilon_{i_\mu})^{d-1} & s^{d-1}_{j_1} & (s^\epsilon_{j_1})^{d-1} & \cdots & s^{d-1}_{j_\nu} & (s^\epsilon_{j_\nu})^{d-1} \\ x_{d+1} & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

Figure 4: The determinant $F_V(\boldsymbol{x};\zeta)$.

Since we have $\binom{n_1+n_2}{\lfloor\frac{d+1}{2}\rfloor} - \binom{n_1}{\lfloor\frac{d+1}{2}\rfloor} - \binom{n_2}{\lfloor\frac{d+1}{2}\rfloor}$ possible subsets U, and for each U there are $(n_1+n_2-\lfloor\frac{d+1}{2}\rfloor)$ vertices in $(U_1\cup U_2)\setminus U$ choose a value $\tau^\star$ for τ that is small enough, so that all

$$(n_1+n_2-\lfloor\tfrac{d+1}{2}\rfloor)\left[\binom{n_1+n_2}{\lfloor\frac{d+1}{2}\rfloor} - \binom{n_1}{\lfloor\frac{d+1}{2}\rfloor} - \binom{n_2}{\lfloor\frac{d+1}{2}\rfloor}\right]$$

possible determinants $H_U(\boldsymbol{u})$ are strictly positive. Call $U_j^\star$, $j=1,2$, the vertex sets we get for $\tau=\tau^\star$, $Q_j^\star$ the corresponding polytopes, and $Q^\star$ the resulting convex hull. Then *for each* $U^\star\subseteq U_1^\star\cup U_2^\star$, where $U^\star\cap U_j^\star\neq\emptyset$, $j=1,2$, the equation $H_{U^\star}(\boldsymbol{x})=0$, $\boldsymbol{x}\in\mathbb{E}^{d+1}$, is the equation of a supporting hyperplane for $Q^\star$ passing through the vertices of $U^\star$ (and those only); hence, $Q^\star$ is $(\lfloor\frac{d+1}{2}\rfloor, U_1^\star)$-bineighborly. □

We assume we have chosen τ to be equal to $\tau^\star$, and, call $U_j^\star$, $Q_j^\star$, $j=1,2$, the corresponding vertex sets and $(d-1)$-polytopes. Perturb the vertex sets $U_1^\star$ and $U_2^\star$, to get the vertex sets V_1 and V_2 by considering vertices on the curves $\boldsymbol{\gamma}_1(t;\zeta)$ and $\boldsymbol{\gamma}_2(t;\zeta)$, with $\zeta>0$. More precisely, define the sets V_1 and V_2 as:

$$V_1=\{\boldsymbol{\gamma}_1(\alpha_1\tau^\star;\zeta),\ldots,\boldsymbol{\gamma}_1(\alpha_{n_1}\tau^\star;\zeta)\},\quad\text{and}$$
$$V_2=\{\boldsymbol{\gamma}_2(\beta_1;\zeta),\ldots\boldsymbol{\gamma}_2(\beta_{n_2};\zeta)\},$$

where $\zeta>0$. Let P_j be the convex hull of the vertices in V_j, $j=1,2$, and notice that P_j is a $\lfloor\frac{d}{2}\rfloor$-neighborly d-polytope. Let $P=CH_{d+1}(\{P_1,P_2\})$, and let $\mathcal{F}_P$ be the set of proper faces of P that are neither faces of P_1 nor faces of P_2. As in the proof of Lemma 4.2, choose $V\subseteq V_1\cup V_2$, such that $V\cap V_j\neq\emptyset$, $j=1,2$, and let $U^\star$ be the set of vertices in $U_1^\star\cup U_2^\star$ that correspond to vertices in V. Let $F_V(\boldsymbol{x};\zeta)$ be the determinant shown in Fig. 4. The equation $F_V(\boldsymbol{x};\zeta)=0$ is the equation of a hyperplane in $\mathbb{E}^{d+1}$ that passes through the points in V. Since for any $\boldsymbol{v} \in (V_1 \cup V_2) \setminus V$, $\lim_{\zeta \to 0^+} F_V(\boldsymbol{v}; \zeta) = F_{U^\star}(\boldsymbol{u}^\star; 0) = H_{U^\star}(\boldsymbol{u}^\star) > 0$, where $\boldsymbol{u}^\star$ is the point in $(U_1^\star \cup U_2^\star) \setminus U^\star$ that corresponds to $\boldsymbol{v}$, we conclude that there exists a value $\zeta_0 > 0$ for ζ, such that, for all $\zeta \in (0, \zeta_0)$, the equation $F_V(\boldsymbol{x}; \zeta) = 0$ represents a supporting hyperplane for P, that passes through the vertices of V, and those only. By choosing ζ to be small enough, so that all possible $(n_1+n_2-\lfloor\frac{d+1}{2}\rfloor)\left[\binom{n_1+n_2}{\lfloor\frac{d+1}{2}\rfloor}-\binom{n_1}{\lfloor\frac{d+1}{2}\rfloor}-\binom{n_2}{\lfloor\frac{d+1}{2}\rfloor}\right]$ determinants $F_V(\boldsymbol{v}; \zeta)$ are positive, the $(d+1)$-polytope P becomes $(\lfloor\frac{d+1}{2}\rfloor, V_1)$-bineighborly; by Theorem 3.1 this establishes the tightness of our bounds for all $f_k(P_1 \oplus P_2)$, $0 \le k \le d-1$.

5 Summary and open problems

In this paper we have computed the maximum number of k-faces, $f_k(P_1 \oplus P_2)$, $0 \le k \le d$, of the Minkowski sum of two d-polytopes P_1 and P_2 as a function of the number of vertices of the two polytopes. Furthermore, we have presented constructions that attain these maximal values. It remains an open problem to extend our results to the Minkowski sum of r d-polytopes in $\mathbb{E}^d$, for $d \ge 4$ and $r \ge 3$. A related open problem is to express the number of k-faces of the Minkowski sum of r d-polytopes in terms of the number of facets of these polytopes. Results in this direction are only known for $d \le 3$. We would like to derive such expressions for $d \ge 4$.

Acknowledgements

The authors would like to thank Efi Fogel for his suggestions for improving the presentation of the material in this paper. The work in this paper has been partially supported by the FP7-REGPOT-2009-1 project "Archimedes Center for Modeling, Analysis and Computation".

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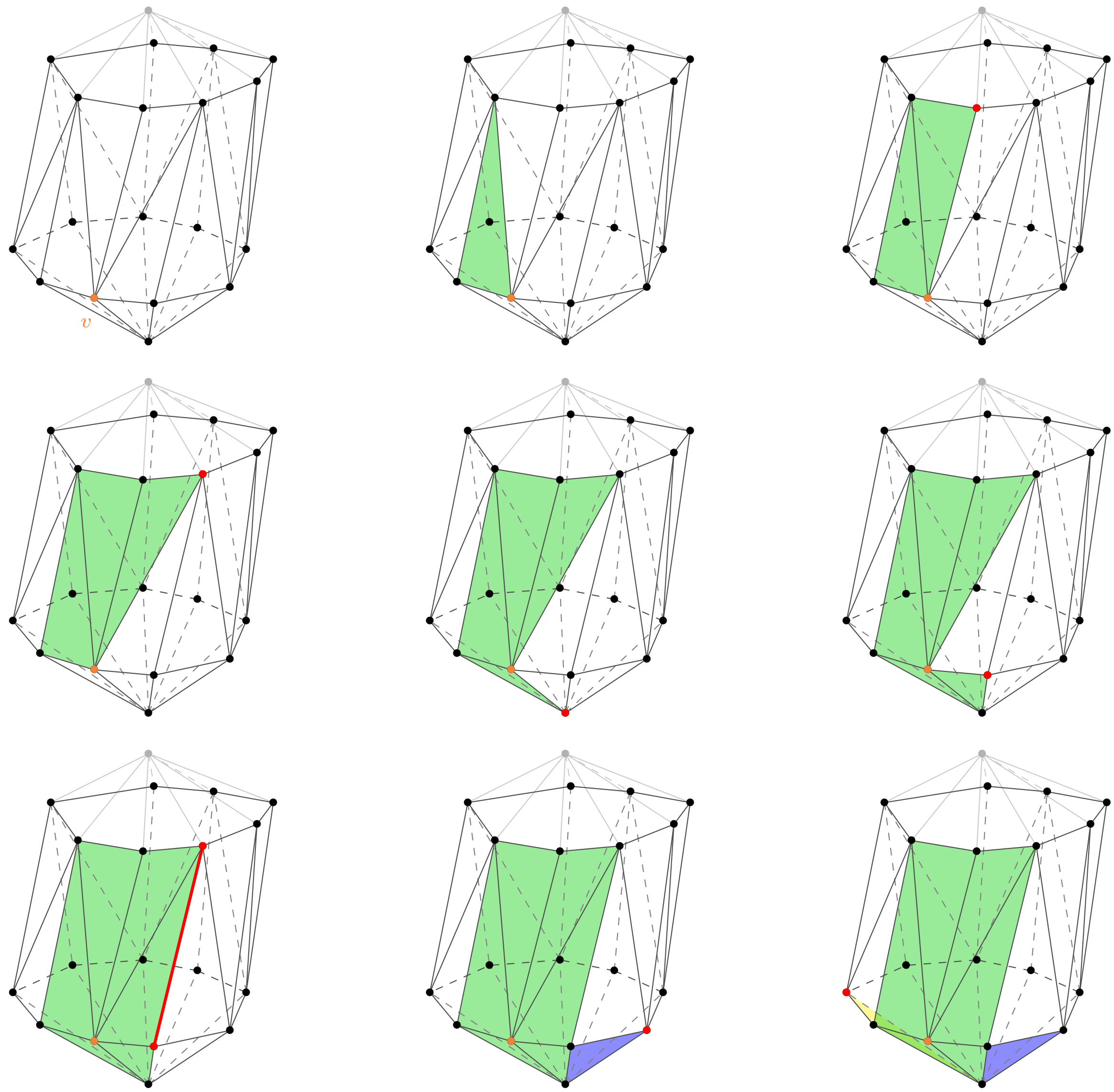

Figure 5: Top left: The complex $\mathcal{K}_1$ (from Fig. 2) with the vertex v shown in orange. Remaining subfigures (from left to right and top to bottom): the first eight steps of the construction of $\mathcal{K}_1$ from a shelling $\mathbb{S}(\mathcal{K}_1) = \{F_1, F_2, \ldots, F_{26}\}$ that shells $\mathrm{star}(v, \mathcal{K}_1)$ first. The facets in green are the facets of $\mathrm{star}(v, \mathcal{K}_1)$. All other facets are shown in either blue or yellow, depending on whether we see their exterior or interior side (w.r.t. the interior of the polytope Q). The minimal new faces at each step of the shelling are shown in red; recall that the minimal new face corresponding to F_1 is $\emptyset$. In all subfigures, the faces of $\mathrm{star}(y_2, \partial Q)$ that do not belong to $\partial Q / y_2 \equiv \partial P_2$ are shown in gray.

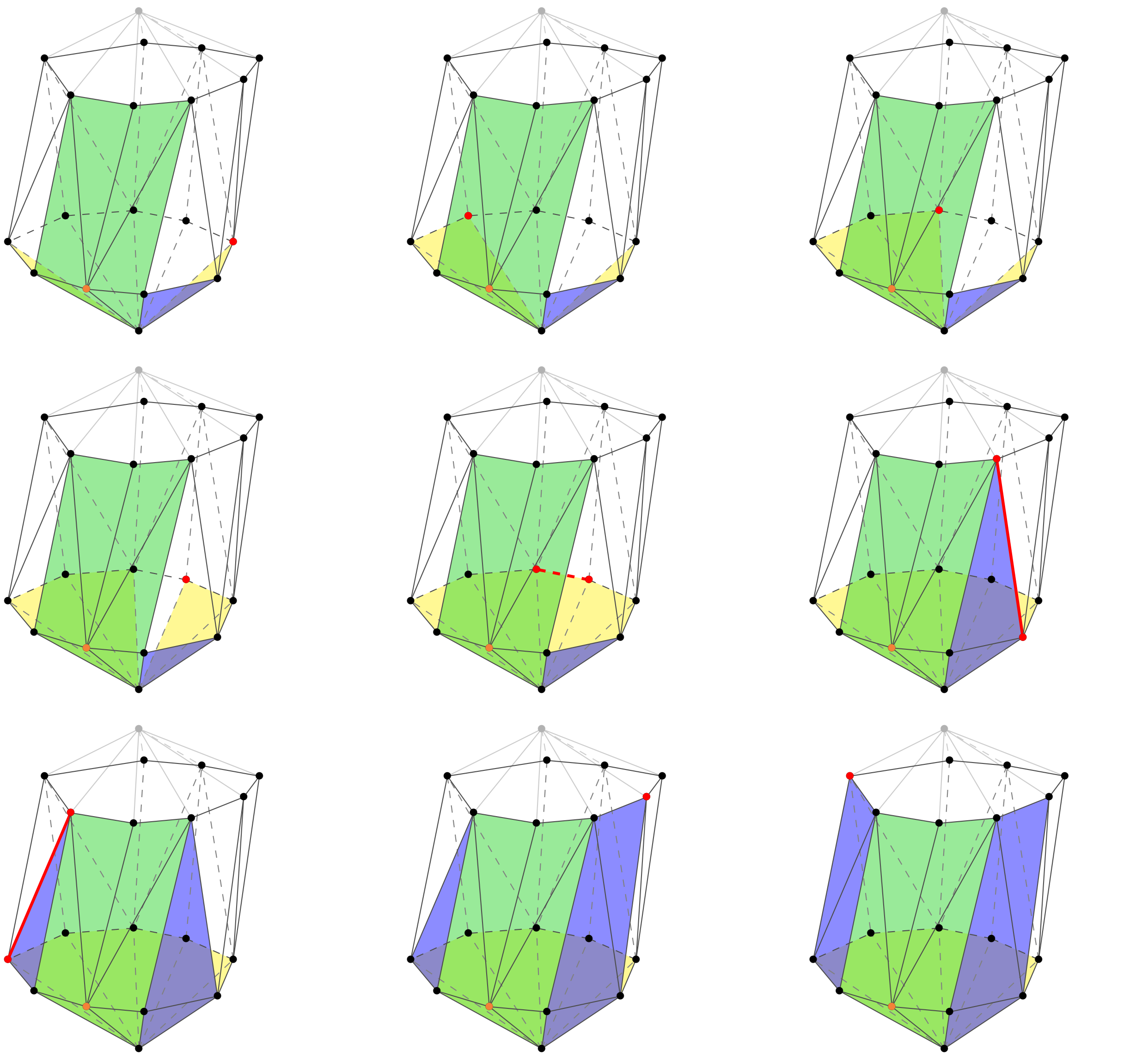

Figure 6: From left to right and top to bottom: The next twelve steps of the construction of $\mathcal{K}_1$ from $\mathbb{S}(\mathcal{K}_1)$. Colors are as in Fig. 5.

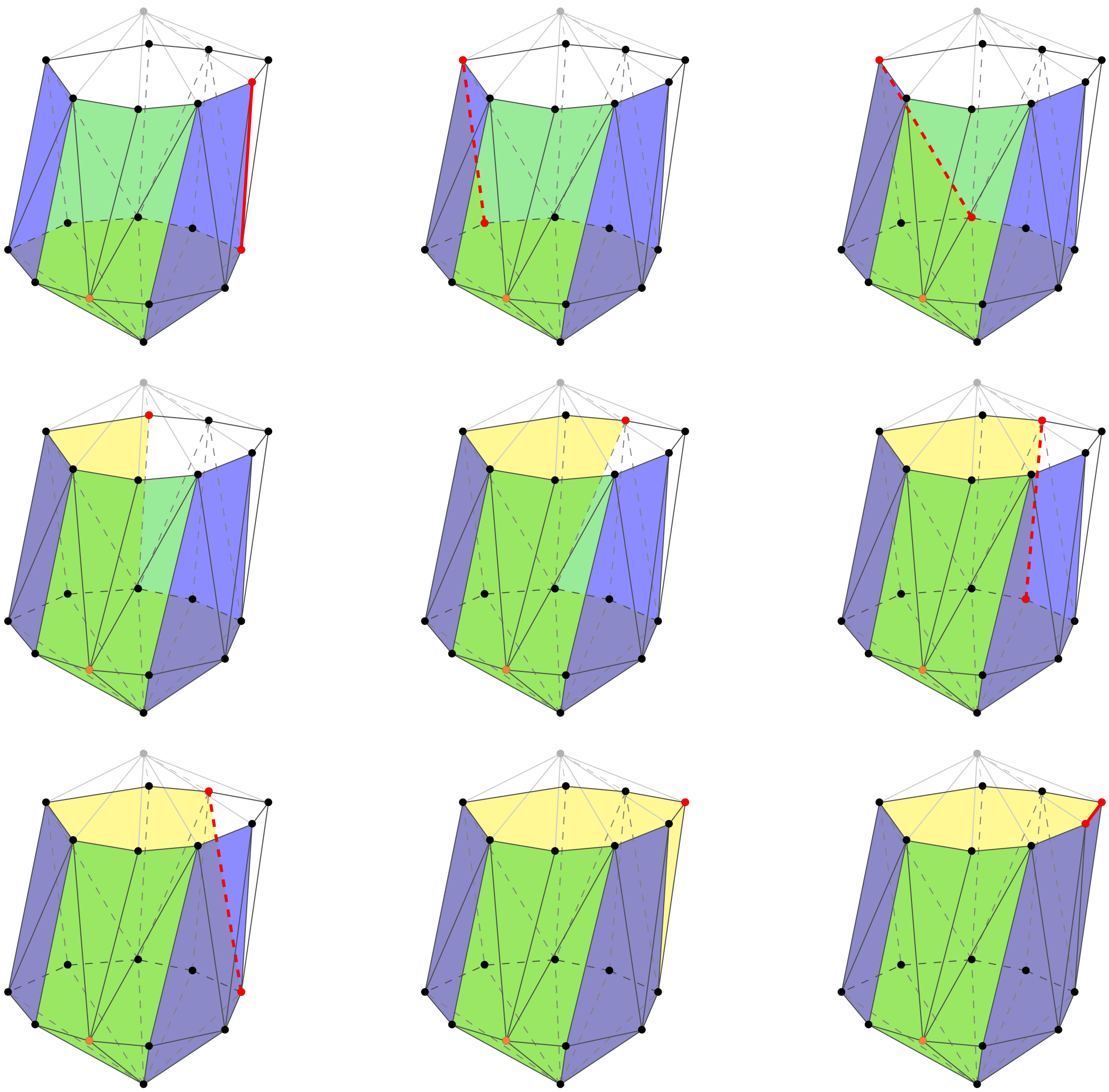

Figure 7: From left to right and top to bottom: The final twelve steps of the construction of $\mathcal{K}_1$ from $\mathbb{S}(\mathcal{K}_1)$. Colors are, again, as in Fig. 5.

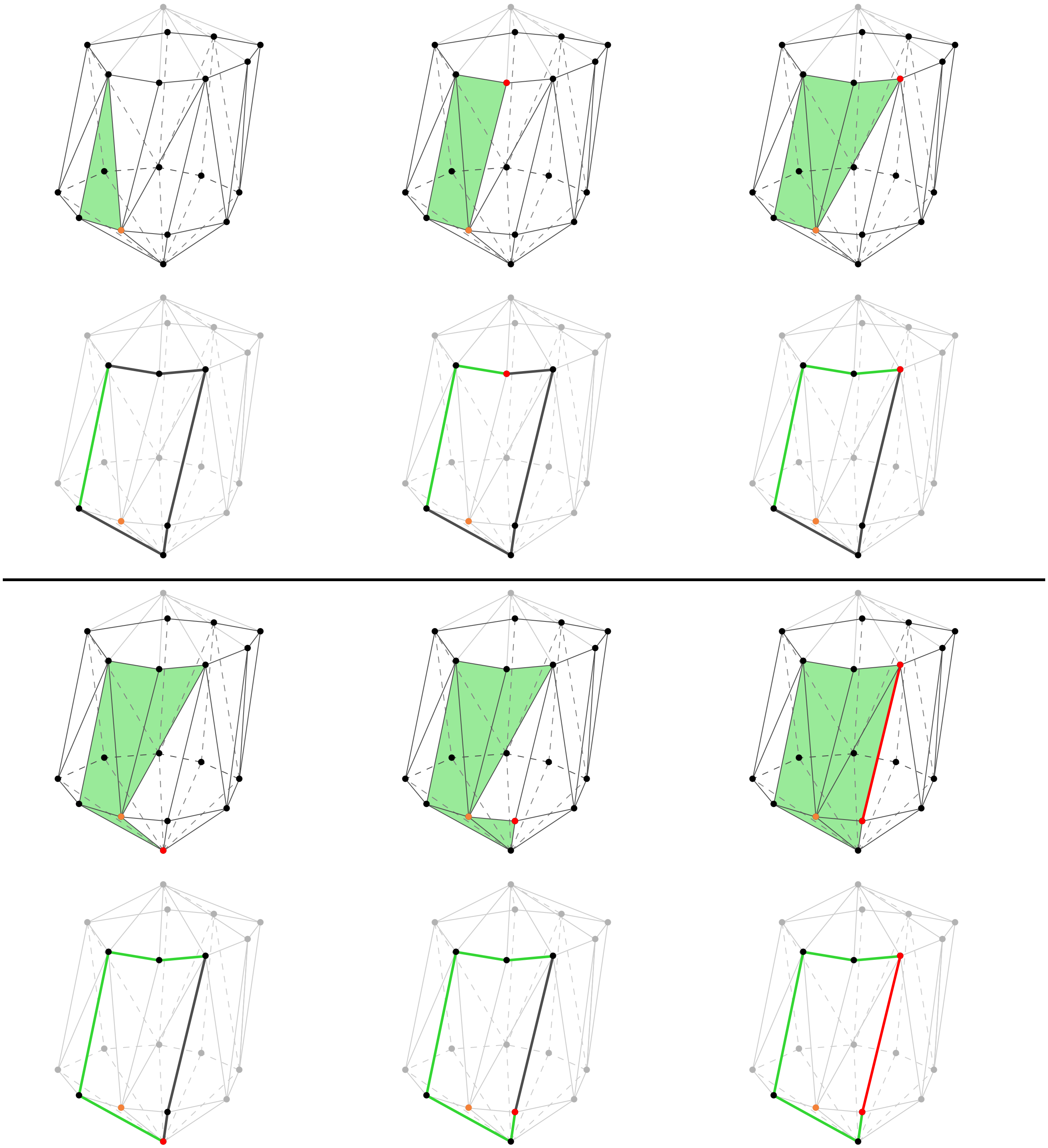

Figure 8: The first six steps of $\mathbb{S}(\mathcal{K}_1)$ and the corresponding steps in the induced shelling $\mathbb{S}(\mathcal{K}_1/v)$ of $\mathcal{K}_1/v$ (recall that $\mathbb{S}(\mathcal{K}_1)$ shells $\text{star}(v, \mathcal{K}_1)$ first). Rows 1 & 3: The steps of $\mathbb{S}(\mathcal{K}_1)$. Rows 2 & 4: The steps of $\mathbb{S}(\mathcal{K}_1/v)$. $\mathcal{K}_1/v$ is shown with green solid segments (the facets of $\mathcal{K}_1/v$, that have not been added yet, are highlighted as black solid segments). The minimal new faces at each step of the shellings $\mathbb{S}(\mathcal{K}_1)$ and $\mathbb{S}(\mathcal{K}_1/v)$ are shown in red. As expected, the minimal new faces, at corresponding steps, coincide.

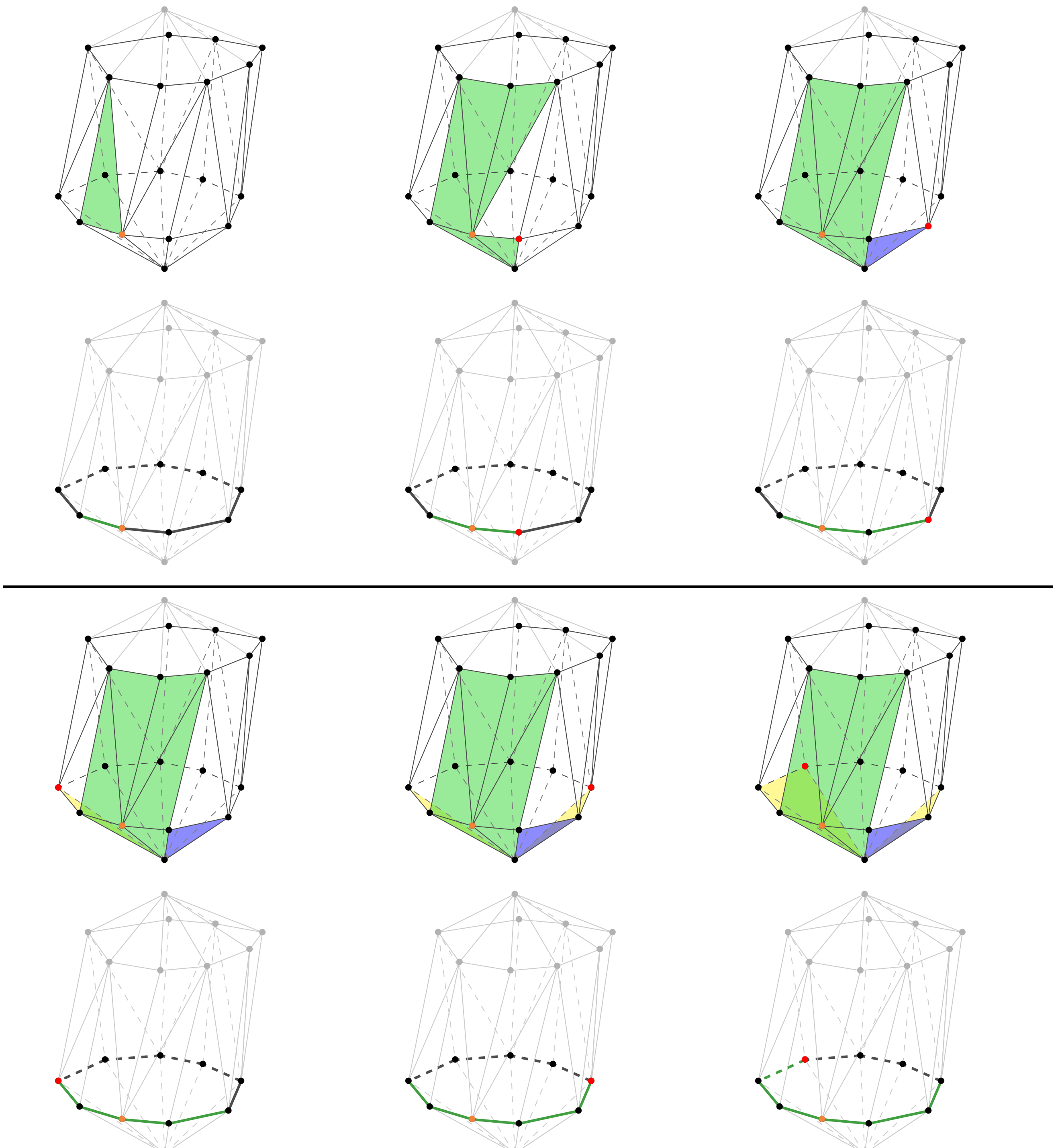

Figure 9: The first six steps of the construction of ∂P_1 from the shelling $\mathbb{S}(\partial P_1)$ induced by $\mathbb{S}(\mathcal{K}_1)$, along with the corresponding steps of the construction of $\mathcal{K}_1$ from $\mathbb{S}(\mathcal{K}_1)$. Rows 1 & 3: the steps of $\mathbb{S}(\mathcal{K}_1)$ that induce facets for $\mathbb{S}(\partial P_1)$. Rows 2 & 4: The corresponding steps of $\mathbb{S}(\partial P_1)$. ∂P_1 is shown with green solid/dashed segments (the facets of ∂P_1, that have not been added yet, are highlighted as black solid/dashed segments). The minimal new faces at each step of the shellings $\mathbb{S}(\mathcal{K}_1)$ and $\mathbb{S}(\partial P_1)$ are shown in red. As expected, the minimal new faces, at corresponding steps, coincide.

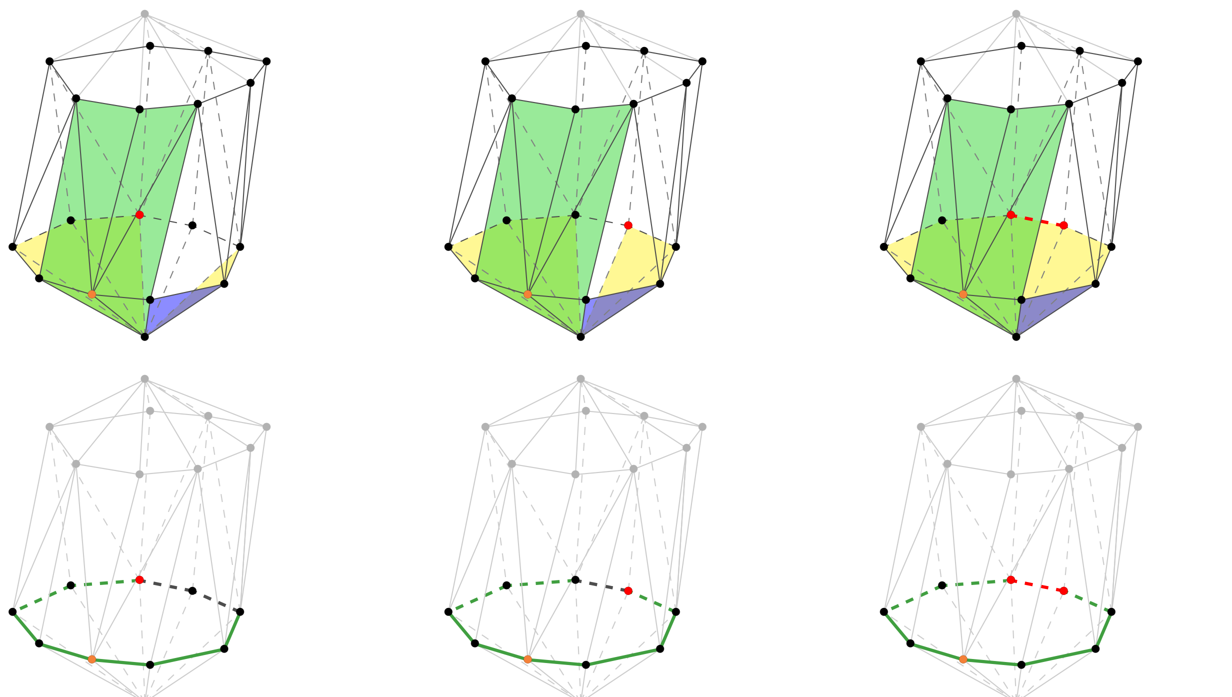

Figure 10: The last three steps of the construction of ∂P_1 from the shelling $\mathbb{S}(\partial P_1)$ induced by $\mathbb{S}(\mathcal{K}_1)$, along with the corresponding steps of the construction of $\mathcal{K}_1$ from $\mathbb{S}(\mathcal{K}_1)$. Top row: The steps of $\mathbb{S}(\mathcal{K}_1)$. Bottom row: The steps of $\mathbb{S}(\mathcal{K}_1/v)$. Colors are as in Fig. 9.