Simplicial functors and stable homotopy theory

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1. Introduction

The problem of constructing a nice smash product of spectra is an old and well-known problem of algebraic topology. This problem has come to mean the following: Find a model category, which is Quillen-equivalent to the model category of spectra, and which has a symmetric monoidal product corresponding to the smash product of spectra. Two solutions were found recently, namely the smash product of S-modules of [EKMM], and the smash product of symmetric spectra of [HSS]. Here we present another solution, the smash product of **SF**, the category of simplicial functors from finite pointed simplicial sets to pointed simplicial sets. Before we describe some interesting special properties of **SF**, we summarize some other results contained in this paper: Although **SF** might not seem a natural object to study, especially to a reader not very familiar with simplicial techniques, it should in fact be thought of as the category of functors from finite pointed CW-complexes to pointed topological spaces which are *pointed* (take one-point spaces to one-point spaces) and *homotopy functors* (take weak equivalences to weak equivalences). It is clear that this latter category has an interesting homotopy theory (it has a natural class of weak equivalences), which cannot come from a model structure (for the trivial reason that colimits and limits do not preserve weak equivalences in general). However, this homotopy theory is equivalent to another one that **SF** has (see section 8, especially for which we construct a model structure Ordinary (Bousfield-Friedlander) spectra can also be viewed as pointed simplicial functors into pointed simplicial sets (only now they are not defined on all finite pointed simplicial sets, but only on spheres—see section 4). As a byproduct, we obtain simpler and more descriptive constructions of the model structures on spectra. As another byproduct, we obtain a model structure on simplicial functors which should be thought of as a model structure on homotopy functors. As a final byproduct, we obtain a model-category version of the part of Goodwillie's calculus of homotopy functors having to do with "the linear approximation of a homotopy functor at $*$" [G1], without the restriction on the relevant homotopy functors of being "stably excisive".

We say a few words about some interesting properties that distinguish simplicial functors from S-modules and symmetric spectra. First of all, an advantage of simplicial functors is their simplicity. Also, the monoids under the smash product of simplicial functors (these are what one would call "algebras over the sphere spectrum", after thinking of simplicial functors as spectra by using the results of this paper) are well known: They are the FSPs, which were introduced in 1985 by Bökstedt [B]. There are interesting constructions with FSPs, e.g., topological cyclic homology, which can be considerably simplified and conceptualized using the smash product of simplicial functors [LS]. Note however that it is not clear that an E_∞-FSP has a commutative model, although the analogous statement is true for S-modules and symmetric spectra. This has, e.g., the disadvantage, that it is not clear how to give a model category structure to (or even how to define) MU-algebras using simplicial functors (although this can be done for modules over any any FSP, and algebras over any commutative FSP, by similar methods as in [Sch]). Returning to examining the special features of simplicial functors, in constrast to S-modules and symmetric spectra, simplicial functors have not only an interesting unstable theory (modeling certain homotopy functors, cf. remark 4.6 and), but interesting "metastable", as well as a whole tower of "higher stable" theories, which are related to the calculus of homotopy functors [G1], [G2], [G3] (this will appear in a future paper). Finally, simplicial functors can be not only smashed, but also composed, with each other. There is a canonical map from the smash product to the composition product, which is an isomorphism in an important special case (proposition 5.13), a fact that provides a very concrete description of the smash product. There is a close connection between simplicial functors and Γ-spaces [L2]. In fact, Γ-spaces may be identified with a special kind of simplicial functors (see convention 2.11 of [L2]) in such a way that the smash product of Γ-spaces corresponds to the smash product of simplicial functors under this identification.

We have tried to make this paper reasonably self-contained. We develop from scratch the part of the theories of simplicial categories, of model categories, and of stable homotopy, that we need. In particular, no background is assumed from the reader, other than familiarity with simplicial sets (and some basic algebraic topology).

This paper is organized as follows. In section 2, we establish some conventions on notation and terminology, and we introduce homotopy cartesian squares of simplicial sets. In section 3, we introduce simplicial categories. In section 4, we introduce spectra and simplicial functors. In section 5, we introduce the smash product of simplicial functors and we prove its category-theoretic properties. In section 6, we introduce model categories. In section 7, we construct a certain model structure on simplicial categories of a certain form, which includes the simplicial categories of spectra and simplicial functors. The resulting model structures for spectra and simplicial functors form stepping stones for the construction of later, more interesting, model structures, as follows. In section 8, we construct a model structure on simplicial functors, which should be thought of as a model structure on certain homotopy functors. In section 9, we construct the stable model structure on simplicial functors, which we later show is Quillen-equivalent to the stable model structure on spectra. The latter was introduced in [BF], but we give a self-contained account of it in section 10, together with all the facts of stable homotopy theory that we need. In section 11, we compare the stable model structures. In section 12, we compare the smash products of spectra and simplicial functors. Finally, in section 13, we give the connection to the calculus of homotopy functors.

2. Preliminaries

We denote the category of pointed simplicial sets by $\mathbf{S}_0$. We write A^+ for the pointed simplicial set obtained from the simplicial set A by adding a disjoint basepoint. We denote the basepoint of any pointed simplicial set K by 0_K, or simply by 0 when no confusion can arise. Let $*$ be the pointed set $\{0\}$, let S^0 be the pointed set $\{0, 1\}$, and let S^1 be the pointed simplicial set $\Delta^1/\partial\Delta^1$. We fix once and for all a countable category $\mathbf{S}_0^{fin}$ equivalent to the full subcategory of $\mathbf{S}_0$ consisting of the finite pointed simplicial sets, such that it contains $*$, S^0, S^1, and all $(\Delta^n)^+$. We choose the smash product in $\mathbf{S}_0$ so that it preserves $\mathbf{S}_0^{fin}$, so that smashing (either on the left or on the right) by S^0 is the identity, and so that smashing by $*$ always yields $*$. Given a simplicial set A and a pointed simplicial set K, we write $A \otimes K$ for $A^+ \wedge K$, and $K \otimes A$ for $K \wedge A^+$.

We work within some fixed universe. A set is *small* if it is in the universe, and it is a *class* if it is a subset of the universe. From now on, unless the contrary is explicitly stated, whenever we say "set" we really mean "small set". All our categories are assumed to have a class of objects and a class of maps, and to have sets of maps between any two fixed objects. When we say that a category has all limits and colimits, we really mean only these indexed by *small* categories (those with a *set* of objects and a *set* of maps). Finally, we need to explain how we use the word "map": For us, a map always preserves all the available structure. For example, if K and L are pointed simplicial sets, then a map $K \to L$ is the same thing as a pointed simplicial map $K \to L$.

Let D be the pointed simplicial set obtained by choosing a basepoint for Δ^1. If A is a simplicial set, A' denotes the singular complex of the realization of A. If $f : A \to uK$ is a map of simplicial sets, where uK denotes the underlying simplicial set of the pointed simplicial set K, *the homotopy fiber* $\mathrm{hf}(f)$ *of* f is the pullback of $A' \to uK' \leftarrow u\,\mathrm{hom}(D, K')$, where the last map is obtained by evaluating on the non-basepoint vertex of D. The basic properties of the homotopy fiber are the following. There is a natural map $F \to \mathrm{hf}(f)$, where F is the fiber of f, which is a weak equivalence if f is a fibration. Further, given a map $K_0 \to K_1$ of pointed simplicial sets and a commutative square

$$\begin{array}{ccccc} & A_0 & \to & A_1 & \\ f_0 & \downarrow & & \downarrow & f_1 \\ & uK_0 & \to & uK_1 & \end{array}$$

of simplicial sets with both horizontal maps weak equivalences, the induced map of homotopy fibers $\mathrm{hf}(f_0) \to \mathrm{hf}(f_1)$ is also a weak equivalence.

A commutative square of simplicial sets

$$\begin{array}{ccccc} & A_0 & \to & A_1 & \\ f_0 & \downarrow & & \downarrow & f_1 \\ & B_0 & \to & B_1 & \end{array}$$

is called *homotopy cartesian* if, for every choice of a basepoint of B_0 (i.e., for every possibility of viewing the bottom horizontal map as a map of pointed simplicial sets), the induced map $\mathrm{hf}(f_0) \to \mathrm{hf}(f_1)$ is a weak equivalence. In case f_0 and f_1 are fibrations, this condition is equivalent to the following: The canonical map from A_0 to the pullback of $B_0 \to B_1 \leftarrow A_1$ is a weak equivalence.

A commutative square of pointed simplicial sets is called homotopy cartesian if and only if so is the underlying square of simplicial sets. In this case, if the underlying square is as above, with $A_i = uK_i$, $B_i = uL_i$, and $K_1 = *$, then we call the sequence $K_0 \to L_0 \to L_1$ a *fibration sequence*.

3. Simplicial categories

In this section we introduce the part of the theory of simplicial categories that we need.

3.1. Definition: [L1, p. 74; see also the bottom of p. 75] Let $\mathbf{C}$ and $\mathbf{D}$ be functors from a small category I to categories, with $\mathbf{C}$ *small*, i.e., taking values in small categories. The maps from $\mathbf{C}$ to $\mathbf{D}$ are themselves objects of a category $hom(\mathbf{C}, \mathbf{D})$. Given two objects X and Y of $hom(\mathbf{C}, \mathbf{D})$, the maps from X to Y are those families f_i, parameterized by the objects i of I, satisfying the following two conditions. First, f_i is a map $X_i \to Y_i$ in the category of functors from $\mathbf{C}_i$ to $\mathbf{D}_i$. Second, given a map $\alpha : i \to j$ in I and an object K in $\mathbf{C}_i$, the two maps from $X\alpha_* K$ (which equals $\alpha_* XK$) to $Y\alpha_* K$ given by f_j and $\alpha_* f_i$ are equal.

3.2. Definition: A *simplicial category* is a functor from Δ^{op} to categories. Given simplicial categories $\mathbf{C}$ and $\mathbf{D}$ with $\mathbf{C}$ small, *the category of simplicial functors* from $\mathbf{C}$ to $\mathbf{D}$ is the category $hom(\mathbf{C}, \mathbf{D})$.

3.3. Convention: For the rest of this paper, unless the contrary is explicitly stated, all simplicial categories $\mathbf{C}$ will have a discrete simplicial class of objects. Thus the class of objects of $\mathbf{C}_n$ is independent of n, and we refer to it as the class of objects of $\mathbf{C}$.

If $\mathbf{C}$ is a simplicial category, we sometimes say "map in $\mathbf{C}$" instead of "map in $\mathbf{C}_0$".

3.4. Definition: A simplicial category is called *pointed*, if there is an object of $\mathbf{C}$ which is *a zero object*, i.e., both initial and final, in $\mathbf{C}_n$ for all n. Thus we may view $\mathbf{C}(K, L)$ as a pointed simplicial set, for all objects K and L of $\mathbf{C}$. A simplicial functor between pointed simplicial categories is called *pointed*, if it takes zero objects to zero objects.

3.5. Example: We define the pointed simplicial category $\mathbf{S}$. Its objects are the pointed simplicial sets. The maps from K to L in $\mathbf{S}_n$ are given by the maps from $\Delta^n \otimes K$ to L in $\mathbf{S}_0$. The composition vu in $\mathbf{S}_n$ is given by the composition $v(\Delta^n \otimes u)(d \otimes K)$ in $\mathbf{S}_0$, where d is the diagonal $\Delta^n \to \Delta^n \times \Delta^n$, and K is the source of u in $\mathbf{S}_n$. Note that $\mathbf{S}$ has a small full pointed simplicial subcategory $\mathbf{S}^{fin}$, whose value at [0] is the category $\mathbf{S}_0^{fin}$ which we have defined in the previous section.

3.6. Remark: According to our conventions, a pointed simplicial category is essentially a category "enriched over $\mathbf{S}_0$". Most of this section, and section 5, could be generalized by using "category theory enriched over a symmetric monoidal category" (see [D]). Our approach does not require enriched category theory, and avoids "coherence conditions". There is also the equivalent approach of defining a simplicial category $\mathbf{C}$ to be a class of objects, a family of simplicial sets $\mathbf{C}(K, L)$ defined for each pair (K, L) of objects, and an assosiative and unital composition. In these terms, a simplicial functor X from the simplicial category $\mathbf{C}$ to the simplicial category $\mathbf{D}$ is a family of maps of the form $\mathbf{C}(K, L) \to \mathbf{D}(XK, XL)$, that preserve composition and units. This is the approach of [Q] (see section II.1), but the definition given there is too restrictive for our purposes. (There it is required that all simplicial categories have functors $?^K$ and $? \otimes K$, behaving like the functors with the same name in proposition 3.16. This rules out the simplicial category **Sph**, which is used in the next section to represent spectra as simplicial functors.)

3.7. Definition: Let $P : \mathbf{S} \times \mathbf{S} \to \mathbf{S}$ be the pointed simplicial functor given on objects by the smash-product (its definition on maps of $\mathbf{S}_n$ involves the diagonal $\Delta^n \to \Delta^n \times \Delta^n$). Given K in $\mathbf{S}$, define $K_* : \mathbf{S} \to \mathbf{S}$ by $K_* L = P(L, K)$. Given a pointed simplicial functor $X : \mathbf{C} \to \mathbf{S}$, define $X \wedge K : \mathbf{C} \to \mathbf{S}$ to be the composition $K_* \circ X$. Given a simplicial set A, define $X \otimes A$ to be $X \wedge A^+$.

3.8. Example: Given a small pointed simplicial category $\mathbf{C}$, *the dual pointed simplicial category* $\mathbf{C}^*$ is defined as follows. Let $\mathbf{C}_0^*$ be the category of pointed simplicial functors from $\mathbf{C}$ to $\mathbf{S}$. Let the objects of $\mathbf{C}^*$ be the objects of $\mathbf{C}_0^*$. Finally, let the maps in $\mathbf{C}_n^*$ from X to Y be the maps in $\mathbf{C}_0^*$ from $X \otimes \Delta^n$ to Y.

3.9. Convention: Given X in $\mathbf{C}^*$, we write $X(n)$ for the functor from $\mathbf{C}_n$ to $\mathbf{S}_n$ it determines. We denote by XK the value of $X(n)$ at K, which does not depend on n. We denote by $X_n K$ the pointed set of n-simplices of XK.

3.10. Definition: Let $\mathbf{E}$ be the category of pointed sets. Given X in $\mathbf{C}^*$ and $[n]$ in Δ, we define a pointed functor $X_n : \mathbf{C}_n \to \mathbf{E}$. The value of X_n on objects has already been defined above. Given $u : K \to L$ in $\mathbf{C}_n$ and x

in X_nK, we define u_*x in X_nL to be $u_*(x,1)$, where we wrote u_* also for the map $XK \otimes \Delta^n \to XL$, and 1 denotes the identity map in Δ^n_n.

3.11. Proposition: *Fix a collection of pointed functors X_n from $\mathbf{C}_n$ to pointed sets, and fix, for all K in $\mathbf{C}$, a choice of assembling the X_nK to a pointed simplicial set. Then this data determines a pointed simplicial functor if and only if, given $\alpha : [m] \to [n]$ in Δ, the induced maps $X_nK \to X_mK$ assemble to a natural map $X_n \to X_m\alpha^*$. Further, given X and Y in $\mathbf{C}^*$, a collection of maps $X_n \to Y_n$ determines a map $X \to Y$ if and only if, for all K in $\mathbf{C}$, the maps assemble to a map of pointed simplicial sets $XK \to YK$.*

Proof. The conclusion follows from the following two observations.

First, given $\alpha : [m] \to [n]$, $u : K \to L$ in $\mathbf{C}_n$, X in $\mathbf{C}_*$, and x in X_mK, the equality $u_*(x,\alpha) = (\alpha^*u)_*(x,1)$ holds. For, since X is simplicial, $\alpha^*(u_*) = (\alpha^*u)_*$. But $\alpha^*(u_*)(x,1) = u_*(x,\alpha)$.

Second, a map $f : X \to Y$ in $\mathbf{C}^*_0$ is a family of maps $XK \otimes \Delta^n \to YK$, satisfying a condition stated in definition 3.1. This condition implies that this family is determined by a family of maps $XK \to YK$, by precomposing with the projections $XK \otimes \Delta^n \to XK$. Equivalently, given $\alpha : [m] \to [n]$ and x in X_mK, we have $f(x,\alpha) = f(x,1)$. □

3.12. Proposition: *All colimits and limits exist in $\mathbf{C}^*_0$, and are computed degreewise.*

Proof. This follows from proposition 3.11. □

3.13. Definition: (Representable simplicial functors) Given an object K of the small pointed simplicial category $\mathbf{C}$, we define, using proposition 3.11, an object K^* of $\mathbf{C}^*$. Given L in $\mathbf{C}$, the pointed simplicial set K^*L is defined to be $\mathbf{C}(K,L)$. The pointed functor $K^*_n : \mathbf{C}_n \to \mathbf{E}$ is defined to be the one represented by K. Then these data assemble to a pointed simplicial functor, and the assignment $K \mapsto K^*$ extends to a pointed simplicial functor $\mathbf{C}^{op} \to \mathbf{C}^*$.

Given X in $\mathbf{C}^*$, define X^Kin $\mathbf{C}^*$ to be the composition $K^* \circ X$.

3.14. Proposition: *Given X in $\mathbf{C}^*$ and K, L in $\mathbf{C}$, the pointed sets $\mathbf{S}_0(L, XK)$ and $\mathbf{C}^*_0(K^* \wedge L, X)$ are isomorphic.*

Proof. By the Yoneda lemma, the maps from $K^*_n \wedge L_n$ to X_n are in bijective correspondence with the maps from L_n to X_nK. The conclusion follows from the description in proposition 3.11 of the maps in $\mathbf{C}^*$. $\square$

3.15. Corollary: *Given X in $\mathbf{C}^*$ and K in $\mathbf{C}$, $XK \cong \mathbf{C}^*(K^*, X)$.*

3.16. Proposition: *Given K, L in $\mathbf{S}$ and X, Y in $\mathbf{C}^*$, there are isomorphisms*

$$\begin{aligned} \mathbf{S}(L, Y^K) &\cong \mathbf{S}(L \wedge K, Y), \quad \textit{and} \\ \mathbf{C}(X, Y^K) &\cong \mathbf{C}^*(X \wedge K, Y) \cong \mathbf{S}(K, \mathbf{C}^*(X, Y)) \end{aligned}$$

natural in all the variables involved. $\square$

3.17. Proposition: *Given X and Y in $\mathbf{C}^*$, the pointed set $\mathbf{C}^*_0(X, Y)$ is isomorphic, via the obvious map, to the limit of the diagram*

$$\prod_K \mathbf{S}_0(XK, YK) \rightrightarrows \prod_u \mathbf{S}_0(XK \otimes \Delta^n, YL),$$

where the first product is indexed by all objects K of $\mathbf{C}$, and the second by all u in the sum over n, K, L of $\mathbf{C}_n(K, L)$. The two maps in the diagram are defined as follows. Given a family of maps $f_M : X(M) \to Y(M)$ and a map $u : K \to L$ in $\mathbf{C}_n$, we have to define two maps $XK \otimes \Delta^n \to YL$. The first is $u_*(f_K \otimes \Delta^n)$ and the second is $f_L u_*$.

Proof. By definition, a map of simplicial functors $f : X \to Y$ is a family of maps $XK \otimes \Delta^n \to YK$, satisfying two conditions. First, for fixed n, this family should give a map of functors $\mathbf{C}_n \to \mathbf{S}_n$. The second condition is found in the proof of proposition 3.11. These conditions are equivalent to the following single condition. Given $u : K \to L$ in $\mathbf{C}_n$, the equality $u_*(f_K \otimes \Delta^n) = f_L u_*$ holds. The conclusion follows immediately from this. $\square$

3.18. Corollary: *Given X, Y in $\mathbf{C}^*$, the pointed simplicial set $\mathbf{C}^*(X, Y)$ is isomorphic, via the obvious map, to the limit of the diagram*

$$\prod_K \mathbf{S}(XK, YK) \rightrightarrows \prod_u \mathbf{S}(XK \otimes \Delta^n, YL).$$

3.19. Definition: The canonical map $K^* \wedge XK \to X$, given X in $\mathbf{C}^*$ and K in $\mathbf{C}$, is the map corresponding to the identity of XK under the isomorphism of proposition 3.14.

3.20. Corollary: *Given X in $\mathbf{C}^*$, the canonical maps $K^* \wedge XK \to X$ induce an isomorphism between X and the colimit of the diagram*

$$\bigvee_u L^* \wedge XK \otimes \Delta^n \rightrightarrows \bigvee_K K^* \wedge XK,$$

where the first map is induced from $u_* : XK \otimes \Delta^n \to XL$, and the second map is induced from $u^* : L^* \otimes \Delta^n \to K^*$.

Proof. Let Z be the colimit. Propositions 3.17, 3.14, and 3.16, imply that the functors $\mathbf{C}_0^*(X, ?)$ and $\mathbf{C}_0^*(Z, ?)$ are isomorphic. □

3.21. Definition: Let $Q : \mathbf{C} \to \mathbf{D}$ be a pointed simplicial functor between small pointed simplicial categories. The pointed simplicial functor $Q_* : \mathbf{C}^* \to \mathbf{D}^*$ takes X to the colimit of the diagram

$$\bigvee_{u \in \bigvee_{n,K,L} \mathbf{C}_n(K,L)} (QL)^* \wedge XK \otimes \Delta^n \rightrightarrows \bigvee_K (QK)^* \wedge XK,$$

where the maps are similar to the ones in the previous corollary.

3.22. Definition: Given pointed simplicial functors $L : \mathbf{C} \to \mathbf{D}$ and $R : \mathbf{D} \to \mathbf{C}$, the pair (L, R) is called an *adjoint pair,* provided that the pointed simplicial functors $\mathbf{C}^{op} \times \mathbf{D} \to \mathbf{S}$, taking (X, Y) to $\mathbf{C}(X, RY)$ and $\mathbf{D}(LX, Y)$, are isomorphic.

3.23. Proposition: *In the situation of definition 3.21, let the pointed simplicial functor $Q^* : \mathbf{D}^* \to \mathbf{C}^*$ be given by precomposing with Q. Then (Q_*, Q^*) is an adjoint pair.*

Proof. By corollaries 3.18, 3.15, and proposition 3.16, the pointed simplicial functors $\mathbf{C}^*(X, Q^*Y)$ and $\mathbf{D}^*(Q_*X, Y)$ are isomorphic. □

3.24. Proposition: *In the situation of proposition 3.23, the pointed simplicial functors Q_*K^* and $(QK)^*$ are isomorphic, for all K in $\mathbf{C}$.*

Proof. By proposition 3.23 and corollary 3.15, the pointed simplicial functors $\mathbf{D}(Q_*K_*, ?)$ and $\mathbf{D}((QK)^*, ?)$ are isomorphic. □

4. Spectra and simplicial functors

In this section we consider two simplicial categories of the form $\mathbf{C}^*$. One of them turns out to be one of the standard models for the simplicial category of spectra (e.g., the spectra of [BF]). The other one, denoted by $\mathbf{SF}$, is closely related to homotopy functors (see remark 4.6).

4.1. Definition: Given K in $\mathbf{S}$, let $\Sigma K = S^1 \wedge K$, and $\Omega K = \mathbf{S}(S^1, K)$. Given a small pointed simplicial category $\mathbf{C}$ and X in $\mathbf{C}^*$, let $\Omega X = X^{S^1}$. A *spectrum* E consists of a sequence of pointed simplicial sets E_n and a sequence of maps $E(n) : \Sigma E_n \to E_{n+1}$, for $n = 0, 1, \ldots$. A map $f : E \to E'$ of spectra is a sequence of maps $f_n : E_n \to E'_n$ such that $f_{n+1} \circ E(n) = E'(n) \circ \Sigma f_n$. Given a spectrum E and a pointed simplicial set K, the pointwise definition of $E \wedge K$ does give a spectrum, thus spectra form a pointed simplicial category $\mathbf{Sp}$.

4.2. Definition: If n is a positive integer and K is in $\mathbf{S}$, define an equivalence relation $\sim$ on the n-fold cartesian product $K^{\times n}$ by $x \sim x'$ if and only if either $x = x'$ or $x_i = 0$ and $x'_j = 0$ for some i and j. Let $K^{\wedge n} = K^{\times n}/\sim$. We denote the class of the sequence $(x_1, x_2, \ldots, x_n)$ in $K^{\wedge n}$ by $x_1 \cdot x_2 \cdots x_n$. Define $K^{\wedge 0} = S^0$. We have an isomorphism $\wedge(K, m, n) : K^{\wedge m} \wedge K^{\wedge n} \to K^{\wedge(m+n)}$, denoted by $x \wedge y \mapsto x \cdot y$, induced by concatenation of sequences. Thus, for any non-negative integers m_1, m_2, and m_3, and any x_1, x_2, and x_3, with x_i in $K^{\wedge m_i}$, the elements $(x_1 \cdot x_2) \cdot x_3$ and $x_1 \cdot (x_2 \cdot x_3)$ of $K^{\wedge(m_1+m_2+m_3)}$ are equal.

We now define a pointed simplicial subcategory $K^{\wedge}$ of $\mathbf{S}$, whose unique zero-object is $*$ and whose non-zero objects are given by the $K^{\wedge n}$, as follows. The pointed simplicial sets of maps from $K^{\wedge n}$ to $K^{\wedge(m+n)}$ are trivial if m is negative, else they are isomorphic to $K^{\wedge m}$, and they are given by the image of $K^{\wedge m}$ in $\mathbf{S}(K^{\wedge n}, K^{\wedge(m+n)})$ under the adjoint to the map $K^{\wedge m} \wedge K^{\wedge n} \to K^{\wedge(m+n)}$ described in the previous paragraph.

Finally, define *the standard n-sphere* S^n recursively, as $S^1 \wedge S^{n-1}$ (note that $\mathbf{S}^{fin}$ contains all S^n). There is an isomorphism $i_n : S^n \to (S^1)^{\wedge n}$, defined by $i_n = \wedge(S^1, 1, n-1) \circ (1_{S^1} \wedge i_{n-1})$. There is a unique pointed simplicial subcategory $\mathbf{Sph}$ of $\mathbf{S}$ with objects all S^n, such that the maps i_n assemble to an isomorphism $\mathbf{Sph} \to (S^1)^{\wedge}$.

4.3. Proposition: *There is an isomorphism (not just an equivalence) of pointed simplicial categories from $\mathbf{Sph}^*$ to $\mathbf{Sp}$, taking the pointed simplicial functor X to the spectrum E with $E_n = XS^n$, and with the map $S^1 \to \mathbf{S}(E_n, E_{n+1})$ given by the composition of the isomorphism $S^1 \to \mathbf{Sph}(S^n, S^{n+1})$ with the map $\mathbf{Sph}(S^n, S^{n+1}) \to \mathbf{S}(XS^n, XS^{n+1})$*

induced by the pointed simplicial functor X. $\square$

From now on, we identify **Sp** and $\mathbf{Sph}^*$, using the previous proposition.

4.4. Definition: Let **SF** be the dual of $\mathbf{S}^{fin}$, and, more generally, let $\mathbf{SF}^{(n)}$ be the dual of the n-fold cartesian product of $\mathbf{S}^{fin}$ with itself. An object X of **SF** can be evaluated on an arbitrary K in **S**, by defining XK as the colimit, over all $L \to K$ with L in $\mathbf{S}^{fin}$, of XL. We obtain a simplicial functor $\mathbf{S} \to \mathbf{S}$ commuting with filtered colimits and, conversely, any such simplicial functor can be canonically identified with one coming from **SF**, in fact, coming from its restriction to **SF**. A map $X \to Y$ in **SF** is called a *a weak equivalence of associated homotopy functors* provided that, for all *fibrant* K in **S**, the map $XK \to YK$ is a weak equivalence (this terminology is explained in remark 4.6). We denote the class of these maps by $\mathbf{SF}_{hw}$.

4.5. Convention: For the remainder of this paper, unless the contrary is explicitly stated, the term "simplicial functor" means "object of **SF**".

4.6. Remark: A functor between two categories, both of which have some fixed class of weak equivalences, is called a *homotopy functor*, provided that it preserves the weak equivalences. Thus the category of homotopy functors from finite pointed simplicial sets to pointed simplicial sets is an interesting category, and so is its full subcategory of pointed homotopy functors (these are the functors preserving zero objects). Note that the category of homotopy functors (between any categories with weak equivalences) has itself a natural class of weak equivalences. It is, however, not clear how to define limits and colimits of homotopy functors, let alone how to put a model structure on them. We already know that **SF** has all limits and colimits, and we shall see in section 8 that there is a model structure on **SF** whose class of weak equivalences is given by $\mathbf{SF}_{hw}$. The connection between **SF** and homotopy functors is explained below.

For the remainder of this remark the term "homotopy functor" means "pointed homotopy functor from $\mathbf{S}_0^{fin}$ to $\mathbf{S}_0$". Any simplicial functor X has an *associated homotopy functor*, defined by $K \mapsto XRK$, where R denotes the simplicial functor "singular complex of the realization". Thus a map of simplicial functors is in $\mathbf{SF}_{hw}$ if and only if it induces a weak equivalence of associated homotopy functors. Further, the construction $F \mapsto \check{F}$ of [W, p. 402] shows that every homotopy functor comes from a simplicial functor, up to weak equivalence. This suggests that the right way to view homotopy functors (at least for purposes of putting a model structure on them) is as simplicial functors.

4.7. Definition: The assignment $K \mapsto K_*$ of definition 3.7 gives a pointed simplicial functor $\mathbf{S}^{fin} \to \mathbf{SF}$ (this involves using the natural isomorphism between $K \wedge L$ and $L \wedge K$ in the special case $L = (\Delta^n)^+$). The pointed simplicial functor $Hom : \mathbf{SF}^{op} \times \mathbf{SF} \to \mathbf{SF}$ is defined by $Hom(X, Y)(K) = \mathbf{SF}(X, Y \circ K_*)$.

4.8. Proposition: *The pointed simplicial functors from* $\mathbf{S}^{fin} \times \mathbf{SF}$ *to* $\mathbf{SF}$*, taking* (K, X) *to* $Hom(K^*, X)$*, resp. to* $X \circ K_*$*, are isomorphic.*

Proof. This follows immediately from corollary 3.15. □

5. The smash product of simplicial functors

In this section we construct a symmetric monoidal pointed simplicial functor $\mathbf{SF} \times \mathbf{SF} \to \mathbf{SF}$, written $(X, Y) \mapsto X \wedge Y$ and called *the smash product of simplicial functors*, satisfying the expected adjoint functor properties.

5.1. Definition: The *external smash product* $\mathbf{SF} \times \mathbf{SF} \to \mathbf{SF}^{(2)}$ is defined by

$$X \tilde{\wedge} Y = P \circ (X \times Y),$$

where P is as in definition 3.7, and $X \times Y : \mathbf{S}^{fin} \times \mathbf{S}^{fin} \to \mathbf{S} \times \mathbf{S}$. Note that P induces a pointed simplicial functor $\mathbf{S}^{fin} \times \mathbf{S}^{fin} \to \mathbf{S}^{fin}$, denoted again by P. The *smash product* $\mathbf{SF} \times \mathbf{SF} \to \mathbf{SF}$ is defined by $X \wedge Y = P_*(X \tilde{\wedge} Y)$, with P_* as in definition 3.21.

5.2. Proposition: *There is an isomorphism*

$$\mathbf{SF}(X, Hom(Y, Z)) \cong \mathbf{SF}(X \wedge Y, Z),$$

natural for X*,* Y*, and* Z *in* $\mathbf{SF}$*.*

Proof. By proposition 3.23, it suffices to show that the left-hand side is naturally isomorphic to $\mathbf{SF}^{(2)}(X \tilde{\wedge} Y, P^* Z)$. Let $[1]'$ be the category with two objects 0 and 1 and two non-identity maps, both from 0 to 1. By corollary 3.18, the left-hand side is the limit of a diagram D defined on $[1]' \times [1]'$, with D_{ij} defined as follows. Let $S_{ij} = \coprod_{m,n} N_i \mathbf{S}^{fin}_m \times N_j \mathbf{S}^{fin}_n$, where "$N$" stands for "nerve", i.e., "N_0" stands for "objects" and "N_1"

stands for "maps". Every element of S_{ij} determines m and n, as well as maps u and v in $\mathbf{S}^{fin}$, with $u : K_0 \otimes \Delta^m \to K_1$, and $v : L_0 \otimes \Delta^n \to L_1$, such that if $i = 0$ then $m = 0$ and u is the identity, and similarly for $j = 0$. Then D_{ij} is the product over all elements of S_{ij} of

$$\mathbf{S}(XK_0 \wedge YL_0 \otimes \Delta^m \otimes \Delta^n, Z(K_1 \wedge L_1)).$$

Note that the limit of a functor F from I to (pointed simplicial) sets injects, via the canonical map, in Fi, provided that i is an object of I that maps to any other object. We use this to identify such a limit with the corresponding subobject of Fi. We want to identify the limit of D as the limit of a functor D' defined on $[1]'$, with D'_i defined as follows. Let $S_i = \coprod_m N_i(\mathbf{S}^{fin}_m \times \mathbf{S}^{fin}_m)$. Every element of S_i determines m, as well as maps u and v of finite pointed simplicial sets, with $u : K_0 \otimes \Delta^m \to K_1$, and $v : L_0 \otimes \Delta^m \to L_1$, such that if $i = 0$ then $m = 0$ and u, v are identity maps. Then D'_i is the product over all elements of S_i of $\mathbf{S}(XK_0 \wedge YL_0 \otimes \Delta^m, Z(K_1 \wedge L_1))$. Thus the first limit is a subobject of the second, and we want to show that this injection is also a surjection. Fix f in $\prod_{(K,L)} \mathbf{S}(XK \wedge YL, Z(K \wedge L))$, and suppose that f is an element of the second limit. Fix $u : K_0 \otimes \Delta^m \to K_1$ and $v : L_0 \otimes \Delta^n \to L_1$. Since f is in the second limit, it satisfies the corresponding naturality statement with respect to $(u, 1_{L_0})$, as well as with respect to $(1_{K_1}, v)$ (here 1_{L_0} denotes the identity map of L_0 in $\mathbf{S}^{fin}_m$, and 1_{K_1} denotes the identity map of K_1 in $\mathbf{S}^{fin}_n$). Thus $f \otimes \Delta^n$ satisfies the corresponding naturality statement with respect to $(u, 1_{L_0})$. Half of the argument that f is in the first limit involves considering the corresponding square, and the square expressing the naturality of f with respect to $(1_{K_1}, v)$, as well as the composition of these two squares. The other half involves using the pairs $(1_{K_0}, v)$, and $(u, 1_{L_1})$. $\square$

5.3. Corollary: *There exists a natural isomorphism*

$$\mathrm{Hom}\,(X \wedge Y, Z) \cong \mathrm{Hom}\,(X, \mathrm{Hom}\,(Y, Z)).$$

Proof. The simplicial functors $\mathrm{Hom}\,(Y, Z) \circ K_*$ and $\mathrm{Hom}\,(Y, Z \circ K_*)$ are naturally isomorphic. The conclusion follows from proposition 5.2. $\square$

We show below that, just like the smash product of pointed sets, this smash product is what we call "strongly symmetric monoidal". This is a property stronger than being symmetric monoidal, and also easier to define (the definition does not involve coherence conditions).

5.4. Definition: Fix a functor $F : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, which is associative, commutative, and unital, up to isomorphism. We say that F is is *strongly symmetric monoidal* provided that, for any non-negative integer n, the n-th iterate $F^{(n)} : \mathbf{C}^n \to \mathbf{C}$ has trivial automorphism group. Here $F^{(n)}$ is defined as follows: It equals the constant functor with value some fixed unit if $n = 0$, it equals the identity if $n = 1$, and, for $n > 1$, $F^{(n)} = F(1_{\mathbf{C}} \times F^{(n-1)})$.

5.5. Definition: Let $\mathbb{S}$ be the object of $\mathbf{SF}$ given by the inclusion of $\mathbf{S}^{fin}$ in $\mathbf{S}$.

5.6. Proposition: *There exists a canonical isomorphism* $(S^0)^* \to \mathbb{S}$. □

5.7. Lemma: *The simplicial functors* $(K \wedge L)^*$ *and* $K^* \wedge L^*$ *are isomorphic, for any* K *and* L *in* $\mathbf{S}^{fin}$.

Proof. This is a special case of proposition 3.24. □

5.8. Remark: Recall from [L2] the smash product of Γ-spaces, and that Γ-spaces may be identified with a special kind of simplicial functors (namely, these simplicial functors which can be evaluated degreewise, cf. convention 2.11 of [L2]) in such a way that representable Γ-spaces correspond naturally to representable simplicial functors (in fact, the Γ-space Γ^n of [L2] corresponds to the simplicial functor K^*, with K a pointed set of cardinality $n + 1$). The smash product of Γ-spaces corresponds to the smash product of simplicial functors under this identification. This is so for representable objects by the above lemma, and every object is a colimit of representables, cf. the proof of proposition 2.16 in [L2].

5.9. Theorem: *The smash product of simplicial functors is strongly symmetric monoidal, with unit* $\mathbb{S}$.

5.10. Lemma: *The smash product of simplicial functors is associative and commutative, up to natural isomorphism, and* $\mathbb{S}$ *acts as a unit, up to natural isomorphism.*

Proof of theorem 5.9. Given lemma 5.10, it remains to show that, given a non-negative integer n and simplicial functors $X_1, \ldots, X_n$, every natural automorphism of $X_1 \wedge \cdots \wedge X_n$ equals the identity. The case $n = 2$ should suffice to explain the proof, so we assume $n = 2$ below, and we write X and Y instead of X_1 and X_2.

Let $\phi = \phi_{(X,Y)}$ be an automorphism of $X \wedge Y$, natural in the simplicial functors X and Y. Consider the special case $X = K^*$and $Y = L^*$ for some K, L. By lemma 5.7 and corollary 3.15, $\mathbf{SF}(K^* \wedge L^*, K^* \wedge L^*) \cong \mathbf{S}(K \wedge L, K \wedge L)$. Let $\phi = f^*$ for some $f : K \wedge L \to K \wedge L$ in $\mathbf{S}^{fin}$. We show that all $k \wedge l$ in $K_0 \wedge L_0$ are fixed under f. Note that this is trivially true if $K = L = S^0$, since ϕ is an automorphism. Let $\hat{k} : S^0 \to K$ and $\hat{l} : S^0 \to L$ correspond to k and l. It suffices to show that $f \circ (\hat{k} \wedge \hat{l}) = \hat{k} \wedge \hat{l}$. But

$$\begin{aligned} f \circ (\hat{k} \wedge \hat{l}) &= (f \circ (\hat{k} \wedge \hat{l}))^* 1_{K \wedge L} \\ &= (\hat{k} \wedge \hat{l})^* f^* 1_{K \wedge L} \\ &= (\hat{k} \wedge \hat{l})^* \phi 1_{K \wedge L} \\ &= (\hat{k}^* \wedge \hat{l}^*) \phi 1_{K \wedge L} \\ &= \phi(\hat{k}^* \wedge \hat{l}^*) 1_{K \wedge L} \\ &= \phi(\hat{k} \wedge \hat{l}) \\ &= \hat{k} \wedge \hat{l}, \end{aligned}$$

where the fourth equality follows from identifying $(K \wedge L)^*$ with $K^* \wedge L^*$ using lemma 5.7, the fifth equality follows from naturality, and the last equality follows since $\phi_{(S^0, S^0)} = 1$.

By definition of $X \wedge Y$, there is a map onto $X \wedge Y$ from the sum, over all (K, L), of $(K \wedge L)^* \wedge XK \wedge YL$. The map

$$(K \wedge L)^*_0 M \wedge X_0 K \wedge Y_0 L \to (X \wedge Y)_0 M$$

can be described as follows, after identifying $(K \wedge L)^*$ and $K^* \wedge L^*$. It takes $f \wedge a \wedge b$ to $(\hat{a} \wedge \hat{b}) f$, where $a \in X_0 K$ and $\hat{a}$ is the corresponding map $K^* \to X$, and similarly for b, and $f \in (K \wedge L)^*_0 M$. Thus every vertex of $(X \wedge Y)M$ is in the image of a map $K^* \wedge L^* \to X \wedge Y$. It follows that ϕ is the identity on vertices.

Given Z in $\mathbf{SF}$, there is a natural map $Z(L^K) \to (ZL)^K$, where $L^K = \mathbf{S}(K, L)$. The adjoint map $Z(L^K) \wedge K \to ZL$ is itself adjoint to the composition of $K \to \mathbf{S}(L^K, L) \to \mathbf{S}(Z(L^K), ZL)$, where the first map is adjoint to the evaluation map (the evaluation map is adjoint to the identity of L^K), and the second map is the one induced by the simplicial functor Z. This map is an isomorphism if Z is representable. Since ϕ is the identity on vertices, it follows that $\phi^{(\Delta^n)^+}$ is the identity on vertices for X and Y representable, i..e, that ϕ is the identity on n-simplices for X and Y representable. It follows that ϕ is the identity on n-simplices for X

and Y of the form (representable)$\otimes\Delta^n$, because a simplex of $\Delta^n \times \Delta^n$ is determined by its vertices.

Finally, for any X and Y, the canonical map $(K \wedge L)^* \wedge XK \wedge YL \to X \wedge Y$ is isomorphic to the smash product of $K^* \wedge XK \to X$ and $L^* \wedge YL \to Y$. Thus every simplex of $X \wedge Y$ is in the image of a map $X' \wedge Y' \to X \wedge Y$, with the primed simplicial functors of the form (representable)$\otimes\Delta^n$. $\square$

Proof of lemma 5.10. We claim that $X \wedge \mathbb{S} \cong X$. This follows from proposition 5.2, proposition 5.6, and the fact that, by proposition 4.8, $\mathrm{Hom}\,((S^0)^*, Z)$ is isomorphic to Z.

To check that $X \wedge Y \cong Y \wedge X$, we check that $\mathbf{SF}(X \wedge Y, Z) \cong \mathbf{SF}(Y \wedge X, Z)$ for all Z in $\mathbf{SF}$. This follows from the isomorphisms $\mathbf{SF}^{(2)}(X \tilde{\wedge} Y, P^* Z) \cong \mathbf{SF}^{(2)}(Y \tilde{\wedge} X, P^* Z \circ T)$ and $P^* Z \cong P^* Z \circ T$, where T is the obvious involution of $\mathbf{S}^{fin} \times \mathbf{S}^{fin}$.

Finally, we compare both $(X \wedge Y) \wedge Z$ and $X \wedge (Y \wedge Z)$ to a more symmetric simplicial functor $X \wedge Y \wedge Z$. We will give only one comparison (the other is similar). Here $X \wedge Y \wedge Z$ is defined as $P^{(3)}_*(X \tilde{\wedge} Y \tilde{\wedge} Z)$, where $P^{(3)}$ is as in definition 5.4 (i.e., it takes (K, L, M) to $K \wedge L \wedge M$) and the external smash product $X \tilde{\wedge} Y \tilde{\wedge} Z$ is defined in the obvious way (to simplify the exposition, we write as if the smash product in $\mathbf{S}$ was associative, instead of associative up to unique natural isomorphism). The isomorphism $\mathbf{SF}((X \wedge Y) \wedge Z, W) \cong \mathbf{SF}(X \wedge Y \wedge Z, W)$ is obtained by observing that

$$\begin{aligned} \mathbf{SF}((X \wedge Y) \wedge Z, W) &\cong \mathbf{SF}(X \wedge Y, \mathrm{Hom}\,(Z, W)) \\ &\cong \mathbf{SF}^{(2)}(X \tilde{\wedge} Y, P^* \mathrm{Hom}\,(Z, W)) \\ &\cong \mathbf{SF}^{(3)}(X \tilde{\wedge} Y \tilde{\wedge} Y, (P^{(3)})^* W), \end{aligned}$$

where the last isomorphism is similar to the one in the proof of proposition 5.2. $\square$

5.11. Definition: We define a natural map $X \wedge Y \to X \circ Y$, which we call the *assembly map*. This map is an isomorphism when Y is representable (proposition 5.13).

We first define a natural map $X(K) \wedge L \to X(K \wedge L)$. This is the map that most resembles other assembly maps in the literature, and it is the special case $Y = \mathbb{S} \wedge L$. Its adjoint $L \to \mathbf{S}(X(K), X(K \wedge L)$ is defined as the composition $L \to \mathbf{S}(K, K \wedge L) \to \mathbf{S}(XK, X(K \wedge L))$, where the first map is adjoint to the identity, and the second map is the one defined by the simplicial functor X. There is a similar map $K \wedge X(L) \to X(K \wedge L)$. We use the name *special assembly map* to refer to any of these two maps.

To handle the general case, by definition of the smash product, it is enough to specify a natural map $XK \wedge YL \to XY(K \wedge L))$. This is defined as the composition

$$XK \wedge YL \to X(K \wedge YL) \to XY(K \wedge L),$$

where the first map is a special assembly map, and the second is given by applying X to a special assembly map.

5.12. Remark: We are now able to identify the monoids in this smash product. Surprisingly enough, these turn out to be well-known (although this is no surprise to the reader familiar with the smash product of Γ-spaces, cf. remark 2.13 of [L2]).

Let us say that a simplicial functor X is a *monoid*, if there are maps $\mu : X \wedge X \to X$ and $\eta : \mathbb{S} \to X$ satisfying the usual associativity and unit conditions. Then μ corresponds to $\widetilde{\mu} : X \widetilde{\wedge} X \to P^* X$, i.e., to a natural map $\widetilde{\mu} : XK \wedge XL \to X(K \wedge L)$. In fact, X is a monoid if and only if it is an FSP, as defined by Bökstedt in [B], under $\widetilde{\mu}$, η, and the special assembly map of definition 5.11.

5.13. Proposition: *The assembly map $X \wedge Y \to X \circ Y$ is an isomorphism, whenever Y is representable.*

Proof. Fix a representable Y. The case X is representable follows from lemma 5.7. The functor $X \circ Y$ preserves all limits and colimits (recall that Y is fixed). The functor $X \wedge Y$ preserves all colimits, since it is a left adjoint. Similarly, both functors commute with smashing by pointed simplicial sets. The conclusion now follows from corollary 3.20. □

5.14. Remark: The composition product $(X, Y) \mapsto X \circ Y$ is associative and unital (with unit $\mathbb{S}$), up to natural isomorphism (it is not strictly associative, because X is originally defined only on $\mathbf{S}^{fin}$), and the assembly map is compatible with associativity and unit isomorphisms. In the language of monoidal categories, the assembly map makes the identity functor of **SF** a lax monoidal functor from the monoidal category $(\mathbf{SF}, \circ, \mathbb{S})$ to the monoidal category $(\mathbf{SF}, \wedge, \mathbb{S})$.

5.15. Remark: If the restriction of the smash product functor P to $\mathbf{Sph} \times \mathbf{Sph}$ would lift to **Sph** (which it does not) then the methods of this section would give a symmetric monoidal product of spectra (such a product seems not to exist).

A symmetric monoidal product on the pointed simplicial category of symmetric spectra is constructed in [HSS]. Let $\mathbf{Sph}^{\Sigma}$ be the smallest pointed simplicial subcategory of $\mathbf{S}$ containing both $\mathbf{Sph}$ and, for all n, the obvious copy of Σ_n inside $\mathbf{S}(S^n, S^n)$. Then the restriction of P to $\mathbf{Sph}^{\Sigma} \times \mathbf{Sph}^{\Sigma}$ does lift to $\mathbf{Sph}^{\Sigma}$, and the methods of this section give a symmetric monoidal product in $(\mathbf{Sph}^{\Sigma})^*$. In fact, the pointed simplicial category $(\mathbf{Sph}^{\Sigma})^*$ is isomorphic to the pointed simplicial category of symmetric spectra, and the symmetric monoidal products correspond under this isomorphism.

A symmetric monoidal product on the pointed simplicial category of Γ-spaces is constructed in [L2]. Let $\mathbf{S}^{dfin}$ be the full pointed simplicial subcategory of $\mathbf{S}^{fin}$ with objects the discrete objects of $\mathbf{S}^{fin}$. Then the restriction of P to $\mathbf{S}^{dfin} \times \mathbf{S}^{dfin}$ does lift to $\mathbf{S}^{dfin}$, and the methods of this section give a symmetric monoidal product in $(\mathbf{S}^{dfin})^*$. In fact, the pointed simplicial category $(\mathbf{S}^{dfin})^*$ is equivalent to the pointed simplicial category of Γ-spaces, and the symmetric monoidal products correspond under this equivalence.

6. Model categories

Model categories were introduced by Quillen in [Q]. In this section, we give an exposition of the parts of the general theory that are needed in this paper.

6.1. Definition: Fix a category $\mathbf{C}$ and a commutative diagram

$$\begin{array}{ccc} A & \rightarrow & C \\ \downarrow & & \downarrow \\ B & \rightarrow & D \end{array}$$

in $\mathbf{C}$. If the diagram is a pushout, then we say that *the map $C \to D$ is a cobase change of the map $A \to B$ (along the map $A \to C$).* If the diagram is a pullback, then we say that *the map $A \to B$ is a base change of the map $C \to D$ (along the map $B \to D$).* If there exists a map $B \to C$ such that both resulting triangles commute, then we say that *the map $A \to B$ has the left lifting property with respect to the map $C \to D$,* and we say also that *the map $C \to D$ has the right lifting property with respect to the map $A \to B$.* Given a class S of maps in $\mathbf{C}$, S^l denotes the class of maps in $\mathbf{C}$ that have the left lifting property with respect to all the maps in S, and S^r denotes the class of maps in $\mathbf{C}$ that have the right lifting property with respect to all the maps in S. If $S = \{f\}$, we abbreviate S^l and S^r by f^l and f^r.

Given a commutative diagram

$$\begin{array}{ccccc} A & \to & C & \to & A \\ \downarrow & & \downarrow & & \downarrow \\ B & \to & D & \to & B \end{array}$$

in $\mathbf{C}$, such that both the composite $A \to A$ and the composite $B \to B$ are identities, we say that *the map $A \to B$ is a retract of the map $C \to D$.*

A limit E_∞ of a diagram $E = E_1 \leftarrow E_2 \leftarrow \cdots$ in $\mathbf{C}$ is called a *sequential limit,* and a colimit E^*_∞ of a diagram $E^* = E^*_1 \to E^*_2 \to \cdots$ in $\mathbf{C}$ is called a *sequential colimit.* The canonical map between $E^{(*)}_1$ and $E^{(*)}_\infty$ will be called the *composition* of the diagram $E^{(*)}$. Let p_n be the map $E_{n+1} \to E_n$, and i_n be the map $E^*_n \to E^*_{n+1}$. We denote the composition of E by $p_1p_2\cdots$, and the composition of E^* by $\cdots i_2i_1$. A class S of maps in $\mathbf{C}$ is called *closed under sequential (co)limits* if, for all diagrams $E^{(*)}$ as above, it contains any composition of $E^{(*)}$ whenever it contains all the maps of $E^{(*)}$.

A class S of maps in $\mathbf{C}$ is called *closed under retracts* if, for all maps f and g in $\mathbf{C}$ with f a retract of g and g in S, the map f is also in S. Finally, S is called *closed under (co)base change* if, for all maps f and g in $\mathbf{C}$ with f a (co)base change of g and g in S, the map f is also in S.

6.2. Proposition: *Let S be any class of maps in a category $\mathbf{C}$. Then S^l is closed under sums, cobase change, sequential colimits, and retracts. Also S^r is closed under products, base change, sequential limits, and retracts.* □

6.3. Proposition: *Let f, p, i be maps in a category $\mathbf{C}$ with $f = pi$. Then $f \in p^l$ implies that f is a retract of i. Also $f \in i^r$ implies that f is a retract of p.* □

6.4. Definition: An object X of a category $\mathbf{C}$ is called *small* if, for every diagram E in $\mathbf{C}$ of the form $E_1 \to E_2 \to \cdots$, the canonical map $\operatorname{colim} \mathbf{C}(X, E) \to \mathbf{C}(X, \operatorname{colim} E)$ is an isomorphism.

Given a class S of maps of a category $\mathbf{C}$ with small domains, *the class S' of cofibrations generated by S* is defined to be the class of all retracts of all compositions $X_0 \to X_1 \to \cdots$ in $\mathbf{C}$ satisfying the following property. For all $n \geq 0$, there exists a set I, a family $\{s_i : A_i \to B_i \mid i \in I\}$ of maps in S indexed by I, a map $f : \coprod_i A_i \to X_n$, and a pushout square expressing $X_n \to X_{n+1}$ as the cobase change of $\coprod_i s_i$ along f.

6.5. Proposition: *(The small object argument) Let* $\mathbf{C}$ *be a category having all colimits and* $f : X \to Y$ *be a map in* $\mathbf{C}$*. Given a set* S *of maps in* $\mathbf{C}$ *with small domains, there exists a factorization* $f = pi$ *in* $\mathbf{C}$*, such that* p *is in* S^r *and* i *is in the class of cofibrations generated by* S*.*

Proof. Given $s \in S$, say A_s is the domain of s and B_s is the codomain of s. Given a map $q : Z \to Y$ in $\mathbf{C}$, let $C(s,q)$ be the set of all pairs $g = (g_1, g_2)$ in $\mathbf{C}(A_s, Z) \times \mathbf{C}(B_s, Y)$ such that $qg_1 = g_2 s$. Consider the following pushout

$$\begin{array}{ccc} \coprod_s \coprod_{g \in C(s,q)} A_g & \to & Z \\ \downarrow & & \downarrow \\ \coprod_s \coprod_{g \in C(s,q)} B_g & \to & W(q) \end{array}$$

where, for $g \in C(s,q)$, $A_g = A_s$ and $B_g = B_s$. The map $A_g \to Z$ is g_1, and the map $A_g \to B_g$ is s. Denote by $i'(q)$ the map $Z \to W(q)$, and by $h(g)$ the map $B_g \to W(q)$. Note that $h(g)s = i'(q)g_1$. By the definition of $C(s,q)$, we have a map $p'(q) : W(q) \to Y$ with $q = p'(q)i'(q)$ and $g_2 = p'(q)h(g)$.

Let $X_0 = X$ and $p_0 = f$. For $n \geq 1$, let $X_n = W(p_{n-1})$, $p_n = p'(p_{n-1})$, and $i_{n-1} = i'(p_{n-1})$. Let $i : X_0 \to X_\infty$ be the composition $\cdots i_1 i_0$. Let $j_n : X_n \to X_\infty$ be the canonical map, so that $j_{n+1} i_n = j_n$ and $i = j_0$. Since $p_n i_{n-1} = p_{n-1}$, there exists $p : X_\infty \to Y$ by $pj_n = p_n$. In particular, $f = pi$.

To show that $p \in S^r$, let $G \in C(s,p)$. Since A_s is small, there exists n with $j_n g_1 = G_1$. Thus, setting $g_2 = G_2$, we obtain $g \in C(s, p_n)$. We claim that $j_{n+1}h(g)$ is the required lifting. For $pj_{n+1}h(g) = p_{n+1}h(g) = g_2 = G_2$, and $j_{n+1}h(g)s = j_{n+1}i_n g_1 = j_n g_1 = G_1$. □

6.6. Proposition: *In the situation of proposition 6.5, suppose also that* S *is countable, that the codomain of every map in* S *is small, that* f *is a map between small objects, and that the set of maps between any two small objects in* $\mathcal{C}$ *is finite. Then the map* i *can be factored as a composition* $X_0 \to X_1 \to \cdots$ *of maps in* S'*, and each* $X_0 \to X_n$ *can be factored as a composition* $X_0 = X_{(n,0)} \to X_{(n,1)} \to \cdots$ *of maps in* S' *between small objects.*

Proof. Recall from the proof of proposition 6.5 that i is the composition of maps in S' of the form $X = X_0 \to X_1 \to \cdots$, with f factoring as a composition $X \to X_n \to Y$. Recall also that there is a construction factoring any $q : Z \to Y$ as $Z \to W(q) \to Y$, such that $X_{n+1} = W(X_n \to Y)$ and such that the map $X_n \to X_{n+1}$ is part of the resulting factorization of $X_n \to Y$. Thus it suffices to prove that, given maps in S' of the form $Z_0 \to Z_1 \to \cdots$ of small objects with colimit Z, and given a map $q : Z \to Y$, then the composition $Z_0 \to Z \to W(q)$ can be factored as a composition $Z_0' \to Z_1' \to \cdots$ of maps in S' between small objects. But, if $\{s_1, s_2, \ldots\}$ is an enumeration of S, then Z_n' can be taken to be the value of $W(Z_n \to Y)$ obtained by taking $S = \{s_1, \ldots, s_n\}$. $\square$

6.7. Definition: Let $\mathbf{C}$ be a category having all finite limits and colimits. Given an ordered triple $\mathbf{M} = (\mathbf{M}_w, \mathbf{M}_f, \mathbf{M}_c)$ of classes of maps of $\mathbf{C}$, we define two more classes of maps by $\mathbf{M}_{af} = \mathbf{M}_w \cap \mathbf{M}_f$ and $\mathbf{M}_{ac} = \mathbf{M}_w \cap \mathbf{M}_c$. We call the maps in these classes, respectively, *weak equivalences, fibrations, cofibrations, acyclic fibrations, and acyclic cofibrations in* $\mathbf{M}$. If the map from an initial object (an initial object exists in $\mathbf{C}$, since the empty diagram has a colimit) to an object X of $\mathbf{C}$ is a cofibration, then we say that X is *cofibrant (in* $\mathbf{M}$*).* If the map to a terminal object from an object X of $\mathbf{C}$ is a fibration, then we say that X is *fibrant (in* $\mathbf{M}$*).*

Given such $\mathbf{C}$ and $\mathbf{M}$, we say that $\mathbf{M}$ *is a closed model structure on* $\mathbf{C}$, if it satisfies the following conditions.

1. For any maps f and g in $\mathbf{C}$ with gf defined, if two of the three maps f, g, and gf are in $\mathbf{M}_w$, so is the third.

2. The classes $\mathbf{M}_w$, $\mathbf{M}_f$, and $\mathbf{M}_c$ are closed under retracts.

3. The inclusions $\mathbf{M}_c \subset (\mathbf{M}_{af})^l$ and $\mathbf{M}_{ac} \subset (\mathbf{M}_f)^l$ hold.

4. For all $f \in \mathbf{C}$ there exist $p \in \mathbf{M}_{af}$, $i \in \mathbf{M}_c$, $q \in \mathbf{M}_f$, and $j \in \mathbf{M}_{ac}$, with $f = pi = qj$.

A *closed model category* is a category $\mathbf{C}$ with some fixed closed model structure. For a closed model category $\mathbf{C}$, the distinguished classes of maps will be denoted simply by $\mathbf{C}_w$, $\mathbf{C}_f$, and so on.

6.8. Proposition: *For any closed model category* $\mathbf{C}$*, the equalities* $\mathbf{C}_{af}^l = \mathbf{C}_c$*,* $\mathbf{C}_f^l = \mathbf{C}_{ac}$*,* $\mathbf{C}_c^r = \mathbf{C}_{af}$*, and* $\mathbf{C}_{ac}^r = \mathbf{C}_f$ *hold.*

Proof. By definition, $\mathbf{C}_{af}^l \supset \mathbf{C}_c$. Conversely, a map f in $\mathbf{C}_{af}^l$ is a retract of a map i in $\mathbf{C}_c$, by proposition 6.3 and condition 4 of definition 6.7. Finally, f is in $\mathbf{C}_c$ by condition 2 of definition 6.7. The proofs of the remaining equalities are similar (note that $T \subset S^l \Leftrightarrow S \subset T^r$). $\square$

6.9. Definition: Let $\mathbf{C}$ be a pointed simplicial category such that $\mathbf{C}_0$ has all limits and colimits. We say that $\mathbf{C}$ *has all limits and colimits,* provided that we have fixed maps $\mathbf{C} \times \mathbf{S} \to \mathbf{C}$, $(X, K) \mapsto X \wedge K$, and $\mathbf{S}^{op} \times \mathbf{C} \to \mathbf{C}$, $(K, X) \mapsto X^K$, and natural isomorphisms $\mathbf{C}(X_0, X_1^K) \cong \mathbf{C}(X_0 \wedge K, X_1) \cong \mathbf{S}(K, \mathbf{C}(X_0, X_1))$.

6.10. Definition: Let $\mathbf{C}$ be a pointed simplicial category having all limits and colimits. Given maps $p : X \to Y$ and $i : W \to Z$ in $\mathbf{C}$, and a map $j : K \to L$ in $\mathbf{S}$, let p^j be the canonical map from X^L to $X^K \times_{Y^K} Y^L$. Let p^i be the canonical map from $\mathbf{C}(Z, X)$ to $\mathbf{C}(W, X) \times_{\mathbf{C}(W,Y)} \mathbf{C}(Z, Y)$. Let $i \overline{\wedge} j$ be the canonical map from $W \wedge L \cup_{W \wedge K} Z \wedge K$ to $Z \wedge L$.

6.11. Definition: Let $\mathbf{C}$ be a pointed simplicial category having all limits and colimits. A *model structure* on $\mathbf{C}$ consists of a closed model structure on $\mathbf{C}_0$, satisfying the following condition. Given a cofibration i and a fibration p in $\mathbf{C}$, the map p^i is a fibration in $\mathbf{S}$ (see example 6.13), which is a weak equivalence if so is i or p.

6.12. Remark: We compare definition 6.11 with the definitions of [Q]. There is a forgetful functor from pointed simplicial categories to non-pointed simplicial categories. There are also obvious versions of definitions 6.9, 6.10 (in the non-pointed case, we write $X \otimes A$ instead of $X \wedge K$, and A is now a (non-pointed) simplicial set) and 6.11, which apply to non-pointed simplicial categories. The forgetful functor from pointed simplicial categories to non-pointed simplicial categories takes model structures to model structures. In the situation of the non-pointed version of definition 6.11, $\mathbf{C}$ becomes a closed simplicial model category, in the sense of section II.2 of [Q]. Conversely, any closed simplicial model category whose underlying category has all limits and colimits (i.e., not just the finite ones) and for which the objects $X \otimes A$ and X^A are defined for all A (i.e., not just for the finite ones) and depend functorially in X and A, is a special case of the non-pointed version of definition 6.11. Any closed simplicial model category is in particular a closed model category, which in turn is in particular a model category, but we will not need any of these weaker structures. All model structures found in this paper are special cases of (the pointed version of) definition 6.11.

6.13. Example: There is *the standard model structure on* $\mathbf{S}$, for which all special classes of maps are defined by checking on underlying simplicial sets (in particular, the cofibrations are the injective maps). This follows immediately from the corresponding result for (non-pointed) simplicial sets [Q, section II.3]. Since we do not need any other model structure on $\mathbf{S}$, we write simply $\mathbf{S}_w$, $\mathbf{S}_f$, etc., for the corresponding classes of maps in $\mathbf{S}$. We need to know [Q, section II.3] that the fibrations can already be defined as $(\mathbf{S}_{ac}^{fin})^r$, where $\mathbf{S}_{ac}^{fin} = \mathbf{S}_{ac} \cap \mathbf{S}_0^{fin}$, and, similarly, $\mathbf{S}_{af} = (\mathbf{S}_c^{fin})^r$.

6.14. Proposition: *Let* $\mathbf{C}$ *be a pointed simplicial category having all limits and colimits, and let* $\mathbf{M}$ *be a closed model structure on* $\mathbf{C}_0$. *Consider the following conditions.*

1. Given a fibration p *in* $\mathbf{C}$ *and a cofibration* j *in* $\mathbf{S}$*, the map* p^j *is a fibration in* $\mathbf{C}$*, which is a weak equivalence if so is* p *or* j.

2. Given a cofibration i *in* $\mathbf{C}$ *and a cofibration* j *in* $\mathbf{S}$*, the map* $i \overline{\wedge} j$ *is a cofibration in* $\mathbf{C}$*, which is a weak equivalence if so is* i *or* j.

Each of these conditions is equivalent to $\mathbf{M}$ *being a model structure on* $\mathbf{C}$.

Proof. The adjoint functor properties of $? \wedge K$ and $?^K$ imply that the statements $j \in (p^i)^l$, $i \in (p^j)^l$, and $i \overline{\wedge} j \in p^l$, are equivalent. The conclusion follows from proposition 6.8 and example 6.13. □

7. The pointwise model structure

In this section we construct a model structure on pointed simplicial categories of the form $\mathbf{C}^*$, called *the pointwise model structure.* The "strict model category structure on spectra" of [BF] is a special case of this construction.

7.1. Definition: Let $\mathbf{C}$ be a small pointed simplicial category. A map $X \to Y$ in $\mathbf{C}^*$ is called a *pointwise weak equivalence*, resp. a *pointwise fibration*, provided that, for all K in $\mathbf{C}$, the map $XK \to YK$ is a weak equivalence, resp. a fibration, in $\mathbf{S}$. We denote the classes of these maps by $\mathbf{C}^*_{pw}$, resp. $\mathbf{C}^*_{pf}$. Let $\mathbf{C}^*_{paf} = \mathbf{C}^*_{pw} \cap \mathbf{C}^*_{pf}$ and $\mathbf{C}^*_{pc} = (\mathbf{C}^*_{paf})^l$.

We define two more classes of maps of $\mathbf{C}^*$, namely $(\mathbf{C}^*_{pc})^g$ and $(\mathbf{C}^*_{pac})^g$ (the notation $(\mathbf{C}^*_{(a)c})^g$ intends to suggest "*generating* pointwise (acyclic) cofibrations in $\mathbf{C}^*$"; see proposition 7.2 below). Define $(\mathbf{C}^*_{p(a)c})^g$ to be the class of all maps of the form $X \wedge f$ with X a representable simplicial functor in $\mathbf{C}^*$ and f a(n acyclic) cofibration of pointed finite simplicial sets.

7.2. Proposition: *(i) The classes* $\mathbf{C}^*_{pw}$*,* $\mathbf{C}^*_{pf}$*, and* $\mathbf{C}^*_{pc}$ *are part of a model structure* $\mathbf{C}^*_p$ *for* $\mathbf{C}^*$.

(ii) The class of cofibrations generated by $(\mathbf{C}^*_{pc})^g$ *equals* $\mathbf{C}^*_{pc}$. *The class of cofibrations generated by* $(\mathbf{C}^*_{pac})^g$ *equals* $\mathbf{C}^*_{pac}$.

7.3. Definition: In case $\mathbf{C} = \mathbf{S}^{fin}$, we write $\mathbf{SF}_p$ instead of $\mathbf{C}^*_p$. We refer to $\mathbf{SF}_p$ as *the pointwise model structure on simplicial functors.* In case $\mathbf{C} = \mathbf{Sph}$, we write $\mathbf{Sp}_p$ instead of $\mathbf{C}^*_p$. We refer to $\mathbf{Sp}_p$ as *the pointwise model structure on spectra.*

The pointwise model structure on spectra is also constructed in [BF] (proposition 2.2), where it is called "the strict model structure", and similarly the pointwise weak equivalences are called "strict weak equivalences", and so on.

Proof of proposition 7.2. The results of section 3 show that $\mathbf{C}^*$ has all limits and colimits, and condition 1 of proposition 6.14 is true in our case. By proposition 6.14, if the four conditions of definition 6.7 are satisfied, then the proof of part (i) is complete. Condition 1 is true. By proposition 6.2, $\mathbf{C}^*_{pc}$ is closed under retracts. Thus condition 2 is also true.

Let the ordered pair of symbols (d, e) be either (c, af) or (ac, f). Fix a map f in $\mathbf{C}^*$ and apply proposition 6.5 with $S = (\mathbf{C}^*_{pd})^g$. We obtain a factorization $f = pi$ with $p \in ((\mathbf{C}^*_{pd})^g)^r$ and $i \in ((\mathbf{C}^*_{pd})^g)'$. By corollary 3.15 and proposition 3.16, the map in $\mathbf{S}$ obtained by evaluating p on any K is in $(\mathbf{S}^{fin}_d)^r$, which equals $\mathbf{S}_e$ by example 6.13. Thus, by definition, p is in $\mathbf{C}^*_{pe}$. By corollary 3.15, proposition 3.16, and example 6.13, $(\mathbf{C}^*_{pc})^g \subset \mathbf{C}^*_{pc}$. Further, $(\mathbf{C}^*_{pac})^g \subset \mathbf{C}^*_{pw}$, and $(\mathbf{C}^*_{ac})^g \subset \mathbf{C}^*_{pc}$, thus $(\mathbf{C}^*_{pac})^g \subset \mathbf{C}^*_{pac}$. By proposition 6.2, $((\mathbf{C}^*_{pc})^g)' \subset \mathbf{C}^*_{pc}$, thus $((\mathbf{C}^*_{pac})^g)' \subset \mathbf{C}^*_{pc}$. Since $((\mathbf{C}^*_{pac})^g)' \subset \mathbf{C}^*_{pw}$, we have $((\mathbf{C}^*_{pac})^g)' \subset \mathbf{C}^*_{pac}$, and the verification of condition 4 is complete.

Before we verify condition 3, we prove that $((\mathbf{C}^*_{pd})^g)' \supset \mathbf{C}^*_{pd}$, completing the proof of part (ii). Let f in $\mathbf{C}^*_{pd}$, and factor $f = pi$ with $i \in ((\mathbf{C}^*_{pd})^g)'$ and $p \in \mathbf{C}^*_{pe}$. In case $d = c$, $f \in \mathbf{C}^*_{pc} = (\mathbf{C}^*_{paf})^l$, also $p \in \mathbf{C}^*_{paf}$, and proposition 6.3 implies that f is a retract of i, thus $f \in ((\mathbf{C}^*_{pc})^g)'$. In case $d = ac$, p is in $\mathbf{C}^*_{pw}$, since so are i and f. Thus $p \in \mathbf{C}^*_{paf}$. But $f \in \mathbf{C}^*_{pac} \subset \mathbf{C}^*_{pc} = (\mathbf{C}^*_{paf})^l$, and proposition 6.3 implies that f is a retract of i, thus $f \in ((\mathbf{C}^*_{pac})^g)'$.

Finally, we verify condition 3. We have to check that $\mathbf{C}^*_{pac} \subset (\mathbf{C}^*_{pf})^l$, and we know that $\mathbf{C}^*_{pac} = ((\mathbf{C}^*_{pac})^g)'$. The conclusion follows from proposition 6.2, and the fact that, by corollary 3.15, proposition 3.16, and example 6.13, $(\mathbf{C}^*_{pac})^g \subset (\mathbf{C}^*_{pf})^l$. □

8. The homotopy-functor model structure

In this section we construct a model structure on $\mathbf{SF}$, *the homotopy-functor model structure of simplicial functors*, for which the weak equivalences are the maps of simplicial functors inducing weak equivalences of associated homotopy functors. This model structure should be thought of as a model structure on pointed homotopy functors from $\mathbf{S}^{fin}$ to $\mathbf{S}$ (cf. remark 4.6).

8.1. Definition: Given a map $f : X \to Y$ in a pointed simplicial model category $\mathbf{C}$, the *mapping cylinder* Z_f of f is defined to be the colimit of $X \otimes \Delta^1 \leftarrow X \to Y$, where the first map is the composition of $X \cong X \otimes \Delta^0$ and $X \otimes \partial^1$ and the second map is f. Consider the colimit of $X \otimes \Delta^1 \leftarrow X \otimes \partial\Delta^1 \to X \vee Y$, where the first map is $X \otimes (\partial\Delta^1 \subset \Delta^1)$, and the second map is defined as follows. It is the composition of $X \otimes \partial\Delta^1 \cong X \vee X$ and $1_X \vee f$, where the inverse isomorphism $X \vee X \cong X \otimes \partial\Delta^1$ takes the first copy of X to the image of $X \otimes \partial^0$. Then these two colimits are naturally isomorphic. Thus there is a canonical map $X \vee Y \to Z_f$, which is a cofibration if X is cofibrant, since it is a cobase change of $X \otimes (\partial\Delta^1 \subset \Delta^1)$, which is a cofibration by proposition 6.14. On the other hand, the canonical map $X \to X \vee Y$ is a cofibration if Y is cofibrant. There is also a canonical map $Z_f \to Y$, induced from the composition of the projection $X \otimes \Delta^1 \to X$ with f, together with 1_Y. This map is a *homotopy equivalence,* i.e., it has a homotopy inverse (which is a section, and is given by the canonical map, and the homotopy in the other direction is given by using the homotopy between 1_{Δ^1} and the map $\Delta^1 \to \Delta^1$ that projects on the image on ∂^1). Thus, if X and Y are cofibrant, we have a factorization of f as $X \to Z_f \to Y$, with the first map a cofibration, and the second a homotopy equivalence. If $\mathbf{C}$ is given by the pointwise structure on $\mathbf{SF}$, and if $X = L^*$, $Y = K^*$, and f is induced from a map $g : K \to L$, we denote the canonical cofibration $X \to Z_f$ by c_g. We define the class $\mathbf{SF}_{hac}^{extra}$ to consist of all $c_g \barwedge h$, with $g \in \mathbf{S}_w^{fin}$ and $h \in \mathbf{S}_c^{fin}$.

8.2. Definition: Let $\mathbf{SF}_{hc} = \mathbf{SF}_{pc}$, $\mathbf{SF}_{hac}^{g} = \mathbf{SF}_{pac}^{g} \cup \mathbf{SF}_{hac}^{extra}$, and $\mathbf{SF}_{hf} = (\mathbf{SF}_{hac}^{g})^r$.

8.3. Theorem: *(i) The classes $\mathbf{SF}_{hw}$, $\mathbf{SF}_{hf}$, and $\mathbf{SF}_{hc}$ form a model structure $\mathbf{SF}_h$ for $\mathbf{SF}$.*

(ii) The class of cofibrations generated by $\mathbf{SF}_{hac}^{g}$ equals $\mathbf{SF}_{hac}$.

8.4. Definition: We refer to $\mathbf{SF}_h$ as *the homotopy-functor model structure of simplicial functors* (cf. remark 4.6).

8.5. Lemma: *The classes of acyclic fibrations for $\mathbf{SF}_h$ and $\mathbf{SF}_p$ coincide.*

Proof of theorem 8.3. Condition 2 of proposition 6.14 is true in our case, by proposition 7.2 and 6.14, since $\mathbf{SF}_{hc} = \mathbf{SF}_{pc}$ (the part where i is an acyclic cofibration follows since $i \overline{\wedge} j$ is in $\mathbf{SF}_{hw}$ if so is i). By proposition 6.14, if the four conditions of definition 6.7 are satisfied, then the proof of part (i) is complete. Condition 1 is true. We already know that $\mathbf{SF}_{hc}$ is closed under retracts, and proposition 6.2 implies that so is $\mathbf{SF}_{hf}$. Thus condition 2 is also true.

Fix a map f in $\mathbf{SF}$. Proposition 7.2 and lemma 8.5 give the factorization by a cofibration and an acyclic fibration. Apply proposition 6.5 with $S = \mathbf{SF}^g_{hac}$. We obtain a factorization $f = pi$ with $p \in (\mathbf{SF}^g_{hac})^r = \mathbf{SF}_{hf}$ and $i \in (\mathbf{SF}^g_{hac})'$. The inequalities $\mathbf{SF}^g_{hac} \subset \mathbf{SF}_{hw}$, and $\mathbf{SF}^g_{hac} \subset \mathbf{SF}_{hc}$ hold. By proposition 6.2, $(\mathbf{SF}^g_{hac})' \subset \mathbf{SF}_{hc}$. Since $(\mathbf{SF}^g_{hac})' \subset \mathbf{SF}_{hw}$, we have $(\mathbf{SF}^g_{hac})' \subset \mathbf{SF}_{hac}$.

Before we verify condition 3, we prove that $(\mathbf{SF}^g_{hac})' \supset \mathbf{SF}_{hac}$, completing the proof of part (ii). Let f in $\mathbf{SF}_{hac}$, and factor $f = pi$ with $i \in (\mathbf{SF}^g_{hac})'$ and $p \in \mathbf{SF}_{hf}$. Then p is in $\mathbf{SF}_{hw}$, since so are i and f. Thus $p \in \mathbf{SF}_{paf}$, by lemma 8.5. But $f \in \mathbf{SF}_{hac} \subset \mathbf{SF}_{pc} = \mathbf{SF}^l_{paf}$, and proposition 6.3 implies that f is a retract of i, thus $f \in (\mathbf{SF}^g_{hac})'$.

Finally, we verify condition 3. We have to check that $\mathbf{SF}_{hac} \subset \mathbf{SF}^l_{hf}$, and we know that $\mathbf{SF}_{hac} = (\mathbf{SF}^g_{hac})'$. The conclusion follows from proposition 6.2, since, by definition, $\mathbf{SF}^g_{hac} \subset \mathbf{SF}^l_{hf}$. □

8.6. Definition: Let R be the object of $\mathbf{SF}$ taking $K \in \mathbf{S}^{fin}$ to the singular complex of the realization of K.

8.7. Proposition: *The class $\mathbf{SF}_{hf}$ consists of these maps $X \to Y$ in $\mathbf{SF}_{pf}$ for which the square*

$$\begin{array}{ccc} XK & \to & XRK \\ \downarrow & & \downarrow \\ YK & \to & YRK \end{array}$$

is homotopy cartesian for each K in $\mathbf{S}$.

Proof of lemma 8.5. This follows immediately from proposition 8.7. □

8.8. Proposition: *The class* $\mathbf{SF}_{hf}$ *consists of these maps* $f : X \to Y$ *in* $\mathbf{SF}_{pf}$ *for which the square*

$$\begin{array}{ccc} XK & \to & XL \\ \downarrow & & \downarrow \\ YK & \to & YL \end{array}$$

is homotopy cartesian for each weak equivalence $K \to L$ *in* $\mathbf{S}^{fin}$.

Proof of proposition 8.7. Fix a map $X \to Y$ is in $\mathbf{SF}_{hf}$. Given K in $\mathbf{S}^{fin}$, let K_∞ be the target of the map i in the factorization of $K \to *$ given by proposition 6.5 with $S = \mathbf{S}^{fin}_{ac}$. We first reduce to proving the version of the conclusion obtained by replacing RK by K_∞: Since i is an acyclic cofibration and RK is fibrant, the map $j : K \to RK$ factors as ki with $k : K_\infty \to RK$. Further, k is a homotopy equivalence, since K_∞ is also fibrant. Thus both $X(k)$ and $Y(k)$ are weak equivalences, and the reduction follows.

Proposition 6.6 gives that K_∞ is the filtered colimit of certain K_n with $K_0 = K$, and each K_n is the filtered colimit of certain finite $K_{(n,m)}$, with $K_{(n,0)} = K$, and all the maps involved are weak equivalences. We have to show that the map from XK to $YK \times_{YK_\infty} XK_\infty$ is a weak equivalence. Since $\mathbf{S}_w$ and $\mathbf{S}_f$ are closed under filtered colimits, and since pullbacks in $\mathbf{S}$ commute with filtered colimits, we may assume that K_∞ equals some $K_{(n,m)}$. The conclusion now follows from proposition 8.8. The general case, where K is not necessarily finite, follows since K is the filtered colimit of its finite subobjects, and since $K \mapsto RK$ commutes with filtered colimits by the definition of R.

Conversely, assume that f gives such a homotopy cartesian square for every K, and fix $K \to L$ in $\mathbf{S}^{fin}_w$. Suppose first that $K \to L$ is in $\mathbf{S}^{fin}_{ac}$. Factor the map $K \to RK$ through the acyclic cofibration which is the composition of $K \to L$ and $L \to RL$. The resulting map $RL \to RK$ is a homotopy equivalence. As above, we get that its associated square is homotopy cartesian. Since, in a composed square, the "source" square is homotopy cartesian if so are the "target" and the composed square, it follows that the square associated to $K \to L$ is homotopy cartesian. In the general case of a map $K \to L$ in $\mathbf{S}^{fin}_w$, similar arguments (using mapping cyclinders to reduce to the case of an acyclic cofibration, and recalling that the map from the mapping cylinder to L is a homotopy equivalence) show that the square associated to $K \to L$ is homotopy cartesian. Thus $f \in \mathbf{SF}_{hf}$, by proposition 8.8. $\square$

Proof of proposition 8.8. Fix $f : X \to Y$ in $\mathbf{SF}_{pf}$ and $g : K \to L$ in $\mathbf{S}_w^{fin}$. Recall that c_g is a cofibration from L^* to Z, where Z is the mapping cylinder of g^*. There is a map from the square in the statement of the proposition to

$$\begin{array}{ccc} \mathcal{F}(Z, X) & \to & XL \\ \downarrow & & \downarrow \\ \mathcal{F}(Z, Y) & \to & YL \end{array}$$

which is the identity on the right column, and on the left column it is induced from the homotopy equivalence $Z \to K^*$. Thus the map is a weak equivalence on all four corners, therefore the square in the statement of the proposition is homotopy cartesian if and only if the square above is, i.e., if and only if the fibration f^{c_g} is acyclic. But, by the proof of proposition 6.14, this happens if and only if $f \in (c_g \overline{\wedge} h)^r$ for all maps h in $\mathbf{S}_c^{fin}$. □

9. The stable model structure of simplicial functors

In this section we construct a model structure on $\mathbf{SF}$, *the stable model structure of simplicial functors*, which we show in section 11 is equivalent to the "standard" model structure of spectra (i.e., to the *stable model structure of spectra* introduced in [BF]).

9.1. Definition: Given K in $\mathbf{S}^{fin}$, the canonical map $K \to \Omega\Sigma K$ is a vertex in $\mathbf{S}(K, \Omega\Sigma K)$, which is canonically isomorphic to $\Omega K^* \Sigma K$. Let s'_K denote the map from $(\Sigma K)^* \wedge S^1$ to K^* corresponding to the above vertex in $\Omega K^* \Sigma K$ under the isomorphisms of corollary 3.15 and proposition 3.16. Let s_K denote the canonical map from $(\Sigma K)^* \wedge S^1$ to the mapping cylinder of s'_K. We define the set $\mathbf{SF}_{sac}^{extra}$ to consist of the maps s_K with K in $\mathbf{S}^{fin}$.

Given X in $\mathbf{SF}$, define TX in $\mathbf{SF}$ by $K \mapsto \Omega X \Sigma K$, so that there is a canonical map $X \to TX$. Define $T^\infty X$ as the colimit of the sequence $X \to TX \to T(TX) \to \cdots$, and define *the stabilization* X^s *of* X as $T^\infty RXR$. A map $f : X \to Y$ in $\mathbf{SF}$ is called a *a stable weak equivalence of simplicial functors* provided that the map f^s is a pointwise weak equivalence of simplicial functors. We denote the class of these maps by $\mathbf{SF}_{sw}$. We set $\mathbf{SF}_{sc} = \mathbf{SF}_{pc}$, $\mathbf{SF}_{sac}^g = \mathbf{SF}_{hac}^g \cup \mathbf{SF}_{sac}^{extra}$, and $\mathbf{SF}_{sf} = (\mathbf{SF}_{sac}^g)^r$.

9.2. Theorem: *(i) The classes* $\mathbf{SF}_{sw}$*,* $\mathbf{SF}_{sf}$*, and* $\mathbf{SF}_{sc}$ *form a model structure* $\mathbf{SF}_s$ *for* $\mathbf{SF}$.

(ii) The class of cofibrations generated by $\mathbf{SF}^g_{sac}$ *equals* $\mathbf{SF}_{sac}$.

9.3. Definition: We refer to $\mathbf{SF}_s$ as *the stable model structure of simplicial functors.*

9.4. Lemma: *The classes of acyclic fibrations for* $\mathbf{SF}_s$ *and* $\mathbf{SF}_p$ *coincide.*

9.5. Definition: Given non-negative integers n and l, we say that a simplicial functor X in $\mathbf{SF}^{(n)}$ is $o(l)$ if there exists an integer c such that, for all integers k and all k-connected $K_1, \ldots, K_n$ in $\mathbf{S}$, the pointed simplicial set $X(K_1, \ldots, K_n)$ is $(lk - c)$-connected. we say that a map f in $\mathbf{SF}^{(n)}$ is $o(l)$ if so is its pointwise homotopy fiber. In case $n = 1$, we say that f is $\widetilde{o}(l)$ if fR is $o(l)$.

9.6. Proposition: *Every map in* $\mathbf{SF}^g_{sac}$ *is* $\widetilde{o}(2)$.

Proof. This is non-trivial only for maps of the form s_K, and it is sufficient to consider the maps s'_K. In case $K = \Delta^n/\partial\Delta^n$, the conclusion follows by estimating the connectivity of the canonical map $\Sigma\Omega\Omega^n RL \to \Omega^n RL$. The general case follows by induction on the skeleta of K. □

9.7. Proposition: *If* f *is* $\widetilde{o}(2)$*, then* f *is a stable weak equivalence.*

Proof. In fact, there exists an integer c such that, for all n, the map obtained by evaluating $T^n RfR$ on any pointed simplicial set K is $(n-c)$-connected. □

Proof of theorem 9.2. The proof of theorem 8.3 applies almost word for word, except that two points need an extra argument. First, the claim that $i \overline{\wedge} j$ is in $\mathbf{SF}_{sw}$ if $i : X \to Y$ is in $\mathbf{SF}_{sac}$ and $j : K \to L$ is in $\mathbf{S}_c$. Second, the claim that $(\mathbf{SF}^g_{sac})' \subset \mathbf{SF}_{sw}$. To prove the second claim, note that the class of stable weak equivalences is closed under retracts and sequential colimits, that every map in $\mathbf{SF}^g_{sac}$ is $\widetilde{o}(2)$, and that the class of maps which are both $\widetilde{o}(2)$ and cofibrations are closed under cobase change and finite sums.

To prove the first claim, note that, even without it, we have a closed model structure. Further, by the proof of proposition 6.5, the class $\mathbf{SF}_{sac}$ consists precisely of the retracts of the filtered colimits of maps which are both $\widetilde{o}(2)$ and cofibrations, and these are closed under smashing with pointed simplicial sets. Thus the map $X \wedge K \to Y \wedge K$ and the map $X \wedge L \to Y \wedge L$ are in $\mathbf{SF}_{sw}$. Since $\mathbf{SF}_{sac}$ is closed under cobase change, the map $X \wedge L \to X \wedge L \cup_{X \wedge K} Y \wedge K$ is also in $\mathbf{SF}_{sw}$, thus so is $i \overline{\wedge} j$. □

9.8. Proposition: *The class $\mathbf{SF}_{sf}$ consists of these maps $f : X \to Y$ in $\mathbf{SF}_{hf}$ for which the square*

$$\begin{array}{ccc} XK & \to & \Omega X \Sigma K \\ \downarrow & & \downarrow \\ YK & \to & \Omega Y \Sigma K \end{array}$$

is homotopy cartesian for each K in $\mathbf{S}^{fin}$.

Proof. The proof is similar to the proof of proposition 8.8. □

Proof of lemma 9.4. Fix a map $f : X \to Y$ in $\mathbf{SF}_{haf} = \mathbf{SF}_{paf}$. The diagram in the statement of proposition 9.8 is homotopy cartesian, since the vertical maps are acyclic fibrations. Thus f is in $\mathbf{SF}_{sf}$ (and therefore in $\mathbf{SF}_{saf}$), by proposition 9.8.

Conversely, fix $f : X \to Y$ in $\mathbf{SF}_{saf}$. The diagram

$$\begin{array}{ccc} XK & \to & YK \\ \downarrow & & \downarrow \\ \Omega X \Sigma K & \to & \Omega Y \Sigma K \\ \downarrow & & \downarrow \\ \Omega X R \Sigma K & \to & \Omega Y R \Sigma K \\ \downarrow & & \downarrow \\ \Omega R X R \Sigma K & \to & \Omega R Y R \Sigma K \\ \downarrow & & \downarrow \\ X^{St} K & \to & Y^{St} K \end{array}$$

consists of four squares, which we label, starting from the top, as a, b, c, and d. Square c is homotopy cartesian, since f is in $\mathbf{SF}_{pf}$, since the functors $X \mapsto \Omega X$ and $X \mapsto RX$ preserve pointwise fibrations, since the functor $X \mapsto \Omega X$ preserves pullbacks, and, finally, since the map $X \to RX$ is in $\mathbf{SF}_{pw}$. Square b is homotopy cartesian, since f is in $\mathbf{SF}_{hf}$, and since the functor $X \mapsto \Omega X$ preserves pointwise fibrations and pullbacks. Square a is homotopy cartesian, since f is in $\mathbf{SF}_{sf}$, by proposition 9.8. Consider the composed square abc. This square, i.e., the one determined, in the obvious way, from the maps $X \to TRXR$, $Y \to TRYR$, and f, is homotopy cartesian. The previous sentence remains true, if we substitute a finite iterate T^n instead of T, by similar arguments. Since fibrations of simplicial sets are closed under sequential colimits, and since pullbacks of simplicial sets commute with sequential colimits, the composed square *abcd* is homotopy cartesian. Since f is in $\mathbf{SF}_{sw}$, the bottom horizontal map is in $\mathbf{S}_w$, thus so is the top horizontal map. Thus f is in $\mathbf{SF}_{pw}$, i.e., f is in $\mathbf{SF}_{paf}$. □

10. The stable model structure of spectra

In this section we give an exposition of the part of stable homotopy theory that we need.

10.1. Convention: All representable simplicial functors in this section are spectra (i.e., the value of $(S^n)^*$ on S^m equals $\mathbf{Sph}(S^n, S^m)$).

10.2. Definition: Let n be a non-negative integer. The canonical map $S^0 \to \Omega\Sigma S^0$ is a vertex in $\Omega\mathbf{Sph}(S^n, S^{n+1}) = \Omega(S^n)^* S^{n+1}$. Let s'_n denote the map from $(S^{n+1})^* \wedge S^1$ to $(S^n)^*$ corresponding to the above vertex in $\Omega(S^n)^* S^{n+1}$ under the isomorphisms of corollary 3.15 and proposition 3.16. Let s_n denote the canonical map from $(S^{n+1})^* \wedge S^1$ to the mapping cylinder of s'_n. We define the set $\mathbf{Sp}_{sac}^{extra}$ to consist of the maps s_n with n a non-negative integer.

A spectrum E can also be viewed as a diagram $\overline{E}_0 \to \overline{E}_1 \to \cdots$ in $\mathbf{S}$, such that $\overline{E}_n$ is given by applying Ω^n to a pointed simplicial set E_n (which is part of the data), and the map $\overline{E}_n \to \overline{E}_{n+1}$ is given by applying Ω^n to a map $E_n \to \Omega E_{n+1}$. In this description, a map $E \to E'$ is a map of associated diagrams, such that the map $\overline{E}_n \to \overline{E'_n}$ is given by applying Ω^n to a map $E_n \to E'_n$. This description shows that, given a spectrum E, there exists a spectrum TE, whose associated diagram is given by forgetting $\overline{E}_0$. There is a natural map $E \to TE$, given for each n by the canonical map $E_n \to \Omega E_{n+1}$. Define the spectrum $T^\infty E$ as the colimit of the diagram $E \to TE \to T(TE) \to \cdots$. A map $f : E \to E'$ in $\mathbf{Sp}$ is called a *a stable weak equivalence* provided that the map $T^\infty Rf$ is a pointwise weak equivalence. We denote the class of these maps by $\mathbf{Sp}_{sw}$. We set $\mathbf{Sp}_{sc} = \mathbf{Sp}_{pc}$, $\mathbf{Sp}_{sac}^g = \mathbf{Sp}_{pac}^g \cup \mathbf{Sp}_{sac}^{extra}$, and $\mathbf{Sp}_{sf} = (\mathbf{Sp}_{sac}^g)^r$.

10.3. Remark: Given a spectrum E, let $E[1]$ denote the canonical spectrum with $(E[1])_n = E_{n+1}$. Then it is *not* true that TE equals $\Omega E[1]$. In fact, if $E' = \Omega E$, then the canonical map $E'_0 \to \Omega E'_1$ is *not* given by applying Ω to the canonical map $\hat{E}(0) : E_0 \to \Omega E_1$ (it is given by postcomposing $\Omega\hat{E}(0)$ with the obvious involution; to see this, write down explicitly the definition of E^K, and specialize to $K = S^1$).

10.4. Theorem: *(i) The classes $\mathbf{Sp}_{sw}$, $\mathbf{Sp}_{sf}$, and $\mathbf{Sp}_{sc}$ form a model structure $\mathbf{Sp}_s$ for $\mathbf{Sp}$.*

(ii) The class of cofibrations generated by $\mathbf{Sp}^g_{sac}$ equals $\mathbf{Sp}_{sac}$.

10.5. Definition: We refer to $\mathbf{Sp}_s$ as *the stable model structure of spectra.*

10.6. Lemma: *The classes of acyclic fibrations for $\mathbf{Sp}_s$ and $\mathbf{Sp}_p$ coincide.*

10.7. Definition: We say that a map f in $\mathbf{Sp}$ is *eventually a pointwise weak equivalence* if there exists a non-negative integer N such that, for all integers $n \geq N$, the map f_n is a weak equivalence.

10.8. Proposition: *Every map in $\mathbf{Sp}^g_{sac}$ is eventually a pointwise weak equivalence.*

Proof. This is non-trivial only for maps of the form s_n, and it is sufficient to consider the maps s'_n. But these are eventually isomorphisms, since $(S^{n+1})^* \wedge S^1$ and $(S^n)^*$ differ, up to isomorphism, only on the value at S^n. □

10.9. Proposition: *If a map of spectra is eventually a pointwise weak equivalence, then it is a stable weak equivalence.* □

Proof of theorem 10.4. The proof of theorem 9.2 applies word for word, after making the obvious changes (this is literally true; here are the changes: replace "$\widetilde{o}(2)$" by "eventually pointwise weak equivalence", and replace $\mathbf{SF}$ by $\mathbf{Sp}$). □

10.10. Proposition: *The class $\mathbf{Sp}_{sf}$ consists of these maps $f : E \to E'$ in $\mathbf{Sp}_{pf}$ for which the square*

$$\begin{array}{ccc} E_n & \to & \Omega E_{n+1} \\ \downarrow & & \downarrow \\ E'_n & \to & \Omega E'_{n+1} \end{array}$$

is homotopy cartesian for each non-negative integer n.

Proof. The proof is similar to the proof of proposition 8.8. □

Proof of lemma 10.6. The proof is similar to the proof of lemma 9.4 (in fact, it is easier than the proof of lemma 9.4, because for spectra we do not have to worry about evaluating on fibrant objects). □

We now prove some standard facts about spectra, which we need in the rest of the paper.

10.11. Definition: An Ω-*spectrum* is a spectrum which is fibrant in $\mathbf{Sp}_s$.

10.12. Proposition: *For any spectrum E, the spectrum $T^\infty RE$ is an Ω-spectrum.* □

10.13. Definition: Let n be a non-negative integer. *The n-skeleton $E^{(n)}$ of the spectrum E* is defined to be the spectrum obtained from E by letting the map $E^{(n)}(m)$ be equal to either $E(m)$ if $m - n < 0$, or the identity map of $\Sigma^{m-n+1}E_n$ if $m - n \geq 0$.

10.14. Proposition: *For any spectrum E and any non-negative integer n, the following hold.*

(i) There is a canonical diagram $E^{(0)} \to E^{(1)} \to \cdots \to E$, which expresses E as the sequential colimit of its skeleta.

(ii) There is a canonical map $(S^n)^* \wedge E_n \to E^{(n)}$, which is eventually an isomorphism. □

10.15. Definition: Let n be an integer. *The n-th homotopy group $\pi_n E$ of the spectrum E* is the abelian group given by the colimit over m of $\pi_{n+m}|E_m|$.

10.16. Proposition: *The following hold.*

(i) There are natural isomorphisms between the n-th homotopy group of the spectrum E and the $(n+m)$-th homotopy group of the pointed topological space $|(T^\infty RE)_m|$, provided that $n+m$ is positive.

(ii) A map of spectra is a stable weak equivalence if and only if it induces an isomorphism on all homotopy groups.

(iii) There are natural isomorphisms between the n-th homotopy group of the Ω-spectrum E and the $(n+m)$-th homotopy group of the pointed topological space $|E_m|$, provided that $n+m$ is positive.

(iv) A map of Ω-spectra is a stable weak equivalence if and only if it is a

pointwise weak equivalence. ☐

10.17. Proposition: *Given $E^0 \to E^1$ an injective map of spectra, with E^2 its cofiber, there is a natural exact sequence*

$$\cdots \to \pi_{n+1}E^2 \to \pi_n E^0 \to \pi_n E^1 \to \pi_n E^2 \to \pi_{n-1}E^0 \to \cdots,$$

where the maps $\pi_n E^i \to \pi_n E^{i+1}$ are the canonical maps.

Proof. Since sequential colimits of abelian groups preserve exact sequences, and since the functor $E \mapsto E^{(n)}$ preserves injections and cofibers, it suffices to show that the required exact sequence exists on all n-skeleta. By proposition 10.14, we may assume that all E^i are given by $(S^n)^* \wedge K^i$ for some K^i in $\mathbf{S}$, and the maps between the E^i are induced by maps between the K^i. The conclusion follows from the Blakers-Massey theorem (the version given in Spanier's book [S] as theorem 9.3.5 is sufficient), which provides a constant c such that, for $m \geq k + c$, the following two statements are true. First, there are natural isomorphisms between $\pi_{k-n}E^i$ and $\pi_{k+m}S^m \wedge K^i$. Second, there is a natural exact sequence

$$\begin{aligned} &\cdots \to \pi_{k+m+1}(S^m \wedge K^2) \to \pi_{k+m}(S^m \wedge K^0) \to \pi_{k+m}(S^m \wedge K^1) \\ &\to \pi_{k+m}(S^m \wedge K^2) \to \pi_{k+m-1}(S^m \wedge K^0) \to \cdots, \end{aligned}$$

where the maps $\pi_{k+m}(S^m \wedge K^i) \to \pi_{k+m}(S^m \wedge K^{i+1})$ are the canonical maps. ☐

10.18. Corollary: *Let $f^i : E_0^i \to E_1^i$ and $g_i : E_i^0 \to E_i^1$ be maps of spectra, for $i = 0, 1$, such that $g_1 f^0 = f^1 g_0$, and the g_i are injective. Then, if any two of f^0, f^1/f^0, and f_1 are stable weak equivalences, so is the third.* ☐

10.19. Definition: Let τ be the natural isomorphism that interchanges the second and third factors in the smash product $K \wedge L \wedge M \wedge N$ in $\mathbf{S}$.

10.20. Definition: The *smash product* of the spectra E and E' is the spectrum $E \wedge E'$ defined as follows. Let $(E \wedge E')_{2n} = E_n \wedge E'_n$, and $(E \wedge E')_{2n+1} = S^1 \wedge (E \wedge E')_{2n}$. Finally, let the map $(E \wedge E')(2n)$ be the identity, and the map $(E \wedge E')(2n+1)$ equal $(E(n) \wedge E'(n)) \circ t$.

10.21. Proposition: *Smashing with a spectrum preserves stable weak equivalences.*

Proof. Smashing with a pointed simplicial set preserves stable weak equivalences: It preserves acyclic cofibrations since $\mathbf{Sp}_s$ is a simplicial model category, and it preserves acyclic fibrations, since these are pointwise weak equivalences. But, as the isomorphisms

$$\pi_k(E \wedge E') \cong \mathrm{colim}_m\, \pi_{k+2m}|E_m \wedge E'_m| \cong \mathrm{colim}_{m,n}\, \pi_{k+m+n}|E_m \wedge E'_n|$$

show, for fixed k and E, the group $\pi_k(E \wedge E')$ depends functorially on a diagram involving only the groups $\pi_k(E_m \wedge E')$. □

11. Comparison of the stable model structures

In this section we compare the stable model structures for spectra and simplicial functors.

11.1. Definition: Let $\mathbf{C}$ and $\mathbf{D}$ be simplicial categories, each with a model structure, and let (L, R) be an adjoint pair of simplicial functors, with $L : \mathbf{C} \to \mathbf{D}$. The pair (L, R) is a *Quillen equivalence,* provided that L preserves cofibrations,that R preserves fibrations, and that the following condition is satisfied. For every cofibrant X in $\mathbf{C}$ and fibrant Y in $\mathbf{D}$, a map $X \to RY$ is a weak equivalence if and only if so is its adjoint.

Quillen equivalences induce adjoint equivalences of homotopy categories, but we will not need this here. One should think of the equivalence relation generated by "being Quillen equivalent to" as "the right way" to partition model structures on simplicial categories, so that each partition consists of model categories that are "really the same".

11.2. Definition: Let i be the inclusion of $\mathbf{Sph}$ in $\mathbf{S}$, and let (i_*, i^*) be the adjoint pair of proposition 3.23.

11.3. Theorem: *The pair (i_*, i^*) is a Quillen equivalence between the stable model structure of simplicial functors and the stable model structure of spectra.*

11.4. Convention: By definition, the model structures $\mathbf{SF}_p$, $\mathbf{SF}_h$, and $\mathbf{SF}_s$ share a common class of cofibrations. We refer to a map in this common class simply as a "cofibration of simplicial functors". We agree to a similar convention for the term "cofibrant simplicial functor", and to similar conventions for cofibrations of spectra.

11.5. Proposition: *The canonical map $E \to i^*((i_*E)R)$ is a stable weak equivalence, for every cofibrant spectrum E.*

11.6. Proposition: *A map $X \to Y$ of simplicial functors is a stable weak equivalence if and only if the induced map $i^*(XR) \to i^*(YR)$ of spectra is a stable weak equivalence.*

Proof of theorem 11.3. By propositions 10.10 and 9.8, i^* maps $\mathbf{SF}_{sf}$ to $\mathbf{Sp}_{sf}$, and, by lemmas 10.6 and 9.4, it maps $\mathbf{SF}_{saf}$, to $\mathbf{Sp}_{saf}$. Proposition 6.8 implies that its left adjoint i_* maps $\mathbf{SF}_{sac}$, to $\mathbf{Sp}_{sac}$, and that it maps $\mathbf{SF}_{sc}$, to $\mathbf{Sp}_{sc}$. It remains to prove that, given a fibrant X in $\mathbf{SF}_s$ and a cofibrant E in $\mathbf{Sp}_s$, a map $E \to i^*X$ is a stable weak equivalence if and only if so is the map $i_*E \to X$.

In case $E \to i^*X$ is a stable weak equivalence, by proposition 11.6, we have to show that $i^*((i_*E)R) \to i^*(XR)$ is a stable weak equivalence. By proposition 11.5, it suffices to show that the map $E \to i^*(XR)$ is a stable weak equivalence. But the map $X \to XR$ is a pointwise weak equivalence, (X is fibrant in $\mathbf{SF}$, and therefore also in $\mathbf{H}$) thus the map $i^*X \to i^*(XR)$ is a pointwise weak equivalence.

In case $i_*E \to X$ is a stable weak equivalence, so is $i^*((i_*E)R) \to i^*(XR)$ (by proposition 11.6), and so is the map $E \to i^*(XR)$ (by proposition 11.5). But, as we have seen above, the map $i^*X \to i^*(XR)$ is a pointwise weak equivalence. $\square$

Proof of proposition 11.5. We treat first the special case $E = (S^n)^* \wedge K$. We claim that the map $E \to i^*((i_*E)R)$ is $o(2)$, where we say that a map f of spectra is $o(2)$ if, for some c, the map f_n is $(2n-c)$-connected. Since smashing with K preserves connectivity, we may assume that $K = S^0$. By proposition 3.24, the associated map $E_{k+n} \to (i^*((i_*E)R))_{k+n}$ is isomorphic to the canonical map $S^k \to \mathbf{S}(S^n, RS^{k+n})$, and the claim follows. Since the class of $o(2)$ maps is included in $\mathbf{Sp}_{sw}$, the special case follows.

In the general case, apply proposition 6.5 to the map $f : * \to E$ with $S = \mathbf{Sp}_c$, and let E' be the target of the resulting i. Thus E' is a cofibrant spectrum, and the map $E' \to E$ is a pointwise weak equivalence. Recall from the proof of proposition 6.5 that E' is built by a combination of cobase changes of finite sums and filtered colimits, and observe that $E \mapsto i^*((i_*E)R)$ commutes with colimits. Recall also that all the cobase changes of finite sums involved are obtained from maps between spectra of the form $(S^n)^* \wedge K$. By a Mayer-Vietoris argument and induction, the map $E' \to i^*((i_*E')R)$ is a filtered colimit of $o(2)$ maps, in particular a weak equivalence of spectra. But i_* preserves pointwise acyclic cofibrations, thus also pointwise weak equivalences between cofibrant objects, by a mapping cylinder argument. Thus the map $i_*E' \to i_*E$ is a pointwise weak equivalence, and therefore so is the map $i^*((i_*E)R) \to i^*((i_*E')R)$. The conclusion follows immediately from this. $\square$

11.7. Proposition: *Given K in $\mathbf{S}^{fin}$ and X in $\mathbf{SF}$, the canonical map $i^*K_*XR \to i^*XRK_*$ is a stable weak equivalence.*

11.8. Proposition: *Given K in $\mathbf{S}^{fin}$ and X in $\mathbf{SF}$, the following hold.*

(i) The map $i^*XRK_* \to i^*T(RXRK_*)$ is a stable weak equivalence.

(ii) The spectrum $i^*X^sK_*$ is an Ω-spectrum.

Proof of proposition 11.6. We first remark that the "only if" part would be trivial, if the functors T for spectra and simplicial functors would correspond under i^*. This is, however, *not* true (in fact, the spectrum i^*TX is canonically isomorphic to $\Omega(i^*X)[1]$, cf. remark 10.3).

Suppose first that $X \to Y$ is a stable weak equivalence. By proposition 11.8(i), the maps $i^*XR \to i^*X^s$ and $i^*YR \to i^*Y^s$ are stable weak equivalences of spectra. It follows that $i^*XR \to i^*YR$ is a stable weak equivalence, since $i^*X^s \to i^*Y^s$ is even a pointwise weak equivalence.

Suppose now that $i^*XR \to i^*YR$ is a stable weak equivalence of spectra. Since smashing by K preserves stable weak equivalences of spectra (for a simple proof of this last fact, see the proof of proposition 10.21), proposition 11.7 implies that $i^*XRK_* \to i^*YRK_*$ is a stable weak equivalence, for each K in $\mathbf{S}^{fin}$. Thus, by proposition 11.8(i), so is the map $i^*T^\infty(RXRK_*) \to i^*T^\infty(RYRK_*)$. In fact, by proposition 10.16(iv) and proposition 11.8(ii) (and since $X^sK_* = (T^\infty RXR)K_*$ is canonically isomorphic to $T^\infty(RXRK_*)$), this map is a pointwise weak equivalence. Thus the map $X^sK \to Y^sK$ is a weak equivalence. $\square$

11.9. Lemma: *The map of proposition 11.7 is $o(2)$, if $X = L^*$ for some L in $\mathbf{S}^{fin}$.*

Proof of proposition 11.7. Call the map f. The proof will be divided in five cases, each more special than the next.

Case 1. X is a finite sum of simplicial functors of the form $K^* \wedge L$ for some K and L in $\mathbf{S}^{fin}$. Then lemma 11.9 implies that f is $o(2)$.

Case 2. X is any sum of simplicial functors of the form $K^* \wedge L$ for some K and L in $\mathbf{S}^{fin}$. Then f is a filtered colimit of maps which are weak equivalences, by case 1.

Case 3. X is a colimit of a sequence $X_0 \to X_1 \to \cdots$, such that $X_0 = *$ and each map is given by a cobase change of a map having the following form: It is a sum of maps of the form $K^* \wedge g$ for some K in $\mathbf{S}^{fin}$ and some cofibration g in $\mathbf{S}^{fin}$. Since stable equivalences are closed under filtered colimits, we may assume that the sequence is in fact finite. The conclusion follows by induction on the length of the sequence, using corollary 10.18 and case 2.

Case 4. X is any cofibrant simplicial functor. By definition, this means that there exists a sequence $X_0 \to X_1 \to \cdots$ satisfying all the conditions of the similar sequence in case 3, except possibly the condition that X_0 is trivial, and with the map $* \to X$ a retract of the composition $X_0 \to X_\infty$. But then X is a retract of X_∞/X_0, and the map $X_n/X_0 \to X_{n+1}/X_0$ is still a cobase change of the same map as $X_n \to X_{n+1}$. The conclusion follows from case 3.

Case 5. General case (X is any simplicial functor). Factor the map $* \to X$ as pj, with j a cofibration and p an acyclic fibration (the model structure is irrelevant, cf. convention 11.4). In particular, p is a pointwise weak equivalence, thus i^*pR is a pointwise weak equivalence of spectra, and the conclusion follows from case 4. $\square$

Before we prove proposition 11.8, we need the following technical lemma. The lemma allows us to view any simplicial functor as a filtered colimit of simplicial functors which are finite in a strong sense.

11.10. Lemma: *Let $\mathbf{A}$ be the partially ordered set given by the finite sets of objects of $\mathbf{S}^{fin}$. Every X in $\mathbf{H}$ determines $X^c : \mathbf{A} \to \mathbf{H}$ satisfying the following two conditions. First, for every $\mathcal{A}$ in $\mathbf{A}$, the object $X^c_{\mathcal{A}}$ is the diagonal of a simplicial object in $\mathbf{H}$, whose value at $[q]$ is*

$$\bigvee K_q^* \wedge \mathbf{S}(K_{q-1}, K_q) \wedge \cdots \wedge \mathbf{S}(K_0, K_1) \wedge X(K_0)$$

where the sum is over $(K_0, \ldots, K_q)$ in $\mathcal{A}^{q+1}$. Second, if X^c_∞ denotes the colimit of X^c, then there exists a pointwise weak equivalence $X^c_\infty \to X$. Finally, the construction $X \mapsto X^c$ is functorial, and the map $X^c_\infty \to X$ is natural.

Proof of proposition 11.8. By lemma 11.10, we may assume that X is of the form $X^c_{\mathcal{A}}$, with the notation as in the lemma. Thus X satisfies the following two conditions. First, the simplicial functor XR is $o(1)$. Second, the canonical map $\Sigma XR \to XR\Sigma$ is $o(2)$. (The proof of these two facts uses that sums, smashing by pointed simplicial sets, and diagonalization, preserve connectivity of pointed simplicial sets. The proof of the second fact uses lemma 11.9. It is essential that the set of K^* involved is finite, so that we may assume that the constants c, involved in the definition of the relevant $o(k)$, are all equal.) Since XRK_* is still $o(1)$, the map $XRK_* \to \Omega R\Sigma XRK_*$ is $o(2)$. Thus, since the map $\Sigma XRK_* \to XRK_*\Sigma$ is still $o(2)$, so is the map $XRK_* \to TRXRK_*$. □

Proof of lemma 11.9. Call the map f. We first prove the following reduction: Given injective maps $L_0 \to L_1$ in $\mathbf{S}^{fin}$, if the lemma is true for L_0 and L_1/L_0, then it is true for L_1. We need the following definition: In case X, Y, and Z are $o(j)$ simplicial functors in $\mathbf{SF}^{(n)}$ and $X \to Y \to Z$ is a sequence with trivial composition, then we say it is an $o(l)$-fibration sequence if the map from X to the pointwise homotopy fiber of $Y \to Z$ is $o(l)$, and we say it is an $o(l)$-cofibration sequence if the map from the pointwise homotopy cofiber of $X \to Y$ to Z is $o(l)$. In case $l = 2j$, the Blakers-Massey theorem (again, [S, theorem 9.3.5] is sufficient) implies that the sequence is an $o(l)$-fibration sequence if and only if it is an $o(l)$-cofibration sequence. (In verifying the above, note that, by definition of $o(l)$, in order to check that a map F in $\mathbf{SF}^{(n)}$ is $o(l)$ it suffices to consider only m-connected $K_1, \ldots, K_n$, for any constant m. If further both the source and the target of F are $o(j)$, then, by taking m large enough, we can arrange that the values of the source and the target of F are highly connected.) Since the sequence $(L_1/L_0)^*R \to L_1^*R \to L_0^*R$ is a fibration sequence, smashing it with K yields an $o(2)$ cofibration sequence in $\mathbf{SF}$. There is a map, given by f, from this sequence to the sequence $(L_1/L_0)^*R(?\wedge K) \to L_1^*R(?\wedge K) \to L_0^*R(?\wedge K)$ in $\mathbf{SF}$. Since each L^* is $o(1)$, the target sequence is an $o(2)$ cofibration sequence (since it is a fibration sequence), and the reduction follows.

The lemma holds for $L = S^n$, since f factors as the composition

$$K\wedge\Omega^n R? \to \Omega^n R(K\wedge\Sigma^n\Omega^n R?) \to \Omega^n R(K\wedge R?),$$

and both maps are $o(2)$. Thus the lemma is true for $\Delta^n/\partial\Delta^n$. The conclusion follows by induction on the simplices of L, using the above reduction. □

Proof of lemma 11.10. Let $\mathbf{I}$ be a small category and F a functor from $\mathbf{I}$ to sets. Given i in $\mathbf{I}$, let $\mathbf{C}(i)$ be the following category. Its objects are the pairs (f, x) such that $f : i_0 \to i$ and $x \in F(i_0)$. There is one morphism from (f, x) to (g, y) for each h in $\mathbf{I}$ such that $gh = f$ and $h_*(x) = y$. Define $F'(i)$ as the nerve of $\mathbf{C}(i)$. There is a map $F'(i) \to F(i)$

taking (f,x) to $f_*(x)$, and is the disjoint union of projections of nerves of categories to their terminal objects, in particular a weak equivalence. Note that

$$F'_q \cong \coprod_{(i_0,\dots,i_q)} i_q^* \times \mathbf{I}(i_{q-1}, i_q) \times \cdots \times \mathbf{I}(i_0, i_1) \times F(i_0).$$

Suppose now that $\mathbf{I}$ is a pointed category and that F is a pointed functor from $\mathbf{I}$ to pointed sets. Let

$$F''_q = \bigvee_{(i_0,\dots,i_q)} i_q^* \wedge \mathbf{I}(i_{q-1}, i_q) \wedge \cdots \wedge \mathbf{I}(i_0, i_1) \wedge F(i_0)$$

so that we have a canonical map $F' \to F''$. Then it is still true that there is a map $F'' \to F$, and that, for fixed i, there is a section $F(i) \to F''(i)$, as well as a map $F''(i) \otimes \Delta^1 \to F''(i)$ which is a homotopy between the identity and the composition $F''(i) \to F(i) \to F''(i)$. All these maps are compatible with the canonical map $F' \to F''$, and exist essentially because $\mathbf{I}$ is a pointed category and F is a pointed functor.

Fix X in $\mathbf{H}$ and apply the above construction with $\mathbf{I} = \mathbf{S}_p^{fin}$ and $F = X_p$ (the functor X_p from $\mathbf{S}_p^{fin}$ to $\mathbf{E}$ is defined in 3.10). The resulting X''_p, evaluated at L, has q-simplices

$$\bigvee K_q^* L \wedge \mathbf{S}_p(K_{q-1}, K_q) \wedge \cdots \wedge \mathbf{S}_p(K_0, K_1) \wedge X_p(K_0)$$

where the sum is over all $(K_0, \dots, K_q)$, and K_q^* denotes the functor from $\mathbf{S}_p^{fin}$ to $\mathbf{E}$ represented by K_q. The conclusion follows immediately from this. □

12. Comparison of the smash products

In this section we compare the smash products of spectra and simplicial functors, cf. theorem 12.5. We also prove that the smash product of simplicial functors is compatible with all three model structures on simplicial functors considered in previous sections, cf. theorems 12.3, 12.4, and 12.6.

12.1. Definition: Given maps $f : X \to Y$ and $g : Z \to W$ in $\mathbf{SF}$, let $f \barwedge g$ be the canonical map from $X \wedge W \cup_{X \wedge Z} Y \wedge W$ to $Z \wedge W$.

12.2. Proposition: *Let f, g, and h be maps in* $\mathbf{SF}$. *If g is a retract of f, then $g \barwedge h$ is a retract of $f \barwedge h$. If f is the composition $\cdots f_1 f_0$,* *then $f \overline{\wedge} h$ is the composition $\cdots (f_1 \overline{\wedge} h)(f_0 \overline{\wedge} h)$. If the map of arrows $f \to g$ corresponds to a cocartesian square (expressing g as a cobase change of f), then the map of arrows $f \overline{\wedge} h \to g \overline{\wedge} h$ also corresponds to a cocartesian square. Finally, if f is the sum of f_s, where $\{f_s\}$ is a family of maps indexed by s in some indexing set S, then $f \overline{\wedge} h$ is the the sum of $f_s \overline{\wedge} h$.* □

12.3. Theorem: *Let f and h be maps in $\mathbf{SF}$ with f a cofibration. Then the map $f \overline{\wedge} h$ is injective, if so is h, and it is a cofibration, if so is h.*

Fix i0 and j0 cofibrations. We want to show that i0sj0 is a cofibration. Fix an acyclic fibration p and a lifting problem i0sj0 to p. We want to solve this lifting problem. By adjointness considerations, similar to the ones we saw in remark ?, this lifting problem is equivalent to a lifting problem of the form i0 to (j0,p). It therefore suffices to show that (j0,p) is an acyclic fibration. Thus we reduced the problem to the special case of solving lifting problems i1 to (j0,p), i.e., i1sj0 to p, where now i1 is a *generating* cofibration. By essentially the same argument (only interchanging the roles of i and j), we reduce the problem to solving lifting problems i1sj1 to p, with both i1 and j1 generating cofibrations. To solve these, observe that i1sj1 is again a generating cofibration: Using lemma-identify-smash-of-representables, we see that (K0*sg0)s(K1*sg1) is isomorphic to (K0sK1)*s(g0sg1).

If (say) i is acyclic, a similar argument reduces the problem to showing isj is acyclic whenever i and j are generating. Since i is generating, it is not just a stable weak equivalence but also tilde o(2) (lemma ?). We show that isj is also tilde o(2). Write the map i as X to Y, and the map j as K*s(M to N), so that isj is isomorphic to (X circle K* to Y circle K*)sg (lemma ? again). Given cofibrations f and g of simply-connected simplicial sets, fsg is as connected as f is (use homology and the fact that the cofiber of fsg is isomorphic to the cofiber of f smash the cofiber of g), thus it suffices to show that (X circle K* to Y circle K*) is tilde o(2). This follows from the fact that K* is o(1)—in fact K* takes n-connected spaces to (n-d)-connected spaces, with d the dimension of K).

Proof. Suppose first that h is injective. Consider first the case where f is in the set $\mathbf{SF}^g_{pc}$ of generating cofibrations, i.e., for some K in $\mathbf{S}^{fin}$ and some cofibration g in $\mathbf{S}^{fin}$, we have $f = K^* \wedge g$. Using proposition 5.13 to write $(X \wedge K^*)L$ as $X(K^*L)$, and the fact that injective maps in $\mathbf{S}$ are closed under cobase change, we conclude that $f \overline{\wedge} h$ is injective.

If f is any cofibration, the conclusion follows from proposition 12.2, using the fact that injective maps in $\mathbf{S}$ are closed under retracts, infinite composition, cobase change, and sums, as well as the definition of the class of cofibrations in $\mathbf{SF}$ (this last class is defined as the class of cofibrations generated by $\mathbf{SF}^g_{pc}$; see also definition 6.4).

Suppose now that h is a cofibration. Consider first the case where both f and h are in the set $\mathbf{SF}^g_{pc}$ of generating cofibrations, i.e., for some K and L in $\mathbf{S}^{fin}$ and some cofibrations g and g' in $\mathbf{S}^{fin}$, we have $f = K^* \wedge g$ and $h = L^* \wedge g'$. By lemma 5.7, $(K^* \wedge L^*)$ is isomorphic to $(K \wedge L)^*$. Thus $f \overline{\wedge} h$ is isomorphic to $(K \wedge L)^* \wedge (g \overline{\wedge} g')$, which is again a generating cofibration.

If f and h are any cofibrations, the conclusion follows from proposition 12.2, using the fact that cofibrations are closed under retracts, infinite composition, cobase change, and sums, as well as the definition of the class of cofibrations in $\mathbf{SF}$. □

12.4. Theorem: *Let f be a map in $\mathbf{SF}$ and X a cofibrant simplicial functor. Then $X \wedge f$ is a pointwise weak equivalence, if so is f, and it is a weak equivalence of homotopy functors, if so is f.*

Proof. If $X = K^*$ for some K in $\mathbf{S}^{fin}$, the conclusion follows from proposition 5.13, together with the fact that K^*L is fibrant if so is L. The special case $X = \bigvee_s K_s^* \wedge L_s$ follows immediately.

If X is any cofibrant simplicial functor, by proceeding as in the proof of proposition 11.7, we can write X as a retract of the colimit X_∞ of a sequence $* = X_0 \to X_1 \to \cdots$, with the map $X_n \to X_{n+1}$ a cobase change of a map $g_n = \bigvee_s K_s^* \wedge h_s$ with h_s a cofibration in $\mathbf{S}^{fin}$. We may assume that $X = X_\infty$, since weak equivalences in $\mathbf{S}$ are closed under retracts. We may assume that X equals some X_n, since weak equivalences in $\mathbf{S}$ are closed under sequential colimits. The conlusion now follows by induction on n, the gluing lemma, and the special case treated in the previous paragraph. (In order to apply the gluing lemma, we need to know that $g \wedge Y$ is injective for any simplicial functor Y, and any cofibration g of simplicial functors. This follows from theorem 12.3, by considering the injective map $* \to Y$.) □

12.5. Theorem: *There is a map of spectra $i^*XR \wedge i^*YR \to i^*(X \wedge Y)R$, natural in the simplicial functors X and Y, which is a stable weak equivalence if one of the factors is cofibrant.*

12.6. Theorem: *Smashing with a cofibrant simplicial functor preserves stable weak equivalences.*

Proof. This follows from theorem 12.5, proposition 11.6, and proposition 10.21. □

12.7. Definition: An S^2*-spectrum* E consists of a sequence of pointed simplicial sets E_{2n} and a sequence of maps $E(2n) : S^2 \wedge E_{2n} \to E_{2n+2}$, for $n = 0, 1, \ldots$. We define maps and stable weak equivalences of such objects, so that forgetting the odd terms of a spectrum gives a functor $E \mapsto E^*$ that preserves stable weak equivalences.

Let X be a simplicial functor. The S^2-spectrum XS_τ is defined as follows. Let $(XS_\tau)_{2n} = XS^{2n}$. Let $(XS_\tau)(2n)$ be the composition

$$S^1 \wedge S^1 \wedge X(S^n \wedge S^n) \to X(S^1 \wedge S^1 \wedge S^n \wedge S^n) \to X(S^1 \wedge S^n \wedge S^1 \wedge S^n),$$

where the first map is the special assembly map, and the second map is induced by τ (for the definition of τ, see 10.19).

12.8. Proposition: *The* S^2*-spectra* $(i^*X)^*$ *and* XS_τ *are naturally isomorphic.*

Proof. Let *the sphere spectrum* S equal $i^*\mathbb{S}$, and the S^2-spectrum S_τ equal $\mathbb{S}(S_\tau)$. The conclusion of the proposition is true in the special case $X = \mathbb{S}$, i.e., S^* and S_τ are isomorphic. The general case follows because there is a functor which associates the S^2-spectrum XE to a simplicial functor X and an S^2-spectrum E, such that $(i^*X)^*$, resp. XS_τ, equals XE with $E = S^*$, resp. $E = S_\tau$. □

The only reason we consider XS_τ (in fact, the only reason we consider S^2-spectra) is to be able to write the map in the statement of theorem 12.5 as a composition of simpler maps. One of these simpler maps is given by the proposition above, and another one is given by the proposition below.

12.9. Proposition: *Given simplicial functors* X *and* Y*, there is a non-trivial natural map* $(i^*XR \wedge i^*YR)^* \to (X \wedge Y)RS_\tau$.

Proof. By definition of $X \wedge Y$, there is a canonical map from $XM \wedge YN$ to $(X \wedge Y)(M \wedge N)$. The map we want is obtained from this map by letting M and N be RS^n (this uses the map $RR \to R$, coming from the adjunction between singular complex and realization). □

12.10. Proposition: *If* X *and* Y *are simplicial functors and* X *is cofibrant, then the map* $(i^*XR \wedge i^*YR)^* \to (X \wedge Y)RS_\tau$ *is a weak equivalence.*

Proof of theorem 12.5. Note that the forgetful functor $E \mapsto E^*$ from spectra to S^2-spectra has a left adjoint L, such that $(LE)_{2n} = E_{2n}$, $(LE)_{2n+1} = S^1 \wedge E_{2n}$, $(LE)(2n) = 1$, and $(LE)(2n+1) = E(2n)$. The conclusion now follows from proposition 12.8, proposition 12.10, and the fact that, for any spectra E and E', the equality $E \wedge E' = L(E \wedge E')^*$ holds. □

12.11. Lemma: *The canonical map $XRM \wedge YRN \to (X \wedge Y)R(M \wedge N)$ is $o(3)$, if $X = K^*$ and $Y = L^*$ for some K and L in $\mathbf{S}^{fin}$.*

Proof of proposition 12.10. We say that a map of S^2-spectra is $o(3)$, if there exists an integer c such that, for all non-negative integers n the map of $(2n)$-terms is $(3n - c)$-connected. It follows immediately that an $o(3)$-map is a stable weak equivalence. Call the map f. The proof will be divided in five cases, each more special than the next.

Case 1. X and Y are a finite sum of simplicial functors of the form $K^* \wedge L$ for some K and L in $\mathbf{S}^{fin}$. Then lemma 12.11 implies that f is $o(3)$.

Case 2. X and Y are any sum of simplicial functors of the form $K^* \wedge L$ for some K and L in $\mathbf{S}^{fin}$. Then f is a filtered colimit of maps which are weak equivalences, by case 1.

Case 3. X and Y are given by colimits of sequences $Z_0 \to Z_1 \to \cdots$, such that $Z_0 = *$ and each map is given by a cobase change of a map having the following form: It is a sum of maps of the form $K^* \wedge g$ for some K in $\mathbf{S}^{fin}$ and some cofibration g in $\mathbf{S}^{fin}$. Since stable equivalences are closed under filtered colimits, we may assume that both compositions are in fact finite. The conclusion follows by a double induction on the length of the compositions, using corollary 10.18 and case 2 (theorem 12.3 guarantees that the injectivity hypothesis in corollary 10.18 is satisfied).

Case 4. X and Y are any cofibrant simplicial functors. The conclusion follows by proceeding as in the proof of proposition 11.7, to write X and Y as retracts of simplicial functors corresponding to the case 3.

Case 5. General case (X and Y are any simplicial functors, and X is cofibrant). Factor the map $* \to Y$ as pj, with j a cofibration and p an acyclic fibration (the model structure is irrelevant, cf. convention 11.4). In particular, p is a pointwise weak equivalence, thus i^*p is a stable weak equivalence of spectra (even a pointwise one). The conclusion follows from theorem 12.4, proposition 10.21, and case 4. □

Proof of lemma 12.11. Call the map f. We first prove the following reduction: Given injective maps $K_0 \to K_1$ in $\mathbf{S}^{fin}$, if the lemma is true for (K_0, L) and $(K_1/K_0, L)$, then it is true for (K_1, L). Since the sequence $(K_1/K_0)^* \to K_1^* \to K_0^*$ is a fibration sequence, external-smashing it with L^* yields an $o(3)$ cofibration sequence in $\mathbf{SF}^{(2)}$ (cf. the proof of lemma 11.9). There is a map, given by f, from this sequence to the sequence

$$(K_1/K_0 \wedge L)^* R(M \wedge N) \to (K_1 \wedge L)^* R(M \wedge N) \to (K_0 \wedge L)^* R(M \wedge N)$$

in $\mathbf{SF}^{(2)}$. Since each $(K_1 \wedge L)^* R(M \wedge N)$ is $o(2)$, the target sequence is an $o(4)$ cofibration sequence (since it is a fibration sequence; cf. the proof of lemma 11.9 again), and the reduction follows.

The lemma holds for $(K, L) = (S^m, S^n)$, since f factors as the composition

$$\Omega^m RM \wedge \Omega^n RN \to \Omega^{m+n} R(S^m \Omega^m RM \wedge S^n \Omega^n RN) \to \Omega^{m+n} R(M \wedge N),$$

and both maps are $o(4)$. Thus the lemma is true for $(\Delta^m/\partial\Delta^m, \Delta^n/\partial\Delta^n)$. The conclusion follows by induction on the simplices of K and L, using the above reduction. □

13. Linear functors

We end this paper by relating the stable model structure of simplicial functors to the calculus of homotopy functors, cf. [G1], [G2], and [G3]. For our purposes, the second section of [G1] is sufficient. In fact, here we are only interested in the part of this calculus having to do with the linear approximation at $*$. We show in proposition 13.4 that, by using simplicial functors instead of homotopy functors (this is "no restriction, up to weak equivalence", cf. remark 4.6), a rigid universal property of this linear approximation is possible. The corresponding results in [G1] are stated for "stably excisive" functors only, a restriction that turns out to be unnecessary.

13.1. Definition: A simplicial functor is *linear*, provided that it is a homotopy functor and that it takes any sequence $K_0 \to K_1 \to K_2$, such that $K_0 \to K_1$ is injective with cofiber $K_1 \to K_2$, to a fibration sequence.

13.2. Proposition: *A simplicial functor is fibrant in the stable model structure if and only if it is linear and pointwise fibrant.*

Proof. Suppose X is linear and pointwise fibrant. Let D be the pointed simplicial set obtained by choosing a basepoint for Δ^1. It suffices to exhibit a weak equivalence from the sequence $XK \to X(D \wedge K) \to X\Sigma K$ to the sequence $XK \to (X\Sigma K)^D \to X\Sigma K$. For, since the source is a fibration sequence, so is the target. But, since $(X\Sigma K)^D \to X\Sigma K$ is a fibration with fiber $\Omega X\Sigma K$, and the resulting map from XK to the fiber is the canonical one, the conclusion follows from proposition 9.8. We exhibit a map which is the identity on the fiber and the base. Choose the contraction of Δ^1 yielding a map $D \wedge D \to D$ (the choice depends on the choice of the basepoint of D), and postcompose with the canonical map $D \to S^1$ to obtain a map $D \wedge D \to S^1$. The required map is the adjoint of the composition $D \wedge X(D \wedge K) \to X(D \wedge D \wedge K) \to X\Sigma K$.

Suppose X is fibrant in the stable model structure, and fix a sequence $K_0 \to K_1 \to K_2$ in $\mathbf{S}^{fin}$, such that $K_0 \to K_1$ is injective with cofiber $K_1 \to K_2$. Note that X is fibrant in $\mathbf{SF}_h$ thus, by proposition 8.7, the map $X \to XR$ is a pointwise weak equivalence. Proposition 11.7 now implies that the canonical map from $i^*X \wedge K = i^*K_*X$ to i^*XK_* is a stable weak equivalence. The proof of proposition 10.17 actually shows that $(T^\infty(i^*X \wedge K_0))_n \to (T^\infty(i^*X \wedge K_1))_n \to (T^\infty(i^*X \wedge K_2))_n$ is a fibration sequence for any n. By propositions 10.16 (iv), and 10.12, and since by proposition 9.8 the spectrum $i^*(XK_*)$ is an Ω-spectrum, this sequence for $n = 0$ is related by a chain of weak equivalences to the sequence $XK_0 \to XK_1 \to XK_2$, therefore this last sequence is also a fibration sequence. $\square$

13.3. Definition: Given a closed model category $\mathbf{D}$, for which there exists a set S of maps in $\mathbf{D}$ with small domains such that the class of acyclic cofibrations equals the class of cofibrations generated by S, and given an object X of $\mathbf{D}$, *the canonical fibrant replacement* $X \to X^f$ is the acyclic cofibration provided by the small object argument applied to the map $X \to *$. Thus X^f is a fibrant object depending functorially on X, and the map $X \to X^f$ is both natural and an acyclic cofibration.

13.4. Proposition: *Given a simplicial functor X, let $X \to PX$ be the canonical fibrant replacement in the stable model structure. Then PX is linear, and $X \to PX$ is a natural stable weak equivalence. Further, if Y is any linear simplicial functor which is pointwise fibrant, any map $X \to Y$ factors through $X \to PX$, and any two factorizations $PX \to Y$ can be connected by a homotopy $PX \otimes \Delta^1 \to Y$ relative to X.*

Proof. This follows immediately from proposition 13.2 and standard properties of simplicial model categories. □

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