1

1. , , . . () , . , – – – . . ¨ .

2. . , «» , , , «» . . – – . :
. , , . , . , , Bolzano - Weierstrass . – , – . , – – : , .

3. . · , , , .

4. – . n_0 δ ϵ. «» , . δ ϵ ( ) , , .

5. – – : , , . . ’ ’ – – . – , – , , , . . , $y = x^n$, $\log x = \int_1^x \frac{1}{t}\,dt$. , , «» .
, , , . . , : (i) ($\log x = \int_1^x \frac{1}{t}\,dt$) , (ii) ( ) ( ) – – , (iii) ($\arctan x = \int_0^x \frac{1}{t^2+1}\,dt$) (iv) ( ) · , .

6. Riemann Darboux. . ′ Darboux – –′ Riemann .

7. . , . · !
– , , – · .

8. . – , – . . . · (*) (**) .

9. ¨ 2007 - 08 ¨ 2008 - 09 . · .
– – : , , . , .
, , .

2009.

2

. ( ) . ( ). . $\lim_{x\to\xi\,(x\neq\xi)}$, x ξ, $\lim_{x\to\xi}$, x ξ. . . .

2010.

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Περιεχόμενα

Κεφάλαιο 1

•

. . . , , , . . .

1.1 .

. , , .

$x+y$, $x-y$, xy $\frac{x}{y}$ $(y\neq 0)$ x, y . (, .) .

. , , , .

$1, 2, 3, \dots$, $0, 1, -1, 2, -2, 3, -3, \dots$, $\frac{m}{n}$, m, n $n\neq 0$. : $\frac{\pm m}{\pm n}$, $\frac{\pm 2m}{\pm 2n}$, $\frac{\pm 3m}{\pm 3n}$, $\dots$. $\frac{m}{n}$ m n, . , $\frac{4}{-3}$ $\frac{-4}{3}$ $\frac{-4}{-3}$ $\frac{4}{3}$. , m $\frac{m}{1}$, .

. , . , , , .

. . , .

$\mathbf{R}$. , $\mathbf{N}, \mathbf{Z}$ $\mathbf{Q}$, , . : 0, $\mathbf{N}$ $0, 1, 2, \dots$ $\mathbf{N}^*$ $1, 2, \dots$.

. , .

. .

1.1 (1) $x\leq y$ $y\leq z$, $x\leq z$. , .
(2) $x\leq y$, $x+z\leq y+z$ $x-z\leq y-z$.
(3) $x\leq y$ $z\leq w$, $x+z\leq y+w$. , .
(4) $x\leq y$ $z>0$, $xz\leq yz$ $\frac{x}{z}\leq\frac{y}{z}$.
(5) $x\leq y$ $z<0$, $xz\geq yz$ $\frac{x}{z}\geq\frac{y}{z}$.
(6) $0<x\leq y$ $0<z\leq w$, $0<xz\leq yw$. , .

x $|x|$

$$|x|=\begin{cases} x, & x\geq 0,\\ -x, & x\leq 0.\end{cases}$$

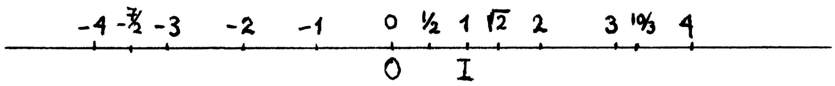
, .

1.2 (1) $|xy| = |x||y|$.
(2) . $\big||x| - |y|\big| \leq |x \pm y| \leq |x| + |y|$.
(3) $y \neq 0$, $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.
(4) $|x| \leq a$ $-a \leq x \leq a$.
(5) $|x| < a$ $-a < x < a$.

x, y $\max\{x, y\}$ $\min\{x, y\}$. : $\max\{x_1, \ldots, x_n\}$ $\min\{x_1, \ldots, x_n\}$. A **R** , A A, **maximum** A $\max A$. , A , **minimum** A $\min A$.

. **R**.

. , 0, , 1. . , x , $x > 0$, , $x < 0$, $|x|$. , . .



Σχήμα 1.1: .

x . « x» « x». , « » « » .
x 0 $|x|$. , , x, y $|x - y|$.
, : $x < y$ x, y $0, 1$, . , 1 0, : $x < y$ y x.
1 0, : $x < y$ y x.
: 1 0. , (,) 1 0.

. $\pm\infty$.

R. $a < b$, $(a, b) = \{x : a < x < b\}$, $(a, b] = \{x : a < x \leq b\}$ $[a, b) = \{x : a \leq x < b\}$. $a \leq b$, $[a, b] = \{x : a \leq x \leq b\}$. a, b. (a, b) $[a, b]$. $(a, +\infty) = \{x : x > a\}$, $(-\infty, b) = \{x : x < b\}$, $[a, +\infty) = \{x : x \geq a\}$ $(-\infty, b] = \{x : x \leq b\}$. ( ) ( ) ( ). , $(-\infty, +\infty) = \mathbf{R}$, .
$+\infty, -\infty$ · $\pm\infty$. $+\infty$ «» «» $-\infty$ «» «» . $<$ $>$, x

$$-\infty < x, \qquad x < +\infty, \qquad -\infty < +\infty.$$

, , «» «» $+\infty$ $-\infty$ − .
$\pm\infty$.

. .

1 b, , , . . . , , . ( ). : $b > 0$ n $n \cdot 1 > b$. $b \leq 0$, :

1.1 b $n > b$.

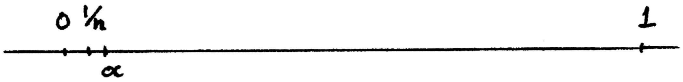
Σχήμα 1.2: $n > b$.

1.1 . n b, $n+1, n+2, n+3, \ldots$ b. :

,
.

$b = \frac{1}{a}$, $a > 0$, 1.1 :

1.3 . $a > 0$ n $\frac{1}{n} < a$.



Σχήμα 1.3: n $\frac{1}{n} < a$.

, $\frac{1}{n}$ a, $\frac{1}{n+1}, \frac{1}{n+2}, \frac{1}{n+3}, \ldots$ a. .

,
.

1.1 .

1.4 x k $k \le x < k+1$.

x $[k, k+1)$, k . $\ldots, [-3,-2), [-2,-1), [-1,0), [0,1), [1,2), [2,3), \ldots$ $(-\infty, +\infty)$.
k $k \le x < k+1$ x $[x]$.

: $[3] = 3$, $[-4] = -4$, $[\frac{8}{5}] = 1$, $[\frac{2}{3}] = 0$, $[-\frac{8}{5}] = -2$.

1.1 1.3 1.4 ′ .

.

. , .

1. r a , $r + a$.
   r $\neq 0$ a , ra .

2. a , p, q, r, s $p + qa = r + sa$, $p = r$ $q = s$.

. , .

1. $x \leq y < 0$ $z \leq w < 0$, $0 < yw \leq xz$.

2. $x \leq y$, $z \leq w$, $t \leq s$ $x + z + t = y + w + s$, $x = y$, $z = w$ $t = s$.
   $0 < x \leq y$, $0 < z \leq w$, $0 < t \leq s$ $xzt = yws$, $x = y$, $z = w$ $t = s$.

3. $|x + y| = |x| + |y|$ $x, y \geq 0$ $x, y \leq 0$.
   $|x + y + z| \leq |x| + |y| + |z|$. , $|x + y + z| = |x| + |y| + |z|$ $x, y, z \geq 0$ $x, y, z \leq 0$.

4. $t \leq x$ $t \leq y$ $t \leq \min\{x, y\}$.
   $t \geq x$ $t \geq y$ $t \geq \max\{x, y\}$.

5. $\max\{x, y\} = \frac{x+y+|x-y|}{2}$ $\min\{x, y\} = \frac{x+y-|x-y|}{2}$.

6. ;

$$[a, b], \quad (a, b), \quad [a, b), \quad \mathbf{N}, \quad \mathbf{Z}, \quad \mathbf{Q}, \quad \left\{\frac{1}{n} : n \quad \right\}.$$

. .

1. 1.1 1.2;

2. .
   $a \leq x \leq b$ $a \leq y \leq b$, $|x - y| \leq b - a$.
   .

3. , x, y, $x + y, x - y, xy$ $\frac{x}{y}$.
   (xy: $x, y > 0$. 0 . ′, ′ 1, y, . ′ ′, . , ;)

. .

1. x .

$$|x + 1| > 2, \quad |x - 1| < |x + 1|, \quad \frac{x}{x + 2} > \frac{x + 3}{3x + 1}, \quad (x - 2)^2 \geq 4,$$

$$|x^2 - 7x| > x^2 - 7x, \quad \frac{(x - 1)(x + 4)}{(x - 7)(x + 5)} > 0, \quad \frac{(x - 1)(x - 3)}{(x - 2)^2} \leq 0.$$

2. x x .

$$(-\infty, 3], \quad (2, +\infty), \quad (3, 7), \quad (-\infty, -2) \cup (1, 4) \cup (7, +\infty),$$

$$[-2, 4] \cup [6, +\infty), \quad [-1, 4) \cup (4, 8], \quad (-\infty, -2] \cup [1, 4) \cup [7, +\infty).$$

. .

1. x $[-x]=-[x]$;

2. k , $[x+k]=[x]+k$.

(: $[y]=m$ m $m \le y < m+1$.)

3. $[x+y]=[x]+[y]$ $[x+y]=[x]+[y]+1$.

$[x+y+z]$.

4. , $0 < x \le 1$, n $\frac{1}{n+1} < x \le \frac{1}{n}$ n x.

5. b $n > b$. n b;

b $n \ge b$;

$a > 0$. n $\frac{1}{n} < a$. n a;

6. $[x]+[x+\frac{1}{2}]=[2x]$, $[x]+[x+\frac{1}{3}]+[x+\frac{2}{3}]=[3x]$, , $[x]+[x+\frac{1}{n}]+\cdots+[x+\frac{n-2}{n}]+[x+\frac{n-1}{n}]=[nx]$ $n \ge 2$.

1.2 .

. .

a^n ( ) n

$$a^n = \underbrace{a\cdots a}_{n},$$

n a. $a \neq 0$, a^0 a^n n

$$a^0 = 1, \quad a^n = \frac{1}{a^{-n}} = \frac{1}{\underbrace{a\cdots a}_{-n}} \qquad (a \neq 0).$$

$(-a)^n = a^n$, n , $(-a)^n = -a^n$, n . , , , n , $a^n > 0$ $a \neq 0$, n , (i) $a^n > 0$ $a > 0$ (ii) $a^n < 0$ $a < 0$.

1.5 .

1.5 n ≥ 2,

$$\boxed{x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).}$$

, n ≥ 3,

$$\boxed{x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1}).}$$

n $1 \cdot 2 \cdots (n-1) \cdot n$ $n!$.

$$n! = 1 \cdot 2 \cdots (n-1) \cdot n.$$

: $1! = 1,\ 2! = 1 \cdot 2 = 2,\ 3! = 1 \cdot 2 \cdot 3 = 6,\ 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24.$

,

$$0! = 1$$

n

$$n! = (n-1)!\, n.$$

$\binom{n}{m}$ m, n $0 \le m \le n$

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}\,.$$

: $\binom{n}{0} = \binom{n}{n} = 1,\ \binom{n}{1} = \binom{n}{n-1} = n,\ \binom{n}{2} = \binom{n}{n-2} = \frac{n(n-1)}{2}\,.$

$1 \le m \le n$, , ,

$$\binom{n}{m} = \frac{n(n-1)\cdots(n-m+1)}{m!}\,.$$

1.6 ***Newton***. x, y n

$$\boxed{(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n\,.}$$

: Newton . $n = 1$ $(x+y)^1 = \binom{1}{0} x^1 + \binom{1}{1} y^1$, $\binom{1}{0} = \binom{1}{1} = 1.$ n $x+y$.

$$\begin{aligned}(x+y)^{n+1} = \binom{n}{0} x^{n+1} + \binom{n}{1} x^n y + \cdots + \binom{n}{n} x y^n \\ + \binom{n}{0} x^n y + \cdots + \binom{n}{n-1} x y^n + \binom{n}{n} y^{n+1}\,.\end{aligned}$$

$\binom{n}{0} = 1 = \binom{n+1}{0}$ $\binom{n}{n} = 1 = \binom{n+1}{n+1}$. , m- $(1 \le m \le n)$ $\binom{n}{m} x^{n-m+1} y^m + \binom{n}{m-1} x^{n-m+1} y^m = \binom{n+1}{m} x^{n-m+1} y^m$ (4) , ,

$$(x+y)^{n+1} = \binom{n+1}{0} x^{n+1} + \binom{n+1}{1} x^n y + \cdots + \binom{n+1}{n} x y^n + \binom{n+1}{n+1} y^{n+1}\,.$$

$n+1$ n.

:

$$\begin{aligned}(x+y)^1 &= x + y\,,\\ (x+y)^2 &= x^2 + 2xy + y^2\,,\\ (x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3\,,\\ (x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\,,\\ (x+y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5\end{aligned}$$

Newton.

1.7 .

1.7 (1) .

$$a^x b^x = (ab)^x\,, \qquad a^x a^y = a^{x+y}\,, \qquad (a^x)^y = (a^y)^x = a^{xy}\,.$$

(2) $0 < a < b$, (i) $a^x < b^x$, $x > 0$, (ii) $a^0 = b^0 = 1$ (iii) $a^x > b^x$, $x < 0$.
(3) $x < y$, (i) $a^x < a^y$, $a > 1$, (ii) $1^x = 1^y = 1$ (iii) $a^x > a^y$, $0 < a < 1$.

: 1.7 · ! , .

(1) : $x > 0$, $a^x b^x = \underbrace{a \cdots a}_{x}\underbrace{b \cdots b}_{x} = \underbrace{(ab) \cdots (ab)}_{x} = (ab)^x$.

$x < 0$, $a, b \neq 0$ $a^x b^x = \frac{1}{\underbrace{a \cdots a}_{-x}}\frac{1}{\underbrace{b \cdots b}_{-x}} = \frac{1}{\underbrace{(ab) \cdots (ab)}_{-x}} = (ab)^x$.

$x = 0$, $a, b \neq 0$ $a^x b^x = 1 \cdot 1 = 1 = (ab)^x$.

: $x, y > 0$, $a^x a^y = \underbrace{a \cdots a}_{x}\underbrace{a \cdots a}_{y} = \underbrace{a \cdots a}_{x+y} = a^{x+y}$.

$x < 0 < y$ $x + y > 0$, $0 < -x < y$, $a^y = \underbrace{a \cdots a}_{y} = \underbrace{a \cdots a}_{-x}\underbrace{a \cdots a}_{y-(-x)} = \underbrace{a \cdots a}_{-x}\underbrace{a \cdots a}_{x+y} = \underbrace{a \cdots a}_{-x} a^{x+y}$, , $a^x a^y = \frac{1}{\underbrace{a \cdots a}_{-x}} a^y = a^{x+y}$.

$x < 0 < y$ $x + y < 0$, $0 < y < -x$, $\underbrace{a \cdots a}_{-x} = \underbrace{a \cdots a}_{y}\underbrace{a \cdots a}_{(-x)-y} = a^y \underbrace{a \cdots a}_{-(x+y)}$, , $a^x a^y = \frac{1}{\underbrace{a \cdots a}_{-x}} a^y = \frac{1}{\underbrace{a \cdots a}_{-(x+y)}} = a^{x+y}$.

$x < 0 < y$ $x + y = 0$, $y = -x$, $a^x a^y = \frac{1}{a^{-x}} a^y = \frac{1}{a^y} a^y = 1 = a^{x+y}$.

$y < 0 < x$ $x < 0 < y$.

$x, y < 0$, $a^x a^y = \frac{1}{\underbrace{a \cdots a}_{-x}}\frac{1}{\underbrace{a \cdots a}_{-y}} = \frac{1}{\underbrace{a \cdots a}_{(-x)+(-y)}} = \frac{1}{\underbrace{a \cdots a}_{-(x+y)}} = a^{x+y}$.

$x = 0$, $a^x a^y = 1 \cdot a^y = a^{0+y} = a^{x+y}$. $y = 0$ $x = 0$.

: $x, y > 0$, $(a^x)^y = \underbrace{a^x \cdots a^x}_{y} = \underbrace{\underbrace{a \cdots a}_{x} \cdots \underbrace{a \cdots a}_{x}}_{y} = \underbrace{a \cdots a}_{xy} = a^{xy}$.

: $x < 0 < y$, $y < 0 < x$, $x, y < 0$, $x = 0$ $y = 0$.

(2) (i) $a < b$ x , $a^x = \underbrace{a \cdots a}_{x} < \underbrace{b \cdots b}_{x} = b^x$. (iii) (i): $a^x = \frac{1}{a^{-x}} > \frac{1}{b^{-x}} = b^x$. (ii) .

(3) (i) $y - x > 0$, , $1 < a$ $y - x$, $a^x = \underbrace{a \cdots a}_{x}\underbrace{1 \cdots 1}_{y-x} < \underbrace{a \cdots a}_{x}\underbrace{a \cdots a}_{y-x} = a^y$. (iii) (ii) .

1.7 . , · .

. .

1.2 , · , . 1.2.

1.2 n $a > 0$, $x^n = a$ x > 0.

1.2 $x^n = a$ $a > 0$. 1.8, , . (1.2) .

1.8 (1) n , $x^n = a$ (i) , , $a > 0$, (ii) , 0, $a = 0$, (iii) , $a < 0$.
(2) n , $x^n = a$ (i) , , $a > 0$, (ii) , 0, $a = 0$, (iii) , , $a < 0$.

n , a $x^n = a$ n- a

$\sqrt[n]{a}$.

n , $a \geq 0$ $x^n = a$ n- a $\sqrt[n]{a}$.

, , $\sqrt[n]{0} = 0$ n $\sqrt[n]{a} > 0$ $a > 0$ n. , $a < 0$ $\sqrt[n]{a} < 0$ n $\sqrt[n]{a}$ n.

$n = 2, 3, 4, \ldots$, $\sqrt[n]{a}$ **, , , . . .** a. $n = 2$ $\sqrt[2]{a}$ $\sqrt{a}$, , a. $n = 3$ $\sqrt[3]{a}$ a.

: (1) $x^4 = 16$, $\sqrt[4]{16} = 2$ $-\sqrt[4]{16} = -2$. , $x^4 = -16$.

(2) $x^5 = 32$, $\sqrt[5]{32} = 2$. $x^5 = -32$, $\sqrt[5]{-32} = -2$. $\sqrt[5]{-32} = -2 = -\sqrt[5]{32}$. $-\sqrt[5]{32}$ $x^5 = -32$, $\sqrt[5]{32}$ $x^5 = 32$. , , :

$$\sqrt[n]{-a} = -\sqrt[n]{a} \qquad (n \ ).$$

.

1.9 n, k. $\sqrt[n]{k}$ k n- .

: . k n- , m $k = m^n$. $\sqrt[n]{k} = \sqrt[n]{m^n} = m$, , .

, $r = \sqrt[n]{k}$ $r = \frac{m}{l}$, m, l > 1. $l > 1$, p l. l, m > 1, p m. $k = r^n = \frac{m^n}{l^n}$, $l^n k = m^n$. p l, $l^n k$, , m^n . , , . , p $m^n = m \cdots m$, m . $l = 1$, $r = m$, , $k = r^n = m^n$ n- .

: (1) $\sqrt{2}$, , m $m^2 = 2$. : $1^2 < 2$ $2^2 > 2$. , $\sqrt[3]{5}$ m $m^3 = 5$.

(2) $\sqrt{2} + \sqrt{3}$. , r, $(\sqrt{2} + \sqrt{3})^2 = r^2$, $\sqrt{6} = \frac{r^2 - 5}{2}$, , $\sqrt{6}$. m $m^2 = 6$.

. .

′ a^r r .

r $r = \frac{m}{n}$, m , n , , m, n , 1. r .

: $\frac{16}{10}$ $\frac{8}{5}$. , $-\frac{6}{4}$ $\frac{-3}{2}$.

r , $r = \frac{m}{n}$,

$$a^r = (\sqrt[n]{a})^m$$

: (i) $a > 0$, $\sqrt[n]{a}$, (ii) $a = 0$, $\sqrt[n]{0} = 0$, $m > 0$, , $r > 0$ $0^r = (\sqrt[n]{0})^m = 0^m = 0$ (iii) $a < 0$, n $\sqrt[n]{a}$. :

a^r (i) $a > 0$**,** (ii) $a = 0$ $r > 0$ (iii) $a < 0$ r .
a^r (i) $a = 0$ $r \leq 0$ (ii) $a < 0$ -
r **.**

: (1) $2^{\frac{3}{4}} = (\sqrt[4]{2})^3$, $2^{\frac{6}{8}} = 2^{\frac{3}{4}} = (\sqrt[4]{2})^3$, $2^{-\frac{3}{4}} = (\sqrt[4]{2})^{-3} = \frac{1}{(\sqrt[4]{2})^3}$, $2^{\frac{6}{2}} = 2^3 = 8$ $2^{-\frac{6}{2}} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$.

(2) $0^{\frac{3}{4}} = 0$ $0^{\frac{3}{5}} = 0$. $0^{-\frac{3}{4}}$, $0^{-\frac{3}{5}}$ 0^0 .

(3) $(-2)^{\frac{5}{3}} = (\sqrt[3]{-2})^5 = (-\sqrt[3]{2})^5 = -(\sqrt[3]{2})^5$, $(-2)^{\frac{10}{6}} = (-2)^{\frac{5}{3}}$ $(-2)^0 = 1$. $(-2)^{\frac{5}{2}}$ $(-2)^{-\frac{14}{4}}$.

n $\frac{1}{n}$. $\frac{1}{n}$ $\frac{1}{n}$. $a^{\frac{1}{n}}$ $(\sqrt[n]{a})^1 = \sqrt[n]{a}$:

$$a^{\frac{1}{n}} = \sqrt[n]{a}\,.$$

, a^r $a<0$: r. , , , . a^r $a<0$ r . $a^r>0$ r $a>0$. , $\frac{m}{n}$ r, $\sqrt[n]{a}>0$, , $a^r = (\sqrt[n]{a})^m > 0$. , 1.7, .

1.7 : 1.7 , . $x=\frac{m}{n}$ $y=\frac{k}{l}$ x y. $(a^x)^n = a^m$: $(a^x)^n = ((\sqrt[n]{a})^m)^n = ((\sqrt[n]{a})^n)^m = a^m$.
(1) : $a,b>0$. $(a^xb^x)^n = (a^x)^n(b^x)^n = a^mb^m = (ab)^m = ((ab)^x)^n$, $a^xb^x, (ab)^x > 0$, $a^xb^x = (ab)^x$.
$a=0$, $x>0$ $a^xb^x = 0\cdot b^x = 0 = 0^x = (ab)^x$. $b=0$.
$a<0$, n . $(a^xb^x)^n = ((ab)^x)^n$, n , $a^xb^x = (ab)^x$. $b<0$.
: $x+y=\frac{p}{q}$ $x+y$. $a>0$, $(a^xa^y)^{nl} = (a^x)^{nl}(a^y)^{nl} = ((a^x)^n)^l((a^y)^l)^n = (a^m)^l(a^k)^n = a^{ml}a^{kn} = a^{ml+kn} = ((\sqrt[q]{a})^q)^{ml+kn} = (\sqrt[q]{a})^{q(ml+kn)} = (\sqrt[q]{a})^{pnl} = ((\sqrt[q]{a})^p)^{nl} = (a^{x+y})^{nl}$, $a^xa^y, a^{x+y} > 0$, $a^xa^y = a^{x+y}$.
$a=0$, $x,y>0$ $a^xa^y = 0\cdot 0 = 0 = a^{x+y}$.
$a<0$, n,l . $(a^xa^y)^{nl} = (a^{x+y})^{nl}$, nl , $a^xa^y = a^{x+y}$.
: $xy = \frac{p}{q}$ xy. $a>0$, $((a^x)^y)^{nl} = \left(((a^x)^y)^l\right)^n = ((a^x)^k)^n = ((a^x)^n)^k = (a^m)^k = a^{mk} = ((\sqrt[q]{a})^q)^{mk} = (\sqrt[q]{a})^{qmk} = (\sqrt[q]{a})^{pnl} = ((\sqrt[q]{a})^p)^{nl} = (a^{xy})^{nl}$, $(a^x)^y, a^{xy} > 0$, $(a^x)^y = a^{xy}$. $(a^y)^x = a^{xy}$.
$a=0$, $x,y>0$ $(a^x)^y = 0^y = 0 = a^{xy}$. $(a^y)^x = a^{xy}$.
$a<0$, n,l . $((a^x)^y)^{nl} = (a^{xy})^{nl}$, nl , $(a^x)^y = a^{xy}$. $(a^y)^x = a^{xy}$.
(2) (i) $x>0$, $m>0$. $(a^x)^n = a^m < b^m = (b^x)^n$, $a^x, b^x > 0$, $a^x < b^x$.
(iii) (ii) .
(3) (i) $x<y$ $n,l>0$, $ml<kn$. $(a^x)^{nl} = ((a^x)^n)^l = (a^m)^l = a^{ml} < a^{kn} = (a^k)^n = ((a^y)^l)^n = (a^y)^{nl}$. $a^x, a^y > 0$, $a^x > a^y$.
(iii) (ii) .

. .

, a^x $a \geq 0$ x .
’ $a>1$.
, s,r,t $s<r<t$, , , $a^s < a^r < a^t$. , , , s,t x $s<x<t$, $a^s < a^x < a^t$. , a^s, a^t a^x . , «» a^x: $a^s < a^x < a^t$ s,t $s<x<t$. .
$s<x$ $t>x$. s,t , , $s<t$ $a>1$, $a^s<a^t$. , , , a^s, , a^t, – . . , ξ, a^s a^t:

Σχήμα 1.4: a^x.

$$a^s < \xi < a^t$$

s, t $s < x < t.$, $\xi' \neq \xi$ $a^s < \xi' < a^t$ s, t $s < x < t.$, , .

1.3 (1) $a > 1$ $x.$ ξ $a^s < \xi < a^t$ s, t $s < x < t.$
(2) $a > 1$ $x.$ ξ $a^s < \xi < a^t$ s, t $s < x < t$ ξ ( ) a^x .

$a > 1$ x , a^x ξ 1.3. , , a^x

$$a^s < a^x < a^t$$

s, t $s < x < t$. 1.3, x .
$a = 1$ x , :

$$1^x = 1.$$

, $0 < a < 1$ x , $\frac{1}{a} > 1$ $-x$. $(\frac{1}{a})^{-x}$:

$$a^x = \left(\frac{1}{a}\right)^{-x}.$$

, x ,

$$0^x = 0.$$

, , , x , a^x (i) $a > 0$ (ii) $a = 0$ $x > 0.$ a^x (i) $a < 0$ x (ii) $a = 0$ x $< 0.$,

a^x (i) $a > 0$ x **,** (ii) $a = 0$ $x > 0$ (iii) $a < 0$ x **.**
a^x (i) $a = 0$ $x \leq 0$ (ii) $a < 0$ x **.**

a^x: x $a > 0$ ($a < 0$), , a^x , $a^x > a^s$ $s < x$, , $a^s > 0$, $a^x > 0.$
, , 1.7, . , .

.

. .

1. n .
   n , $x^n < y^n$ $x < y.$
   n , $x^n < y^n$ $|x| < |y|.$

2. x, y 0, $x^2 + xy + y^2 > 0$ $x^4 + x^3y + x^2y^2 + xy^3 + y^4 > 0.$;
   $x^3 + x^2y + xy^2 + y^3 > 0$ $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5 > 0$; ;

3. n
   (i) $1 + 2 + \cdots + n = \frac{1}{2}\, n(n+1),$
   (ii) $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}\, n(n+1)(2n+1),$
   (iii) $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}\, n^2(n+1)^2.$

4. Newton **Pascal**;

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

$1 \le m \le n$, $\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$. Pascal;

5. $\binom{n}{m}$ n m ;
$\binom{n}{m}$ m 0 n;
Pascal;

6. n .

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n\,,$$

$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} = 0.$$

. .

1. n , $\sqrt[n]{a^n} = a$.
n , $\sqrt[n]{a^n} = |a|$.

2. $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ $a, b \ge 0$. $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ $a = 0$ $b = 0$.

3. :

$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}\,, \qquad \sqrt[n]{\sqrt[m]{a}} = \sqrt[m]{\sqrt[n]{a}} = \sqrt[nm]{a}\,.$$

; $a < 0$ $b < 0$;

4. n $0 \le a < b$, $\sqrt[n]{a} < \sqrt[n]{b}$.
n $a < b$, $\sqrt[n]{a} < \sqrt[n]{b}$.

5. (*) $\sqrt[105]{105}$ $\sqrt[106]{106}$.

6. $\sqrt[7]{129}$, $3\sqrt{5+\sqrt{2}}$ $\sqrt[3]{2} + \sqrt{5}$.

7. $a > 0$, $\sqrt{a\sqrt{a\cdots\sqrt{a\sqrt{a}}}} = \frac{a}{\sqrt[2^n]{a}}$ n .

8. $\sqrt{a}$.
(: $1 < a$. , 0, 1 a, . x , $x^2 = a$.)
$\sqrt[4]{a}$ $\sqrt[8]{a}$.

.   .

1. $(-2)^0$, 0^0, $(-3)^{\frac{7}{3}}$, $(-2)^{\frac{16}{12}}$, $(-2)^{-\frac{10}{12}}$;
   $(-8)^{\frac{4}{3}}$, $(-1)^{\frac{14}{6}}$.

2. $\left((-1)^{\frac{2}{3}}\right)^{\frac{5}{2}} = (-1)^{\frac{2}{3}\cdot\frac{5}{2}}$; ; 1.7;

3. r $(-2)^r < 0$;

.   .

1. $2^{-\sqrt{2}}$, $(-2)^{\sqrt{2}}$, $0^{-\sqrt{2}}$, $0^{\sqrt{2}}$;

2. $(\sqrt[17]{3})^{24} < 3^{\sqrt{2}} < (\sqrt[17]{3})^{25}$.

3. $2^{\sqrt{3}}$ $3^{\frac{2}{\sqrt{5}}}$.

4. $[10 \cdot 2^{\sqrt{2}}]$ $[100 \cdot 2^{\sqrt{2}}]$.

5. $\left((-1)^2\right)^{\sqrt{3}} = (-1)^{2\sqrt{3}}$; 1.7;
   x $\left((-1)^x\right)^{\sqrt{3}} = (-1)^{x\sqrt{3}}$;

1.3 .

$a > 0$ $\neq 1$. : y $a^x = y$ (x) ;
x $a^x > 0$, , $a^x = y$, $y > 0$. 1.4, , , , y.

1.4 $a > 0$ $a \neq 1$. $y > 0$ x

$$a^x = y.$$

1.4 $a^x = y$. . , x_1, x_2 $a^x = y$ (y) 1.7 , $x_1 \neq x_2$, $a^{x_1} \neq a^{x_2}$.
$a = 1$, $a^x = y$, . , $1^x = 1$ x, y 1 ′ : . $a = 0$. $0^x = y$ $y = 0$ ′ : . $a < 0$ a^x x a^x x .
$a^x = y$ y a

$$\log_a y.$$

, :

$$\boxed{x = \log_a y \qquad a^x = y.}$$

1.10 $a > 0$ $\neq 1$.
(1) $\log_a(yz) = \log_a y + \log_a z$ $y, z > 0$.
(2) $\log_a \frac{y}{z} = \log_a y - \log_a z$ $y, z > 0$.
(3) $\log_a(y^z) = z \log_a y$ $y > 0$ z.
(4) $\log_a 1 = 0$ $\log_a a = 1$.
(5) $0 < y < z$. (i) $\log_a y < \log_a z$, $a > 1$, (ii) $\log_a y > \log_a z$, $0 < a < 1$.

: (1) $x = \log_a y$ $w = \log_a z$, $a^x = y$ $a^w = z$. $a^{x+w} = a^x a^w = yz$, $\log_a(yz) = x + w = \log_a y + \log_a z$.
(2) $\log_a \frac{y}{z} + \log_a z = \log_a(\frac{y}{z}z) = \log_a y$ $\log_a \frac{y}{z} = \log_a y - \log_a z$.
(3) $x = \log_a y$, $a^x = y$. $a^{zx} = (a^x)^z = y^z$, , $\log_a(y^z) = zx = z\log_a y$.
(4) $\log_a 1 = 0$ $a^0 = 1$ $\log_a a = 1$ $a^1 = a$.
(5) $0 < y < z$. $x = \log_a y$ $w = \log_a z$, $y = a^x$ $z = a^w$. $a^x < a^w$, $a > 1$, $x < w$, $0 < a < 1$, $x > w$.

1.11 $a, b > 0$ $a, b \neq 1$.

$$\log_b y = \frac{1}{\log_a b} \log_a y$$

$y > 0$.

: $a, b > 0$ $a, b \neq 1$. $x = \log_b y$ $w = \log_a b$, $b^x = y$ $a^w = b$. $a^{wx} = (a^w)^x = b^x = y$. $\log_a y = wx = \log_a b \log_b y$.

.

1. $\log_2 4$, $\log_{\frac{1}{2}} 2$, $\log_{\frac{1}{2}} 4$.
2. $\log_2 3$ $\log_3 4$.
3. $\log_2 3 \cdot \log_3 5 \cdot \log_5 7 \cdot \log_7 10 \cdot \log_{10} 8$.
4. $a > 0$, $a \neq 1$. $a^{\log_a y} = y$ $y > 0$.
5. $\log_2 3$;
6. $a > 0$, $a \neq 1$. $\log_{\frac{1}{a}} y = -\log_a y$ $y > 0$.
7. $a > 0$, $a \neq 1$. $\log_{a^z}(y^z) = \log_a y$ $y > 0$ $z \neq 0$.

1.4 . .

. .

1 , ′ ( ) ′ ( ). x $|x|$, , $x > 0$, ( ) $x < 0$. x , : x , .
π 1. , $\frac{\pi}{2}$ B, π ′, $\frac{3\pi}{2}$ ′ 2π . x $[0, 2\pi]$ ″ x 2π . x $[k2\pi, (k+1)2\pi]$, k : 2π. , x, x ( ) 2π.

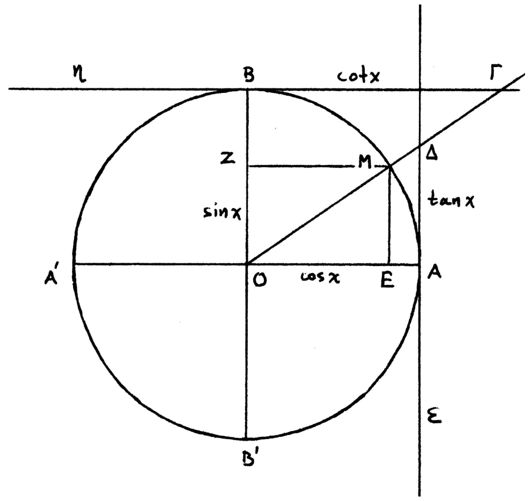
.

x , ,

1. ′.

$$\cos x = \pm$$

$+$, , $-$, .



Σχήμα 1.5: .

2. ′.

$$\sin x = \pm$$

$+$, , $-$, .

3. ο .

$$\tan x = \pm$$

$+$, , $-$, .

4. ο .

$$\cot x = \pm$$

$+$, , $-$, .

$\tan x$ ′, , $x = \frac{\pi}{2} + k\pi$ $(k \in \mathbf{Z})$. , $\cot x$ ′, , $x = k\pi$ $(k \in \mathbf{Z})$. $(\cos x, \sin x)$ ′ ′ . $\cos x$, $\sin x$, $\tan x$ $\cot x$, , , , x. x. , .

: (1) $\cos 0 = 1$, $\sin 0 = 0$, $\tan 0 = 0$. $\cot 0$.

(2) $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$, $\cot\frac{\pi}{2} = 0$. $\tan\frac{\pi}{2}$.

(3) $\cos\pi = -1$, $\sin\pi = 0$, $\tan\pi = 0$. $\cot\pi$.

(4) $\cos\frac{3\pi}{2} = 0$, $\sin\frac{3\pi}{2} = -1$, $\cot\frac{3\pi}{2} = 0$. $\tan\frac{3\pi}{2}$.

$\cos x > 0$, x $(-\frac{\pi}{2} + k2\pi, \frac{\pi}{2} + k2\pi)$ $(k \in \mathbf{Z})$, $\cos x < 0$, x $(\frac{\pi}{2} + k2\pi, \frac{3\pi}{2} + k2\pi)$ $(k \in \mathbf{Z})$. , $\sin x > 0$, x $(k2\pi, \pi + k2\pi)$ $(k \in \mathbf{Z})$, $\sin x < 0$, x $(\pi + k2\pi, 2\pi + k2\pi)$ $(k \in \mathbf{Z})$.

, ,

$$-1 \le \cos x \le 1, \qquad -1 \le \sin x \le 1$$

x. 1.12 · .

1.12 (1) $(\sin x)^2 + (\cos x)^2 = 1$.
(2) $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$.
(3) $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$, $\tan(-x) = -\tan x$, $\cot(-x) = -\cot x$.
(4) $\cos(\frac{\pi}{2}-x) = \sin x$, $\sin(\frac{\pi}{2}-x) = \cos x$, $\tan(\frac{\pi}{2}-x) = \cot x$, $\cot(\frac{\pi}{2}-x) = \tan x$.
(5) $\cos(x+\pi) = -\cos x$, $\sin(x+\pi) = -\sin x$, $\tan(x+\pi) = \tan x$, $\cot(x+\pi) = \cot x$.
(6) $\cos(x+y) = \cos x \cos y - \sin x \sin y$, $\sin(x+y) = \sin x \cos y + \cos x \sin y$.
(7) $\cos x - \cos y = -2\sin\frac{x-y}{2}\sin\frac{x+y}{2}$, $\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$.
(8) k . (i) $\cos x > \cos x'$, $k2\pi \le x < x' \le \pi + k2\pi$, (ii) $\cos x < \cos x'$, $\pi + k2\pi \le x < x' \le 2\pi + k2\pi$.
(9) k . (i) $\sin x < \sin x'$, $-\frac{\pi}{2} + k2\pi \le x < x' \le \frac{\pi}{2} + k2\pi$, (ii) $\sin x > \sin x'$, $\frac{\pi}{2} + k2\pi \le x < x' \le \frac{3\pi}{2} + k2\pi$.

: 1.12 .
(1) .
(2) , $-=-$, , $\tan x = \frac{\sin x}{\cos x}$. , , $-=-$, , $\cot x = \frac{\cos x}{\sin x}$.
(3) $x, -x$ $'$.
(4) $x, \frac{\pi}{2} - x$.
(5) $x, x+\pi$.
(6) , $x, -y$ $x+y$. ( ) ( ) . , ,

$$\sqrt{(\cos(x+y)-1)^2 + (\sin(x+y)-0)^2} = \sqrt{(\cos x - \cos(-y))^2 + (\sin x - \sin(-y))^2}\,.$$

, (1) (3), (6). , (3) (4):

$$\begin{aligned} \sin(x+y) &= \cos\left(\frac{\pi}{2} - (x+y)\right) = \cos\left(\left(\frac{\pi}{2} - x\right) + (-y)\right) \\ &= \cos\left(\frac{\pi}{2} - x\right)\cos(-y) - \sin\left(\frac{\pi}{2} - x\right)\sin(-y) \\ &= \sin x \cos y + \cos x \sin y. \end{aligned}$$

(7)

$$\cos x = \cos\left(\frac{x+y}{2} + \frac{x-y}{2}\right) = \cos\frac{x+y}{2}\cos\frac{x-y}{2} - \sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\cos y = \cos\left(\frac{x+y}{2} - \frac{x-y}{2}\right) = \cos\frac{x+y}{2}\cos\frac{x-y}{2} + \sin\frac{x+y}{2}\sin\frac{x-y}{2}\,.$$

,

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}\,.$$

(7).
(8) $k2\pi \le x < x' \le \pi + k2\pi$, , $'$ x, x' $'$, $\cos x > \cos x'$. $\pi + k2\pi \le x < x' \le 2\pi + k2\pi$, , $'$ $'$, $\cos x < \cos x'$.
(9) $-\frac{\pi}{2} + k2\pi \le x < x' \le \frac{\pi}{2} + k2\pi$, , $'$ x, x' $'$, $\sin x < \sin x'$. $\frac{\pi}{2} + k2\pi \le x < x' \le \frac{3\pi}{2} + k2\pi$, , $'$ $'$, $\sin x > \sin x'$.

. .

. . .

1. y $[-1,1]$. ′ y ′ ′ .

$$\arccos y$$

$[0,\pi]$, .

2. y $[-1,1]$. ′ y ′ ′ .

$$\arcsin y$$

$[-\frac{\pi}{2},\frac{\pi}{2}]$.

3. y. o y. ′.

$$\arctan y$$

$(-\frac{\pi}{2},\frac{\pi}{2})$.

4. y. o y. ′.

$$\operatorname{arccot} y$$

$(0,\pi)$.

: (1) $\arccos 1 = 0$, $\arccos 0 = \frac{\pi}{2}$, $\arccos(-1) = \pi$.

(2) $\arcsin 1 = \frac{\pi}{2}$, $\arcsin 0 = 0$, $\arcsin(-1) = -\frac{\pi}{2}$.

(3) $\arctan 0 = 0$ $\operatorname{arccot} 0 = \frac{\pi}{2}$.

$\arccos y$, $\arcsin y$, $\arctan y$ $\operatorname{arccot} y$, , **-**, **-**, **-** **-** y. y.
y $[-1,1]$ $\arccos y$ $[0,\pi]$ $\cos x = y$.

$$\boxed{x = \arccos y \qquad \cos x = y \ \ 0 \le x \le \pi.}$$

$\cos x = y$ $[-\pi,0]$, $-\arccos y$. $\cos x = y$ **R** $\arccos y + k2\pi$ $(k \in \mathbf{Z})$ $-\arccos y + k2\pi$ $(k \in \mathbf{Z})$.
, , y $[-1,1]$ $\arcsin y$ $[-\frac{\pi}{2},\frac{\pi}{2}]$ $\sin x = y$.

$$\boxed{x = \arcsin y \qquad \sin x = y \ \ -\frac{\pi}{2} \le x \le \frac{\pi}{2}.}$$

$\sin x = y$ $[\frac{\pi}{2},\frac{3\pi}{2}]$, $\pi - \arcsin y$. $\sin x = y$ **R** $\arcsin y + k2\pi$ $(k \in \mathbf{Z})$ $\pi - \arcsin y + k2\pi$ $(k \in \mathbf{Z})$.
y $\arctan y$ $(-\frac{\pi}{2},\frac{\pi}{2})$ $\tan x = y$.

$$\boxed{x = \arctan y \qquad \tan x = y \ \ -\frac{\pi}{2} < x < \frac{\pi}{2}.}$$

$\tan x = y$ **R** $\arctan y + k\pi$ $(k \in \mathbf{Z})$.
y $\operatorname{arccot} y$ $(0,\pi)$ $\cot x = y$.

$$\boxed{x = \operatorname{arccot} y \qquad \cot x = y \ \ 0 < x < \pi.}$$

$\cot x = y$ **R** $\operatorname{arccot} y + k\pi$ $(k \in \mathbf{Z})$.

1.13 (1) $-1 \le y < y' \le 1$. $\arccos y > \arccos y'$ $\arcsin y < \arcsin y'$.
(2) $y < y'$. $\arctan y < \arctan y'$ $\operatorname{arccot} y > \operatorname{arccot} y'$.

: (1) , ′ ′ y, y' , ′ ′. ′, M′ , , $\arccos y > \arccos y'$.
, , ′ ′ y, y' , ′ ′. ′, ′ , , $\arcsin y < \arcsin y'$.
(2) , ′ y, y' , ′ ′. ′, ′ , , $\arctan y < \arctan y'$.
, ′ y, y' , ′ ′. ′, ′ , , $\operatorname{arccot} y > \operatorname{arccot} y'$.

. ( ) . , , . , . . : $\cos x$ $\sin x$ 8 10.

, , .
. π 1. 2 « » π. π 8 10.

.

. .

1. $\frac{\pi}{6}$, $\frac{\pi}{4}$ $\frac{\pi}{3}$.

2. .

$$\cos x = \frac{1}{2}, \quad \sin x = -\frac{1}{2}, \quad \cos x = -\frac{1}{\sqrt{2}}, \quad \sin x = \frac{\sqrt{3}}{2},$$

$$\tan x = 0, \quad \cot x = -1, \quad \tan x = -\sqrt{3}, \quad \cot x = \sqrt{3}.$$

3. $|a\cos x + b\sin x| \le \sqrt{a^2+b^2}$.

4. a, b $a^2+b^2=1$ q $[0, 2\pi)$

$$\cos q = a \qquad \sin q = b.$$

5. a, b 0. , $p > 0$ q

$$a\cos x + b\sin x = p\cos(x-q)$$

x.
(: $a\cos x + b\sin x = \sqrt{a^2+b^2}\big(\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x\big)$.)

6.
(i) $\cos y = \cos x$ $y = x + k2\pi$ $y = -x + k2\pi$ $(k \in \mathbf{Z})$,
(ii) $\sin y = \sin x$ $y = x + k2\pi$ $y = \pi - x + k2\pi$ $(k \in \mathbf{Z})$,
(iii) $\tan y = \tan x$ $y = x + k\pi$ $(k \in \mathbf{Z})$,
(iv) $\cot y = \cot x$ $y = x + k\pi$ $(k \in \mathbf{Z})$.

7.
$$1 + (\tan x)^2 = \frac{1}{(\cos x)^2}, \qquad 1 + (\cot x)^2 = \frac{1}{(\sin x)^2}.$$

8.
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}\,, \qquad \cot(x+y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}\,.$$

9.
$$\cos(2x) = (\cos x)^2 - (\sin x)^2 = 2(\cos x)^2 - 1 = 1 - 2(\sin x)^2\,,$$
$$\sin(2x) = 2\sin x \cos x,$$
$$\tan(2x) = \frac{2\tan x}{1-(\tan x)^2}\,, \qquad \cot(2x) = \frac{(\cot x)^2 - 1}{2\cot x}\,.$$

10.
$$\cos x = \frac{1-(\tan\frac{x}{2})^2}{1+(\tan\frac{x}{2})^2}\,, \qquad \sin x = \frac{2\tan\frac{x}{2}}{1+(\tan\frac{x}{2})^2}\,,$$
$$\tan x = \frac{2\tan\frac{x}{2}}{1-(\tan\frac{x}{2})^2}\,, \qquad \cot x = \frac{1-(\tan\frac{x}{2})^2}{2\tan\frac{x}{2}}\,.$$

11.
$$2\sin x \sin y = \cos(x-y) - \cos(x+y), \quad 2\cos x \cos y = \cos(x-y) + \cos(x+y),$$
$$2\sin x \cos y = \sin(x-y) + \sin(x+y).$$

12. ,
$$\cos x + \cos(2x) + \cos(3x) + \cdots + \cos(nx) = \frac{\sin(\frac{nx}{2})\cos(\frac{(n+1)x}{2})}{\sin\frac{x}{2}}\,,$$
$$\sin x + \sin(2x) + \sin(3x) + \cdots + \sin(nx) = \frac{\sin(\frac{nx}{2})\sin(\frac{(n+1)x}{2})}{\sin\frac{x}{2}}\,.$$

(: $\sin\frac{x}{2}$.)

. .

1. $0,\ \pm\frac{1}{2}\,,\ \pm\frac{\sqrt{2}}{2}\,,\ \pm\frac{\sqrt{3}}{2}\ \ \pm 1.$

2.
$$\arccos y + \arcsin y = \frac{\pi}{2} \quad (-1 \le y \le 1), \qquad \arctan y + \operatorname{arccot}\, y = \frac{\pi}{2}\,.$$

3. y $y = \cos(\arccos y)$ $y = \sin(\arcsin y)$;
y $y = \tan(\arctan y)$ $y = \cot(\operatorname{arccot} y)$;

4. $\arccos(\cos x) = x$ x $[0, \pi]$.
, $\arccos(\cos x)$, x $[k\pi, (k+1)\pi]$, k ;
$\arcsin(\sin x)$, $\arctan(\tan x)$ $\operatorname{arccot}(\cot x)$;

Κεφάλαιο 2

.

(). , . : « ϵ n_0» , ... e π.

2.1 .

( ) : , , . · 1, 2, 3 . :

$$x_1, x_2, \ldots, x_n, \ldots, \qquad y_1, y_2, \ldots, y_n, \ldots, \qquad z_1, z_2, \ldots, z_n, \ldots.$$

$$(x_n), \qquad (y_n), \qquad (z_n).$$

n (,) , : , ′. x_{n+1} x_n x_{n-1} x_n.

: (1) $(\frac{1}{n})$, $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$.

(2) (n), $1, 2, 3, 4, \ldots, n, \ldots$.

(3) (1), $1, 1, 1, \ldots, 1, \ldots$.

(4) $\big((-1)^{n-1}\big)$, $1, -1, 1, -1, \ldots, 1, -1, \ldots$.

(5) $\big(\frac{1}{10^n}\big)$, $\frac{1}{10}, \frac{1}{10^2}, \frac{1}{10^3}, \ldots, \frac{1}{10^n}, \ldots$.

(6) n- n, $1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, \ldots$.

. «» : , ′. , «»· − − n- . (,) . , , (,). , . $\{1\}$. , , · $1, 1, 1, \ldots$. , , (, ,) ( ).

(x_n) $x_{n+1} \geq x_n$ n. (x_n) $x_{n+1} > x_n$ n. (x_n) $x_{n+1} \leq x_n$ n. , (x_n) $x_{n+1} < x_n$ n. . : , .

(x_n) , c $x_n = c$ n. . .

(x_n) , u $x_n \leq u$ n. , (x_n) , l $l \leq x_n$ n. (x_n) , l, u $l \leq x_n \leq u$ n , , $[l, u]$.

: (1) (c) , , . $[c,c]$.

(2) $(\frac{1}{n})$ $[0,1]$.

(3) $\left(\frac{(-1)^{n-1}}{n}\right)$ $[-\frac{1}{2},1]$.

(4) $(\frac{n-1}{n})$ $[0,1]$.

(5) $\left((-1)^{n-1}\right)$ $[-1,1]$.

(6) $\left(\frac{(1+(-1)^{n-1})n}{2}\right)$, $1,0,3,0,5,0,7,0,\dots$, . ≤ 0 . , , u . $\frac{(1+(-1)^{n-1})n}{2} \leq u$ n, , $n=2k-1$, $2k-1 \leq u$ k. , $k \leq \frac{u+1}{2}$ k, 1.1.

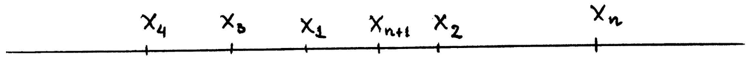
(7) $-1,0,-3,0,-5,0,-7,0,\dots$, , · l , $\geq l$, $\leq -l$, .

(8) $\left((-1)^{n-1}n\right)$, $1,-2,3,-4,5,-6,\dots$, . u l $(-1)^{n-1}n \leq u$ n $l \leq (-1)^{n-1}n$ n, , n, .

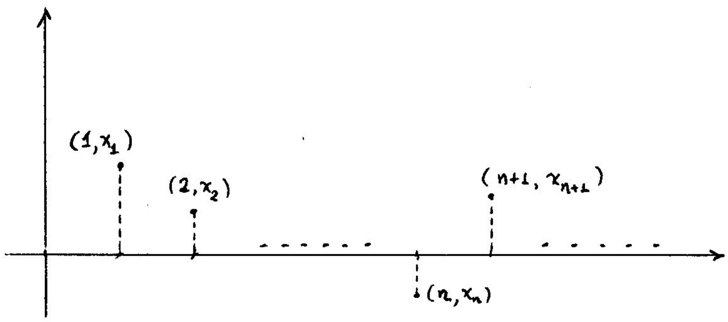
(x_n) , x_n $[l,u]$. (x_n) 0. $[-M,M]$ $[l,u]$, $M=\max\{u,-l\}$. , (x_n) , M $|x_n| \leq M$ n.

u (x_n), $> u$ (x_n). , , . , l (x_n), $< l$ (x_n). , , .

. . (x_n) , , n , x_n «» . , (x_n) , x_n «» .



Σχήμα 2.1: .



Σχήμα 2.2: .

(x_n) , , 0. n x_n. (n,x_n) (x_n). , x_n , (n,x_n) . , «» : n , (n,x_n) . , (x_n) , , n , (n,x_n) ( ).

, (x_n) , (n, x_n) ( ). , (x_n) , (n, x_n) .

.

. .

1. .

$$\Big(\frac{2n-1}{3n+2}\Big),\quad \Big(\frac{\sqrt{n}}{n+1}\Big),\quad \Big(\frac{1-(-1)^n}{n^3}\Big),\quad \Big(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}\Big),$$

$$(2^{n!}),\quad \Big(\frac{(-1)^{n+1}}{n!}\Big),\quad \Big(\frac{(-1)^{n-1}}{2\cdot 4\cdot 6\cdots 2n}\Big),\quad \Big(\frac{n^2-3n+1}{2^n n!}\Big),$$

$$\Big(\frac{(2x)^{n-1}}{(2n-1)^5}\Big),\quad \Big(\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}\Big),\quad \Big(\frac{(-1)^n x^{2n-1}}{1\cdot 3\cdot 5\cdots(2n-1)}\Big).$$

2. :

$$1,\ 4,\ 9,\ 16,\ 25,\ 36.$$

, ; (i) 49. (ii) 24. (iii) .

3. .

$$(n),\quad (-n),\quad ((-1)^{n-1}),\quad \Big(\frac{1+(-1)^{n-1}}{2}\Big),\quad \Big(\frac{a+b}{2}+(-1)^{n-1}\frac{a-b}{2}\Big).$$

(: .)

4. $\big(n-2[\frac{n}{2}]\big)$, $\big(n-3[\frac{n}{3}]\big)$ $\big(n-4[\frac{n}{4}]\big)$. , m , $\big(n-m[\frac{n}{m}]\big)$.

5. . a, b, p, q, p, q 0. (x_n) :

$$x_1 = a, x_2 = b \qquad x_{n+2} = px_{n+1} + qx_n \ (n \ge 1).$$

n- x_n .

1: $p \neq 0, q = 0$. $x_n = bp^{n-2}$ $n \ge 2$.

2: $p = 0, q \neq 0$. $x_n = aq^{\frac{n-1}{2}}$, n , $x_n = bq^{\frac{n-2}{2}}$, n .

3: $p \neq 0, q \neq 0$.

$$x^2 = px + q.$$

(i) $\Delta = p^2 + 4q > 0$, () , $\rho_1 = \frac{p+\sqrt{\Delta}}{2}$ $\rho_2 = \frac{p-\sqrt{\Delta}}{2}$. κ, λ $\kappa + \lambda = a$ $\kappa\rho_1 + \lambda\rho_2 = b$.

$$x_n = \kappa{\rho_1}^{n-1} + \lambda{\rho_2}^{n-1}$$

$n \ge 1$.

(ii) $\Delta = p^2 + 4q = 0$, , $\rho = \frac{p}{2}$. κ, λ $\kappa = a$ $\kappa\rho + \lambda\rho = b$.

$$x_n = \kappa\rho^{n-1} + \lambda(n-1)\rho^{n-1}$$

$n \geq 1$.

(iii) $\Delta = p^2 + 4q < 0$ ($q < 0$), () , $\rho_1 = \frac{p+i\sqrt{-\Delta}}{2}$ $\rho_2 = \frac{p-i\sqrt{-\Delta}}{2}$. $\rho = \sqrt{-q} > 0$ $\left(\frac{p}{2\rho}\right)^2 + \left(\frac{\sqrt{-\Delta}}{2\rho}\right)^2 = 1$, θ $[0, 2\pi)$ $\cos\theta = \frac{p}{2\rho}$ $\sin\theta = \frac{\sqrt{-\Delta}}{2\rho}$. $\rho_1 = \rho(\cos\theta + i\sin\theta)$ $\rho_2 = \rho(\cos\theta - i\sin\theta)$. $\rho^2\cos(2\theta) = p\rho\cos\theta + q$ $\rho^2\sin(2\theta) = p\rho\sin\theta$. κ, λ $\kappa = a$ $\rho(\kappa\cos\theta + \lambda\sin\theta) = b$. ,

$$x_n = \rho^{n-1}\big(\kappa\cos((n-1)\theta) + \lambda\sin((n-1)\theta)\big)$$

$n \geq 1$.

n- ( )

$$x_1 = x_2 = 1$$

.

$$x_{n+2} = 3x_n \ (n \geq 1), \qquad x_{n+2} = x_{n+1} + x_n \ (n \geq 1),$$

$$x_{n+2} = 2x_{n+1} - x_n \ (n \geq 1), \qquad x_{n+2} = x_{n+1} - x_n \ (n \geq 1).$$

, $x_1 = x_2 = 1$ $x_{n+2} = x_{n+1} + x_n$ $(n \geq 1)$, **Fibonnaci** $1, 1, 2, 3, 5, 8, 13$.

. .

1. (x_n) , .

2. (x_n) $(-x_n)$, .

3. ; ;

, , . .

$$(n), \quad ((-1)^{n-1}), \quad ((-1)^{n-1}n), \quad \left(\frac{(-1)^{n-1}}{n}\right), \quad (2^n), \quad \left(\frac{1}{2^n}\right),$$

$$\left(\frac{8n-1}{n^2+n+1}\right), \quad \left(\binom{n+15}{16}\right), \quad \left(\frac{8^n}{n!}\right), \quad \left(2\left[\frac{n}{2}\right]\right), \quad \left(n - 3\left[\frac{n}{3}\right]\right).$$

. .

1. ; ; ;

2. − − − .

3. (x_n) $(-x_n)$, .

2.2 .

: (1) $(\frac{1}{n})$.

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{100}, \frac{1}{101}, \dots, \frac{1}{100000}, \dots, \frac{1}{100000000}, \dots .$$

, n « », $\frac{1}{n}$ « ». , , , , n « », $\frac{1}{n}$ « » 0. ( ).

(2) , , $(\frac{n-1}{n})$:

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{99}{100}, \frac{100}{101}, \dots, \frac{99999}{100000}, \dots, \frac{99999999}{100000000}, \dots .$$

n « », $\frac{n-1}{n}$ 1 « ». , $\frac{n-1}{n}$ 1 $|\frac{n-1}{n} - 1| = 1 - \frac{n-1}{n} = \frac{1}{n}$, , n , $\frac{1}{n}$. ′ , , , n « », $\frac{n-1}{n}$ « » 1.

(x_n) :

n « », x_n
x « ».

, / . , ( ) . , , ( ) , «» .

x_n x « » $|x_n - x|$ « ». « n » « n ». , , :

$|x_n - x|$ n .

. :

: $(\frac{1}{n})$. $\frac{1}{n}$ 0 $|\frac{1}{n} - 0| = \frac{1}{n}$, , n . ; , 0,000132. $\frac{1}{n}$ 0,000132 n ; : n $\frac{1}{n}$ 0,000132; : $\frac{1}{n} < 0,000132$ $n > \frac{1000000}{132}$. $n > \frac{1000000}{132}$; 7576 $> \frac{1000000}{132}$ (7575 $\leq \frac{1000000}{132}$) , , $n \geq 7576$, $\frac{1}{n} < 0,000132$. , 0,0000000000132. , $n \geq 75757575758$, $\frac{1}{n} < 0,0000000000132$. , , : (0,000132 0,0000000000132) (7576 75757575758). , , . . , , . , ϵ, , n_0 , , $n \geq n_0$, $\frac{1}{n} < \epsilon$. , , : $\epsilon > 0$, , n_0 $\frac{1}{n_0}, \frac{1}{n_0+1}, \frac{1}{n_0+2}, \dots$ $< \epsilon$. (ϵ) n_0 ϵ. n_0 (ϵ) $\frac{1}{n} < \epsilon$; , . $\frac{1}{n} < \epsilon$ $n > \frac{1}{\epsilon}$ (, n) :

$> a$ $n_0 = [a] + 1$, $a \geq 0$, $n_0 = 1$, $a < 0$.

: > -1 1, $> \frac{8}{3}$ ($2 < \frac{8}{3} < 3$) $3 = [\frac{8}{3}] + 1$ > 2 $3 = 2 + 1 = [2] + 1$.

($\frac{1}{\epsilon} \geq 0$) n_0 $n_0 = [\frac{1}{\epsilon}] + 1$. ϵ, ( ) n_0 ϵ, ϵ n_0.

. (x_n) x x x (x_n) $|x_n - x|$ $\epsilon > 0$ n n_0 , , $\epsilon > 0$ n_0 $|x_n - x| < \epsilon$ $n \geq n_0$. (x_n) x

$$x_n \to x \qquad \lim x_n = x \qquad \lim_{n \to +\infty} x_n = x.$$

. (x_n) x $\epsilon > 0$ n_0 $|x_n - x| < \epsilon$ $n \geq n_0$, , $\epsilon > 0$ n_0 $n \geq n_0$ $|x_n - x| < \epsilon$, , $\epsilon > 0$ n_0 $n \geq n_0$ $|x_n - x| < \epsilon$.

Σχήμα 2.3: $x - \epsilon < x_n < x + \epsilon$ $n \geq n_0$.

(x_n) , (x_n) .

$\lim_{n\to+\infty} x_n = x$; n, n (,), $\lim_{n\to+\infty} x_n = x$ x_n , , x n .

: (1) , $\lim_{n\to+\infty} \frac{1}{n} = 0$ $\lim_{n\to+\infty} \frac{n-1}{n} = 1$.

(2) $\big((-1)^{n-1}\big)$.

$$1, -1, 1, -1, \ldots, 1, -1, 1, -1, \ldots .$$

n , $(-1)^{n-1}$ «» 1 -1 1 . , $\big((-1)^{n-1}\big)$. «» (, ,) 1 «» (, ,) -1.

$$\boxed{\big((-1)^{n-1}\big)\ .}$$

, , $\big((-1)^{n-1}\big)$. , , $\big((-1)^{n-1}\big)$ x. $\epsilon > 0$ n_0 $|(-1)^{n-1} - x| < \epsilon$ $n \geq n_0$. n_0 $n \geq n_0$ $n \geq n_0$. , $n \geq n_0$ $|-1-x| < \epsilon$ $n \geq n_0$ $|1-x| < \epsilon$. $\epsilon > 0$ $|-1-x| < \epsilon$ $|1-x| < \epsilon$. , $|-1-x| < \frac{1}{2}$ $|1-x| < \frac{1}{2}$. , , x! $\big((-1)^{n-1}\big)$ x 2.15 ≥ 1 ≤ -1.

(3) $\big(\frac{(-1)^{n-1}}{n}\big)$, $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$. n- 0 $\big|\frac{(-1)^{n-1}}{n} - 0\big| = \frac{1}{n}$, , , $\epsilon > 0$ n_0 $\big|\frac{(-1)^{n-1}}{n} - 0\big| = \frac{1}{n} < \epsilon$ $n \geq n_0$. , $\lim_{n\to+\infty} \frac{(-1)^{n-1}}{n} = 0$.

(4) (c), $c, c, c, c, \ldots$.

$$\boxed{\lim_{n\to+\infty} c = c.}$$

, c $|c - c| = 0$. , $\epsilon > 0$ $n_0 = 1$ $|c - c| = 0 < \epsilon$ $n \geq n_0 = 1$.

(5) $(\frac{1}{n})$. $a > 0$ $(\frac{1}{n^a})$, $1, \frac{1}{2^a}, \frac{1}{3^a}, \frac{1}{4^a}, \ldots$.

$$\boxed{\lim_{n\to+\infty} \frac{1}{n^a} = 0 \qquad (a > 0).}$$

$\epsilon > 0$ n_0 $|\frac{1}{n^a} - 0| < \epsilon$ $n \geq n_0$. $|\frac{1}{n^a} - 0| < \epsilon$ $\frac{1}{n^a} < \epsilon$ $n^a > \frac{1}{\epsilon}$ $n > (\frac{1}{\epsilon})^{\frac{1}{a}}$. $(\frac{1}{\epsilon})^{\frac{1}{a}} \geq 0$, $n_0 = [(\frac{1}{\epsilon})^{\frac{1}{a}}] + 1$, $n \geq n_0$ $n > (\frac{1}{\epsilon})^{\frac{1}{a}}$, , $|\frac{1}{n^a} - 0| < \epsilon$.

: $\lim_{n\to+\infty} \frac{1}{n^2} = 0$, $\lim_{n\to+\infty} \frac{1}{\sqrt{n}} = 0$, $\lim_{n\to+\infty} \frac{1}{\sqrt[3]{n}} = 0$.

$\lim_{n\to+\infty} x_n = x$, . $\epsilon > 0$ «» , $|x_n - x| < \epsilon$ $n > a$. : « $_1$ $_2$» , , « $_1 \Longleftarrow$ $_2$». $n > a$ $n_0 = [a] + 1$, $a \geq 0$, $n_0 = 1$,

$a < 0$, $n \geq n_0$ $n > a$, , $|x_n - x| < \epsilon$.

: (1) $\lim_{n\to+\infty} \frac{1}{n^2+n} = 0$.

$\epsilon > 0$ n_0 $|\frac{1}{n^2+n} - 0| < \epsilon$ $n \geq n_0$. $|\frac{1}{n^2+n} - 0| < \epsilon$ $\frac{1}{n^2+n} < \epsilon$ $n^2 + n - \frac{1}{\epsilon} > 0$ $n > -\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{4}{\epsilon}}$ $n < -\frac{1}{2} - \frac{1}{2}\sqrt{1+\frac{4}{\epsilon}}$ (: n) $n > -\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{4}{\epsilon}}$. $-\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{4}{\epsilon}} \geq 0$. , , $n_0 = \left[-\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{4}{\epsilon}}\right] + 1$, $n \geq n_0$ $n > -\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{4}{\epsilon}}$, , $|\frac{1}{n^2+n} - 0| < \epsilon$.

. $|\frac{1}{n^2+n} - 0| < \epsilon$ $\frac{1}{n^2+n} < \epsilon$ ($\frac{1}{n^2+n} \leq \frac{1}{n}$) $\frac{1}{n} < \epsilon$ $n > \frac{1}{\epsilon}$. $\frac{1}{\epsilon} \geq 0$, $n_0 = [\frac{1}{\epsilon}] + 1$, $n \geq n_0$ $n > \frac{1}{\epsilon}$, , $|\frac{1}{n^2+n} - 0| < \epsilon$.

(2) (x_n), $x_n = \frac{3+(-1)^n}{2n} = \begin{cases} \frac{2}{n}, & n \ , \\ \frac{1}{n}, & n \ . \end{cases}$ $1, 1, \frac{1}{3}, \frac{1}{2}, \frac{1}{5}, \frac{1}{3}, \frac{1}{7}, \frac{1}{4}, \ldots$.

$\lim_{n\to+\infty} x_n = 0$. $\epsilon > 0$. $|x_n - 0| < \epsilon$ ($x_n \geq 0$) $x_n < \epsilon$ ($x_n \leq \frac{2}{n}$) $\frac{2}{n} < \epsilon$ $n > \frac{2}{\epsilon}$. $\frac{2}{\epsilon} \geq 0$, $n_0 = [\frac{2}{\epsilon}] + 1$, $n \geq n_0$ $n > \frac{2}{\epsilon}$, , $|x_n - 0| < \epsilon$.

(x_n) , > 0 0· 1 .

(3) $\lim_{n\to+\infty} \frac{\sin n}{n} = 0$.

$\epsilon > 0$. $|\frac{\sin n}{n} - 0| < \epsilon$ $\frac{|\sin n|}{n} < \epsilon$ ($\frac{|\sin n|}{n} \leq \frac{1}{n}$) $\frac{1}{n} < \epsilon$ $n > \frac{1}{\epsilon}$. $\frac{1}{\epsilon} \geq 0$, $n_0 = [\frac{1}{\epsilon}] + 1$, $n \geq n_0$ $n > \frac{1}{\epsilon}$, , $|\frac{\sin n}{n} - 0| < \epsilon$.

(4) . (a^n), $a, a^2, a^3, a^4, \ldots$. a.

$a = 1$, (1) 1. , $a = 0$, (0) 0.

$a \leq -1$ (: $a = -1$), $a \leq -1$, $a^2 \geq 1$, $a^3 \leq -1$, $a^4 \geq 1, \ldots$. , ≥ 1 ≤ -1 , 2.15, .

$0 < |a| < 1$, 0. $a = \pm\frac{1}{2}$ $a = \pm\frac{1}{10}$. $a = \frac{1}{2}$, $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots$, $a = -\frac{1}{10}$, $-\frac{1}{10}, \frac{1}{10^2}, -\frac{1}{10^3}, \frac{1}{10^4}, \ldots$. , 0.

, , $0 < |a| < 1$ $\epsilon > 0$. $|a^n - 0| < \epsilon$ $|a|^n < \epsilon$ $n > \log_{|a|} \epsilon$. , $\log_{|a|} \epsilon \geq 0$, $0 < \epsilon \leq 1$, $\log_{|a|} \epsilon < 0$, $\epsilon > 1$. , $n_0 = [\log_{|a|} \epsilon] + 1$, $\epsilon \leq 1$, $n_0 = 1$, $\epsilon > 1$, $n \geq n_0$ $n > \log_{|a|} \epsilon$, , $|a^n - 0| < \epsilon$.

$a > 1$. .

.

1. (x_n), $x_n = \frac{3+(-1)^n}{2n}$, . $x_n > x_{n+1}$ n $x_n < x_{n+1}$ $n \geq 3$.

2. . ϵ n_0 .

$$\lim_{n\to+\infty} \frac{3}{4}, \quad \lim_{n\to+\infty} n^{-\frac{8}{3}}, \quad \lim_{n\to+\infty} \frac{1}{n\sqrt{n}}, \quad \lim_{n\to+\infty} \frac{3^n}{4^n}, \quad \lim_{n\to+\infty} \frac{(-1)^n 8^n}{3^{2n}}.$$

3. :

$$\lim_{n\to+\infty} \frac{n-2}{3n+4} = \frac{1}{3}, \quad \lim_{n\to+\infty} \frac{3n}{n+3} = 2, \quad \lim_{n\to+\infty} \frac{\sqrt{n}}{n+1} = 0.$$

; , n- n . , , «» n (, $n = 1000, 10000$). ϵ n_0 .

4. (, $\epsilon > 0$ n_0 ϵ,), .

$$\lim_{n\to+\infty}\frac{1}{\sqrt{n}}=0,\quad \lim_{n\to+\infty}\frac{1}{n^5}=0,\quad \lim_{n\to+\infty}\frac{1}{10^n}=0,\quad \lim_{n\to+\infty}\Big(-\frac{1}{3}\Big)^n=0,$$

$$\lim_{n\to+\infty}\frac{1}{n+1}=0,\quad \lim_{n\to+\infty}\frac{3n-1}{4n+5}=\frac{3}{4},\quad \lim_{n\to+\infty}\frac{1}{n^2+1}=0,$$

$$\lim_{n\to+\infty}\frac{1}{\sqrt{n}+1}=0,\quad \lim_{n\to+\infty}\frac{n^2-n+1}{3n^2+2}=\frac{1}{3},\quad \lim_{n\to+\infty}\frac{2\sqrt{n}+3}{2-3\sqrt{n}}=-\frac{2}{3},$$

$$\lim_{n\to+\infty}\frac{\cos n}{n\sqrt{n}}=0,\quad \lim_{n\to+\infty}\frac{\cos(2n)+\sqrt{n}}{\sqrt{n}}=1,\quad \lim_{n\to+\infty}\frac{1}{2^n+3n}=0.$$

5. $\lim_{n\to+\infty} x_n = x$, (x_n) x, : $\epsilon > 0$ n_0 $n \geq n_0$ $|x_n - x| \geq \epsilon$.
;

2.3 $\pm\infty$.

: (1) (n^2)

$$1, 4, 9, 16, 25, 36, 49, 64, \ldots .$$

n- n^2 « » n « ». , n^2 « » n « ».

(2) (n)

$$1, 2, 3, 4, 5, 6, 7, 8, \ldots .$$

n- n « » n « ».

(x_n) :

x_n « » n
« ».

, « » « », , « » « ». :

x_n
n .

: (n^2) : n^2 n . ; , , 35000. n^2 35000 n ; : n n^2 35000; : $n^2 > 35000$ $n > \sqrt{35000}$. n $> \sqrt{35000}$; 188 $> \sqrt{35000}$ (187 $\leq \sqrt{35000}$), , $n \geq 188$, $n^2 > 35000$. , , 35000000000. , $n \geq 187083$, $n^2 > 35000000000$. , (35000 35000000000) (188 187083), . , M, , n_0, , $n \geq n_0$, $n^2 > M$. $n^2 > M$ $n > \sqrt{M}$. , , 1.1: $M > 0$ n_0 $n_0, n_0+1, n_0+2, \ldots$ $> \sqrt{M}$. (M) n_0 M. n_0 (M) $n > \sqrt{M}$; : $\sqrt{M} \geq 0$, n_0 $n_0 = [\sqrt{M}] + 1$.

. (x_n) $+\infty$ $+\infty$ $+\infty$ (x_n) x_n $M > 0$ n n_0 , , $M > 0$ n_0 $x_n > M$ $n \geq n_0$. (x_n) $+\infty$

$$x_n \to +\infty \qquad \lim x_n = +\infty \qquad \lim_{n\to+\infty} x_n = +\infty.$$

. (x_n) $+\infty$ $M > 0$ n_0 $x_n > M$ $n \geq n_0$, , $M > 0$ n_0 $n \geq n_0$ $x_n > M$, , $M > 0$ n_0 $n \geq n_0$ $x_n > M$.

Σχήμα 2.4: $x_n > M$ $n \geq n_0$.

, (x_n) $-\infty$ $-\infty$ $-\infty$ (x_n) x_n $-M < 0$ n n_0 , , $M > 0$ n_0 $x_n < -M$ $n \geq n_0$. (x_n) $-\infty$

$$x_n \to -\infty \qquad \lim x_n = -\infty \qquad \lim_{n\to+\infty} x_n = -\infty\,.$$

$\lim_{n\to+\infty} x_n = +\infty$ $\lim_{n\to+\infty} x_n = -\infty$; $\lim_{n\to+\infty} x_n = +\infty$ $-\infty$ x_n, , , , n .

: (1) , $\lim_{n\to+\infty} n^2 = +\infty$ $\lim_{n\to+\infty} n = +\infty$.

(2) $-n < -M$ $n > M$. , $M > 0$ n_0 $-n < -M$ $n \geq n_0$ $M > 0$ n_0 $-n^2 < -M$ $n \geq n_0$. $\lim_{n\to+\infty}(-n^2) = -\infty$ $\lim_{n\to+\infty}(-n) = -\infty$.

(3) (x_n), $x_n = \frac{\big(3-(-1)^{n-1}\big)n}{2} = \begin{cases} n, & n \ , \\ 2n, & n \ . \end{cases}$, , $1, 4, 3, 8, 5, 12, 7, 16, \ldots$. $\lim_{n\to+\infty} x_n = +\infty$. $M > 0$. $x_n > M$ ($x_n \geq n$) $n > M$. , $n_0 = [M] + 1$, $n \geq n_0$ $n > M$, , $x_n > M$. (x_n) $+\infty$. 1 .

(4) $\big((-1)^{n-1}\big)$. 2.15, $+\infty$ $-\infty$· ≥ 1 ≤ -1. , , , , $\big((-1)^{n-1}\big)$ $+\infty$. $M > 0$ n_0 $(-1)^{n-1} > M$ $n \geq n_0$. , n_0 $(-1)^{n-1} > 1$ $n \geq n_0$. , , ! $\big((-1)^{n-1}\big)$ $-\infty$. , $\big((-1)^{n-1}\big)$ · , $+\infty$, $-\infty$.

$$\boxed{\lim_{n\to+\infty} (-1)^{n-1} \quad .}$$

(5) (n) (n^2). $a > 0$ (n^a), $1, 2^a, 3^a, 4^a, \ldots$. :

$$\boxed{\lim_{n\to+\infty} n^a = +\infty \qquad (a > 0).}$$

$M > 0$ n_0 $n \geq n_0$ $n^a > M$. $n^a > M$ $n > M^{\frac{1}{a}}$. , $n_0 = [M^{\frac{1}{a}}] + 1$, $n \geq n_0$ $n > M^{\frac{1}{a}}$, , $n^a > M$.

: $\lim_{n\to+\infty} n^2 = +\infty$, $\lim_{n\to+\infty} \sqrt[5]{n} = +\infty$.

(6) $a > 1$. $(\log_a n)$, $\log_a 1 = 0, \log_a 2, \log_a 3, \log_a 4, \dots$. :

$$\boxed{\lim_{n\to+\infty} \log_a n = +\infty \qquad (a > 1).}$$

$M > 0$ $\log_a n > M$ $n > a^M$. $n_0 = [a^M] + 1$, $n \geq n_0$ $n > a^M$, , $\log_a n > M$.

(7) . $a > 1$ (a^n). $+\infty$. $a = 2$ $a = 10$ $2, 2^2, 2^3, 2^4, \dots$ $10, 10^2, 10^3, 10^4 \dots$. $+\infty$.

, , $M > 0$ $a^n > M$ $n > \log_a M$. , $\log_a M \geq 0$, $M \geq 1$, $\log_a M < 0$, $M < 1$. , $n_0 = [\log_a M] + 1$, $M \geq 1$, $n_0 = 1$, $M < 1$, $n \geq n_0$ $n > \log_a M$, , $a^n > M$.

, 2.15, $a \leq -1$, (a^n) · ≥ 1 ≤ -1. , $+\infty$ $-\infty$, , .

:

$$\boxed{\lim_{n\to+\infty} a^n \begin{cases} = +\infty, & a > 1, \\ = 1, & a = 1, \\ = 0, & -1 < a < 1, \\ \;\;, & a \leq -1. \end{cases}}$$

(8) $\big((-1)^{n-1} n\big)$, $1, -2, 3, -4, 5, -6, 7, \dots$, · ($a \leq -1$), ≥ 1 ≤ -1.

(9) $\lim_{n\to+\infty} \frac{n^2+n}{n+3} = +\infty$.

$M > 0$. $\frac{n^2+n}{n+3} > M$ $n^2 + n > Mn + 3M$ $n^2 + (1-M)n - 3M > 0$ $n > \frac{M-1}{2} + \frac{1}{2}\sqrt{(M-1)^2 + 12M}$ $n < \frac{M-1}{2} - \frac{1}{2}\sqrt{(M-1)^2 + 12M}$ (:) $n > \frac{M-1}{2} + \frac{1}{2}\sqrt{(M-1)^2 + 12M}$. $\frac{M-1}{2} + \frac{1}{2}\sqrt{(M-1)^2 + 12M} \geq 0$. , $n_0 = \left[\frac{M-1}{2} + \frac{1}{2}\sqrt{(M-1)^2 + 12M}\right] + 1$, $n \geq n_0$ $n > \frac{M-1}{2} + \frac{1}{2}\sqrt{(M-1)^2 + 12M}$, , $\frac{n^2+n}{n+3} > M$.

. $\frac{n^2+n}{n+3} > M$ ($\frac{n^2+n}{n+3} \geq \frac{n^2}{n+3n} = \frac{n}{4}$) $\frac{n}{4} > M$ $n > 4M$. $n_0 = [4M] + 1$, $n \geq n_0$ $n > 4M$, , $\frac{n^2+n}{n+3} > M$.

(10) $\lim_{n\to+\infty}(n^7 + 2n^2 - n) = +\infty$.

$M > 0$. $n^7 + 2n^2 - n > M$ ($n^7 + 2n^2 - n \geq n^7$) $n^7 > M$ $n > \sqrt[7]{M}$. $n_0 = \left[\sqrt[7]{M}\right] + 1$, $n \geq n_0$ $n > \sqrt[7]{M}$, , $n^7 + 2n^2 - n > M$. . $n^7 + 2n^2 - n > M$ ($n^7 + 2n^2 - n \geq 2n^2$) $2n^2 > M$ $n > \sqrt{\frac{M}{2}}$. $n_0 = \left[\sqrt{\frac{M}{2}}\right] + 1$, $n \geq n_0$ $n > \sqrt{\frac{M}{2}}$, , $n^7 + 2n^2 - n > M$.

$\to$, lim, $\lim_{n\to+\infty}$ – $\pm\infty$. : , $\pm\infty$.

$+\infty$ ( ) «» ( ) «» . , $+\infty$. , , , . «» $+\infty$, . $+\infty$ « ». , $-\infty$ « ». , (!), x « x ». , . , ΄ .

.

1. (x_n) $x_n = \frac{(3-(-1)^{n-1})n}{2}$. $x_n > x_{n+1}$ n $x_n < x_{n+1}$ n.

2. , , . M n_0 .

$$\lim_{n\to+\infty} \frac{n^2}{\sqrt{n}}\,, \quad \lim_{n\to+\infty} \log_3 n, \quad \lim_{n\to+\infty} \frac{2^{2n}}{3^n}\,, \quad \lim_{n\to+\infty} (-3)^{2n}\,, \quad \lim_{n\to+\infty} (-3)^{3n}\,.$$

3. $x \neq -1$, , $\lim_{n\to+\infty} \frac{(1-x)^n}{(1+x)^n}$.

4. M n_0 , $+\infty$ $-\infty$. «» ( ) (;) n.

$$(n^2-18n-4), \quad \Big(7n-\frac{1}{30}n^2\Big), \quad \Big(\frac{n}{30\sqrt{n}+1}\Big), \quad \Big(\frac{1-n}{1+\sqrt{n}}\Big), \quad \Big(\frac{n^2+1}{n+100}\Big).$$

5. (, $M > 0$ n_0 M,), .

$$\lim_{n\to+\infty} n^4 = +\infty, \quad \lim_{n\to+\infty} (-\sqrt[3]{n}) = -\infty, \quad \lim_{n\to+\infty} 3^n = +\infty,$$

$$\lim_{n\to+\infty} \log_2\Big(\frac{1}{n}\Big) = -\infty, \quad \lim_{n\to+\infty} (n^2-2n) = +\infty, \quad \lim_{n\to+\infty} \frac{n^2-5}{2n+1} = +\infty,$$

$$\lim_{n\to+\infty} \big(n+(-1)^{n-1}\big) = +\infty, \quad \lim_{n\to+\infty} (-2n^5+n^3) = -\infty,$$

$$\lim_{n\to+\infty} (n+2^n) = +\infty, \quad \lim_{n\to+\infty} n(2+\sin n) = +\infty.$$

6. $\lim_{n\to+\infty} x_n = +\infty$, (x_n) $+\infty$, : $M > 0$ n_0 $n \geq n_0$ $x_n \leq M$.

;

$\lim_{n\to+\infty} x_n = -\infty$, (x_n) $-\infty$, : $M > 0$ n_0 $n \geq n_0$ $x_n \geq -M$.

;

2.4 .

. .

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$, $-2, 5, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$. $(\frac{1}{n})$ 0. 0.

2.1 , .

. (x_n) x_n n , , . , , (,), .

: (x_n) (y_n) . , N, M $x_N = y_M$, $x_{N+1} = y_{M+1}$, $x_{N+2} = y_{M+2}$. , , $\lim_{n\to+\infty} x_n = \rho$. $\lim_{n\to+\infty} y_n = \rho$.

$\epsilon > 0$, $n_0{}'$ $|x_n - \rho| < \epsilon$ $n \geq n_0{}'$. $n_0{}'' = \max\{n_0{}', N\}$, $n_0{}'' \geq n_0{}'$ $n_0{}'' \geq N$. , $n_0 = n_0{}'' - N + M$. n_0 $n_0{}'' - N$ M . , $n \geq n_0$, $n - M \geq 0$, $y_n = y_{M+(n-M)} = x_{N+(n-M)}$ $N + (n-M) \geq N + (n_0 - M) = n_0{}'' \geq n_0{}'$, , $|y_n - \rho| = |x_{N+(n-M)} - \rho| < \epsilon$. n_0 $|y_n - \rho| < \epsilon$ $n \geq n_0$. $\lim_{n\to+\infty} y_n = \rho$.

, $\lim_{n\to+\infty} x_n = +\infty$ $-\infty$, $\lim_{n\to+\infty} y_n = +\infty$ $-\infty$, .

: (x_n) x $+\infty$ $-\infty$.

(x_n) $x_1, x_2, x_3, x_4, \ldots$. $x_2, x_3, x_4, x_5, \ldots$, (x_{n+1}). , x $+\infty$ $-\infty$. $x_3, x_4, x_5, x_6, \ldots$, (x_{n+2}). , , k ,

$$\lim_{n\to+\infty} x_{n+k} = \lim_{n\to+\infty} x_n \,.$$

: $\lim_{n\to+\infty} \frac{1}{n+3} = 0$, $\lim_{n\to+\infty} \log_2(n+8) = +\infty$, $\lim_{n\to+\infty} \frac{1}{2^{n+1}} = 0$.

. .

$\pm\infty$:

$$-(+\infty) = -\infty\,, \qquad -(-\infty) = +\infty\,.$$

$+\infty$, $-(+\infty)$, . $-(+\infty)$ «» $-\infty$. $-(-\infty)$ «» $+\infty$.

(x_n) $(-x_n)$, n- n- .

2.2 . (x_n) , $(-x_n)$

$$\boxed{\lim_{n\to+\infty}(-x_n) = -\lim_{n\to+\infty} x_n \,.}$$

: $\lim_{n\to+\infty} x_n = x$. $\epsilon > 0$, n_0 $|x_n - x| < \epsilon$ $n \geq n_0$. $|(-x_n)-(-x)| = |x-x_n| = |x_n-x|$. $|(-x_n)-(-x)| < \epsilon$ $n \geq n_0$, , $\lim_{n\to+\infty}(-x_n) = -x = -\lim_{n\to+\infty} x_n$.

$\lim_{n\to+\infty} x_n = +\infty$. $M > 0$, n_0 $x_n > M$ $n \geq n_0$. $-x_n < -M$ $n \geq n_0$, , $\lim_{n\to+\infty}(-x_n) = -\infty = -(+\infty) = -\lim_{n\to+\infty} x_n$.

, , $\lim_{n\to+\infty} x_n = -\infty$, $\lim_{n\to+\infty}(-x_n) = +\infty = -(-\infty) = -\lim_{n\to+\infty} x_n$.

$\pm\infty$:

$$(+\infty) + x = +\infty, \qquad x + (+\infty) = +\infty, \qquad (+\infty) + (+\infty) = +\infty,$$

$$(-\infty) + x = -\infty, \qquad x + (-\infty) = -\infty, \qquad (-\infty) + (-\infty) = -\infty.$$

,

$$(+\infty) + (-\infty), \qquad (-\infty) + (+\infty)$$

.

$(+\infty) + (+\infty)$ «» $+\infty$: , , . $(+\infty) + x$ «» $+\infty$: x (, x) , , . .

$(+\infty) + (-\infty)$. , . . : (i) 2^{50} , -2^{49} $2^{50} + (-2^{49}) = 2^{49}$, (ii) 2^{49} , -2^{50} $2^{49} + (-2^{50}) = -2^{49}$, (iii) 2^{50} , -2^{50} $2^{50} + (-2^{50}) = 0$.

$(x_n), (y_n)$ $(x_n + y_n)$ n- n- .

2.3 . (x_n), (y_n) $\lim_{n\to+\infty} x_n + \lim_{n\to+\infty} y_n$, $(x_n + y_n)$

$$\boxed{\lim_{n\to+\infty}(x_n + y_n) = \lim_{n\to+\infty} x_n + \lim_{n\to+\infty} y_n \,.}$$

: $\lim_{n\to+\infty} x_n = x$ $\lim_{n\to+\infty} y_n = y$. $\epsilon > 0$, n_0' $|x_n - x| < \frac{\epsilon}{2}$ $n \geq n_0'$ n_0'' $|y_n - y| < \frac{\epsilon}{2}$ $n \geq n_0''$. $n_0 = \max\{n_0', n_0''\}$, $n_0 \geq n_0'$ $n_0 \geq n_0''$. $|x_n - x| < \frac{\epsilon}{2}$ $|y_n - y| < \frac{\epsilon}{2}$ $n \geq n_0$. $|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y|$. $|(x_n + y_n) - (x + y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ $n \geq n_0$, , $\lim_{n\to+\infty}(x_n + y_n) = x + y = \lim_{n\to+\infty} x_n + \lim_{n\to+\infty} y_n$.

$\lim_{n\to+\infty} x_n = +\infty$ $\lim_{n\to+\infty} y_n = +\infty$. $M > 0$, n_0' $x_n > \frac{M}{2}$ $n \geq n_0'$ n_0'' $y_n > \frac{M}{2}$ $n \geq n_0''$. $n_0 = \max\{n_0', n_0''\}$, $n_0 \geq n_0'$ $n_0 \geq n_0''$. $x_n > \frac{M}{2}$ $y_n > \frac{M}{2}$ $n \geq n_0$. $x_n + y_n > \frac{M}{2} + \frac{M}{2} = M$ $n \geq n_0$, , $\lim_{n\to+\infty}(x_n + y_n) = +\infty = (+\infty) + (+\infty) = \lim_{n\to+\infty} x_n + \lim_{n\to+\infty} y_n$.

$\lim_{n\to+\infty} x_n = +\infty$ $\lim_{n\to+\infty} y_n = y$. $M > 0$, n_0' $x_n > M - y + 1$ $n \geq n_0'$ n_0'' $|y_n - y| < 1$ $n \geq n_0''$. $|y_n - y| < 1$ $y_n > y - 1$ $n \geq n_0''$. $n_0 = \max\{n_0', n_0''\}$, $n_0 \geq n_0'$ $n_0 \geq n_0''$. $x_n > M - y + 1$ $y_n > y - 1$ $n \geq n_0$. $x_n + y_n > (M - y + 1) + (y - 1) = M$ $n \geq n_0$, , $\lim_{n\to+\infty}(x_n + y_n) = +\infty = (+\infty) + y = \lim_{n\to+\infty} x_n + \lim_{n\to+\infty} y_n$.

.

: (1) $\lim_{n\to+\infty}\left(\frac{1}{n} + \frac{(-1)^n}{n}\right) = \lim_{n\to+\infty}\frac{1}{n} + \lim_{n\to+\infty}\frac{(-1)^n}{n} = 0 + 0 = 0.$

(2) $\lim_{n\to+\infty}\frac{n^2+1}{n} = \lim_{n\to+\infty}(n + \frac{1}{n}) = \lim_{n\to+\infty} n + \lim_{n\to+\infty}\frac{1}{n} = (+\infty) + 0 = +\infty.$

(3) $\lim_{n\to+\infty}(-n - \sqrt{n}) = \lim_{n\to+\infty}(-n) + \lim_{n\to+\infty}(-\sqrt{n}) = (-\infty) + (-\infty) = -\infty.$

$+\infty, -\infty$:

: (1) $\lim_{n\to+\infty}(n + 3) = \lim_{n\to+\infty} n + \lim_{n\to+\infty} 3 = (+\infty) + 3 = +\infty$, $\lim_{n\to+\infty}(-n) = -\infty$ $\lim_{n\to+\infty}\big((n + 3) + (-n)\big) = \lim_{n\to+\infty} 3 = 3.$
, 3 .

(2) $\lim_{n\to+\infty} 2n = +\infty$, $\lim_{n\to+\infty}(-n) = -\infty$ $\lim_{n\to+\infty}\big(2n + (-n)\big) = \lim_{n\to+\infty} n = +\infty.$

(3) $\lim_{n\to+\infty} n = +\infty$, $\lim_{n\to+\infty}(-2n) = -\infty$ $\lim_{n\to+\infty}\big(n + (-2n)\big) = \lim_{n\to+\infty}(-n) = -\infty.$

(4) $n + (-1)^{n-1} \geq n - 1$ n $\lim_{n\to+\infty}(n - 1) = +\infty$. (2.11), $\lim_{n\to+\infty}\big(n + (-1)^{n-1}\big) = +\infty$. , $\lim_{n\to+\infty}(-n) = -\infty$ $\lim_{n\to+\infty}\big(\big(n + (-1)^{n-1}\big) + (-n)\big) = \lim_{n\to+\infty}(-1)^{n-1}$.

$$(+\infty) - x = +\infty, \qquad x - (-\infty) = +\infty, \qquad (+\infty) - (-\infty) = +\infty,$$

$$(-\infty) - x = -\infty, \qquad x - (+\infty) = -\infty, \qquad (-\infty) - (+\infty) = -\infty,$$

, .

$$(+\infty) - (+\infty), \qquad (-\infty) - (-\infty)$$

.

«» $+\infty$ $(-\infty)$ () () x x (, x).

$(x_n), (y_n)$, $(x_n - y_n)$, , , , $x_n - y_n = x_n + (-y_n)$.

2.4 . (x_n), (y_n) $\lim_{n\to+\infty} x_n - \lim_{n\to+\infty} y_n$, $(x_n - y_n)$

$$\boxed{\lim_{n\to+\infty}(x_n - y_n) = \lim_{n\to+\infty} x_n - \lim_{n\to+\infty} y_n \,.}$$

$\pm\infty$.

$$(\pm\infty)\cdot x = \pm\infty, \quad x\cdot(\pm\infty) = \pm\infty \qquad (x>0),$$

$$(\pm\infty)\cdot x = \mp\infty, \quad x\cdot(\pm\infty) = \mp\infty \qquad (x<0),$$

$$(\pm\infty)\cdot(\pm\infty) = +\infty\,, \qquad (\pm\infty)\cdot(\mp\infty) = -\infty.$$

$$(\pm\infty)\cdot 0, \qquad 0\cdot(\pm\infty)$$

.

$(+\infty)\cdot(+\infty)$ «» $+\infty$: , , . $(+\infty)\cdot x = +\infty$ ($x>0$) : x (, x) , , . .

$(+\infty)\cdot 0$ (, 0)· . : (i) 2^{100} , $\frac{1}{2^{50}}$ $2^{100}\cdot\frac{1}{2^{50}} = 2^{50}$, (ii) 2^{50} , $\frac{1}{2^{100}}$ $2^{50}\cdot\frac{1}{2^{100}} = \frac{1}{2^{50}}$, (iii) 2^{100} , $\frac{1}{2^{100}}$ $2^{100}\cdot\frac{1}{2^{100}} = 1$.

$(x_n),(y_n)$ (x_ny_n) n- n- .

2.5 . (x_n), (y_n) $\lim_{n\to+\infty} x_n \lim_{n\to+\infty} y_n$, (x_ny_n)

$$\boxed{\lim_{n\to+\infty} x_ny_n = \lim_{n\to+\infty} x_n \lim_{n\to+\infty} y_n \,.}$$

: $\lim_{n\to+\infty} x_n = x$ $\lim_{n\to+\infty} y_n = y$. $\epsilon>0$, $n_0{}'$ $|x_n - x| < \frac{\epsilon}{3|y|+1}$ $n \geq n_0{}'$ $n_0{}''$ $|y_n - y| < \min\{\frac{\epsilon}{3|x|+1},\frac{1}{3}\}$ $n\geq n_0{}''$. $n_0 = \max\{n_0{}', n_0{}''\}$, $n_0 \geq n_0{}'$ $n_0 \geq n_0{}''$. $|x_n - x| < \frac{\epsilon}{3|y|+1}$ $|y_n - y| < \min\{\frac{\epsilon}{3|x|+1},\frac{1}{3}\}$ $n\geq n_0$. $|x_ny_n - xy| = |(x_n-x)(y_n-y) + x(y_n-y) + y(x_n-x)| \leq |x_n-x||y_n-y| + |x||y_n-y| + |y||x_n-x|$. $|x_ny_n - xy| \leq \frac{\epsilon}{3|y|+1}\frac{1}{3} + |x|\frac{\epsilon}{3|x|+1} + |y|\frac{\epsilon}{3|y|+1} < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon$ $n\geq n_0$, , $\lim_{n\to+\infty} x_ny_n = xy = \lim_{n\to+\infty} x_n \lim_{n\to+\infty} y_n$.

$\lim_{n\to+\infty} x_n = +\infty$ $\lim_{n\to+\infty} y_n = +\infty$. $M>0$, $n_0{}'$ $x_n > \sqrt{M}$ $n\geq n_0{}'$ $n_0{}''$ $y_n > \sqrt{M}$ $n \geq n_0{}''$. $n_0 = \max\{n_0{}', n_0{}''\}$, $n_0\geq n_0{}'$ $n_0 \geq n_0{}''$. $x_n > \sqrt{M}$ $y_n > \sqrt{M}$ $n\geq n_0$. $x_ny_n > \sqrt{M}\sqrt{M} = M$ $n\geq n_0$, , $\lim_{n\to+\infty} x_ny_n = +\infty = (+\infty)(+\infty) = \lim_{n\to+\infty} x_n \lim_{n\to+\infty} y_n$.

$\lim_{n\to+\infty} x_n = +\infty$ $\lim_{n\to+\infty} y_n = y$. $M>0$, $n_0{}'$ $x_n > \frac{2M}{y}$ $n\geq n_0{}'$ $n_0{}''$ $|y_n - y| < \frac{y}{2}$ $n\geq n_0{}''$. $n_0 = \max\{n_0{}', n_0{}''\}$, $n_0\geq n_0{}'$ $n_0\geq n_0{}''$. $x_n > \frac{2M}{y}$ $|y_n - y| < \frac{y}{2}$ $n\geq n_0$. $|y_n - y| < \frac{y}{2}$ $y_n > y - \frac{y}{2} = \frac{y}{2}$ $n \geq n_0$. $x_ny_n > \frac{2M}{y}\frac{y}{2} = M$ $n\geq n_0$, , $\lim_{n\to+\infty} x_ny_n = +\infty = (+\infty)y = \lim_{n\to+\infty} x_n \lim_{n\to+\infty} y_n$.

. .

: (1) $\lim_{n\to+\infty}\frac{(-1)^n}{n^2} = \lim_{n\to+\infty}\frac{1}{n}\frac{(-1)^n}{n} = \lim_{n\to+\infty}\frac{1}{n}\cdot\lim_{n\to+\infty}\frac{(-1)^n}{n} = 0\cdot 0 = 0.$

(2) $\lim_{n\to+\infty}\frac{n-1}{n^2} = \lim_{n\to+\infty}\frac{n-1}{n}\frac{1}{n} = \lim_{n\to+\infty}\frac{n-1}{n}\lim_{n\to+\infty}\frac{1}{n} = 1\cdot 0 = 0.$

(3) $\lim_{n\to+\infty}(n-n^2) = \lim_{n\to+\infty} n(1-n) = \lim_{n\to+\infty} n \lim_{n\to+\infty}(1-n) = (+\infty)\cdot(-\infty) = -\infty.$ $\lim(n-n^2)$.

(4) . (x_n) c. $\lim_{n\to+\infty} x_n$ $c\cdot\lim_{n\to+\infty} x_n$,

$$\boxed{\lim_{n\to+\infty} cx_n = c\lim_{n\to+\infty} x_n \qquad (c\neq 0 \quad \lim_{n\to+\infty} x_n = \pm\infty).}$$

(x_n) (c) $c.$

(5) $a>0.$
$c>0,\ \lim_{n\to+\infty} cn^a = c\lim_{n\to+\infty} n^a = c\cdot(+\infty) = +\infty.$
$c<0,\ \lim_{n\to+\infty} cn^a = c\lim_{n\to+\infty} n^a = c\cdot(+\infty) = -\infty.$

(6) $a>0,\ \lim_{n\to+\infty} cn^{-a} = c\lim_{n\to+\infty} n^{-a} = c\cdot 0 = 0.$

(7) $n.$ $a_0+a_1x+\cdots+a_Nx^N$ $(\ a_N\neq 0 \ \ N\geq 1).$

$$\boxed{\lim_{n\to+\infty}\big(a_0+a_1n+\cdots+a_Nn^N\big) = a_N\cdot(+\infty) = \begin{cases} +\infty, & a_N>0,\\ -\infty, & a_N<0.\end{cases}}$$

a_Nn^N ,

$$a_0+a_1n+\cdots+a_Nn^N = a_Nn^N\Big(\frac{a_0}{a_N}\frac{1}{n^N}+\frac{a_1}{a_N}\frac{1}{n^{N-1}}+\cdots+\frac{a_{N-1}}{a_N}\frac{1}{n}+1\Big),$$

1, 0. $\lim_{n\to+\infty}(a_0+a_1n+\cdots+a_Nn^N) = \lim_{n\to+\infty} a_Nn^N\cdot 1 = a_N\cdot(+\infty).$
n .

$$\boxed{\lim_{n\to+\infty}\big(a_0+a_1n+\cdots+a_Nn^N\big) = \lim_{n\to+\infty} a_Nn^N\,.}$$

, : $\lim_{n\to+\infty}(3n^2-5n+2) = \lim_{n\to+\infty} 3n^2 = +\infty$ $\lim_{n\to+\infty}(-\frac{1}{2}n^5+4n^4-n^3) = \lim_{n\to+\infty}(-\frac{1}{2}n^5) = -\infty.$

(8) . a $(1+a+a^2+\cdots+a^{n-1}+a^n),$ $1+a, 1+a+a^2, 1+a+a^2+a^3, \ldots$
. :

$$\boxed{\lim_{n\to+\infty}(1+a+a^2+\cdots+a^n)\begin{cases} =+\infty, & a\geq 1,\\ =\frac{1}{1-a}, & -1<a<1,\\ \ , & a\leq -1.\end{cases}}$$

.

$a>1,\ \lim_{n\to+\infty}(1+a+a^2+\cdots+a^n) = \lim_{n\to+\infty}\frac{a^{n+1}-1}{a-1} = \frac{1}{a-1}\lim_{n\to+\infty}(a^{n+1}-1) = \frac{1}{a-1}((+\infty)-1) = +\infty.$

$a=1,\ \lim_{n\to+\infty}(1+a+a^2+\cdots+a^n) = \lim_{n\to+\infty}(n+1) = +\infty.$

$-1<a<1,\ \lim_{n\to+\infty}(1+a+a^2+\cdots+a^n) = \lim_{n\to+\infty}\frac{1-a^{n+1}}{1-a} = \frac{1}{1-a}\lim_{n\to+\infty}(1-a^{n+1}) = \frac{1}{1-a}(1-0) = \frac{1}{1-a}.$

, $a\leq -1.$ $x_n = 1+a+a^2+\cdots+a^n$ $a^{n+1}-1 = (a-1)(1+a+a^2+\cdots+a^n)$ $a^{n+1} = (a-1)x_n+1.$ $\lim_{n\to+\infty}(1+a+a^2+\cdots+a^n) = \lim_{n\to+\infty} x_n$, $\lim_{n\to+\infty} a^{n+1} = (a-1)\lim_{n\to+\infty} x_n + 1.$, $\lim_{n\to+\infty} a^{n+1}$, $\lim_{n\to+\infty}(1+a+a^2+\cdots+a^n)$.

$\pm\infty,$ $(\pm\infty)^k$ k :

$$(+\infty)^k = \underbrace{(+\infty)\cdots(+\infty)}_{k} = +\infty, \qquad (-\infty)^k = \underbrace{(-\infty)\cdots(-\infty)}_{k} = \pm\infty.$$

$(-\infty)^k = \pm\infty$ +, k , −, k .

2.6 $\lim_{n\to+\infty} x_n$ k ,

$$\boxed{\lim_{n\to+\infty} {x_n}^k = (\lim_{n\to+\infty} x_n)^k\,.}$$

, $\lim_{n\to+\infty} {x_n}^k = \lim_{n\to+\infty}(\underbrace{x_n\cdots x_n}_{k}) = \underbrace{\lim_{n\to+\infty} x_n \cdots \lim_{n\to+\infty} x_n}_{k} = (\lim_{n\to+\infty} x_n)^k\,.$

: (1) $\lim_{n\to+\infty}(\frac{n-1}{n})^3 = (\lim_{n\to+\infty}\frac{n-1}{n})^3 = 1^3 = 1.$

(2) $\lim_{n\to+\infty}(n^5-2n^2+n-7)^8 = \big(\lim_{n\to+\infty}(n^5-2n^2+n-7)\big)^8 = (+\infty)^8 = +\infty.$

(3) $\lim_{n\to+\infty}(-2n^3+n^2+2n-7)^4 = \big(\lim_{n\to+\infty}(-2n^3+n^2+2n-7)\big)^4 = (-\infty)^4 = +\infty.$

(4) $\lim_{n\to+\infty}(-n^3+2n-1)^5 = \big(\lim_{n\to+\infty}(-n^3+2n-1)\big)^5 = (-\infty)^5 = -\infty.$

$+\infty, 0,$.

: (1) $\lim_{n\to+\infty} n = +\infty$, $\lim_{n\to+\infty}\frac{7}{n} = 0$ $\lim_{n\to+\infty} n\cdot\frac{7}{n} = \lim_{n\to+\infty} 7 = 7.$
, 7.

(2) $\lim_{n\to+\infty} n^2 = +\infty$, $\lim_{n\to+\infty}\frac{1}{n} = 0$ $\lim_{n\to+\infty} n^2\cdot\frac{1}{n} = \lim_{n\to+\infty} n = +\infty.$

(3) $\lim_{n\to+\infty} n = +\infty$, $\lim_{n\to+\infty}\frac{1}{n^2} = 0$ $\lim_{n\to+\infty} n\cdot\frac{1}{n^2} = \lim_{n\to+\infty}\frac{1}{n} = 0.$

(4) $\lim_{n\to+\infty} n = +\infty$, $\lim_{n\to+\infty}\frac{(-1)^{n-1}}{n} = 0$ $\lim_{n\to+\infty} n\cdot\frac{(-1)^{n-1}}{n} = \lim_{n\to+\infty}(-1)^{n-1}$
.

$$\frac{1}{+\infty} = 0, \qquad \frac{1}{-\infty} = 0.$$

$$\frac{1}{0}$$

.

$\frac{1}{\pm\infty}$ «» 0 : , , (0) (, 0).
$\frac{1}{0}$. 0 (, 0) . , , , . , , .
(x_n) $(\frac{1}{x_n})$, , $x_n \neq 0$ n.

2.7 **,** . $x_n \neq 0$ n. (x_n) $\frac{1}{\lim_{n\to+\infty} x_n}$ (, $\lim_{n\to+\infty} x_n \neq 0$), $\left(\frac{1}{x_n}\right)$

$$\boxed{\lim_{n\to+\infty}\frac{1}{x_n} = \frac{1}{\lim_{n\to+\infty} x_n}\,.}$$

: $\lim_{n\to+\infty} x_n = x$. $\epsilon > 0$, n_0 $|x_n - x| < \min\{\frac{x^2\epsilon}{2}, \frac{x}{2}\}$ $n \geq n_0$. $|x_n - x| < \frac{x^2\epsilon}{2}$ $|x_n - x| < \frac{x}{2}$ $n \geq n_0$. $|x_n - x| < \frac{x}{2}$ $x_n > x - \frac{x}{2} = \frac{x}{2}$ $n \geq n_0$. , $\left|\frac{1}{x_n} - \frac{1}{x}\right| = \frac{|x_n - x|}{x_n x} < \frac{\frac{x^2\epsilon}{2}}{\frac{x}{2}x} = \epsilon$ $n \geq n_0$, $\lim_{n\to+\infty} \frac{1}{x_n} = \frac{1}{x} = \frac{1}{\lim_{n\to+\infty} x_n}$.

$\lim_{n\to+\infty} x_n = +\infty$. $\epsilon > 0$, n_0 $x_n > \frac{1}{\epsilon}$ $n \geq n_0$. $0 < \frac{1}{x_n} < \epsilon$, , $\left|\frac{1}{x_n} - 0\right| = \frac{1}{x_n} < \epsilon$ $n \geq n_0$. $\lim_{n\to+\infty} \frac{1}{x_n} = 0 = \frac{1}{+\infty} = \frac{1}{\lim_{n\to+\infty} x_n}$.

$-\infty$.

: (1) $a > 1$, $\lim_{n\to+\infty} \frac{1}{\log_a n} = \frac{1}{\lim_{n\to+\infty} \log_a n} = \frac{1}{+\infty} = 0$.

(2) $\lim_{n\to+\infty} x_n = 0$. $\lim_{n\to+\infty} \frac{(-1)^{n-1}}{n} = 0$, $\lim_{n\to+\infty} \frac{n}{(-1)^{n-1}} = \lim_{n\to+\infty} (-1)^{n-1} n$.

. 2.7. , · 2.8.

2.8 , . (1) $\lim_{n\to+\infty} x_n = 0$ (x_n) , $\lim_{n\to+\infty} \frac{1}{x_n} = +\infty$.
(2) $\lim_{n\to+\infty} x_n = 0$ (x_n) , $\lim_{n\to+\infty} \frac{1}{x_n} = -\infty$.

: (1) $\lim_{n\to+\infty} x_n = 0$ $x_n > 0$ n. $M > 0$, n_0 $0 < x_n < \frac{1}{M}$ $n \geq n_0$. $\frac{1}{x_n} > M$ $n \geq n_0$, , $\lim_{n\to+\infty} \frac{1}{x_n} = +\infty$.
(2) (1).

.

$$\frac{\pm\infty}{x} = \pm\infty \quad (x > 0), \qquad \frac{\pm\infty}{x} = \mp\infty \quad (x < 0),$$

$$\frac{x}{\pm\infty} = 0.$$

$$\frac{x}{0}, \qquad \frac{\pm\infty}{0}, \qquad \frac{\pm\infty}{\pm\infty}, \qquad \frac{\pm\infty}{\mp\infty}$$

. $\frac{x}{0} = x \cdot \frac{1}{0}$ $\frac{1}{0}$. $\frac{\pm\infty}{0} = (\pm\infty) \cdot \frac{1}{0}$ $\frac{1}{0}$. $\frac{\pm\infty}{\pm\infty} = (\pm\infty) \cdot \frac{1}{\pm\infty} = (\pm\infty) \cdot 0$.

$+\infty$ $(-\infty)$ () () x x (, x).
(x_n) (y_n), $(\frac{x_n}{y_n})$, , $\frac{x_n}{y_n} = x_n \cdot \frac{1}{y_n}$.

2.9 . $y_n \neq 0$ n. (x_n), (y_n) $\frac{\lim_{n\to+\infty} x_n}{\lim_{n\to+\infty} y_n}$, $\left(\frac{x_n}{y_n}\right)$

$$\boxed{\lim_{n\to+\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to+\infty} x_n}{\lim_{n\to+\infty} y_n}.}$$

: (1) n. $\frac{a_0 + a_1 x + \cdots + a_N x^N}{b_0 + b_1 x + \cdots + b_M x^M}$, $a_N \neq 0$, $b_M \neq 0$.

$$\boxed{\lim_{n\to+\infty} \frac{a_0 + a_1 n + \cdots + a_N n^N}{b_0 + b_1 n + \cdots + b_M n^M} = \begin{cases} \frac{a_N}{b_M} \cdot (+\infty), & N > M, \\ \frac{a_N}{b_M}, & N = M, \\ 0, & N < M. \end{cases}}$$

,

$$\frac{a_0+a_1n+\cdots+a_Nn^N}{b_0+b_1n+\cdots+b_Mn^M}=\frac{a_N}{b_M}n^{N-M}\frac{\frac{a_0}{a_N}\frac{1}{n^N}+\frac{a_1}{a_N}\frac{1}{n^{N-1}}+\cdots+\frac{a_{N-1}}{a_N}\frac{1}{n}+1}{\frac{b_0}{b_M}\frac{1}{n^M}+\frac{b_1}{b_M}\frac{1}{n^{M-1}}+\cdots+\frac{b_{M-1}}{b_M}\frac{1}{n}+1}$$

, , 1. $\lim_{n\to+\infty}\frac{a_0+a_1n+\cdots+a_Nn^N}{b_0+b_1n+\cdots+b_Mn^M}=\frac{a_N}{b_M}\cdot\lim_{n\to+\infty}n^{N-M}\cdot\frac{1}{1}=\frac{a_N}{b_M}\cdot\lim_{n\to+\infty}n^{N-M}$. , , n .

$$\boxed{\lim_{n\to+\infty}\frac{a_0+a_1n+\cdots+a_Nn^N}{b_0+b_1n+\cdots+b_Mn^M}=\lim_{n\to+\infty}\frac{a_Nn^N}{b_Mn^M}\,.}$$

, : $\lim_{n\to+\infty}\frac{n^3-2n^2+n+1}{2n^2-3n-1}=\lim_{n\to+\infty}\frac{n^3}{2n^2}=+\infty$, $\lim_{n\to+\infty}\frac{-n^2+n}{n+2}=\lim_{n\to+\infty}\frac{-n^2}{n}=-\infty$, $\lim_{n\to+\infty}\frac{n^4-n^3-7}{n^4+n+1}=\lim_{n\to+\infty}\frac{n^4}{n^4}=1$, $\lim_{n\to+\infty}\frac{-n^2+n+4}{n^3+n^2+5n+6}=\lim_{n\to+\infty}\frac{-n^2}{n^3}=0$.

(2) $\lim_{n\to+\infty}\left(\frac{-2n^3+n^2+n+1}{2n+3}\right)^7=\left(\lim_{n\to+\infty}\frac{-2n^3+n^2+n+1}{2n+3}\right)^7=(-\infty)^7=-\infty$.

(3) $\lim_{n\to+\infty}\left(\frac{n^3+n+7}{-3n^3+n^2+1}\right)^3=\left(\lim_{n\to+\infty}\frac{n^3+n+7}{-3n^3+n^2+1}\right)^3=(-\frac{1}{3})^3=-\frac{1}{27}$.

, $\frac{0}{0}$ $\frac{+\infty}{+\infty}$.

: (1) $\lim_{n\to+\infty}\frac{-2}{n}=0$, $\lim_{n\to+\infty}\frac{1}{n}=0$ $\lim_{n\to+\infty}\frac{\frac{-2}{n}}{\frac{1}{n}}=\lim_{n\to+\infty}(-2)=-2$. -2 .

(2) $\lim_{n\to+\infty}\frac{1}{n}=0$, $\lim_{n\to+\infty}\frac{1}{n^2}=0$ $\lim_{n\to+\infty}\frac{\frac{1}{n}}{\frac{1}{n^2}}=\lim_{n\to+\infty}n=+\infty$.

(3) $\lim_{n\to+\infty}\frac{1}{n^2}=0$, $\lim_{n\to+\infty}\frac{1}{n}=0$ $\lim_{n\to+\infty}\frac{\frac{1}{n^2}}{\frac{1}{n}}=\lim_{n\to+\infty}\frac{1}{n}=0$.

(4) $\lim_{n\to+\infty}(5n)=+\infty$, $\lim_{n\to+\infty}n=+\infty$ $\lim_{n\to+\infty}\frac{5n}{n}=5$. 5 .

(5) $\lim_{n\to+\infty}n^2=+\infty$, $\lim_{n\to+\infty}n=+\infty$ $\lim_{n\to+\infty}\frac{n^2}{n}=\lim_{n\to+\infty}n=+\infty$.

(6) $\lim_{n\to+\infty}n=+\infty$, $\lim_{n\to+\infty}n^2=+\infty$ $\lim_{n\to+\infty}\frac{n}{n^2}=\lim_{n\to+\infty}\frac{1}{n}=0$.

$$|+\infty|=+\infty,\qquad |-\infty|=+\infty.$$

$|\pm\infty|=+\infty$: , , . , , $(|x_n|)$ (x_n).

2.10 (x_n) , $(|x_n|)$

$$\boxed{\lim_{n\to+\infty}|x_n|=\big|\lim_{n\to+\infty}x_n\big|\,.}$$

: $\lim_{n\to+\infty}x_n=x$. $\epsilon>0$, n_0 $|x_n-x|<\epsilon$ $n\ge n_0$. $\big||x_n|-|x|\big|\le|x_n-x|$, $\big||x_n|-|x|\big|<\epsilon$ $n\ge n_0$. $\lim_{n\to+\infty}|x_n|=|x|=|\lim_{n\to+\infty}x_n|$.

$\lim_{n\to+\infty} x_n = +\infty$ $\lim_{n\to+\infty} x_n = -\infty$. $M > 0$, n_0 $x_n > M$ $x_n < -M$, , $n \geq n_0$. $|x_n| > M$ $n \geq n_0$, , $\lim_{n\to+\infty} |x_n| = +\infty = |\pm\infty| = |\lim_{n\to+\infty} x_n|$.

: (1) $(5-n)$ $4, 3, 2, 1, 0, -1, -2, -3, \ldots$ $\lim_{n\to+\infty}(5-n) = -\infty$. $(|5-n|)$ $4, 3, 2, 1, 0, 1, 2, 3, \ldots$ $\lim_{n\to+\infty} |5-n| = |\lim_{n\to+\infty}(5-n)| = |-\infty| = +\infty$.

(2) 2.10 . $\lim_{n\to+\infty} \left|(-1)^{n-1}\right| = \lim_{n\to+\infty} 1 = 1$. , $\lim_{n\to+\infty}(-1)^{n-1}$.

. .

2.11 $x_n \leq y_n$ n.
(1) $\lim_{n\to+\infty} x_n = +\infty$, $\lim_{n\to+\infty} y_n = +\infty$.
(2) $\lim_{n\to+\infty} y_n = -\infty$, $\lim_{n\to+\infty} x_n = -\infty$.

: (1) $M > 0$. $\lim_{n\to+\infty} x_n = +\infty$, n_0 $x_n > M$ $n \geq n_0$, $y_n \geq x_n$, $y_n > M$ $n \geq n_0$. , $\lim_{n\to+\infty} y_n = +\infty$.
(2) (1).

: (1) $n+(-1)^{n-1} \geq n-1$ $\lim_{n\to+\infty}(n-1) = +\infty$ $\lim_{n\to+\infty} \left(n+(-1)^{n-1}\right) = +\infty$.

(2) $\frac{n^2+2n+1}{n+2} \geq n$ $\lim_{n\to+\infty} n = +\infty$ $\lim_{n\to+\infty} \frac{n^2+2n+1}{n+2} = +\infty$.

(3) $[\sqrt{n}] > \sqrt{n} - 1$ $\lim_{n\to+\infty}(\sqrt{n} - 1) = +\infty$ $\lim_{n\to+\infty}[\sqrt{n}] = +\infty$.

2.12 $x_n \leq y_n$ n. $\lim_{n\to+\infty} x_n = x$ $\lim_{n\to+\infty} y_n = y$, $x \leq y$.

: ( ) $x > y$. $\epsilon = \frac{x-y}{2} > 0$, $\lim_{n\to+\infty} x_n = x$ $\lim_{n\to+\infty} y_n = y$ n_0' $|x_n - x| < \frac{x-y}{2}$ $n \geq n_0'$ n_0'' $|y_n - y| < \frac{x-y}{2}$ $n \geq n_0''$. $n_0 = \max\{n_0', n_0''\}$, $n_0 \geq n_0'$ $n_0 \geq n_0''$. $|x_n - x| < \frac{x-y}{2}$ $|y_n - y| < \frac{x-y}{2}$ $n \geq n_0$. $x_n > x - \frac{x-y}{2} = \frac{x+y}{2}$ $y_n < y + \frac{x-y}{2} = \frac{x+y}{2}$ $n \geq n_0$. $x_n > \frac{x+y}{2} > y_n$ $n \geq n_0$ $x_n \leq y_n$ n.

: (1) $x_n \geq a$ n $\lim_{n\to+\infty} x_n = x$, $x \geq a$.
, (a) , $a \leq x_n$ n $\lim_{n\to+\infty} a = a$ $\lim_{n\to+\infty} x_n = x$, $a \leq x$.

(2) $x_n \leq b$ n $\lim_{n\to+\infty} x_n = x$, $x \leq b$.
(1).

(3) (x_n) $[a, b]$ $\lim_{n\to+\infty} x_n = x$, x $[a, b]$.
(1) (2).

$x_n < y_n$ n $\lim_{n\to+\infty} x_n = x$, $\lim_{n\to+\infty} y_n = y$, $x < y$. $x_n < y_n$ $x_n \leq y_n$ n, – 2.12 – $x \leq y$. , , $x = y$.

: $-\frac{1}{n} < \frac{1}{n}$ n, $\lim_{n\to+\infty}(-\frac{1}{n}) = 0$ $\lim_{n\to+\infty} \frac{1}{n} = 0$.

2.13 . $x_n \leq y_n \leq z_n$ n. $\lim_{n\to+\infty} x_n = \rho$ $\lim_{n\to+\infty} z_n = \rho$, $\lim_{n\to+\infty} y_n = \rho$.

: $\epsilon > 0$, n_0' $|x_n - \rho| < \epsilon$ $n \geq n_0'$ n_0'' $|z_n - \rho| < \epsilon$ $n \geq n_0''$. $n_0 = \max\{n_0', n_0''\}$, $n_0 \geq n_0'$ $n_0 \geq n_0''$. $|x_n - \rho| < \epsilon$ $|z_n - \rho| < \epsilon$ $n \geq n_0$. $x_n > \rho - \epsilon$ $z_n < \rho + \epsilon$ $n \geq n_0$. $\rho - \epsilon < x_n \leq y_n \leq z_n < \rho + \epsilon$, , $|y_n - \rho| < \epsilon$ $n \geq n_0$. $\lim_{n\to+\infty} y_n = \rho$.

: (1) $-\frac{1}{n} \leq \frac{(-1)^{n-1}}{n} \leq \frac{1}{n}$, $\lim_{n\to+\infty}(-\frac{1}{n}) = 0$ $\lim_{n\to+\infty}\frac{1}{n} = 0$ $\lim_{n\to+\infty}\frac{(-1)^{n-1}}{n} = 0$.

(2) $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ n. $\lim_{n\to+\infty}(-\frac{1}{n}) = \lim_{n\to+\infty}\frac{1}{n} = 0$, $\lim_{n\to+\infty}\frac{\sin n}{n} = 0$.

(3) $\frac{\sqrt{n}-1}{\sqrt{n}} < \frac{[\sqrt{n}]}{\sqrt{n}} \leq \frac{\sqrt{n}}{\sqrt{n}} = 1$, $\lim_{n\to+\infty}\frac{\sqrt{n}-1}{\sqrt{n}} = \lim_{n\to+\infty}\big(1-\frac{1}{\sqrt{n}}\big) = 1$ $\lim_{n\to+\infty}1 = 1$ $\lim_{n\to+\infty}\frac{[\sqrt{n}]}{\sqrt{n}} = 1$.

. .

2.14 (1) $\lim_{n\to+\infty}x_n < u$, $x_n < u$.
(2) $\lim_{n\to+\infty}x_n > l$, $x_n > l$.

: (1) $\lim_{n\to+\infty}x_n < u$ $\lim_{n\to+\infty}x_n = x$, $x < u$. $\epsilon = u-x > 0$, n_0 $|x_n-x| < u-x$ $n \geq n_0$. $x_n < x+(u-x) = u$ $n \geq n_0$.

$\lim_{n\to+\infty}x_n = -\infty$ $M > 0$ $\geq -u$ – , $M = -u$, $u < 0$, $M = 1$, $u \geq 0$. n_0 $x_n < -M \leq u$ $n \geq n_0$.

(2) (1), $\lim_{n\to+\infty}x_n = x > l$ $\lim_{n\to+\infty}x_n = +\infty$.

: (1) $\frac{24n^7+323n^5-17n^2+135}{n^8-n^6+2n^3-1} < \frac{1}{1000}$ n, $\lim_{n\to+\infty}\frac{24n^7+323n^5-17n^2+135}{n^8-n^6+2n^3-1} = \lim_{n\to+\infty}\frac{24n^7}{n^8} = 0$, $0 < \frac{1}{1000}$, n n . , , n_0 $n \geq n_0$. n_0 . , , !

(2) : $\frac{n^3-2n+37}{4n^3+1} \geq \frac{2}{3}$ n;
$\lim_{n\to+\infty}\frac{n^3-2n+37}{4n^3+1} = \lim_{n\to+\infty}\frac{n^3}{4n^3} = \frac{1}{4}$. $\frac{1}{4} < \frac{2}{3}$, $\frac{n^3-2n+37}{4n^3+1} < \frac{2}{3}$ n . n.

2.15, 2.14, , , .

2.15 l, u .
(1) (x_n) $\geq u$ $\lim_{n\to+\infty}x_n$, $\lim_{n\to+\infty}x_n \geq u$.
(2) (x_n) $\leq l$ $\lim_{n\to+\infty}x_n$, $\lim_{n\to+\infty}x_n \leq l$.
(3) $l < u$ (x_n) $\geq u$ $\leq l$, (x_n) .

, (1) , $\lim_{n\to+\infty}x_n < u$, $x_n < u$, , (x_n) $\geq u$. , (2), $\lim_{n\to+\infty}x_n > l$, $x_n > l$, , (x_n) $\leq l$. , (3) , $\lim_{n\to+\infty}x_n$ $\geq u$ $\leq l$, $l < u$, .

: (1) $a \leq -1$, (a^n) , ≥ 1 ≤ -1.

(2) $\big((-1)^{n-1}n\big)$, ≥ 1 ≤ -1.

(3) $\big(n-3[\frac{n}{3}]\big)$, $n-3[\frac{n}{3}] = 0$ $n = 3k$ $(k \in \mathbf{Z})$ $n-3[\frac{n}{3}] = 1$ $n = 3k+1$ $(k \in \mathbf{Z})$.

. .

$(\frac{1}{n})$, $\big(\frac{(-1)^{n-1}}{n}\big)$ $(\frac{n-1}{n})$, , . – – 2.16.

2.16 , .

: $\lim_{n\to+\infty} x_n = x$. $\epsilon = 1$, n_0 $|x_n - x| < 1$ $n \geq n_0$. $|x_n - x| < 1$ $|x_n| = |(x_n - x) + x| \leq |x_n - x| + |x| < 1 + |x|$. $|x_n| < 1 + |x|$ $n \geq n_0$. $M = \max\{|x_1|, \ldots, |x_{n_0-1}|, 1 + |x|\}$, $|x_n| \leq M$ n.

: 2.16. H $\big((-1)^{n-1}\big)$.

2.17 (1) $+\infty$, .
(2) $-\infty$, .

: (1) $\lim_{n\to+\infty} x_n = +\infty$. $M = 1$, n_0 $x_n > 1$ $n \geq n_0$. $l = \min\{x_1, \ldots, x_{n_0-1}, 1\}$ $x_n \geq l$ n. , u, n_0 $x_n > u$ $n \geq n_0$, , u (x_n). (x_n) .
(2) , , (1).

: (1) (1) 2.17 . $\big(\frac{(1+(-1)^{n-1})n}{2}\big)$, $1, 0, 3, 0, 5, 0, 7, \ldots$, . , $+\infty$ ≤ 0.

(2) , (2) 2.17 . $-1, 0, -3, 0, -5, 0, -7, \ldots$, $-\infty$.

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. .

1. (x_n) . ι

$$x_{n+1} = -x_n + 2, \quad x_{n+3} = x_n - 3, \quad x_{n+1} = {x_n}^2 - 3, \quad x_{n+2} = -{x_n}^2 + 3,$$

$$x_{n+1} = {x_n}^2 + 3, \quad x_{n+2} = x_{n+1} + {x_n}^3 .$$

(: , .)

. .

1. , , , , .

$$\lim_{n\to+\infty}\Big(2n^3+3n+\frac{1}{n}\Big), \quad \lim_{n\to+\infty}\Big(n+\frac{1}{n}+2\frac{(-1)^{n-1}}{n}\Big), \quad \lim_{n\to+\infty}\frac{n^2-n+3}{n},$$

$$\lim_{n\to+\infty}\Big(1+\frac{1}{n}\Big)^3, \quad \lim_{n\to+\infty}\Big(-n+\frac{(-1)^{n-1}}{n}\Big)^9, \quad \lim_{n\to+\infty}\frac{(n+1)^{27}(n+3)^{79}}{(2n+1)^{106}},$$

$$\lim_{n\to+\infty}\frac{-1+\frac{1}{n^2}}{2+\frac{(-1)^{n-1}}{n}}, \quad \lim_{n\to+\infty}\frac{-n+\frac{1}{n}}{2+\frac{(-1)^{n-1}}{n}}, \quad \lim_{n\to+\infty}\frac{1+\frac{(-1)^{n-1}}{n}}{n+3\log_{10}n},$$

$$\lim_{n\to+\infty}\frac{n+(-1)^n\sqrt{n}}{2+(-1)^n}, \quad \lim_{n\to+\infty}\frac{1}{\frac{1}{n}+\frac{1}{n^2}}, \quad \lim_{n\to+\infty}\frac{1}{\frac{1}{n}+\frac{(-1)^{n-1}}{n^2}},$$

$$\lim_{n\to+\infty}\frac{1+2\cdot 10^n}{5+3\cdot 10^n}, \quad \lim_{n\to+\infty}\frac{3^n+(-2)^n}{3^{n+1}+2^{n+1}}, \quad \lim_{n\to+\infty}\frac{\log_2 n+3}{-2\log_{10}n+15}.$$

2. n.

$$\lim_{n\to+\infty}(3n^2-4n+5),\quad \lim_{n\to+\infty}(n^2-4n^5+1),\quad \lim_{n\to+\infty}\big((1-n)^5+n^4\big),$$

$$\lim_{n\to+\infty}\frac{3n^2-5n}{5n^2+2n-6},\quad \lim_{n\to+\infty}\frac{-2n^5+4n^2}{3n^7+n^3-10},\quad \lim_{n\to+\infty}\frac{3n^2+4n}{2n-1},$$

$$\lim_{n\to+\infty}\Big(\frac{2n-3}{3n+7}\Big)^4,\quad \lim_{n\to+\infty}\Big(\frac{-n^2+n+1}{3n+1}\Big)^3,\quad \lim_{n\to+\infty}\Big(\frac{-n^2+n+1}{3n+1}\Big)^4,$$

$$\lim_{n\to+\infty}\Big(\frac{n^2}{n+1}-n-1\Big),\quad \lim_{n\to+\infty}\Big(\frac{n(n+1)}{n+4}-\frac{4n^3}{4n^2+1}\Big).$$

3. , , .

$$\lim_{n\to+\infty}(1+2+2^2+\cdots+2^n),\quad \lim_{n\to+\infty}\big(1-2+2^2+\cdots+(-1)^n2^n\big),$$

$$\lim_{n\to+\infty}\big(1-1+1-1+\cdots+(-1)^n\big),\quad \lim_{n\to+\infty}\Big(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\Big),$$

$$\lim_{n\to+\infty}\Big(\frac{2^7}{3^7}+\frac{2^8}{3^8}+\cdots+\frac{2^{n+6}}{3^{n+6}}\Big),\quad \lim_{n\to+\infty}\Big(\frac{2^n}{3^n}+\frac{2^{n+1}}{3^{n+1}}+\cdots+\frac{2^{2n}}{3^{2n}}\Big),$$

$$\lim_{n\to+\infty}\frac{1+2+\cdots+2^n}{1+3+\cdots+3^n},\quad \lim_{n\to+\infty}\frac{1-3+\cdots+(-1)^n3^n}{1-2+\cdots+(-1)^n2^n}.$$

4. ;

$$\lim_{n\to+\infty}\Big((-1)^{n-1}+\frac{10}{n^3}\Big),\quad \lim_{n\to+\infty}\frac{1+(-1)^{n-1}n}{n},\quad \lim_{n\to+\infty}\Big(\frac{2}{n}+(-1)^{n-1}n\Big),$$

$$\lim_{n\to+\infty}(-1)^{n-1}\frac{n}{n+1},\quad \lim_{n\to+\infty}\frac{1}{(-1)^{n-1}+\frac{1}{\log_3 n}},\quad \lim_{n\to+\infty}\frac{1}{\frac{(-1)^{n-1}}{n}+\frac{1}{n^2}}.$$

(: .)

5. $x_n\neq-1$ n $x\neq-1$. $\lim_{n\to+\infty}x_n=x$ $\lim_{n\to+\infty}\frac{x_n}{1+x_n}=\frac{x}{1+x}$.
(: , $y_n=\frac{x_n}{1+x_n}$ $y=\frac{x}{1+x}$.)

6. (x_n), (y_n) (x_n+y_n) .
(x_n), (y_n) (x_ny_n) .

7. (x_n+y_n) (x_n), (y_n) , , ¨; .
(x_ny_n) (x_n), (y_n) , , ¨; .

8. (x_n), (y_n) $\lim_{n\to+\infty}x_n=0$, $\lim_{n\to+\infty}y_n=+\infty$ $\lim_{n\to+\infty}(x_ny_n)$.

9. $\lim_{n\to+\infty}|x_n|=0$, $\lim_{n\to+\infty}x_n=0$.
(: ϵ n_0.)

10. ;

$$\lim_{n\to+\infty}\Big(n\cdot\frac{1}{n}\Big)=\lim_{n\to+\infty}\Big(\underbrace{\frac{1}{n}+\cdots+\frac{1}{n}}_{n}\Big)=\underbrace{0+\cdots+0}_{n}=0.$$

$$\lim_{n\to+\infty}\Big(1+\frac{1}{n}\Big)^n=\lim_{n\to+\infty}\underbrace{\Big(1+\frac{1}{n}\Big)\cdots\Big(1+\frac{1}{n}\Big)}_{n}=\underbrace{1\cdots1}_{n}=1.$$

11. : s. , , 12 . , ( ) , 48 . ′,
(*i*) « ».
(*ii*) « ».

12. (**) $v\frac{km}{hr}$. , $d\,km$, $\frac{d}{v}\,hr$. , , :
< , , . , , . ′. , , ( ) , >.
.

. .

1. 2.11, 2.12 2.13.

2. $\lim_{n\to+\infty} x_n$.
(*i*) $1 < x_n \le \frac{n^2+3n}{n^2+1}$ n.
(*ii*) $\frac{\log_{10} n-2}{2\log_{10} n+4} < x_n < \frac{3+n}{1+2n}$ n.
(*iii*) $x_n \le 15n + 6n^2 - n^3$ n.

3. , $\lim_{n\to+\infty}\big(2n+(-1)^{n-1}n\big)$. , $\lim_{n\to+\infty}\big(2n+(-1)^{n-1}n\big)=+\infty$.
$\lim_{n\to+\infty}(2n+n\sin n)=+\infty$.

4.
$$\lim_{n\to+\infty}[nx]=\begin{cases}+\infty\,, & x>0,\\ 0\,, & x=0,\\ -\infty\,, & x<0.\end{cases}$$
(: $[a]\le a<[a]+1$.)

$$\lim_{n\to+\infty}([nx]-[ny])=\begin{cases}+\infty, & x>y,\\ 0, & x=y,\\ -\infty, & x<y.\end{cases}$$

5. $\lim_{n\to+\infty} x_n=+\infty$ (y_n) . $\lim_{n\to+\infty}(x_n+y_n)=+\infty$.
(: l (y_n). … .)
$\lim_{n\to+\infty} x_n=-\infty$ (y_n) . $\lim_{n\to+\infty}(x_n+y_n)=-\infty$.

6. $\lim_{n\to+\infty} x_n=0$ (y_n) . $\lim_{n\to+\infty} x_ny_n=0$.
(: $|y_n|\le M$ n, $-M|x_n|\le x_ny_n\le M|x_n|$ n.)

7. $\lim_{n\to+\infty} x_n = +\infty$ $-\infty$ (y_n) . $\lim_{n\to+\infty} x_n y_n = +\infty$ $-\infty$, .
$\lim_{n\to+\infty} x_n = +\infty$ $-\infty$ (y_n) . $\lim_{n\to+\infty} x_n y_n = -\infty$ $+\infty$, .

8. .

$$\lim_{n\to+\infty} 2^{-2n+(-1)^{n-1}n} = 0, \qquad \lim_{n\to+\infty} \Big(\frac{1}{2} + \frac{(-1)^{n-1}}{4}\Big)^n = 0.$$

9. (*)

$$\lim_{n\to+\infty} \Big(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n-1}} + \frac{1}{\sqrt{n^2+n}}\Big) = 1.$$

(: $\frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+k}} \leq \frac{1}{\sqrt{n^2+1}}$ k $1 \leq k \leq n$.)

10. Fibonnaci (x_n), $x_1 = x_2 = 1$ $x_{n+2} = x_{n+1} + x_n$ $(n \geq 1)$.
$x_n \geq \frac{n}{2}$ n. (x_n);

11. $0 \leq a < 1$ (x_n) : $|x_{n+1}| \leq a|x_n|$ n. $\lim_{n\to+\infty} x_n = 0$.
(: $|x_2| \leq a|x_1|$, $|x_3| \leq a^2|x_1|$, $|x_4| \leq a^3|x_1|$.)
$a > 1$ (x_n) : $x_{n+1} \geq a x_n$ n $x_1 > 0$. $\lim_{n\to+\infty} x_n = +\infty$.

12. , (x_n) $[a,b]$ $\lim_{n\to+\infty} x_n = x$, x $[a,b]$. x (x_n), (a,b);
(: $(0,2)$ $(\frac{1}{n})$ $(2-\frac{1}{n})$.)
x (x_n) (a,b);

13. 2.14 2.15.

14. , .
$-n^5 + 4n^3 < -100$ n ;
$n^7 - 35n^6 + n^3 - 47n < 84$ n;
$\frac{3}{2} < \frac{7n^3-n+5}{4n^3+n^2+35} < 2$ n ;
$\frac{2n^4-n^3+7}{-n^3+n^2+3} \leq -78$ n ;
$\frac{2n^3-n^2+7n+1}{n^3+n^2+3} \leq 1$ n;

15. 2.15, :
(i) (1) $\neq 1$.
(ii) $(\frac{1}{n})$ $\neq 0$.
(iii) (n) $\neq +\infty$.

16. 2.15, .

$$\lim_{n\to+\infty} 2^{(-1)^{n-1}}, \quad \lim_{n\to+\infty} 2^{(-1)^{n-1}n}, \quad \lim_{n\to+\infty} \Big(1 + \frac{(-1)^{n-1}}{2}\Big)^n.$$

4.

17. (*) $\lim_{n\to+\infty} x_n = x$ (x_n) $> x$. (x_n).
(: $x < x_k$ k. $x_n < x_k$. , , $x_n < x_k$ $n \geq n_0$, ;)

. .

1. 2.16 2.17.

2. , .

$$(\log_3 n),\quad (2^n),\quad \Big(\frac{2^n}{3^n}\Big),\quad \Big(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\Big),\quad \Big(\frac{2^{n+1}+3^{n+1}}{2^n+3^n}\Big),$$

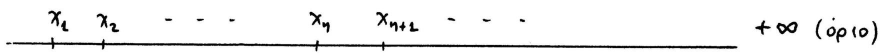
$$\Big(\frac{4n^3-2n-1}{7n^4+n^3+5n^2+2}\Big),\quad \Big(\frac{3-(\log_2 n)^3}{1+\log_2 n+(\log_2 n)^2}\Big).$$

3. .

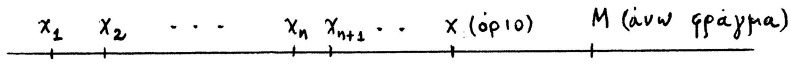
$$\Big((-2)^n\frac{n^3-3}{2n+1}\Big),\quad \big(2^n+3^n(-1)^{n-1}\big),\quad \big((2^n+n)(-1)^{n-1}+2^n-n\big).$$

2.5 . e, π.

(x_n) , , n , x_n . . (x_n) − , (n) (n^2) − x_n . (x_n) − , $(\frac{n-1}{n})$ − x_n , , , . (x_n) $+\infty$. , , x_n (x_n). , , (x_n).



Σχήμα 2.5: .



Σχήμα 2.6: .

(x_n). , n , x_n . (x_n) , x_n , (x_n) , x_n , . (x_n) $-\infty$ x_n (x_n) (x_n).
:

2.1 . :
(1) , *(i)* , $+\infty$, *(ii)* , .
(2) , *(i)* , $-\infty$, *(ii)* , .

2.1 . x (x_n), $x_n \le x$, , (x_n) , $< x$. x (x_n) , (x_n), :

,
. , ,
.

2.1 ′ .

2.1 . : . : , . : , , ( ) n- ( ) . 2.1 (,).

2.1 . , , ( ) . 1 .

: (x_n) :

$$x_1 = 1, \qquad x_{n+1} = \sqrt{2x_n} \quad (n \ge 1).$$

(x_n) $1, \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ . . , $x_1 \le x_2$ n $x_n \le x_{n+1}$. ′ 1 . , , : $x_n \le x_{n+1}$ $2x_n \le 2x_{n+1}$ $\sqrt{2x_n} \le \sqrt{2x_{n+1}}$ $x_{n+1} \le x_{n+2}$. $x_n \le x_{n+1}$ n, , , .

$x_n \le x_{n+1}$ $x_n \le \sqrt{2x_n}$, $x_n \le 2$ n. (x_n) , , .

x (x_n). ${x_{n+1}}^2 = 2x_n$, $x^2 = 2x$, $x = 0$ $x = 2$. $x = 0$ 1, ≥ 1 , , ≥ 1. $\lim_{n\to+\infty} x_n = 2$.

(x_n) . $x_n \le x_{n+1}$ $x_n \le \sqrt{2x_n}$ ($x_n \ge 0$ n) $x_n \le 2$. , $x_n \le 2$ n (x_n) , , 2. . $x_1 \le 2$. $x_n \le 2$ n. $x_{n+1} = \sqrt{2x_n} \le \sqrt{2\cdot 2} = 2$, , $x_n \le 2$ n.

!

: K. (!) , $(1+1)K = 2K$. , , $(1+\frac{1}{2})(1+\frac{1}{2})K = (1+\frac{1}{2})^2 K$. $\frac{100}{3}$, . $(1+\frac{1}{3})(1+\frac{1}{3})(1+\frac{1}{3})K = (1+\frac{1}{3})^3 K$.

, n- $n-1$ $(1+\frac{1}{n})^n K$.

, . , $x_n = (1+\frac{1}{n})^n$. , K, . , (x_n) , , 4. , , .

(x_n) . ′ .

2.1 n $x \ge -1$ $(1+x)^n \ge 1 + nx$.

$(1+x)^n \ge 1+nx$ **Bernoulli** n.

, $(1+\frac{1}{n})^n \le (1+\frac{1}{n+1})^{n+1}$ $(\frac{n+1}{n})^n \le (\frac{n+2}{n+1})^{n+1}$ $\frac{n}{n+1}(\frac{n+1}{n})^{n+1} \le (\frac{n+2}{n+1})^{n+1}$ $\frac{n}{n+1} \le (\frac{n^2+2n}{n^2+2n+1})^{n+1}$ $\frac{n}{n+1} \le (1-\frac{1}{n^2+2n+1})^{n+1}$ 2.1. , $(1-\frac{1}{n^2+2n+1})^{n+1} \ge 1-\frac{n+1}{n^2+2n+1} = 1-\frac{1}{n+1} = \frac{n}{n+1}$ . .

$(1+\frac{1}{n})^n < 4$, $\frac{1}{2} < \left(\frac{\sqrt{n}}{\sqrt{n+1}}\right)^n$. 1.1 $\left(\frac{\sqrt{n}}{\sqrt{n+1}}\right)^n = \left(1-\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}}\right)^n \ge 1-n\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}} > 1-n\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}} = 1-\sqrt{n}(\sqrt{n+1}-\sqrt{n}) = 1-\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} > 1-\frac{\sqrt{n}}{2\sqrt{n}} = \frac{1}{2}$.

e $\left((1+\frac{1}{n})^n\right)$.

$$\boxed{e = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n.}$$

e, π, .

e , , . , $a_0 + a_1x + \cdots + a_Nx^N = 0$. : $\frac{m}{n}$ (m,n) , , $m + (-n)x = 0$ . . , $\sqrt{2}$ $(-2) + 1x^2 = 0.$. , e . , 10, e .

e $y > 0$

$$\log y \qquad \ln y$$

$\log_e y$. , , 1.10 1.11.

2.18 (1) $\log(yz) = \log y + \log z$ $y, z > 0$.
(2) $\log\frac{y}{z} = \log y - \log z$ $y, z > 0$.
(3) $\log(y^z) = z\log y$ $y > 0$ z.
(4) $\log 1 = 0$ $\log e = 1$.
(5) $0 < y < z$, $\log y < \log z$.

2.19 $a > 0$, $a \neq 1$.

$$\log_a y = \frac{\log y}{\log a}$$

$y > 0$.

: .

π 1. « » , 2π, .

, 1, $P_4, P_8, P_{16}, \ldots$ $4, 8, 16, \ldots$. , P_{2^n} 2^n . P_{2^n} $P_{2^{n+1}}$: $P_{2^{n+1}}$ 2^n P_{2^n} 2^n P_{2^n}, $2^n + 2^n = 2^{n+1}$. p_n P_{2^n}, $p_2 = 4\sqrt{2}$. p_{n+1} p_n .

$$p_{n+1} = \frac{2p_n}{\sqrt{2 + \sqrt{4 - \frac{{p_n}^2}{4^n}}}}$$

.

, , $Q_4, Q_8, Q_{16}, \ldots$ $4, 8, 16, \ldots$. , Q_{2^n} 2^n , P_{2^n} . q_n Q_{2^n} , , , $q_2 = 8$ q_n p_n

$$q_n = \frac{p_n}{\sqrt{1 - \frac{{p_n}^2}{4^{n+1}}}}\,.$$

,

$$q_{n+1} = \frac{4q_n}{2 + \sqrt{4 + \frac{{q_n}^2}{4^n}}}\,.$$

(p_n) (q_n) . , $p_{n+1} = \frac{2p_n}{\sqrt{2+\sqrt{4-\frac{{p_n}^2}{4^n}}}} > \frac{2p_n}{\sqrt{2+\sqrt{4}}} = p_n$ $q_{n+1} = \frac{4q_n}{2+\sqrt{4+\frac{{q_n}^2}{4^n}}} < \frac{4q_n}{2+\sqrt{4}} = q_n$. $q_n > \frac{p_n}{\sqrt{1}} = p_n$. , , :

$$p_2 < p_3 < \cdots < p_n < p_{n+1} < \cdots < q_{n+1} < q_n < \cdots < q_3 < q_2\,.$$

(p_n) , , (q_2,), . , ′ 2^n . , ,

–
′ 2^n .

«» , n , , , «» .
, , π 1, π (p_n). ,

$$\boxed{\pi = \frac{1}{2}\lim_{n\to+\infty} p_n\,.}$$

, $\lim_{n\to+\infty} q_n = \lim_{n\to+\infty} \frac{p_n}{\sqrt{1-\frac{p_n{}^2}{4^{n+1}}}} = \frac{2\pi}{\sqrt{1-4\pi^2\cdot 0}} = 2\pi.$ (p_n) (q_n) , ,
π

$$\boxed{\pi = \frac{1}{2}\lim_{n\to+\infty} q_n\,.}$$

, ,

$$\boxed{p_2 < p_3 < \cdots < p_n < \cdots \; < 2\pi < \; \cdots < q_n < \cdots < q_3 < q_2\,.}$$

e $\big((1+\frac{1}{n})^n\big)$. π , . , , , , . , , . (q_n)
$q_2 = 8$ $q_{n+1} = \frac{4q_n}{2+\sqrt{4+\frac{q_n{}^2}{4^n}}}$ $(n \geq 2)$, . – – , ≥ 0.
(q_n) π (q_n): $\pi = \frac{1}{2}\lim_{n\to+\infty} q_n\,.$ (p_n) (q_n) (q_n) .

.

1. $a > 1$. $\left(\frac{a^n}{n}\right)$.

   (i) $\left(\frac{a^n}{n}\right)$ – – $\frac{a^{n+1}}{n+1} = \frac{an}{n+1}\frac{a^n}{n}$ n. , , $\left(\frac{a^n}{n}\right)$.

   (ii) b $1 < b < a$ 2.1 $b^n \geq 1+(b-1)n$ n. $\left(\frac{a^n}{n}\right)$ $\frac{a^n}{n} = \left(\frac{a}{b}\right)^n \frac{b^n}{n}$.

2. $a > 1$. $(\sqrt[n]{a})$ – – $\left(\sqrt[2n]{a}\right)^2 = \sqrt[n]{a}$ n. $(\sqrt[n]{a})$.
   $a = 1$ $0 < a < 1$;

3. (*) $0 \leq a \leq b$. , $\lim_{n\to+\infty} \sqrt[n]{a^n+b^n} = b$.
   (: $b = \sqrt[n]{b^n} \leq \sqrt[n]{a^n+b^n} \leq \sqrt[n]{2b^n} = b\,\sqrt[n]{2}\,$.)

. e.

1. .

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n+3}, \quad \lim_{n\to+\infty}\left(1+\frac{1}{n+2}\right)^{n}, \quad \lim_{n\to+\infty}\left(1+\frac{1}{n+2}\right)^{3n+5},$$

$$\lim_{n\to+\infty}\left(1-\frac{1}{n}\right)^{n}\ (*), \quad \lim_{n\to+\infty}\left(1+\frac{2}{n}\right)^{n}\ (*), \quad \lim_{n\to+\infty}\left(1-\frac{2}{n}\right)^{n}\ (*).$$

(: $1-\frac{1}{n} = \frac{n-1}{n} = \frac{1}{\frac{n}{n-1}} = \frac{1}{1+\frac{1}{n-1}}$ $1+\frac{2}{n} = \frac{n+2}{n} = \frac{n+1}{n}\frac{n+2}{n+1} = \big(1+\frac{1}{n}\big)\big(1+\frac{1}{n+1}\big)$.)

2. (*) $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n^2}=+\infty.$
(: $(1+\frac{1}{n})^n \geq (1+\frac{1}{1})^1 = 2$ n.)

. .

1. $x_1 = 1$ $x_{n+1} = x_n + \frac{1}{x_n{}^2}$ n. (x_n) .
(: . .)

2. $7x_{n+1} = x_n{}^3 + 6$ n. , x_1 , (x_n) .
(: (x_n), . x_1 , . .)

3. $4x_{n+1} = x_n{}^2 + 3$ n. , x_1 , (x_n) .

4. $0 < x_1 < 1$ $x_{n+1} = 1 - \sqrt{1-x_n}$ n. (x_n) .

5. $\lambda > 0$, $x_1 > 0$ $x_{n+1} = \sqrt{\lambda + x_n}$ n. , x_1 , (x_n) .

6. $x_1 > 0$ $x_{n+1} = \frac{6+6x_n}{7+x_n}$ n. , x_1 , (x_n) .

7. $a, x_1 > 0$ $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ n. (x_n) , , .
a, 1.2 $n = 2$.

8. $x_1 = x_2 = 1$ $\frac{1}{x_{n+2}} = \frac{1}{x_{n+1}} + \frac{1}{x_n}$ n. (x_n) .

9. (x_n) (y_n) :

$$0 < x_1 \leq y_1\,, \qquad x_{n+1} = \sqrt{x_n y_n}\,, \quad y_{n+1} = \frac{x_n + y_n}{2} \quad (n \geq 1).$$

(x_n) , (y_n) $x_n \leq y_n$ n.
(x_n) (y_n) .

10. (x_n) (y_n) :

$$0 < x_1 \leq y_1\,, \qquad x_{n+1} = \frac{2x_n y_n}{x_n + y_n}\,, \quad y_{n+1} = \sqrt{x_n y_n} \quad (n \geq 1).$$

(x_n) , (y_n) $x_n \leq y_n$ n.
(x_n) (y_n) .

Κεφάλαιο 3

.

. , ...

3.1 .

: (1) , , p V

$$pV = c\,,$$

c . , *Boyle* p V . , p V , , V p:

$$V = \frac{c}{p}\,, \qquad p = \frac{c}{V}\,.$$

«V p» «p V». p , ( ) , V , .
.

(2) ^{o}C, l

$$l = (1 +)l_0,$$

l_0 0^oC , , . l , l l .
l

$$= \frac{1}{}\Big(\frac{l}{l_0} - 1\Big).$$

(3) ,

$$= \sqrt{^2 + ^2 - 2 \cos}\,.$$

. 0 π $|-|$ $+$. , $-1 \leq \cos \leq 1$, $^2 + ^2 - 2\cos \leq ^2 + ^2 + 2 = (+)^2$
$^2 + ^2 - 2\cos \geq ^2 + ^2 - 2 = (-)^2$. , $|-| \leq \leq +$.

$$= \arccos\frac{^2 + ^2 - ^2}{2}\,.$$

, . 0 π, , . , $= \sqrt{^2 + ^2 - 2\cos}$ cos $|-|$ $+$

$$\cos = \frac{^2 + ^2 - ^2}{2}\,,$$

$| - | \leq \ \leq \ +$ $\ ^2 + \ ^2 - 2 \leq \ ^2 \leq \ ^2 + \ ^2 + 2$ $-1 \leq \frac{^2 + ^2 - ^2}{2} \leq 1.$ $[0, \pi]$
$\cos \ = \frac{^2 + ^2 - ^2}{2}$. , : 0 π ( ) − − $-\pi$ 0 2π.

(4) ( ) π. , s r

$$s = 2\pi r, \qquad r = \frac{1}{2\pi}\, s.$$

′ 2^n . , , p_n 1, , 1, 2π π 1. , , r ′
2^n , , , rp_n . r $\lim_{n\to+\infty}(rp_n) = r\lim_{n\to+\infty} p_n = r2\pi$.
s r, r s, , , r s, s r.

(5) A r , A r

$$A = \pi r^2\,.$$

V r , V r

$$V = \frac{4}{3}\pi r^3\,.$$

(6) : « t (k) q ». q t:

$$q = e^{-kt} q_0\,,$$

q_0 0. , 8. , , $q-$ − $t-$.

$$t = \frac{1}{k}\log\frac{q_0}{q}\,,$$

q t.

3.2 , , .

x − , x, − − − x y − , y, − y
x

$$y = f(x) \qquad y = F(x) \qquad y = g(x)$$

. f , , f $y = f(x)$. x y .
$y = f(x)$, , y x. , .

:

$$y = x^3 + \sin(e^{2x} + \log_2(x+3))$$

y x,

$$f(x) \qquad x^3 + \sin(e^{2x} + \log_2(x+3)),$$

f

$$y = f(x)$$

, , $y = x^3 + \sin(e^{2x} + \log_2(x+3))$.

, , « f $y = f(x)$» « $y = f(x)$». ′ .

x y . , , . , , . : $u = f(v)$, $t = f(x)$, $y = f(t)$.

.

, , .

: . , . – – ( ) – .

. .

: .

: $V = \frac{c}{p}$. p , $(0, +\infty)$. , . , , $V = \frac{c}{p}$, , $y = \frac{c}{x}$, , 0 (p x), $(-\infty, 0) \cup (0, +\infty)$.

, – – .
, $y = \frac{c}{x}$, x – – x, $(-\infty, 0) \cup (0, +\infty)$.
. $y = f(x)$. , . b ( ) a $f(a) = b$. ,

$y = f(x)$ y $f(x) = y$ x
.

: $y = 3x - 1$. $(-\infty, +\infty)$.
$3x - 1 = y$ x y . y $x = \frac{y+1}{3}$. y , $(-\infty, +\infty)$.

, . , . , .

: (1) $y = 3x - 1$. $(-\infty, +\infty)$ $[-2, 5)$ $(-\infty, +\infty)$.
$3x - 1 = y$ x y $[-2, 5)$. $x = \frac{y+1}{3}$ y $[-2, 5)$, $-2 \le \frac{y+1}{3} < 5$, , $-7 \le y < 14$. y $[-7, 14)$ $[-2, 5)$, , $[-2, 5)$ $[-7, 14)$.

(2) $y = \frac{x}{x-1}$ $(-\infty, 1) \cup (1, +\infty)$.
$\frac{x}{x-1} = y$ x y . $\frac{x}{x-1} = y$ $(y-1)x = y$, , $y = 1$, , $y \ne 1$, $x = \frac{y}{y-1}$. , , , $\frac{y}{y-1} \ne 1$. , , $y \ne y - 1$.
$y \ne 1$, , $(-\infty, 1) \cup (1, +\infty)$.
, , $(-\infty, 1)$ $(1, +\infty)$.
$(1, +\infty)$ $\frac{x}{x-1} = y$ x y $(1, +\infty)$. , $y = 1$, $x = \frac{y}{y-1}$ $(1, +\infty)$, $\frac{y}{y-1} > 1$. $\frac{1}{y-1} > 0$ $y > 1$. $y > 1$ $(1, +\infty)$, , $(1, +\infty)$ $(1, +\infty)$.
$(-\infty, 1)$ $(-\infty, 1)$.

(3) $y = e^{-2x}$ $(-\infty, +\infty)$.
y $e^{-2x} = y$ x . $y \le 0$ $e^{-2x} = y$ $y > 0$ $x = -\frac{1}{2}\log y$. $y > 0$, $(0, +\infty)$.

(4) $y = \sqrt{1 + \frac{1}{x}}$. x $1 + \frac{1}{x} \ge 0$, , $x \le -1$ $x > 0$. $(-\infty, -1] \cup (0, +\infty)$.
y $\sqrt{1 + \frac{1}{x}} = y$ x . , $\sqrt{1 + \frac{1}{x}} = y$, $y < 0$. $y \ge 0$, $\sqrt{1 + \frac{1}{x}} = y$ $(y^2 - 1)x = 1$. $y = 1$, , $y \ge 0$ $y \ne 1$, $x = \frac{1}{y^2-1}$. , $\frac{1}{y^2-1} \le -1$

$\frac{1}{y^2-1} > 0$. , $y \geq 0, y \neq 1$. , $[0,1) \cup (1,+\infty)$.
$(0,+\infty)$. y $\sqrt{1+\frac{1}{x}} = y$ x $(0,+\infty)$. , $y < 0$ $y = 1$, $x = \frac{1}{y^2-1}$ $(0,+\infty)$, $\frac{1}{y^2-1} > 0$. $y^2 > 1$, $y > 1$. $(0,+\infty)$ $(1,+\infty)$.
, y $\sqrt{1+\frac{1}{x}} = y$ x $(-\infty,-1]$, $(-\infty,-1]$ $[0,1)$.

.

1. .

$$y = \frac{2}{3}x - 4, \quad y = x^2 - 4x + 3, \quad y = \frac{2x-1}{x+4}, \quad y = \frac{x^2-1}{x^2+1}, \quad y = \frac{x^2+1}{x^2-1},$$

$$y = 2^x, \quad y = \log_{10} x + 4, \quad y = e^{2x} - 2e^x + 3, \quad y = \frac{e^x+1}{e^x-1} \quad y = \frac{x}{\sqrt{x}-1}.$$

2. .
(i) $y = x^2 - 4x + 3$ $(-\infty,1]$, $(1,3]$, $(3,+\infty)$, $(-\infty,2]$, $[2,+\infty)$.
(ii) $y = \frac{2x-1}{x+4}$ $(-\infty,-4)$, $(-4,+\infty)$.
(iii) $y = \frac{x^2+1}{x^2-1}$ $(-\infty,-1)$, $(-1,1)$, $(1,+\infty)$.
(iv) $y = \frac{e^x+1}{e^x-1}$ $(-\infty,0)$, $(0,+\infty)$.
(v) $y = \frac{x}{\sqrt{x}-1}$ $[0,1)$, $(1,+\infty)$.

3.3 .

y x . , , :

$$y = x^2, \qquad y = \sin x, \qquad xy = 2, \qquad y^2 - x^3 = 0.$$

y $x\cdot$ y x . y $x\cdot$ $xy = 2$ y y y x. . y x

$$y = \frac{2}{x}.$$

y x. , . $y^2 - x^3 = 0$ y, . $x < 0$ y. $x = 0$ y, $y = 0$. $x > 0$ y : $y = \sqrt{x^3} = x^{\frac{3}{2}}$ $y = -\sqrt{x^3} = -x^{\frac{3}{2}}$. $y^2 - x^3 = 0$,

$$y = x^{\frac{3}{2}}, \qquad y = -x^{\frac{3}{2}}$$

$[0,+\infty)$.
. , : . : . $y^2 - x^3 = 0$ () , ,

$$y = \pm x^{\frac{3}{2}}.$$

.

1. . x y; − ; − ; .

$$x+y=1, \quad x^2-2yx+1=0, \quad \frac{y-x}{y+x}=-2, \quad x^3+y^3=0, \quad x^2-y^2=0,$$

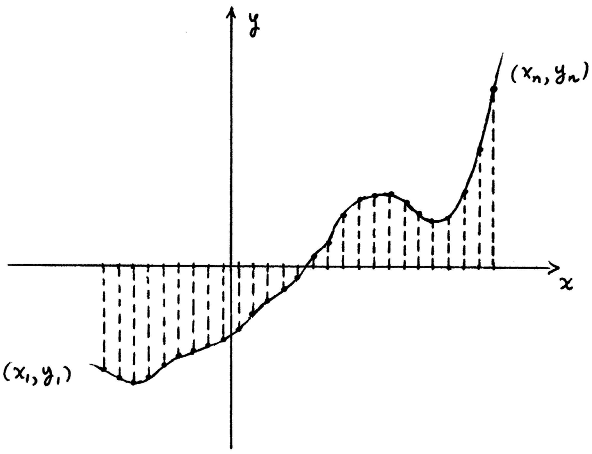
$$(xy)^2=1, \quad e^{(x-1)y^2}=x, \quad y^4-2xy^2+x^2=1, \quad \sin(x+y)=1.$$

3.4 .

$y=f(x)$, , .
, , 0 . x $y=f(x)$ y (x y). x- y-. , x $y=f(x)$ $(x,y)=(x,f(x))$. , x $y=f(x)$, $(x,y)=(x,f(x))$. $(x,y)=(x,f(x))$.

$$f=\{(x,f(x)):x \quad f\}.$$

, .
x, , . $(x_1,f(x_1)),(x_2,f(x_2)),\ \ldots\ ,(x_n,f(x_n))$ n x . ¨, .

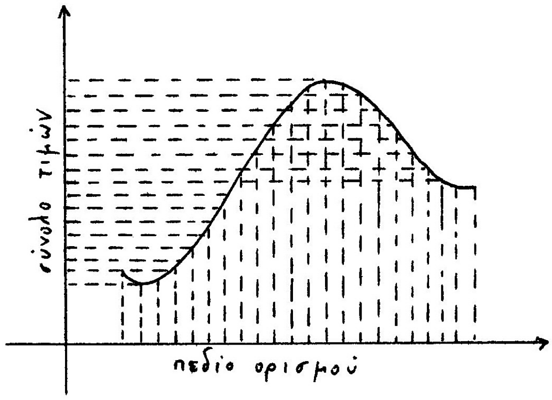


Σχήμα 3.1: .

$y=f(x)$, . ′ $y=f(x)$. $(x,f(x))$. x- x y- $y=f(x)$. ,

$y=f(x)$ -
x- -
y-.

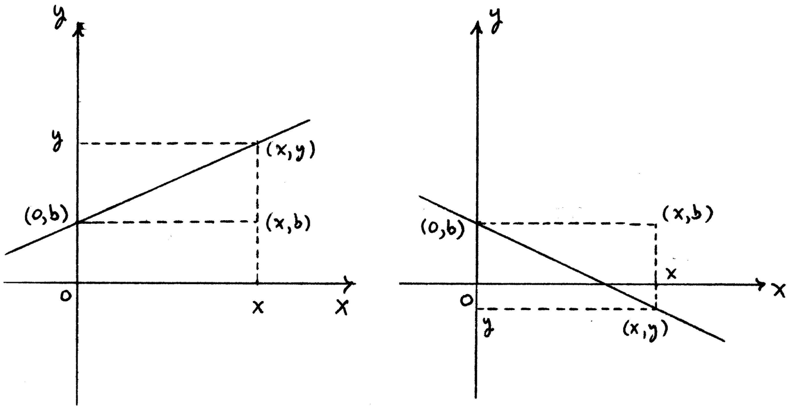
:



Σχήμα 3.2: – .

$$\boxed{y = ax + b,}$$

a, b , $(-\infty, +\infty)$.
$(0, b)$. (x, y) $y = ax + b$, , $y - b = a(x - 0)$. $(0, b)$ (x, y) $\frac{y-b}{x-0} = a$. , (x, y) l $(0, b)$ a. . (x, y) l $\frac{y-b}{x-0}$ $(0, b)$ (x, y) a, $\frac{y-b}{x-0} = a$, $y = ax + b$, , (x, y) . , , l.



Σχήμα 3.3: $y = ax + b$. : $a > 0$ $a < 0$.

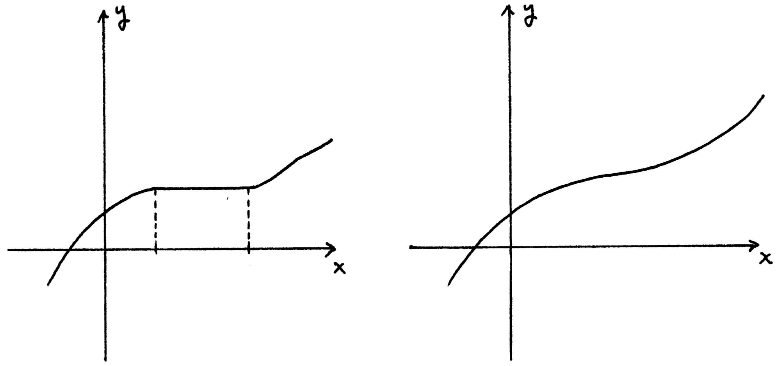
$a > 0$, l l , $a < 0$, l l . $a = 0$, l l . (, $a \neq 0$) l y- y-, $(-\infty, +\infty)$. (, $y = 3x - 1$ 3.2.) (, $a = 0$) l , y- b, $\{b\}$.

$y = f(x)$ I x_1, x_2 I $x_1 < x_2$ $f(x_1) \leq f(x_2)$. $x_1 < x_2$ $f(x_1) < f(x_2)$, I.

, $y = f(x)$ I x_1, x_2 I $x_1 < x_2$ $f(x_1) \geq f(x_2)$. $x_1 < x_2$ $f(x_1) > f(x_2)$, I.

.

. . , .



Σχήμα 3.4: .

y $y = f(x)$ x ( ). $f(x) = y$ x y ( ).

: $y = ax + b$ $(-\infty, +\infty)$, $a > 0$, , $a < 0$. $a = 0$, $(-\infty, +\infty)$, x y b.

$y = f(x)$ A u $f(x) \leq u$ x A. u A. $y = f(x)$ A l $f(x) \geq l$ x A l A. , $y = f(x)$ A A, u l $l \leq f(x) \leq u$ x A.

u $y = f(x)$ A, $u' \geq u$ A. , l $y = f(x)$ A, $l' \leq l$ A.

u $y = f(x)$ A A $y = u$. , l $y = f(x)$ A A $y = l$. , , u l , , $y = f(x)$ A, A $y = u$ $y = l$.

, , u $y = f(x)$ A A $(-\infty, u]$ l $y = f(x)$ A A $[l, +\infty)$. , u l $y = f(x)$ A, , A $[l, u]$.

:

$$\boxed{y = x^2}$$

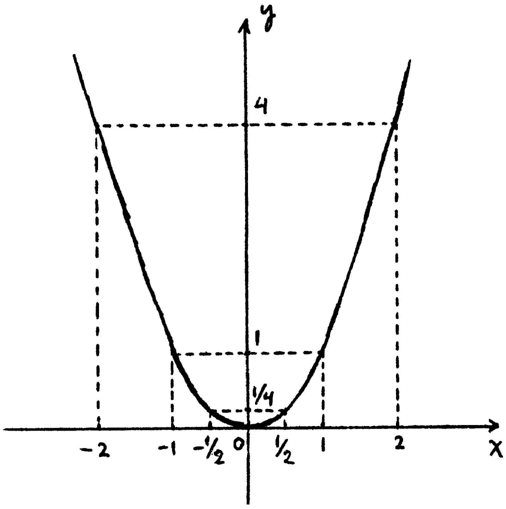
$(-\infty, +\infty)$.

$y = x^2$ $[0, +\infty)$ $(-\infty, 0]$. $[0, +\infty)$ $(-\infty, 0]$.

y $x^2 = y$ x $[0, +\infty)$. : $y < 0$, , $y \geq 0$, $x = \sqrt{y}$ $[0, +\infty)$. $[0, +\infty)$ $[0, +\infty)$.

, $x^2 = y$ x , $y < 0$, $x = -\sqrt{y}$ $(-\infty, 0]$, $y \geq 0$. $(-\infty, 0]$ $[0, +\infty)$.

$y = x^2$ $(-\infty, +\infty)$ 0 $[0, +\infty)$. , $y = x^2$ $(-\infty, +\infty)$ $(-\infty, 0]$, $[0, +\infty)$, $[0, +\infty)$ $(-\infty, u]$.



Σχήμα 3.5: $y = x^2$.

$[0, +\infty)$. $(0,0)$ $(0,0)$, $(\frac{1}{2}, \frac{1}{4})$, $(1,1)$, $(2,4)$. x- $[0, +\infty)$ y- $[0, +\infty)$, $[0, +\infty)$. . , x , (x, x^2), $y = x^2$, .

, $(-\infty, 0]$. , $(0,0)$ $(0,0)$, $(-2,4)$, $(-1,1)$, $(-\frac{1}{2}, \frac{1}{4})$. x- $(-\infty, 0]$ y- $(-\infty, 0]$, $[0, +\infty)$. . , x , (x, x^2), $y = x^2$, .

$y = x^2$ **()** . .

$y = f(x)$ $f(-x) = f(x)$ x . (a,b) , $b = f(a)$, $b = f(-a)$, $(-a, b)$. (a,b) $(-a,b)$ y-. , , , y- .

$y = f(x)$ y- **.**

$y = x^2$, y-.

:

$$\boxed{y = x^3}$$

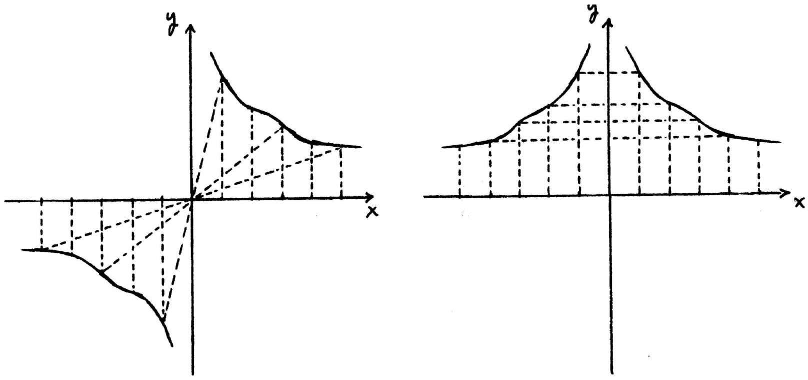
$(-\infty, +\infty)$.

$'$ $(-\infty, +\infty)$. $(-2,-8)$, $(-1,-1)$, $(-\frac{1}{2}, -\frac{1}{8})$, $(0,0)$, $(\frac{1}{2}, \frac{1}{8})$, $(1,1)$, $(2,8)$.

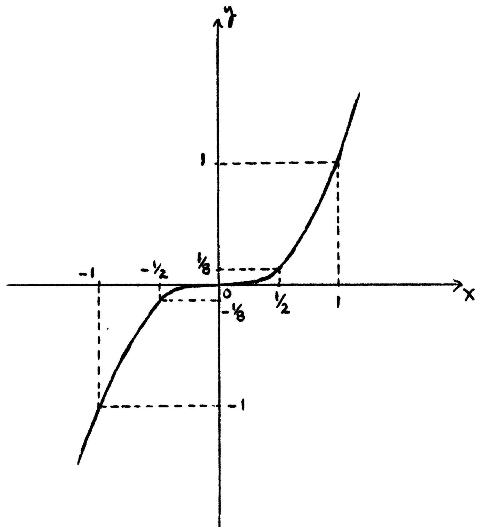
$y = x^3$, $x^3 = y$ x. y , $x = \sqrt[3]{y}$, y , , $(-\infty, +\infty)$.

$y = x^3$ $(-\infty, +\infty)$, $(-\infty, +\infty)$.

x- , $(-\infty, +\infty)$, y- , $(-\infty, +\infty)$. . , x , (x, x^3) , x , (x, x^3) .



Σχήμα 3.6: .



Σχήμα 3.7: $y = x^3$.

$y = x^3$.

$y = f(x)$ $f(-x) = -f(x)$ x . (a, b) $y = f(x)$, $b = f(a)$, $-b = f(-a)$, $(-a, -b)$ $y = f(x)$. (a, b) $(-a, -b)$ $(0, 0)$. , , $(0, 0)$.

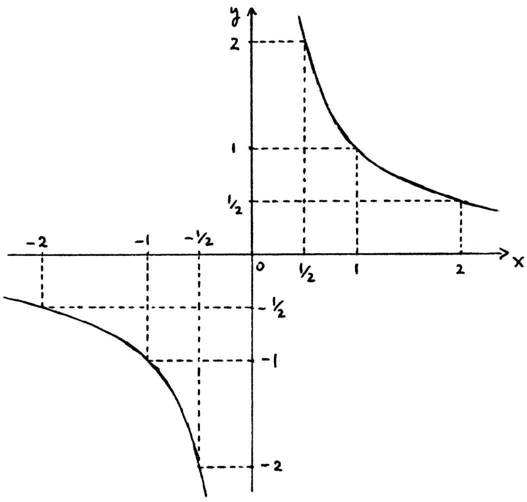
$y = f(x)$
$(0, 0)$.

$y = x^3$, $(0,0)$.

:

$$\boxed{y = \frac{1}{x}}$$

$(-\infty, 0) \cup (0, +\infty)$.
$y = \frac{1}{x}$ $(-\infty, 0)$, $(0, +\infty)$. ′ .
$\frac{1}{x} = y$ x. $y \leq 0$, $(0, +\infty)$, $y > 0$, $x = \frac{1}{y}$ $(0, +\infty)$. $(0, +\infty)$ $(0, +\infty)$.
, $\frac{1}{x} = y$ x $(-\infty, 0)$, $y \geq 0$, $x = \frac{1}{y}$ $(-\infty, 0)$, $y < 0$. $(-\infty, 0)$ $(-\infty, 0)$.
$y = \frac{1}{x}$ $(0, +\infty)$, , 0 $(0, +\infty)$. $(-\infty, 0)$, , 0 $(-\infty, 0)$.



Σχήμα 3.8: $y = \frac{1}{x}$.

$(0, +\infty)$. $(\frac{1}{2}, 2)$, $(1, 1)$, $(2, \frac{1}{2})$. x- $(0, +\infty)$ y- $(0, +\infty)$, $(0, +\infty)$. , x , $(x, \frac{1}{x})$ ($y = \frac{1}{x}$) , x , $(x, \frac{1}{x})$. , y- x-.
, $(-\infty, 0)$. $(-2, -\frac{1}{2})$, $(-1, -1)$, $(-\frac{1}{2}, -2)$. x- $(-\infty, 0)$ y- $(-\infty, 0)$, $(-\infty, 0)$. , x , $(x, \frac{1}{x})$, x , $(x, \frac{1}{x})$. , x- y-.
$y = \frac{1}{x}$, , .
$y = \frac{1}{x}$, $(0,0)$: $(0,0)$.
. (a, b) , $b = \frac{1}{a}$, $a = \frac{1}{b}$, (b, a) . (a, b) (b, a) $y = x$, . ,
, . , , .

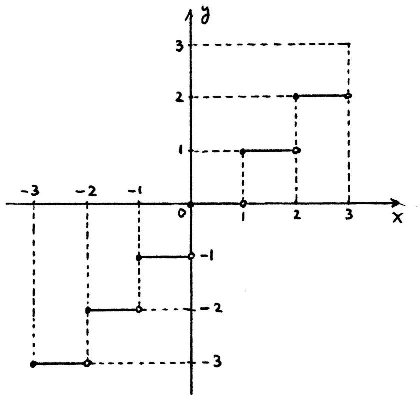
. , .

$y = \frac{1}{x}$. $y = \frac{1}{x}$: $(-\infty, 0)$ $(0, +\infty)$. 0 .
«. . . , ». «» . ′ . – – 6.7.
, ′ . , .

:

$$\boxed{y = [x],}$$

x, $(-\infty, +\infty)$.
$y = [x]$ $[k, k+1)$, k : $y = k$ x $[k, k+1)$. $[k, k+1)$. ,
$y = [x]$ $[k, k+1)$, .



Σχήμα 3.9: $y = [x]$.

, $y = [x]$ ($(-\infty, +\infty)$) .

.

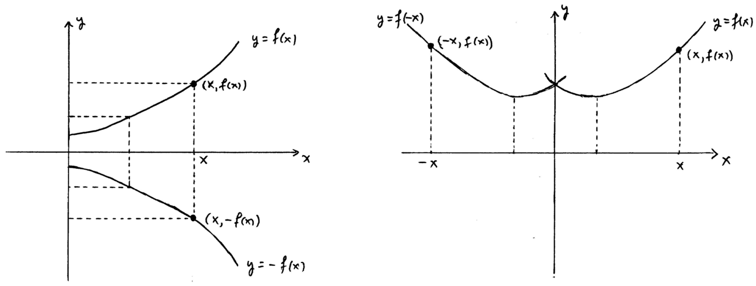
. . , , .
′ $y = f(x)$, , .
$y = f(x)$, , . . , x- y- .
.
. , $[0, +\infty)$. y- $(0, 0)$.

. $y = f(x)$. $y = f(x)$.

(1) $(x, -f(x))$ $y = -f(x)$ x- $(x, f(x))$ $y = f(x)$. :

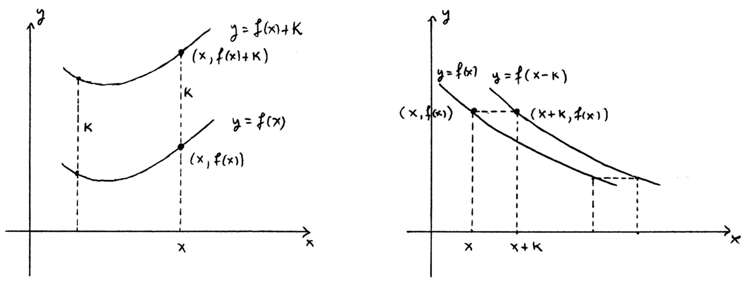
$y = -f(x)$ x-
$y = f(x)$.

(2) $(-x, f(x)) = (x', f(-x'))$ $y = f(-x)$ y- $(x, f(x))$ $y = f(x)$. :



Σχήμα 3.10: $y = -f(x)$ $y = f(-x)$.

$y = f(-x)$ y-
$y = f(x)$.



Σχήμα 3.11: $y = f(x) + \kappa$ $y = f(x - \kappa)$.

(3) κ. κ (a, b) $(a, b + \kappa)$.
$(x, f(x) + \kappa)$ $y = f(x) + \kappa$ κ $(x, f(x))$ $y = f(x)$. :

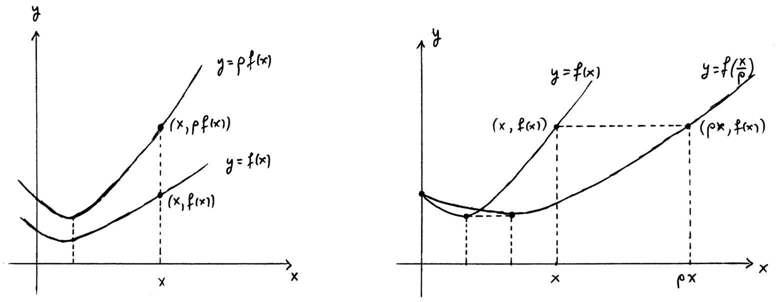
$y = f(x) + \kappa$ κ $y = f(x)$.

(4) κ. κ (a, b) $(a + \kappa, b)$.
$(x + \kappa, f(x)) = (x', f(x' - \kappa))$ $y = f(x - \kappa)$ κ $(x, f(x))$ $y = f(x)$.
:

$y = f(x - \kappa)$ κ
$y = f(x)$.

(5) ρ . ρ (a, b) $(a, \rho b)$.
$(x, \rho f(x))$ $y = \rho f(x)$ ρ $(x, f(x))$ $y = f(x)$. :

$y = \rho f(x)$ ρ
$y = f(x)$.



Σχήμα 3.12: $y = \rho f(x)$ $y = f(\frac{x}{\rho})$.

(6) ρ . ρ (a, b) $(\rho a, b)$. $(\rho x, f(x)) = \big(x', f\big(\frac{x'}{\rho}\big)\big)$ $y = f\big(\frac{x}{\rho}\big)$ ρ $(x, f(x))$ $y = f(x)$. :

$y = f(\frac{x}{\rho})$ ρ $y = f(x)$.

.

1. .

$$y = |x|, \quad y = \frac{|x|}{x}, \quad y = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases} \quad y = x - [x],$$

$$y = (-1)^{[x]}, \quad y = x(-1)^{[x]}, \quad y = (-1)^{[\frac{1}{x}]}, \quad y = x(-1)^{[\frac{1}{x}]}.$$

; ; ; . ; ;

2. $y = \sqrt{-x^2}$ $y = \sqrt{-x^2 - 1}$.

3. $y = x^2$, :

$$y = 3x^2, \quad y = x^2 - 4, \quad y = (x+4)^2, \quad y = (3x+4)^2, \quad y = 4 - (3x+4)^2.$$

, , . , , ( ) $(-\infty, +\infty)$.

4. $a \neq 0$ b, c. $y = x^2$,

$$y = ax^2 + bx + c.$$

(: $ax^2 + bx + c = a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}$.) $(-\infty, +\infty)$;

5. $y = \frac{1}{x}$, :

$$y = \frac{1}{x} + 2, \quad y = \frac{1}{x+2}, \quad y = \frac{1}{3x+2}, \quad y = \frac{3}{x+2}.$$

6. a, b, c, d $c \neq 0$. $y = \frac{1}{x}$,

$$y = \frac{ax+b}{cx+d}.$$

(: $\frac{ax+b}{cx+d} = \frac{\frac{a}{c}(cx+d)+b-\frac{ad}{c}}{cx+d} = \frac{bc-ad}{c^2}\frac{1}{x+\frac{d}{c}} + \frac{a}{c}$.)

$y = \frac{ax+b}{cx+d}$.

$y = \frac{2x+3}{3x-1}$. , , .

7. κ $\rho > 0$. $y = f(x)$, $y = -f(x)$, $y = f(-x)$, $y = f(x) + \kappa$, $y = f(x-\kappa)$, $y = \rho f(x)$ $y = f(\frac{x}{\rho})$;

8. $y = x^2 - 3x + 2$, , $y = |x^2 - 3x + 2|$ $y = |x|^2 - 3|x| + 2$. ;

:

(i) $y = f(x)$ $y = |f(x)|$.

(ii) $y = f(x)$ $y = f(|x|)$.

3.5 .

$y = f(x)$, ¨, , y x , , x y. , , ¨ y $y = f(x)$ x . , y $f(x) = y$ x . ¨, f, f^{-1}

$$x = f^{-1}(y).$$

:

$$\boxed{y = f(x) \qquad\qquad x = f^{-1}(y)}$$

x y f.

.

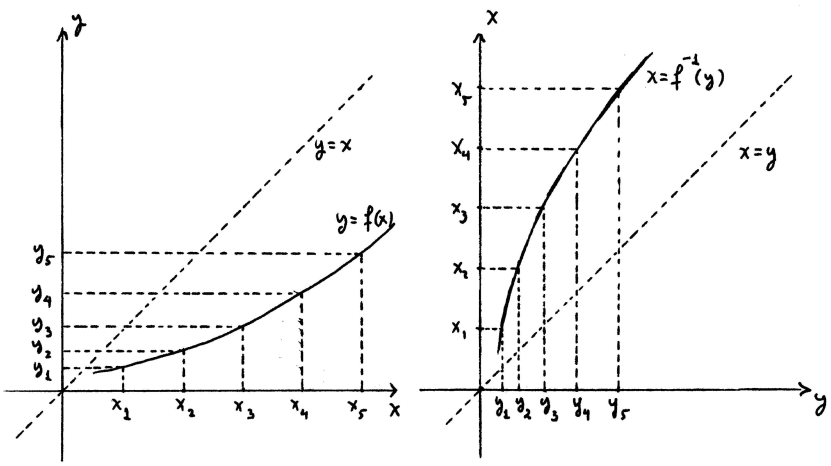
$f(x) = y$ x. y x, $y = f(x_1) = f(x_2)$ $x_1 = x_2$, -- . -- , , .

«$y = f(x)$ $x = f^{-1}(y)$» : (x, y) $y = f(x)$ (y, x) $x = f^{-1}(y)$. , (x, y) (y, x) , $y = x$, :

-

.

$x = f^{-1}(y)$ $y = f(x)$, . , $x = f^{-1}(y)$ y- x-. . , $x = f^{-1}(y)$ $y = f(x)$, x- y- .



Σχήμα 3.13: $x = f^{-1}(y)$.

x y, $y = f^{-1}(x)$, $x = f^{-1}(y)$ $y = f(x)$. , , :
x- y- .
:

,

, .

, $y = f(x)$ y_1, y_2 $x = f^{-1}(y)$ $y_1 < y_2$. $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$ $x = f^{-1}(y)$ $f(x_1) = y_1$, $f(x_2) = y_2$. $x_1 = x_2$, , , $y_1 = f(x_1) = f(x_2) = y_2$. $x_1 > x_2$, , $y = f(x)$, $y_1 = f(x_1) > f(x_2) = y_2$. , $x_1 < x_2$, , $f^{-1}(y_1) < f^{-1}(y_2)$.

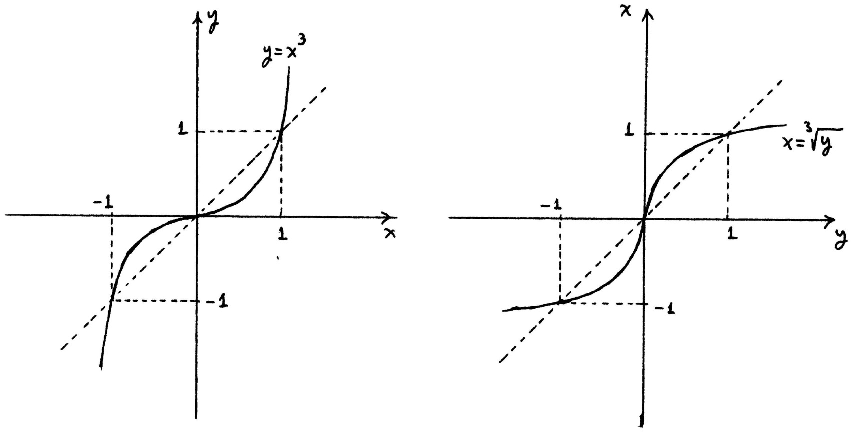
$y = f(x)$ $x = f^{-1}(y)$. $y = f(x)$, $x = f^{-1}(y)$, ,
.

.

,
.

, $y = f(x)$, $x = f^{-1}(y)$, , .

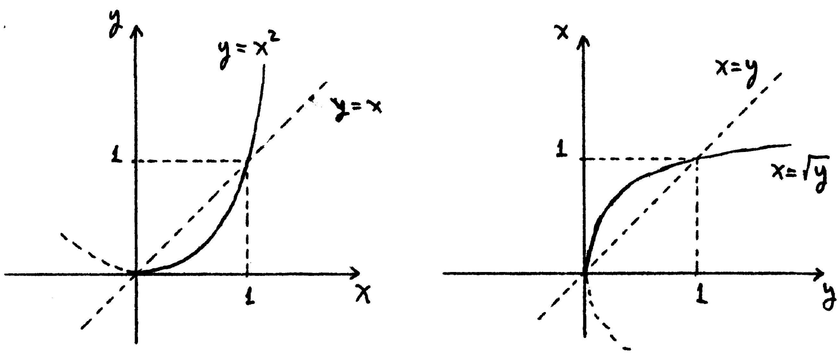
: $y = x^3$.
$y = x^3$, $(-\infty, +\infty)$, $(-\infty, +\infty)$. , $x = \sqrt[3]{y} = y^{\frac{1}{3}}$, $(-\infty, +\infty)$ $(-\infty, +\infty)$. . , .
, , $y = x^{\frac{1}{3}}$.



Σχήμα 3.14: $x = \sqrt[3]{y}$.

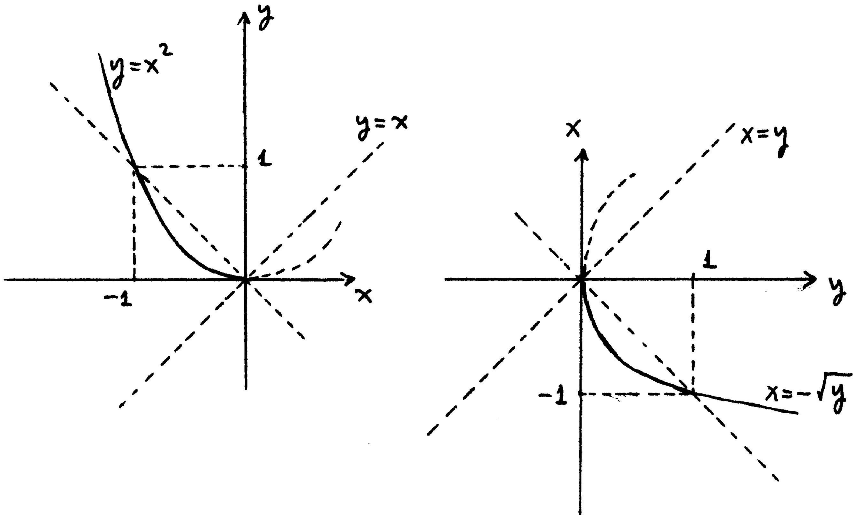
$y = f(x)$ --, . , I $y = f(x)$, x I, $y = f(x)$ --: x_1 , x_2 I $f(x_1) = f(x_2)$ $x_1 = x_2$. $y = f(x)$ I ( ) ( ) I. $y = f(x)$ -- , , . , , I .

: $y = x^2$ $(-\infty, +\infty)$.
$y = x^2$ -- $y > 0$ $x^2 = y$: $x = \sqrt{y}$ $x = -\sqrt{y}$.



Σχήμα 3.15: $x = \sqrt{y}$.

, $[0, +\infty)$ $y = x^2$ − , , -- − $[0, +\infty)$. $x = \sqrt{y}$, $[0, +\infty)$ $[0, +\infty)$. ( ) , .
, $(-\infty, 0]$ $y = x^2$ − , , -- − $[0, +\infty)$. $x = -\sqrt{y}$, $[0, +\infty)$ $(-\infty, 0]$. ( ) , .



Σχήμα 3.16: $x = -\sqrt{y}$.

3.3 $y = x^2$ $(-\infty, +\infty)$ $[0, +\infty)$ $x = \pm\sqrt{y}$ $[0, +\infty)$ $(-\infty, +\infty)$.

.

1. $y = \frac{1}{3x+1}$.

   , .

   , , .

2. a, b, c, d $c \neq 0$ $y = \frac{ax+b}{cx+d}$ 6 . , .

   $y = \frac{2x+3}{3x-1}$.

3. $y = x^2 + 4x + 1$.

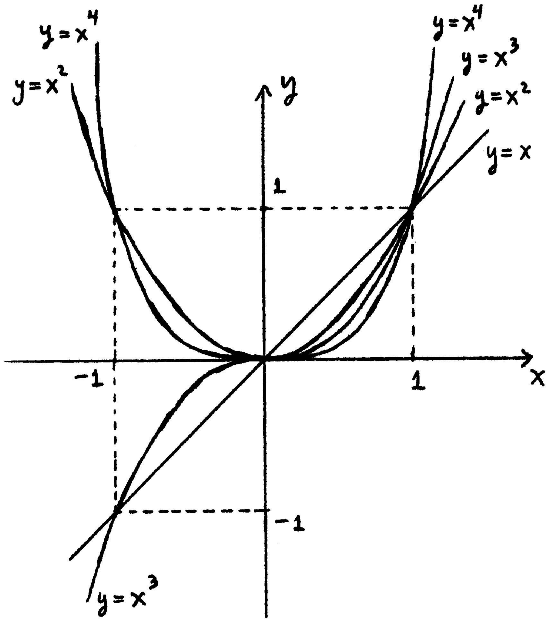
   , .

   ;

   , «» . , .

   $y = x^2 + 4x + 1$;

3.6 .



Σχήμα 3.17: $y = x^n$.

$$\boxed{y = a_0 + a_1 x + \cdots + a_N x^N\,.}$$

$a_N \neq 0$, N . , , $(-\infty, +\infty)$.

$$\boxed{y = x^n}$$

(,) n.

$y = x^3$, n , $y = x^n$, $(-\infty, +\infty)$. , $(0,0)$, $(-1,-1)$, $(0,0)$, $(1,1)$.

, $y = x^2$, n , $y = x^n$, $[0, +\infty)$ $[0, +\infty)$ $(-\infty, 0]$ $[0, +\infty)$. , y-, $(-1,1)$, $(0,0)$, $(1,1)$ $(0,0)$ $(0,0)$.

, $y = x^n$ $(0,0)$, $(1,1)$. $(0,1)$ $y = x$, $y = x^2$, $y = x^3$, ... $(1, +\infty)$. , , x $(0,1)$ x x- : $x > x^2 > x^3 > \cdots$. x $(1, +\infty)$ x : $x < x^2 < x^3 < \cdots$.

$$\boxed{y = \frac{a_0 + a_1 x + \cdots + a_N x^N}{b_0 + b_1 x + \cdots + b_M x^M}\,.}$$

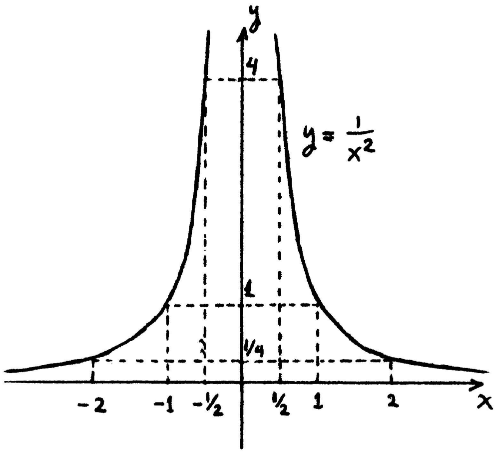
x , $\leq M$.

.

,

$$\boxed{y = \frac{1}{x^n} = x^{-n}}$$

(,) n. $(-\infty, 0) \cup (0, +\infty)$
n , $y = \frac{1}{x^n}$, $(-\infty, 0)$ $(-\infty, 0)$ $(0, +\infty)$ $(0, +\infty)$. $(-\infty, 0)$
$(-1, -1)$ x- y-. , $(0, +\infty)$ $(1, 1)$ y- x-. $(0, 0)$.



Σχήμα 3.18: $y = \frac{1}{x^2}$.

n , $y = \frac{1}{x^n}$, $(-\infty, 0)$ $(0, +\infty)$ $(0, +\infty)$ $(0, +\infty)$. $(-\infty, 0)$
$(-1, 1)$ x- y-. , $(0, +\infty)$ $(1, 1)$ y- x-. y-.

.

1. $y = \frac{\frac{1}{x+1} + \frac{1}{x-1}}{\frac{1}{x} + \frac{1}{x-2}}$ $y = \frac{x^2(x-2)}{(x-1)^2(x+1)}$; ; ;

2. $y = \frac{1}{x^n}$ n;

3. $y = \frac{1}{x^2}$, $y = \frac{1}{x^3}$ $y = \frac{1}{x^4}$, .

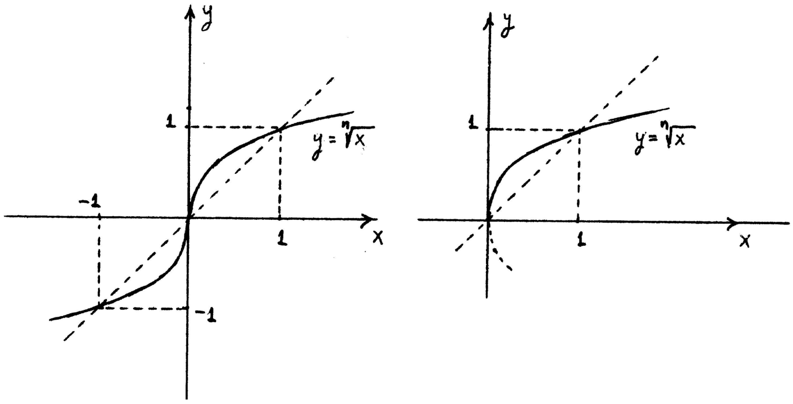
$$y = \frac{1}{(x-1)^2}\,, \quad y = \frac{1}{(2-3x)^3} + 4, \quad y = -\frac{3}{(2x+1)^4} + 2.$$

, , .

3.7 .

n. $y = x^n$ $(-\infty, +\infty)$ $(-\infty, +\infty)$, $x = \sqrt[n]{y}$.

$$\boxed{y = \sqrt[n]{x}}$$



Σχήμα 3.19: $y = \sqrt[n]{x}$. : n, n.

$(-\infty, +\infty)$ $(-\infty, +\infty)$. $y = \sqrt[n]{x}$ $(-1, -1)$, $(0, 0)$, $(1, 1)$ x- $(-\infty, +\infty)$ y- $(-\infty, +\infty)$. .
n. $y = x^n$ $[0, +\infty)$ $[0, +\infty)$, $x = \sqrt[n]{y}$

$$y = \sqrt[n]{x}$$

$[0, +\infty)$ $[0, +\infty)$. $y = \sqrt[n]{x}$ $(0, 0)$, $(1, 1)$ x- $[0, +\infty)$ y- $[0, +\infty)$. $(0, 0)$.
$y = \sqrt[n]{x}$. , , . :

$$y = \sqrt[4]{x} + \sqrt[3]{\frac{x^2 + 1 + \sqrt{x}}{x - 1}}\,.$$

, .

$$p_0(x) + p_1(x)y + \cdots + p_N(x)y^N = 0$$

y, $p_0(x), p_1(x), \ldots, p_N(x)$, $N \geq 1$ $p_N(x)$. , , $y = g(x)$, «» ,

$$p_0(x) + p_1(x)g(x) + \cdots + p_N(x)g(x)^N = 0$$

x $y = g(x)$. $y = g(x)$. ¨ $y = g(x)$: . 5. , , $y = g(x)$.

: (1) $y = p(x)$ $(-\infty, +\infty)$.
, $y = p(x)$ $-p(x) + 1y = 0$, $-p(x)$, 1 .
(2) $y = \frac{p(x)}{q(x)}$, $p(x), q(x)$, .
$y = \frac{p(x)}{q(x)}$ $-p(x) + q(x)y = 0$, $-p(x)$, $q(x)$.

.

: $y = \sqrt[n]{x}$ $(-\infty, +\infty)$, n , $[0, +\infty)$, n . .

, $y = \sqrt[n]{x}$ $-x + 1y^n = 0$, $-x, 0, \ldots, 0, 1$.

.

. , , , , , .

.

1. $y = \sqrt{x}$, $y = \sqrt[4]{x}$, $y = \sqrt[3]{x}$, $y = \sqrt[5]{x}$, .

$$y = \sqrt{x-1}, \quad y = -\sqrt[4]{2-3x} + 3, \quad y = 2 + \sqrt[3]{2x+1}, \quad y = \sqrt[5]{3-x}.$$

.

2.

$$y = \sqrt{\frac{x}{x-1}} + \sqrt{x+1}, \quad y = \sqrt{x} + \sqrt[3]{x}, \quad y = \frac{\sqrt{x}+2}{\sqrt{3x+1}-4}$$

, «» . ;

3. n $p(x)$, $q(x)$. $y = \sqrt[n]{\frac{p(x)}{q(x)}}$.

4. n ≥ 2, $y = \sqrt[n]{x}$.

(: $y = \sqrt[n]{x} = \frac{p(x)}{q(x)}$, , $p(x)$ $q(x)$.)

3.8 .

$$\boxed{y = x^a}$$

$[0, +\infty)$, $a > 0$, $(0, +\infty)$, $a < 0$. a. $y = x^a$ a . , , , a (, , a) $y = x^a$ x, $y = x^a$ $(-\infty, 0)$. , , a , , $y = x^a$.

$y = x^a$, $x^a = y$ x. $y < 0$, . $y = 0$, $x = 0$, $a > 0$, , $a < 0$. $y > 0$, $x = y^{\frac{1}{a}}$. , $a > 0$, $y = x^a$ $[0, +\infty)$, $a < 0$, $(0, +\infty)$.

$y = x^a$ $[0, +\infty)$, $a > 0$, $(0, +\infty)$, $a < 0$.

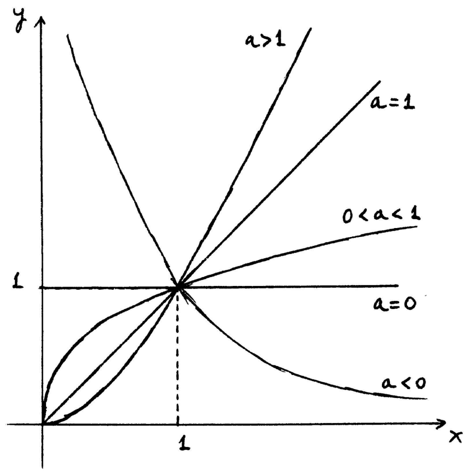
$a > 0$, $y = x^a$ $(0,0)$, $(1,1)$ x- $[0, +\infty)$ – – y- $[0, +\infty)$ – . $(0,0)$.

, $a < 0$, $y = x^a$ $(1,1)$ x- $(0, +\infty)$ y- $(0, +\infty)$. y- x-.

$y = x^a$ $y = x^b$ $a < b$, $(1,1)$, $(0,1)$ $y = x^a$ $y = x^b$ $(1, +\infty)$ $y = x^a$ $y = x^b$.

$y = x^a$ $x = y^{\frac{1}{a}}$. a $\frac{1}{a}$ > 0 < 0.

.



Σχήμα 3.20: $y = x^a$.

1. ;

$$y = x^0\,, \quad y = x^3\,, \quad y = x^{-3}\,, \quad y = x^{\frac{4}{6}}\,, \quad y = x^{-\frac{4}{6}}\,,$$

$$y = x^{\frac{6}{4}}\,, \quad y = x^{-\frac{6}{4}}\,, \quad y = x^{\sqrt{2}}\,, \quad y = x^{-\sqrt{2}}\,.$$

.

2. $y = x^{\sqrt{2}}$, $y = x^{-\sqrt{2}}$, .

$$y = (2x-3)^{\sqrt{2}}\,, \quad y = 2-(2-3x)^{\sqrt{2}}, \quad y = (1-x)^{-\sqrt{2}}, \quad y = 3+(2x+1)^{\sqrt{2}}.$$

.

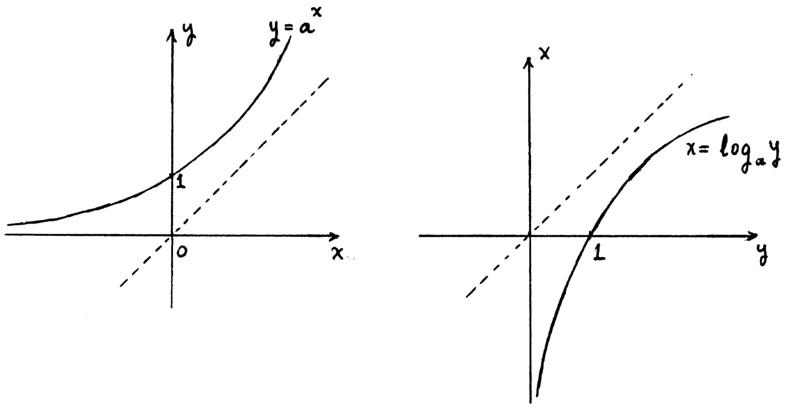
3.9 .

$a > 0$

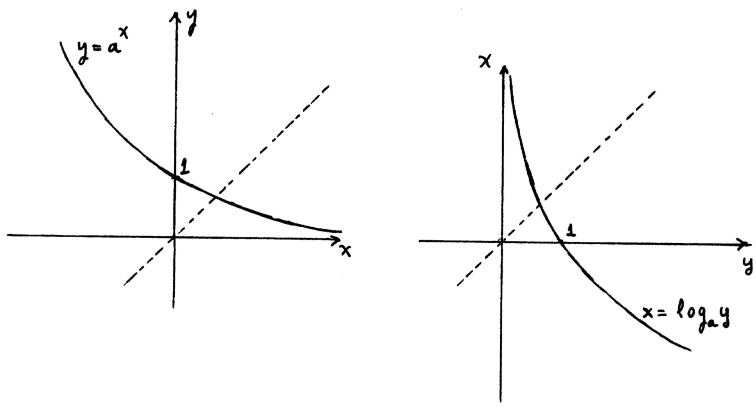
$$\boxed{y = a^x}$$

$(-\infty, +\infty)$ a.
$a = 1$, , $y = 1^x = 1$, $\{1\}$.
$a > 1$ $0 < a < 1$, $y = a^x$ $(0, +\infty)$. , $a^x = y$ x , $y \leq 0$, $x = \log_a y$, $y > 0$. , , $a > 1$, , $0 < a < 1$.
$y = a^x$ $(0, 1)$, $(1, a)$. $a > 1$, x- $(-\infty, +\infty)$ – – y- $(0, +\infty)$ – . x- . , $0 < a < 1$, x-.
$a = 1$, $y = a^x$, , , , .
$0 < a < 1$ $a > 1$, $y = a^x$, . $x^a = y$ x $x = \log_a y$. ,

$$\boxed{y = \log_a x\,.}$$



Σχήμα 3.21: $y = a^x$ $x = \log_a y$ $a > 1$.



Σχήμα 3.22: $y = a^x$ $x = \log_a y$ $0 < a < 1$.

$y = \log_a x$, $y = a^x$, a. $(0, +\infty)$ $(-\infty, +\infty)$.
$a > 1$, $y = \log_a x$, $0 < a < 1$, .
$y = \log_a x$ $(1, 0)$, $(a, 1)$. $a > 1$, y- , $0 < a < 1$, y- .

•

1. .

$$y = 3e^{-x} - 2\,, \quad y = 1 + 2^{3-x}\,, \quad y = e^{|x|}\,, \quad y = e^{-|x|}\,,$$

$$y=\log(-x),\quad y=\log|x|,\quad y=\log_{\frac{1}{2}}(2-x),\quad y=\log_{10}(2x-1).$$

2. . , , .

$$y=\log\frac{x-1}{x+1}\,,\quad y=\log\frac{1-x}{1+x}\,,\quad y=\log(1-x^2),\quad y=\log(x^2-1).$$

;

, .

; «» ;

3.10 .

. .

$y=f(x)$ $T>0$

$$f(x\pm T)=f(x)$$

x . , , ¨ , x $y=f(x)$, $x\pm T$. T $y=f(x)$.

: (1) $y=\cos x$ $y=\sin x$ 2π, $\cos(x\pm 2\pi)=\cos x$ $\sin(x\pm 2\pi)=\sin x$.

(2) $y=\tan x$ $y=\cot x$ π, $\tan(x\pm\pi)=\tan x$ $\cot(x\pm\pi)=\cot x$.

$y=f(x)$ T. $f(x-T)=f(x)$ $y=f(x-T)$ $y=f(x)$, , . , , $y=f(x+T)$ $y=f(x)$. $\pm T$ f f. . $y=f(x)$ T. a $y=f(x)$ $[a,a+T]$. k $y=f(x)$ $[a+kT,a+(k+1)T]$ kT $[a,a+T]$. ,

$y=f(x)$ T.

$[a,a+T]$ T.

.

1.

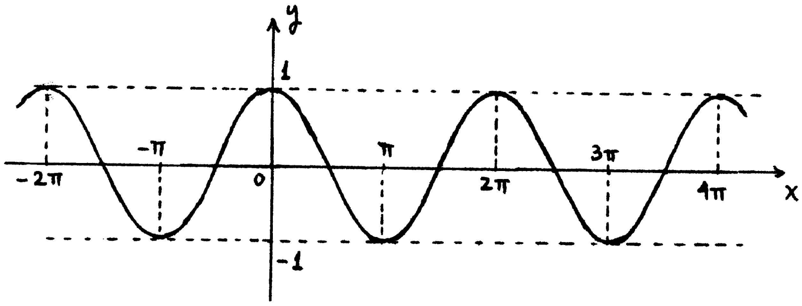
$$\boxed{y=\cos x.}$$

$(-\infty,+\infty)$ $[-1,1]$. 2π . $[-\pi,0]$ $[0,\pi]$. $[-1,1]$. $[-\pi,\pi]$: $(-\pi,-1)$ $(0,1)$ $(0,1)$ $(\pi,-1)$ $(-\frac{\pi}{2},0)$, $(\frac{\pi}{2},0)$.

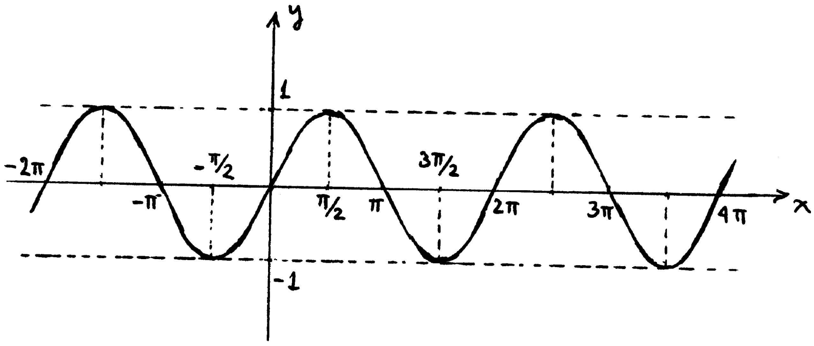
2.

$$\boxed{y=\sin x.}$$

$(-\infty,+\infty)$ $[-1,1]$. 2π . $[-\frac{\pi}{2},\frac{\pi}{2}]$ $[\frac{\pi}{2},\frac{3\pi}{2}]$. $[-1,1]$. $[-\frac{\pi}{2},\frac{3\pi}{2}]$: $(-\frac{\pi}{2},-1)$ $(\frac{\pi}{2},1)$ $(\frac{\pi}{2},1)$ $(\frac{3\pi}{2},-1)$ $(0,0)$, $(\pi,0)$.



Σχήμα 3.23: $y = \cos x$.



Σχήμα 3.24: $y = \sin x$.

3.

$$\boxed{y = \tan x.}$$

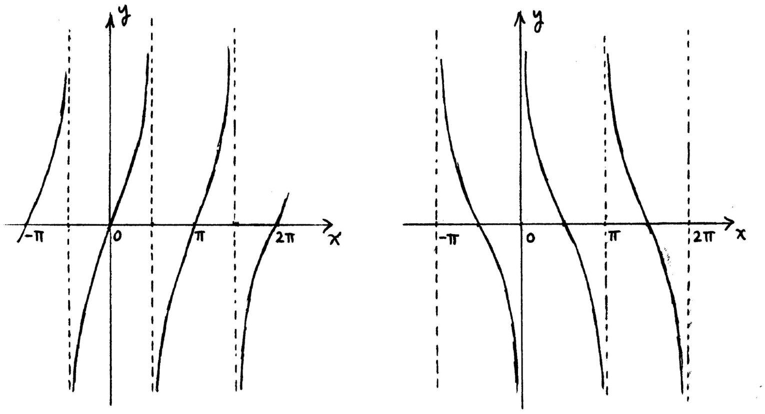
$(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$ $(k \in \mathbf{Z})$. $(-\infty, +\infty)$. π. $(-\frac{\pi}{2}, \frac{\pi}{2})$ $(-\infty, +\infty)$. $(-\frac{\pi}{2}, \frac{\pi}{2})$: $(0, 0)$ $x = -\frac{\pi}{2}$ $x = \frac{\pi}{2}$.

4.

$$\boxed{y = \cot x.}$$

$(k\pi, (k+1)\pi)$ $(k \in \mathbf{Z})$. $(-\infty, +\infty)$. π. $(0, \pi)$ $(-\infty, +\infty)$. $(0, \pi)$: $(\frac{\pi}{2}, 0)$ $x = 0$ $x = \pi$.

$y = \cos x$ $y = \sin x$ $(-\infty, +\infty)$ 1 -1.



Σχήμα 3.25: $y = \tan x$ $y = \cot x$.

. .

, . .
$y = \cos x$ $[0, \pi]$ $[-1, 1]$. , $x = \arccos y$, $[-1, 1]$ $[0, \pi]$.
$y = \sin x$ $[-\frac{\pi}{2}, \frac{\pi}{2}]$ $[-1, 1]$. $x = \arcsin y$, $[-1, 1]$ $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
$y = \tan x$ $(-\frac{\pi}{2}, \frac{\pi}{2})$ $(-\infty, +\infty)$. $x = \arctan y$, $(-\infty, +\infty)$ $(-\frac{\pi}{2}, \frac{\pi}{2})$.
$y = \cot x$ $(0, \pi)$ $(-\infty, +\infty)$. $x = \operatorname{arccot} y$, $(-\infty, +\infty)$ $(0, \pi)$.
, , , , x y, :

1. -

$$\boxed{y = \arccos x.}$$

$[-1, 1]$ $[0, \pi]$. $[-1, 1]$ $(-1, \pi)$ $(1, 0)$ $(0, \frac{\pi}{2})$.

2. -

$$\boxed{y = \arcsin x.}$$

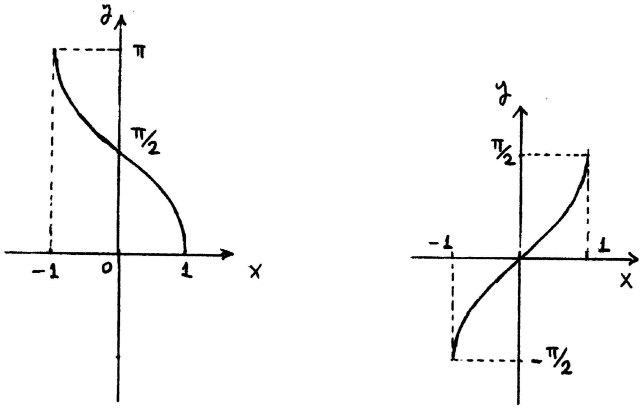
$[-1, 1]$ $[-\frac{\pi}{2}, \frac{\pi}{2}]$. $[-1, 1]$ $(-1, -\frac{\pi}{2})$ $(1, \frac{\pi}{2})$ $(0, 0)$.

3. -

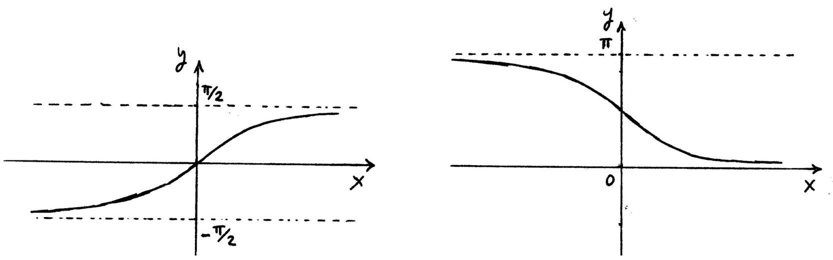
$$\boxed{y = \arctan x.}$$

$(-\infty, +\infty)$ $(-\frac{\pi}{2}, \frac{\pi}{2})$. $(-\infty, +\infty)$ $y = -\frac{\pi}{2}$ $y = \frac{\pi}{2}$ $(0, 0)$.

4. -



Σχήμα 3.26: $y = \arccos x$ $y = \arcsin x$.



Σχήμα 3.27: $y = \arctan x$ $y = \operatorname{arccot} x$.

$$\boxed{y = \operatorname{arccot} x.}$$

$(-\infty, +\infty)$ $(0, \pi)$. $(-\infty, +\infty)$ $y = \pi$ $y = 0$
$(0, \frac{\pi}{2})$.

•

1. .

$$y = \cos(2x), \quad y = \tan\left(\frac{x}{2} - 1\right), \quad y = 1 + 2\sin(1 - 3x), \quad y = \cot(1 - x),$$

$$x = 2\arccos(2y + 1), \quad x = \frac{\pi}{2} + \arctan(1 - y), \quad x = \arctan\left(\frac{y + 1}{2}\right).$$

2. 5 1.4

$$y = a\cos x + b\sin x.$$

$$y = \cos x + \sin x, \quad y = \sqrt{3}\cos x + \sin x, \quad y = \sqrt{3}\cos x - \sin x.$$

3. .

$$y = \sqrt{\sin x}\,, \quad y = \frac{1}{1+\sin x}\,, \quad y = \log(\sin x), \quad y = \arcsin\frac{x}{x-1}\,.$$

4. $y = \arcsin x$ $y = \arctan x$; ;
(: $x = \sin y$ $x = \tan y$.)

5.

$$y = \arccos(\cos x),\ y = \arcsin(\sin x),\ y = \arctan(\tan x),\ y = \operatorname{arccot}(\cot x).$$

6. $y = x\sin x$ $[0, +\infty)$. , $-x \le x\sin x \le x$, $y = x\sin x$ $y = -x$ $y = x$ $\sin x = 1$, $x = \frac{\pi}{2} + k2\pi\,(k = 0, 1, 2, \dots)$, $y = x\sin x$ «» $y = x$ $\sin x = -1$, $x = \frac{3\pi}{2} + k2\pi\,(k = 0, 1, 2, \dots)$, $y = x\sin x$ «» $y = -x$. $y = x\sin x$ x-;

$y = x\sin x$ $[0, +\infty)$, $y = x\sin x$, $(-\infty, 0]$.

7. $y = \sin\frac{1}{x}$ $(0, +\infty)$. $y = \sin\frac{1}{x}$ $y = -1$ $y = 1$.

$\sin\frac{1}{x} = 1$ $\sin\frac{1}{x} = -1$ $(0, +\infty)$. $(0, +\infty)$ «» 0 $y = \sin\frac{1}{x}$. $\sin\frac{1}{x} = 1$ $y = \sin\frac{1}{x}$ «» $y = 1$ $\sin\frac{1}{x} = -1$ «» $y = -1$. $y = \sin\frac{1}{x}$ x-;

$y = \sin\frac{1}{x}$ $(0, +\infty)$, $y = \sin\frac{1}{x}$, $(-\infty, 0)$.

8. , .

$$y = x^2\sin x, \qquad y = \sqrt{x}\sin x, \qquad y = \frac{1}{x}\sin\frac{1}{x}\,.$$

3.11 .

. .

x

$$\cosh x = \frac{e^x + e^{-x}}{2}\,, \qquad \sinh x = \frac{e^x - e^{-x}}{2}\,,$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}\,, \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}\,.$$

, , x, .

3.1 (1) $(\cosh x)^2 - (\sinh x)^2 = 1$.
(2) $\tanh x = \frac{\sinh x}{\cosh x}$, $\coth x = \frac{\cosh x}{\sinh x}$.
(3) $\cosh(-x) = \cosh x$, $\sinh(-x) = -\sinh x$, $\tanh(-x) = -\tanh x$, $\coth(-x) = -\coth x$.

(4) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$, $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.
(5) $\cosh x - \cosh y = 2\sinh\frac{x-y}{2}\sinh\frac{x+y}{2}$, $\sinh x - \sinh y = 2\sinh\frac{x-y}{2}\cosh\frac{x+y}{2}$.
(6) (i) $1 \leq \cosh x < \cosh x'$, $0 \leq x < x'$ (ii) $1 \leq \cosh x < \cosh x'$, $x' < x \leq 0$.
(7) $\sinh x < \sinh x'$, $x < x'$.

3.1 .
3.1 1.12. $\cosh x = \frac{e^x+e^{-x}}{2}$ $\sinh x = \frac{e^x-e^{-x}}{2}$.

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

$e^{ix} = \cos x + i \sin x$ $e^{-ix} = \cos x - i \sin x$. $'$.

$$\boxed{y = \cosh x = \frac{e^x+e^{-x}}{2}}$$

$(-\infty, +\infty)$. . 3.1 $y = \cosh x$ $[0, +\infty)$ $(-\infty, 0]$.
$y = \cosh x$ $\frac{e^x+e^{-x}}{2} = y$ x $e^{2x} - 2ye^x + 1 = 0$. $t = e^x$, $t^2 - 2yt + 1 = 0$ $= 4y^2 - 4$. .
(i) $y^2 < 1$. $t^2 - 2yt + 1 = 0$.
(ii) $y^2 = 1$. $y = -1$, $t^2 - 2yt + 1 = 0$ $e^x = t = -1$, , . $y = 1$, $t^2 - 2yt + 1 = 0$ $e^x = t = 1$ $x = 0$ $\cosh x = 1$.
(iii) $y^2 > 1$. $t^2 - 2yt + 1 = 0$ () $2y$ 1. $y < -1$, , . , $y > 1$, $t^2 - 2yt + 1 = 0$, . 1, 1 0 1. $t = y \pm \sqrt{y^2-1}$ $0 < y - \sqrt{y^2-1} < 1 < y + \sqrt{y^2-1}$. , $x = \log(y \pm \sqrt{y^2-1})$, . , $\log(y - \sqrt{y^2-1}) < 0 < \log(y + \sqrt{y^2-1})$.
, , $y = \cosh x$, y $\cosh x = y$, $[1, +\infty)$. , y $[1, +\infty)$ $\cosh x = y$ $(-\infty, 0]$ $[0, +\infty)$. $[1, +\infty)$ $(-\infty, 0]$ $[0, +\infty)$.
, $y = \cosh x$, $(0, 1)$ $(0, 1)$. y-.

$$\boxed{y = \sinh x = \frac{e^x-e^{-x}}{2}}$$

$(-\infty, +\infty)$. . $y = \sinh x$ $(-\infty, +\infty)$.
$y = \sinh x$ $\frac{e^x-e^{-x}}{2} = y$ x $e^{2x} - 2ye^x - 1 = 0$. $t = e^x$ $t^2 - 2yt - 1 = 0$ $= 4y^2 + 4$. , $t^2 - 2yt - 1 = 0$ () $2y$ -1, . $t = y \pm \sqrt{y^2+1}$ $y - \sqrt{y^2+1} < 0 < y + \sqrt{y^2+1}$. , , $x = \log(y + \sqrt{y^2+1})$ $\sinh x = y$.
, , $y = \sinh x$, y $\sinh x = y$, $(-\infty, +\infty)$.
$y = \sinh x$. $(0, 0)$ $(0, 0)$.

. .

$y = \cosh x$ $y = \sinh x$.

$y = \cosh x$ -- $(-\infty, +\infty)$· $y > 1$ () $\cosh x = y$. , $[0, +\infty)$ $[1, +\infty)$. $[1, +\infty)$ $[0, +\infty)$. y $[1, +\infty)$ $\cosh x = y$ $[0, +\infty)$. , $x = \log(y + \sqrt{y^2 - 1})$. , x y,

$$y = \log(x + \sqrt{x^2 - 1}).$$

′ $y = \cos x$, $y = \arccos x$ -, $\log(x + \sqrt{x^2 - 1})$ arccosh x - x,

$$\boxed{y = \operatorname{arc}\cosh x.}$$

$y = \operatorname{arc}\cosh x$ x- $[1, +\infty)$ y- $[0, +\infty)$. $(1, 0)$. , $y = \cosh x$ $(-\infty, 0]$ $[1, +\infty)$, $[1, +\infty)$ $(-\infty, 0]$ $y = \log(x - \sqrt{x^2 - 1})$. , $\log(x - \sqrt{x^2 - 1}) = -\log(x + \sqrt{x^2 - 1}) = -\operatorname{arc}\cosh x$, $y = -\operatorname{arc}\cosh x$.

$y = \sinh x$ $(-\infty, +\infty)$ $(-\infty, +\infty)$. $(-\infty, +\infty)$ $(-\infty, +\infty)$, ,

$$y = \log(x + \sqrt{x^2 + 1}).$$

′ , $\log(x + \sqrt{x^2 + 1})$ arcsinh x - x,

$$\boxed{y = \operatorname{arc}\sinh x.}$$

$y = \operatorname{arc}\sinh x$ $(0, 0)$.

.

1.

$$y = \tanh x, \qquad y = \coth x.$$

, .

, .

2.

$$1 - (\tanh x)^2 = \frac{1}{(\cosh x)^2}, \qquad (\coth x)^2 - 1 = \frac{1}{(\sinh x)^2}.$$

3.

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}, \qquad \coth(x + y) = \frac{\coth x \coth y + 1}{\coth x + \coth y}.$$

4.

$$\cosh(2x) = (\cosh x)^2 + (\sinh x)^2 = 2(\cosh x)^2 - 1 = 1 + 2(\sinh x)^2,$$

$$\sinh(2x) = 2 \sinh x \cosh x,$$

$$\tanh(2x) = \frac{2 \tanh x}{1 + (\tanh x)^2}, \qquad \coth(2x) = \frac{(\coth x)^2 + 1}{2 \coth x}.$$

5.

$$\cosh x = \frac{1 + (\tanh \frac{x}{2})^2}{1 - (\tanh \frac{x}{2})^2}\,, \qquad \sinh x = \frac{2 \tanh \frac{x}{2}}{1 - (\tanh \frac{x}{2})^2}\,,$$

$$\tanh x = \frac{2 \tanh \frac{x}{2}}{1 + (\tanh \frac{x}{2})^2}\,, \qquad \coth x = \frac{1 + (\tanh \frac{x}{2})^2}{2 \tanh \frac{x}{2}}\,.$$

Κεφάλαιο 4

.

: « ϵ δ» , ... , , . .

4.1 , .

1. $y = f(x)$ $(a,\xi)\cup(\xi,b)$ ξ. $y = f(x)$, , ξ.

1_α : = .

: (1)

$$y = \frac{3x^2 - x - 2}{x - 1}$$

$(-\infty,1)\cup(1,+\infty)$. $x \neq 1$

$$y = 3x + 2.$$

: $y = \frac{3x^2-x-2}{x-1}$ $y = 3x+2$. $(-\infty,1)\cup(1,+\infty)$ $(-\infty,+\infty)$. , $(-\infty,1)\cup(1,+\infty)$.

, x « » 1 , , 1, $y = \frac{3x^2-x-2}{x-1} = 3x+2$ « » 5. . , $x = 1,00023$, $y = 5,00069$, $x = 1,000000035$, $y = 5,000000105$, $x = 0,9999999999913$, $y = 4,9999999999739$.

(2) , $y = 3x+2$.

, , , x « » 1 1, $y = 3x+2$ « » 5. ′ . , , . $y = \frac{3x^2-x-2}{x-1}$, x « » 1 1. $y = 3x+2$, x « » 1 1. , , x « » 1 1.

$y = f(x)$, , ξ, $(a,\xi)\cup(\xi,b)$:

« » ξ,
$f(x)$ «
» η.

, , . ′ .

$f(x)$ « » η, $|f(x)-\eta|$ « ». « x ξ $\neq \xi$», « $|x-\xi|$ ».

:

$|f(x)-\eta|$ $\epsilon > 0$ x $|x-\xi|$ -
$\delta > 0$ $\neq 0$.

, ,

$\epsilon > 0$ $\delta > 0$,
$0 < |x-\xi| < \delta$ x -
, $|f(x)-\eta| < \epsilon$.

: (1) $y = \frac{3x^2-x-2}{x-1}$. ( ) $\epsilon > 0$ $\delta > 0$, $0 < |x-1| < \delta$ x , $|\frac{3x^2-x-2}{x-1} - 5| < \epsilon$. ′ , $0 < |x-1| < \delta$, x , 1 x x. $\delta > 0$, $0 < |x-1| < \delta$, $|\frac{3x^2-x-2}{x-1} - 5| < \epsilon$. , ¨ $x \neq 1$, $|\frac{3x^2-x-2}{x-1} - 5| < \epsilon$ $|(3x+2)-5| < \epsilon$ $3|x-1| < \epsilon$ $|x-1| < \frac{\epsilon}{3}$. , $\delta = \frac{\epsilon}{3}$, $0 < |x-1| < \delta$ $|x-1| < \frac{\epsilon}{3}$ $|\frac{3x^2-x-2}{x-1} - 5| < \epsilon$.

(2) $y = x^2+3$. $(-\infty, +\infty)$, , 0. , x 0 0, $y = x^2+3$ 3. , $x = 0,004$, $y = 3,000016$, $x = -0,000005$, $y = 3,000000000025$. , $\epsilon > 0$ $\delta > 0$, $0 < |x-0| < \delta$ x , $|(x^2+3)-3| < \epsilon$. , , $0 < |x| < \delta$, $x^2 < \epsilon$. , $x^2 < \epsilon$ $|x| < \sqrt{\epsilon}$. , $\delta = \sqrt{\epsilon}$, $0 < |x| < \delta$ $|x| < \sqrt{\epsilon}$ $|(x^2+3)-3| < \epsilon$.

, .

$\epsilon > 0$ $\delta > 0$, $0 < |x-\xi| < \delta$ x , $|f(x)-\eta| < \epsilon$. : $\epsilon > 0$ $\delta > 0$ $0 < |x-\xi| < \delta$ x $|f(x)-\eta| < \epsilon$. : $\epsilon > 0$ $\delta > 0$ $|f(x)-\eta| < \epsilon$ x $0 < |x-\xi| < \delta$.

$$\lim_{x\to\xi} f(x) = \eta$$

$y = f(x)$ η η η x ξ.

$\lim_{x\to\xi} f(x) = \eta$ x ξ $\neq \xi$. ξ.

$\lim_{n\to+\infty} x_n = x$. $|x_n - x| < \epsilon$ ϵ «» $|x_n - x|$ $n \geq n_0$ n_0 «» n $|x_n - x| < \epsilon$. $\lim_{x\to\xi} f(x) = \eta$, $|f(x)-\eta| < \epsilon$ ϵ «» $|f(x)-\eta|$ $0 < |x-\xi| < \delta$ δ «» $|x-\xi|$ $|f(x)-\eta| < \epsilon$. .

$\lim_{x\to\xi} f(x) = \eta$, , ( ). $\epsilon > 0$ ( ) «» , $|f(x)-\eta| < \epsilon$ $0 < |x-\xi| < \delta$, ′ x . , : « $_1$ $_2$» , , « $_1 \Longleftarrow$ $_2$». x $y = f(x)$ $0 < |x-\xi| < \delta$, , $|f(x)-\eta| < \epsilon$.

, $|f(x)-\eta| < \epsilon$ $0 < |x-\xi| < \delta$, $x > a$ (a). $\xi > a$. , δ ξ a, $\delta = \xi - a > 0$, $x > a$ $0 < |x-\xi| < \delta$, , x $0 < |x-\xi| < \delta$ $x > a$, , , $|f(x)-\eta| < \epsilon$.

, , $|f(x)-\eta| < \epsilon$, $x < b$ (b). $\xi < b$, δ ξ b, $\delta = b - \xi > 0$, $x < b$ $0 < |x-\xi| < \delta$, , x $0 < |x-\xi| < \delta$ $x < b$, , , $|f(x)-\eta| < \epsilon$.

, $|f(x)-\eta| < \epsilon$ $a < x < b$ (a, b). $a < \xi < b$, δ ξ a b, $\delta = \min\{\xi - a, b - \xi\}$, $a < x < b$ $0 < |x-\xi| < \delta$, , x $0 < |x-\xi| < \delta$

$a < x < b$, , , $|f(x) - \eta| < \epsilon$.

: (1) $y = c$ ξ

$$\boxed{\lim_{x\to\xi} c = c.}$$

$\epsilon > 0$ x c c $|c - c| = 0$. , , $\delta > 0$ (, $\delta = 1$) x (x) $0 < |x - \xi| < \delta$ $|c - c| = 0 < \epsilon$.

(2) $a > 0$. $y = |x - \xi|^a$ $(-\infty, +\infty)$, ξ. (, $a = 1$ $a = 2$ $a = \frac{1}{2}$) , x ξ $\neq \xi$, $y = |x - \xi|^a$ 0. , , $\lim_{x\to\xi} |x - \xi|^a = 0$.

$\epsilon > 0$ $\delta > 0$ x (x) $0 < |x - \xi| < \delta$ $\big||x - \xi|^a - 0\big| < \epsilon$.

$\big||x - \xi|^a - 0\big| < \epsilon$ $|x - \xi|^a < \epsilon$ $|x - \xi| < \epsilon^{\frac{1}{a}}$ $0 < |x - \xi| < \epsilon^{\frac{1}{a}}$. , $\delta = \epsilon^{\frac{1}{a}}$, x $0 < |x - \xi| < \delta$ $0 < |x - \xi| < \epsilon^{\frac{1}{a}}$, , $\big||x - \xi|^a - 0\big| < \epsilon$. :

$$\boxed{\lim_{x\to\xi} |x - \xi|^a = 0 \qquad (a > 0).}$$

: $\lim_{x\to\xi} |x - \xi| = 0$, $\lim_{x\to\xi} (x - \xi)^2 = 0$, $\lim_{x\to\xi} \sqrt{|x - \xi|} = 0$.

(3) $y = x^2 + 3$ $(-\infty, +\infty)$, 1. x 1 $y = x^2 + 3$ 4. $\lim_{x\to 1}(x^2 + 3) = 4$.

$\epsilon > 0$ $\delta > 0$ x (x) $0 < |x - 1| < \delta$ $|(x^2 + 3) - 4| < \epsilon$.

$|(x^2 + 3) - 4| < \epsilon$ $|x^2 - 1| < \epsilon$ $1 - \epsilon < x^2 < 1 + \epsilon$. , $1 - \epsilon$.

$\epsilon > 1$ $1 - \epsilon < x^2 < 1 + \epsilon$ ($x^2 \geq 0$) $x^2 < 1 + \epsilon$ $|x| < \sqrt{1 + \epsilon}$ $-\sqrt{1 + \epsilon} < x < \sqrt{1 + \epsilon}$. 1 $-\sqrt{1 + \epsilon}$ $\sqrt{1 + \epsilon}$, , δ 1 , $\delta = \min\{1 + \sqrt{1 + \epsilon}, \sqrt{1 + \epsilon} - 1\} = \sqrt{1 + \epsilon} - 1$, x $0 < |x - 1| < \delta$ $-\sqrt{1 + \epsilon} < x < \sqrt{1 + \epsilon}$, ( ) $1 - \epsilon < x^2 < 1 + \epsilon$.

$0 < \epsilon \leq 1$ $1 - \epsilon < x^2 < 1 + \epsilon$ $\sqrt{1 - \epsilon} < |x| < \sqrt{1 + \epsilon}$ $-\sqrt{1 + \epsilon} < x < -\sqrt{1 - \epsilon}$ $\sqrt{1 - \epsilon} < x < \sqrt{1 + \epsilon}$ (: x 1, 1) $\sqrt{1 - \epsilon} < x < \sqrt{1 + \epsilon}$. 1 $\sqrt{1 - \epsilon}$ $\sqrt{1 + \epsilon}$, , $\delta = \min\{1 - \sqrt{1 - \epsilon}, \sqrt{1 + \epsilon} - 1\} = \sqrt{1 + \epsilon} - 1$, x $0 < |x - 1| < \delta$ $\sqrt{1 - \epsilon} < x < \sqrt{1 + \epsilon}$, ( ) $1 - \epsilon < x^2 < 1 + \epsilon$.

, , $\delta = \sqrt{1 + \epsilon} - 1 > 0$ x (x) $0 < |x - 1| < \delta$ $1 - \epsilon < x^2 < 1 + \epsilon$, ( ) $|(x^2 + 3) - 4| < \epsilon$.

$\lim_{x\to 1}(x^2 + 3) = 4$.

(4) $y = \frac{1}{x+1}$ $(-\infty, -1) \cup (-1, +\infty)$, 1 · , $(-1, 1) \cup (1, +\infty)$. x 1 $y = \frac{1}{x+1}$ $\frac{1}{2}$. , $\lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$.

$\epsilon > 0$ $\delta > 0$ x ($x \neq -1$) $0 < |x - 1| < \delta$ $\left|\frac{1}{x+1} - \frac{1}{2}\right| < \epsilon$.

$\left|\frac{1}{x+1} - \frac{1}{2}\right| < \epsilon$ $\frac{1}{2} - \epsilon < \frac{1}{x+1} < \frac{1}{2} + \epsilon$. , $\frac{1}{2} - \epsilon$, ϵ.

$0 < \epsilon < \frac{1}{2}$, $\frac{1}{2} - \epsilon < \frac{1}{x+1} < \frac{1}{2} + \epsilon$ $\frac{2}{1+2\epsilon} < x + 1 < \frac{2}{1-2\epsilon}$ $\frac{1-2\epsilon}{1+2\epsilon} < x < \frac{1+2\epsilon}{1-2\epsilon}$. 1 $\frac{1-2\epsilon}{1+2\epsilon}$ $\frac{1+2\epsilon}{1-2\epsilon}$, , $\delta = \min\left\{1 - \frac{1-2\epsilon}{1+2\epsilon}, \frac{1+2\epsilon}{1-2\epsilon} - 1\right\} = \min\left\{\frac{4\epsilon}{1+2\epsilon}, \frac{4\epsilon}{1-2\epsilon}\right\} = \frac{4\epsilon}{1+2\epsilon}$, x $0 < |x - 1| < \delta$ $\frac{1-2\epsilon}{1+2\epsilon} < x < \frac{1+2\epsilon}{1-2\epsilon}$, , $\frac{1}{2} - \epsilon < \frac{1}{x+1} < \frac{1}{2} + \epsilon$.

$\epsilon = \frac{1}{2}$, $\frac{1}{2} - \epsilon < \frac{1}{x+1} < \frac{1}{2} + \epsilon$ $0 < \frac{1}{x+1} < \frac{1}{2} + \epsilon$ $\frac{2}{1+2\epsilon} < x + 1$ $x > \frac{1-2\epsilon}{1+2\epsilon}$. $1 > \frac{1-2\epsilon}{1+2\epsilon}$, $\delta = 1 - \frac{1-2\epsilon}{1+2\epsilon} = \frac{4\epsilon}{1+2\epsilon}$, x $0 < |x - 1| < \delta$ $x > \frac{1-2\epsilon}{1+2\epsilon}$, $\frac{1}{2} - \epsilon < \frac{1}{x+1} < \frac{1}{2} + \epsilon$.

, $\epsilon > \frac{1}{2}$, $\frac{1}{2} - \epsilon < \frac{1}{x+1} < \frac{1}{2} + \epsilon$ $x+1 < \frac{2}{1-2\epsilon}$ $x+1 > \frac{2}{1+2\epsilon}$ $x < \frac{1+2\epsilon}{1-2\epsilon}$ $x > \frac{1-2\epsilon}{1+2\epsilon}$ (: x 1, 1) $x > \frac{1-2\epsilon}{1+2\epsilon}$. 1 $> \frac{1-2\epsilon}{1+2\epsilon}$, , $\delta = 1 - \frac{1-2\epsilon}{1+2\epsilon} = \frac{4\epsilon}{1+2\epsilon}$, x $0 < |x-1| < \delta$ $x > \frac{1-2\epsilon}{1+2\epsilon}$, , $\frac{1}{2} - \epsilon < \frac{1}{x+1} < \frac{1}{2} + \epsilon$.

, $\delta = \frac{4\epsilon}{1+2\epsilon} > 0$ $x \neq -1$ $0 < |x-1| < \delta$ $\frac{1}{2} - \epsilon < \frac{1}{x+1} < \frac{1}{2} + \epsilon$, ( ) $\left|\frac{1}{x+1} - \frac{1}{2}\right| < \epsilon$.

$\lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$.

1_β : $= +\infty$.

: $y = \frac{1}{(x-1)^2}$ $(-\infty, 1) \cup (1, +\infty)$. , x 1 $\neq 1$, $y = \frac{1}{(x-1)^2}$. : $x = 1,0003$, $y = 11111111,11\ldots$, $x = 0,9999997$, $y = 11111111111111,11\ldots$.

$y = f(x)$, , ξ, $(a, \xi) \cup (\xi, b)$:

« » ξ**,**
$f(x)$ **«**
».

$f(x)$ « », $f(x)$ « ». « x ξ $\neq \xi$», « $|x - \xi|$ ». :

$f(x)$ $M > 0$ x
$|x - \xi|$ -
$\delta > 0$ $\neq 0$**.**

, ,

$M > 0$ $\delta > 0$ **,**
$0 < |x - \xi| < \delta$ x -
, $f(x) > M$**.**

: $y = \frac{1}{(x-1)^2}$. ( ) $M > 0$ $\delta > 0$, $0 < |x-1| < \delta$ x , $\frac{1}{(x-1)^2} > M$. ′ , $0 < |x-1| < \delta$, x , 1 x x. $\delta > 0$, $0 < |x-1| < \delta$, $\frac{1}{(x-1)^2} > M$. $\frac{1}{(x-1)^2} > M$ $(x-1)^2 < \frac{1}{M}$ $|x-1| < \frac{1}{\sqrt{M}}$. , $\delta = \frac{1}{\sqrt{M}}$, $0 < |x-1| < \delta$ $|x-1| < \frac{1}{\sqrt{M}}$ $\frac{1}{(x-1)^2} > M$.

, .

$M > 0$ $\delta > 0$, $0 < |x - \xi| < \delta$ x , $f(x) > M$. : $M > 0$ $\delta > 0$ $0 < |x - \xi| < \delta$ x $f(x) > M$. : $M > 0$ $\delta > 0$ $f(x) > M$ x $0 < |x - \xi| < \delta$.

$$\lim_{x\to\xi} f(x) = +\infty$$

$y = f(x)$ $+\infty$ $+\infty$ $+\infty$ x ξ.

: (1) $a > 0$. $y = |x - \xi|^{-a} = \frac{1}{|x-\xi|^a}$ $(-\infty, \xi) \cup (\xi, +\infty)$, ξ. , $a = 1$, $a = 2$ ( ) $a = \frac{1}{2}$, , x ξ $\neq \xi$, $y = |x - \xi|^{-a}$. , $\lim_{x\to\xi} |x - \xi|^{-a} = +\infty$. , .

$M > 0$ $\delta > 0$ $|x-\xi|^{-a} > M$ x ($x \neq \xi$) $0 < |x-\xi| < \delta$.
$|x-\xi|^{-a} > M$ $0 < |x-\xi| < M^{-\frac{1}{a}}$, , $\delta = M^{-\frac{1}{a}}$, x $0 < |x-\xi| < \delta$ $0 < |x-\xi| < M^{-\frac{1}{a}}$, , $|x-\xi|^{-a} > M$. :

$$\boxed{\lim_{x\to\xi} |x-\xi|^{-a} = +\infty \qquad (a > 0).}$$

: $\lim_{x\to\xi} \frac{1}{|x-\xi|} = +\infty$, $\lim_{x\to\xi} \frac{1}{(x-\xi)^2} = +\infty$, $\lim_{x\to\xi} \frac{1}{\sqrt{|x-\xi|}} = +\infty$.

(2) $y = \frac{x+2}{(x+1)^2}$ $(-\infty,-1)\cup(-1,+\infty)$ -1. x -1 $\neq -1$, $y = \frac{x+2}{(x+1)^2}$, , $\lim_{x\to-1} \frac{x+2}{(x+1)^2} = +\infty$. .

$M > 0$ $\delta > 0$ $\frac{x+2}{(x+1)^2} > M$ x ($x \neq -1$) $0 < |x+1| < \delta$.

$x \neq -1$, $\frac{x+2}{(x+1)^2} > M$ $(x+1)^2 < \frac{x+2}{M}$ $x^2 + (2-\frac{1}{M})x + (1-\frac{2}{M}) < 0$ $\frac{1-2M-\sqrt{1+4M}}{2M} < x < \frac{1-2M+\sqrt{1+4M}}{2M}$. -1 $\frac{1-2M-\sqrt{1+4M}}{2M}$ $\frac{1-2M+\sqrt{1+4M}}{2M}$, , $\delta = \min\left\{-1-\frac{1-2M-\sqrt{1+4M}}{2M}, \frac{1-2M+\sqrt{1+4M}}{2M}+1\right\} = \min\left\{\frac{\sqrt{1+4M}-1}{2M}, \frac{\sqrt{1+4M}+1}{2M}\right\} = \frac{\sqrt{1+4M}-1}{2M}$, x $0 < |x+1| < \delta$ $\frac{1-2M-\sqrt{1+4M}}{2M} < x < \frac{1-2M+\sqrt{1+4M}}{2M}$. x $0 < |x+1| < \delta$ ( ) $\frac{x+2}{(x+1)^2} > M$.

$\lim_{x\to-1} \frac{x+2}{(x+1)^2} = +\infty$.

1_γ : $= -\infty$.

«» . $y = f(x)$ ( ). ′ .

$M > 0$ $\delta > 0$, $0 < |x-\xi| < \delta$ x , $f(x) < -M$. : $M > 0$ $\delta > 0$ $0 < |x-\xi| < \delta$ x $f(x) < -M$. : $M > 0$ $\delta > 0$ $f(x) < -M$ x $0 < |x-\xi| < \delta$.

$$\lim_{x\to\xi} f(x) = -\infty$$

$y = f(x)$ $-\infty$ $-\infty$ $-\infty$ x ξ.

$f(x) < -M$ $-f(x) > M$, , $\lim_{x\to\xi} f(x) = -\infty$ $\lim_{x\to\xi}\big(-f(x)\big) = +\infty$. . 1_β 1_γ .

: (1) $\lim_{x\to-1}\left(-\frac{x+2}{(x+1)^2}\right) = -\infty$.
(2) $\lim_{x\to\xi}\left(-|x-\xi|^{-a}\right) = -\infty \quad (a > 0)$.

2. $y = f(x)$ $(a,\xi)\cup(\xi,b)$ ξ. , , , ξ. , , x ξ , $< \xi$ $> \xi$. . . , , .

2_α : = .

$\epsilon > 0$ $\delta > 0$ $|f(x)-\eta| < \epsilon$ x $\xi < x < \xi+\delta$.

$$\lim_{x\to\xi+} f(x) = \eta$$

$y = f(x)$ η η η x ξ .

, $\epsilon > 0$ $\delta > 0$ $|f(x)-\eta| < \epsilon$ x $\xi-\delta < x < \xi$.

$$\lim_{x\to\xi-} f(x) = \eta$$

$y = f(x)$ η η η x ξ .

2_β : $= +\infty$.

$M > 0$ $\delta > 0$ $f(x) > M$ x $\xi < x < \xi + \delta$.

$$\lim_{x \to \xi+} f(x) = +\infty$$

$y = f(x)$ $+\infty$ $+\infty$ $+\infty$ x ξ .

, $M > 0$ $\delta > 0$ $f(x) > M$ x $\xi - \delta < x < \xi$.

$$\lim_{x \to \xi-} f(x) = +\infty$$

$y = f(x)$ $+\infty$ $+\infty$ $+\infty$ x ξ .

2_γ : $= -\infty$.

$M > 0$ $\delta > 0$ $f(x) < -M$ x $\xi < x < \xi + \delta$.

$$\lim_{x \to \xi+} f(x) = -\infty$$

$y = f(x)$ $-\infty$ $-\infty$ $-\infty$ x ξ .

, $M > 0$ $\delta > 0$ $f(x) < -M$ x $\xi - \delta < x < \xi$.

$$\lim_{x \to \xi-} f(x) = -\infty$$

$y = f(x)$ $-\infty$ $-\infty$ $-\infty$ x ξ .

2 $\lim_{x\to\xi+} f(x)$ $\lim_{x\to\xi-} f(x)$ $y = f(x)$ ξ, . $\lim_{x\to\xi} f(x)$, , $\lim_{x\to\xi+} f(x)$ $\lim_{x\to\xi-} f(x)$. 4.1.

4.1 $y = f(x)$ *()* $(a, \xi) \cup (\xi, b)$ ξ. $\lim_{x\to\xi} f(x)$, $\lim_{x\to\xi+} f(x)$ $\lim_{x\to\xi-} f(x)$ *:*

$$\lim_{x \to \xi} f(x) = \lim_{x \to \xi+} f(x) = \lim_{x \to \xi-} f(x).$$

, $\lim_{x\to\xi+} f(x)$ $\lim_{x\to\xi-} f(x)$, $\lim_{x\to\xi} f(x)$.

: . $+\infty$ $-\infty$.

$\lim_{x\to\xi} f(x) = \eta$. $\epsilon > 0$, $\delta > 0$ $|f(x) - \eta| < \epsilon$ x $y = f(x)$ $0 < |x - \xi| < \delta$, , $\xi - \delta < x < \xi$ $\xi < x < \xi + \delta$. $|f(x) - \eta| < \epsilon$ x $y = f(x)$ $\xi - \delta < x < \xi$ $|f(x) - \eta| < \epsilon$ x $y = f(x)$ $\xi < x < \xi + \delta$. $\lim_{x\to\xi-} f(x) = \eta$ $\lim_{x\to\xi+} f(x) = \eta$.

$\lim_{x\to\xi-} f(x) = \lim_{x\to\xi+} f(x) = \eta$ $\epsilon > 0$. $\lim_{x\to\xi-} f(x) = \eta$, $\delta' > 0$ $|f(x) - \eta| < \epsilon$ x $y = f(x)$ $\xi - \delta' < x < \xi$. , $\lim_{x\to\xi+} f(x) = \eta$, $\delta'' > 0$ $|f(x) - \eta| < \epsilon$ x $y = f(x)$ $\xi < x < \xi + \delta''$. $\delta = \min\{\delta', \delta''\}$, $\delta \le \delta'$ $\delta \le \delta''$. $|f(x) - \eta| < \epsilon$ x $y = f(x)$ $\xi - \delta < x < \xi$ $\xi < x < \xi + \delta$. $|f(x) - \eta| < \epsilon$ x $y = f(x)$ $0 < |x - \xi| < \delta$, , $\lim_{x\to\xi} f(x) = \eta$.

: (1) $a > 0$. $y = |x - \xi|^a$ $(-\infty, +\infty)$, , $\lim_{x\to\xi} |x - \xi|^a = 0$. $\lim_{x\to\xi+} |x - \xi|^a = 0$ $\lim_{x\to\xi-} |x - \xi|^a = 0$.

(2) $a > 0$. $y = |x-\xi|^{-a}$ $(-\infty,\xi)\cup(\xi,+\infty)$ $\lim_{x\to\xi}|x-\xi|^{-a} = +\infty$. , $\lim_{x\to\xi+}|x-\xi|^{-a} = +\infty$ $\lim_{x\to\xi-}|x-\xi|^{-a} = +\infty$.

(3) $y = f(x) = \begin{cases} 2x+1, & x > 0, \\ x^2+1, & x \le 0, \end{cases}$ $(-\infty,+\infty)$. $y = f(x)$ 0 0. , $\lim_{x\to 0-} f(x) = 1$ $\lim_{x\to 0+} f(x) = 1$, $\lim_{x\to 0} f(x) = 1$.

$\epsilon > 0$ $(-\infty, 0)$. $|f(x)-1| < \epsilon$ ($x < 0$) $|(x^2+1)-1| < \epsilon$ $x^2 < \epsilon$ ($x < 0$) $-\sqrt{\epsilon} < x < 0$. , , $\delta = \sqrt{\epsilon}$, x (x) $-\delta < x < 0$ $-\sqrt{\epsilon} < x < 0$, , $|f(x)-1| < \epsilon$. $\lim_{x\to 0-} f(x) = 1$.

$\epsilon > 0$ $(0,+\infty)$. $|f(x)-1| < \epsilon$ ($x > 0$) $|(2x+1)-1| < \epsilon$ $2x < \epsilon$ ($x > 0$) $0 < x < \frac{\epsilon}{2}$. $\delta = \frac{\epsilon}{2}$, x (x) $0 < x < \delta$ $0 < x < \frac{\epsilon}{2}$, , $|f(x)-1| < \epsilon$. $\lim_{x\to 0+} f(x) = 1$.

4.1 .

$y = f(x)$ () $(a,\xi)\cup(\xi,b)$. $\lim_{x\to\xi+} f(x)$ $\lim_{x\to\xi-} f(x)$
,
$\lim_{x\to\xi} f(x)$.

: (1) $y = \frac{|x|}{x}$ $(-\infty,0)\cup(0,+\infty)$.

$(0,+\infty)$ $\frac{|x|}{x} = 1$. $\epsilon > 0$. 1 $(0,+\infty)$, $\delta > 0$ $\left|\frac{|x|}{x} - 1\right| = |1-1| = 0 < \epsilon$ x ($x \ne 0$) $0 < x < \delta$. $\lim_{x\to 0+}\frac{|x|}{x} = 1$.

, $(-\infty,0)$ $\frac{|x|}{x} = -1$. $\epsilon > 0$. -1 $(-\infty,0)$, $\delta > 0$ $\left|\frac{|x|}{x} - (-1)\right| = |(-1)-(-1)| = 0 < \epsilon$ x ($x \ne 0$) $-\delta < x < 0$. $\lim_{x\to 0-}\frac{|x|}{x} = -1$.

$$\lim_{x\to 0-}\frac{|x|}{x} = -1, \qquad \lim_{x\to 0+}\frac{|x|}{x} = 1.$$

, , . , $\lim_{x\to 0}\frac{|x|}{x}$.

(2) $y = \frac{1}{x-\xi}$ $(-\infty,\xi)\cup(\xi,+\infty)$. $\lim_{x\to\xi+}\frac{1}{x-\xi} = +\infty$ $\lim_{x\to\xi-}\frac{1}{x-\xi} = -\infty$, $\lim_{x\to\xi}\frac{1}{x-\xi}$.

$M > 0$ $(\xi,+\infty)$. $\frac{1}{x-\xi} > M$ ($x > \xi$) $\xi < x < \xi + \frac{1}{M}$. $\delta = \frac{1}{M}$, x ($x \ne \xi$) $\xi < x < \xi+\delta$ $\xi < x < \xi + \frac{1}{M}$, , $\frac{1}{x-\xi} > M$. $\lim_{x\to\xi+}\frac{1}{x-\xi} = +\infty$.

$M > 0$ $(-\infty,\xi)$. $\frac{1}{x-\xi} < -M$ ($x < \xi$) $\xi - \frac{1}{M} < x < \xi$. $\delta = \frac{1}{M}$, x ($x \ne \xi$) $\xi-\delta < x < \xi$ $\xi - \frac{1}{M} < x < \xi$, , $\frac{1}{x-\xi} < -M$. $\lim_{x\to\xi-}\frac{1}{x-\xi} = -\infty$.

$$\lim_{x\to\xi-}\frac{1}{x-\xi} = -\infty, \qquad \lim_{x\to\xi+}\frac{1}{x-\xi} = +\infty.$$

3. $y = f(x)$ (ξ,b) (a,ξ). $y = f(x)$, , ξ. x $y = f(x)$, x ξ $\ne \xi$ x ξ $> \xi$. , : $x \to \xi$ $x \to \xi+$.
´ o $\lim_{x\to\xi} f(x)$ $\lim_{x\to\xi+} f(x)$, 2,

$$\lim_{x\to\xi} f(x) = \lim_{x\to\xi+} f(x).$$

$y = f(x)$ (a,ξ) (ξ,b). $y = f(x)$, , ξ. , x
$y = f(x)$, $x \to \xi$ $x \to \xi-$.
′ o $\lim_{x\to\xi} f(x)$ $\lim_{x\to\xi-} f(x)$

$$\lim_{x\to\xi} f(x) = \lim_{x\to\xi-} f(x).$$

: $a > 0$.

(1) $y = (x-\xi)^a$ $[\xi,+\infty)$ $(x-\xi)^a = |x-\xi|^a$. $\lim_{x\to\xi+} |x-\xi|^a = 0$. $\lim_{x\to\xi}(x-\xi)^a = \lim_{x\to\xi+}(x-\xi)^a = \lim_{x\to\xi+} |x-\xi|^a = 0$.

(2) $y = (x-\xi)^{-a}$ $(\xi,+\infty)$ $(x-\xi)^{-a} = |x-\xi|^{-a}$. $\lim_{x\to\xi+} |x-\xi|^{-a} = +\infty$. $\lim_{x\to\xi}(x-\xi)^{-a} = \lim_{x\to\xi+}(x-\xi)^{-a} = \lim_{x\to\xi+} |x-\xi|^{-a} = +\infty$.

(3) $y = (\xi-x)^a$ $(-\infty,\xi]$ $(\xi-x)^a = |x-\xi|^a$ $(-\infty,\xi]$. $\lim_{x\to\xi-} |x-\xi|^a = 0$. $\lim_{x\to\xi}(\xi-x)^a = \lim_{x\to\xi-}(\xi-x)^a = \lim_{x\to\xi-} |x-\xi|^a = 0$.

(4) $y = (\xi-x)^{-a}$ $(-\infty,\xi)$ $(\xi-x)^{-a} = |x-\xi|^{-a}$ $(-\infty,\xi)$. $\lim_{x\to\xi-} |x-\xi|^{-a} = +\infty$. $\lim_{x\to\xi}(\xi-x)^{-a} = \lim_{x\to\xi-}(\xi-x)^{-a} = \lim_{x\to\xi-} |x-\xi|^{-a} = +\infty$.

4. $y = f(x)$ $(a,+\infty)$, . x $f(x)$.

4_α : = .

$\epsilon > 0$ $N > 0$ $|f(x)-\eta| < \epsilon$ x $x > N$.

$$\lim_{x\to+\infty} f(x) = \eta$$

$y = f(x)$ η η η x $+\infty$.

4_β : $= +\infty$.

$M > 0$ $N > 0$ $f(x) > M$ x $x > N$.

$$\lim_{x\to+\infty} f(x) = +\infty$$

$y = f(x)$ $+\infty$ $+\infty$ $+\infty$ x $+\infty$.

4_γ : $= -\infty$.

, $M > 0$ $N > 0$ $f(x) < -M$ x $x > N$.

$$\lim_{x\to+\infty} f(x) = -\infty$$

$y = f(x)$ $-\infty$ $-\infty$ $-\infty$ x $+\infty$.

: (1) $y = \frac{x+1}{x+3}$ $(-\infty,-3)\cup(-3,+\infty)$, $+\infty$. , , x , $y = \frac{x+1}{x+3}$ 1. , , $\lim_{x\to+\infty} \frac{x+1}{x+3} = 1$. .

$\epsilon > 0$ $N > 0$ $\left|\frac{x+1}{x+3} - 1\right| < \epsilon$ x ($x \neq -3$) $x > N$.

$\left|\frac{x+1}{x+3} - 1\right| < \epsilon$ $\frac{2}{|x+3|} < \epsilon$ $|x+3| > \frac{2}{\epsilon}$ $x < -3 - \frac{2}{\epsilon}$ $x > -3 + \frac{2}{\epsilon}$ (: x) $x > -3 + \frac{2}{\epsilon}$. $N > 0$ $\geq -3 + \frac{2}{\epsilon}$. $-3 + \frac{2}{\epsilon} > 0$, , $\epsilon < \frac{2}{3}$, $N = -3 + \frac{2}{\epsilon}$

, $-3+\frac{2}{\epsilon}\leq 0$, , $\epsilon\geq\frac{2}{3}$, $N=1$. $x>N$ ($N\geq -3+\frac{2}{\epsilon}$) $x>-3+\frac{2}{\epsilon}$, , $\left|\frac{x+1}{x+3}-1\right|<\epsilon$.

$\lim_{x\to+\infty}\frac{x+1}{x+3}=1$.

(2) $y=x-\frac{7}{x}$ $(-\infty,0)\cup(0,+\infty)$, $+\infty$. x, $y=x-\frac{7}{x}$. $\lim_{x\to+\infty}\left(x-\frac{7}{x}\right)=+\infty$.

$M>0$ $N>0$ $x-\frac{7}{x}>M$ x ($x\neq 0$) $x>N$.

$x-\frac{7}{x}>M$ $\frac{x^2-Mx-7}{x}>0$ $x(x^2-Mx-7)>0$ $\frac{M-\sqrt{M^2+28}}{2}<x<0$ $x>\frac{M+\sqrt{M^2+28}}{2}$ (: x) $x>\frac{M+\sqrt{M^2+28}}{2}$. $N=\frac{M+\sqrt{M^2+28}}{2}>0$, x ($x\neq 0$) $x>N$ $x>\frac{M+\sqrt{M^2+28}}{2}$, , $x-\frac{7}{x}>M$.

$\lim_{x\to+\infty}\left(x-\frac{7}{x}\right)=+\infty$.

(3) $y=c$. $\epsilon>0$ $N>0$ $|y-c|=|c-c|=0<\epsilon$ x (x) $x>N$. :

$$\boxed{\lim_{x\to+\infty}c=c.}$$

(4) $a>0$. $y=x^a$ $[0,+\infty)$. $\lim_{x\to+\infty}x^a=+\infty$.

$M>0$ $N>0$ $x^a>M$ x $x>N$.

$x>0$, $x^a>M$ $x>M^{\frac{1}{a}}$, , $N=M^{\frac{1}{a}}>0$, $x>N$ $x>M^{\frac{1}{a}}$, , $x^a>M$. :

$$\boxed{\lim_{x\to+\infty}x^a=+\infty\qquad(a>0).}$$

: $\lim_{x\to+\infty}x=+\infty$, $\lim_{x\to+\infty}x^2=+\infty$, $\lim_{x\to+\infty}\sqrt{x}=+\infty$, $\lim_{x\to+\infty}\sqrt[5]{x}=+\infty$.

(5) $a>0$. $y=x^{-a}$ $(0,+\infty)$. $\lim_{x\to+\infty}x^{-a}=0$.

$\epsilon>0$ $N>0$ $|x^{-a}-0|<\epsilon$ x $x>N$.

$x>0$, $|x^{-a}-0|<\epsilon$ $x>\epsilon^{-\frac{1}{a}}$, , $N=\epsilon^{-\frac{1}{a}}>0$, x $x>N$ $x>\epsilon^{-\frac{1}{a}}$, , $|x^{-a}-0|<\epsilon$. :

$$\boxed{\lim_{x\to+\infty}x^{-a}=0\qquad(a>0).}$$

: $\lim_{x\to+\infty}\frac{1}{x}=0$, $\lim_{x\to+\infty}\frac{1}{\sqrt{x}}=0$, $\lim_{x\to+\infty}\frac{1}{x\sqrt[3]{x}}=0$.

5. $y=f(x)$ $(-\infty,b)$, . x $f(x)$.

5_α: = .

$\epsilon>0$ $N>0$ $|f(x)-\eta|<\epsilon$ x $x<-N$.

$$\lim_{x\to-\infty}f(x)=\eta$$

$y=f(x)$ η η η x $-\infty$.

5_β: $=+\infty$.

$M>0$ $N>0$ $f(x)>M$ x $x<-N$.

$$\lim_{x\to-\infty}f(x)=+\infty$$

$y = f(x)$ $+\infty$ $+\infty$ $+\infty$ x $-\infty$.

$5_\gamma :$ $= -\infty$.

, $M > 0$ $N > 0$ $f(x) < -M$ x $x < -N$.

$$\lim_{x\to-\infty} f(x) = -\infty$$

$y = f(x)$ $-\infty$ $-\infty$ $-\infty$ x $-\infty$.

: (1) $y = c$. $\epsilon > 0$ $N > 0$ $|y-c| = |c-c| = 0 < \epsilon$ x (x) $x < -N$.
:

$$\boxed{\lim_{x\to-\infty} c = c.}$$

(2) $y = \frac{3-x}{7+x}$ $(-\infty,-7)\cup(-7,+\infty)$, $-\infty$. x, $y = \frac{3-x}{7+x}$ -1, $\lim_{x\to-\infty}\frac{3-x}{7+x} = -1$. .

$\epsilon > 0$ $N > 0$ $\left|\frac{3-x}{7+x} - (-1)\right| < \epsilon$ x ($x \neq -7$) $x < -N$.

$\left|\frac{3-x}{7+x} - (-1)\right| < \epsilon$ $\frac{10}{|7+x|} < \epsilon$ $|x+7| > \frac{10}{\epsilon}$ $x < -7 - \frac{10}{\epsilon}$ $x > -7 + \frac{10}{\epsilon}$ (: x) $x < -7 - \frac{10}{\epsilon}$. $N = 7 + \frac{10}{\epsilon} > 0$, x ($x \neq -7$) $x < -N$ $x < -7 - \frac{10}{\epsilon}$, , $\left|\frac{3-x}{7+x} - (-1)\right| < \epsilon$.

$\lim_{x\to-\infty}\frac{3-x}{7+x} = -1$.

.

1. ;

$$\lim_{x\to1}\frac{1}{\sqrt{x^2-2x}}, \qquad \lim_{x\to0}\sqrt{-x^2}, \qquad \lim_{x\to1-}\log(x-1),$$

$$\lim_{x\to1+}\sqrt{1-x^2}, \qquad \lim_{x\to+\infty}\frac{1}{\log(3-x)}, \qquad \lim_{x\to-\infty}\sqrt{4+3x-x^2}\,.$$

2. ; «¨» . «» x .

$$\lim_{x\to2}(x^2+1), \quad \lim_{x\to1}\frac{x+2}{x+1}, \quad \lim_{x\to1}\frac{1}{(x-1)^2}, \quad \lim_{x\to1-}\frac{1}{x^2-1}, \quad \lim_{x\to1+}\frac{1}{x^2-1},$$

$$\lim_{x\to+\infty}\frac{1}{x^3}, \quad \lim_{x\to+\infty}\frac{1}{\sqrt{x+5}}, \quad \lim_{x\to-\infty}\frac{1}{\sqrt{x^2+5}}, \quad \lim_{x\to-\infty}\sqrt{x^2-3}\,.$$

3. .

$$\lim_{x\to1}3x = 3, \quad \lim_{x\to2}\left(\frac{x}{2}-7\right) = -6, \quad \lim_{x\to1}\frac{1}{x} = 1, \quad \lim_{x\to1}x^2 = 1,$$

$$\lim_{x\to2}\frac{x+1}{x-1} = 3, \quad \lim_{x\to1}\log x = 0, \quad \lim_{x\to0}e^x = 1, \quad \lim_{x\to2}\sqrt{x} = \sqrt{2}\,,$$

$$\lim_{x\to1}\left|\frac{x}{x-1}\right| = +\infty, \quad \lim_{x\to-3}\left(\frac{x+1}{x+3}\right)^2 = +\infty, \quad \lim_{x\to2}\frac{1-x}{(x-2)^2} = -\infty,$$

$$\lim_{x\to-1+}(3x-2)=-5,\quad \lim_{x\to 2+}x^2=4,\quad \lim_{x\to 0-}\frac{x}{x+1}=0,$$

$$\lim_{x\to 1+}\frac{-3}{x-1}=-\infty,\quad \lim_{x\to 2+}\frac{x}{2-x}=-\infty,\quad \lim_{x\to 1-}\frac{x+2}{1-x}=+\infty,$$

$$\lim_{x\to 0+}\log x=-\infty,\quad \lim_{x\to+\infty}e^x=+\infty,\quad \lim_{x\to+\infty}\log x=+\infty,$$

$$\lim_{x\to-\infty}e^x=0,\quad \lim_{x\to\pm\infty}\frac{x-3}{2x+1}=\frac{1}{2},\quad \lim_{x\to\pm\infty}(3x^2+x)=+\infty,$$

$$\lim_{x\to+\infty}\frac{2-x^2}{x+1}=-\infty,\quad \lim_{x\to-\infty}\frac{2-x^2}{x+1}=+\infty.$$

4. 1. .

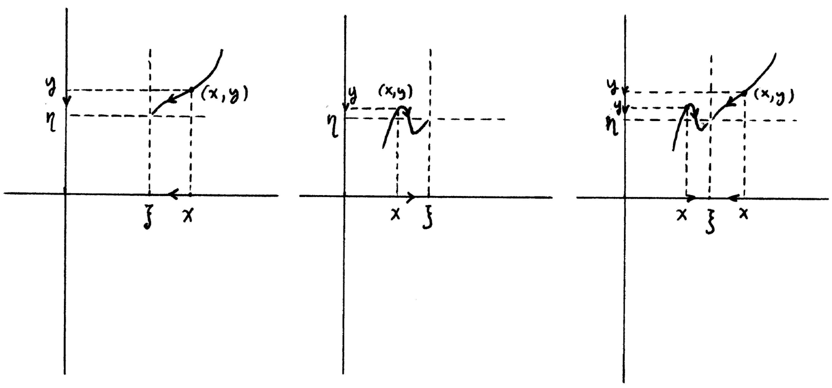
$x\to 1$, , ;

$$y=\begin{cases}2x+3, & x>1,\\ 1-2x, & x<1,\end{cases}\qquad y=\begin{cases}2x-1, & x>1,\\ \frac{x}{x-1}, & x<1,\end{cases}$$

$$y=\begin{cases}\sqrt{x}, & x\geq 1,\\ x^2, & x<1,\end{cases}\qquad y=\begin{cases}\frac{2x}{x-1}, & x>1,\\ (x-1)^{-2}, & x<1.\end{cases}$$

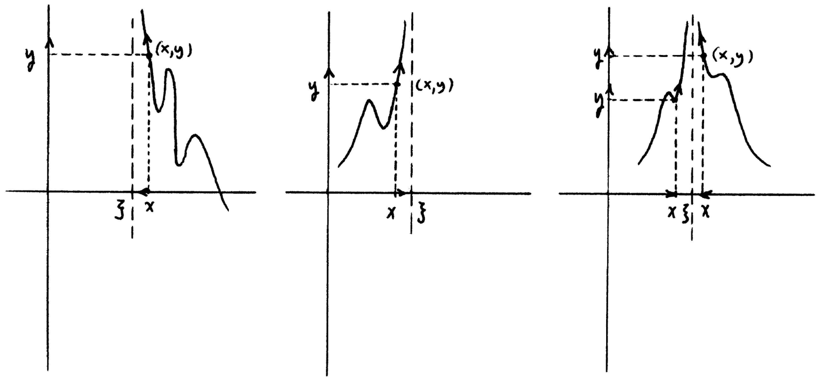
4.2 .

«» .

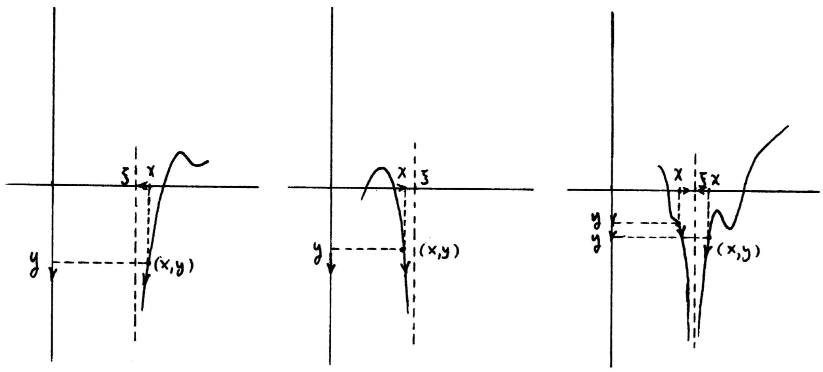


Σχήμα 4.1: $\lim_{x\to\xi}f(x)=\eta$.

$y=f(x)$ $(a,\xi)\cup(\xi,b)$ x ξ (a,ξ) (ξ,b). x x- $f(x)$ y- , , $(x,f(x))$. $f(x)$ $(x,f(x))$. , $f(x)$ η x ξ $\neq\xi$, , $\lim_{x\to\xi}f(x)=\eta$ $(x,f(x))$ (ξ,η) $x=\xi$. , $f(x)$ x ξ $\neq\xi$, , $\lim_{x\to\xi}f(x)=+\infty$ $(x,f(x))$ $x=\xi$. : $\lim_{x\to\xi}f(x)=-\infty$ $(x,f(x))$ $x=\xi$. , $\lim_{x\to\xi}f(x)=\pm\infty$, $x=\xi$ $y=f(x)$.



Σχήμα 4.2: $\lim_{x\to\xi} f(x) = +\infty$. .



Σχήμα 4.3: $\lim_{x\to\xi} f(x) = -\infty$. .

$\lim_{x\to\xi\pm} f(x)$. $\lim_{x\to\xi+} f(x)$ $(x, f(x))$ $x = \xi$ $\lim_{x\to\xi-} f(x)$ $(x, f(x))$ $x = \xi$.

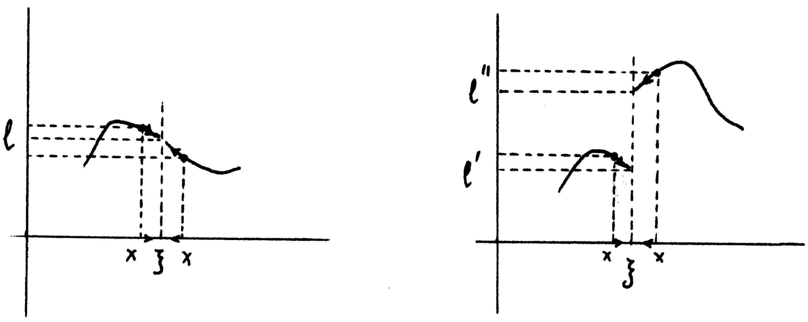
$x = \xi$ $y = f(x)$ $\lim_{x\to\xi\pm} f(x) = \pm\infty$.

, , $\lim_{x\to\xi+} f(x)$ $\lim_{x\to\xi-} f(x)$, $\lim_{x\to\xi} f(x)$.

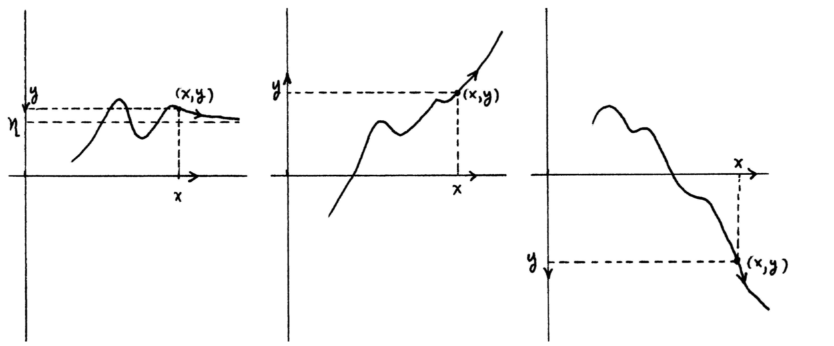
$y = f(x)$ $(a, +\infty)$ x x-. $f(x)$ y- $(x, y) = (x, f(x))$. , $f(x)$ η , , $\lim_{x\to+\infty} f(x) = \eta$ $(x, f(x))$ $y = \eta$. $y = \eta$ ($+\infty$) $y = f(x)$. : $\lim_{x\to+\infty} f(x) = +\infty$ $(x, f(x))$. : $\lim_{x\to+\infty} f(x) = -\infty$ $(x, f(x))$.

, $y = f(x)$ $(-\infty, b)$, $\lim_{x\to-\infty} f(x) = \eta$ $(x, f(x))$ $y = \eta$. ′ $y = \eta$ ($-\infty$) $y = f(x)$. : $\lim_{x\to-\infty} f(x) = +\infty$ $(x, f(x))$. : $\lim_{x\to-\infty} f(x) = -\infty$ $(x, f(x))$.

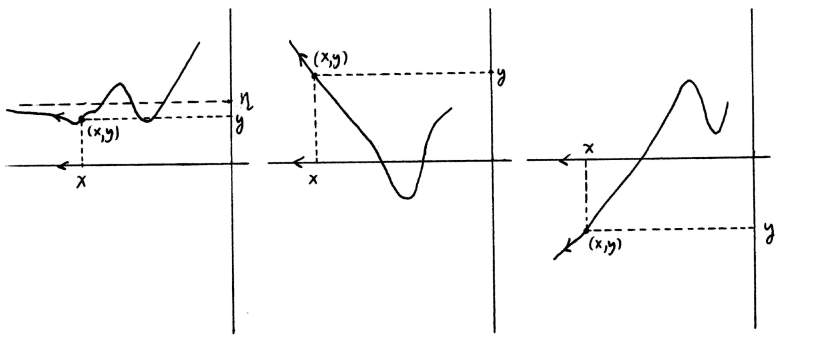
«» − ϵ δ − $\lim_{x\to\xi} f(x) = \eta$ $y = f(x)$. $\epsilon > 0$ $\delta > 0$ x $(\xi - \delta, \xi) \cup (\xi, \xi + \delta)$ $f(x)$ $(x, f(x))$, $\eta - \epsilon$ $\eta + \epsilon$, ,



Σχήμα 4.4: $\lim_{x\to\xi-} f(x) = \lim_{x\to\xi+} f(x)$ $\lim_{x\to\xi-} f(x) \neq \lim_{x\to\xi+} f(x)$.

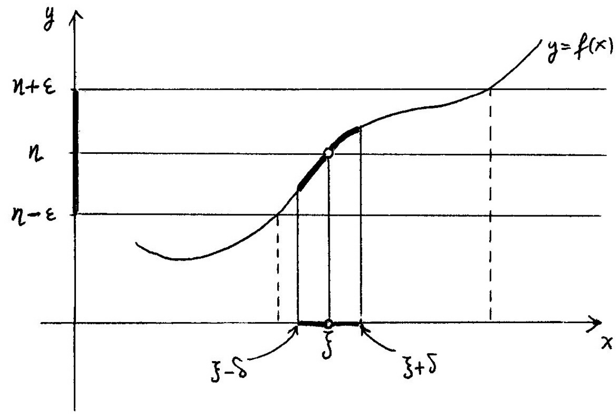


Σχήμα 4.5: $\lim_{x\to+\infty} f(x) = \eta$, $\lim_{x\to+\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = -\infty$.



Σχήμα 4.6: $\lim_{x\to-\infty} f(x) = \eta$, $\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to-\infty} f(x) = -\infty$.

$(\xi-\delta,\xi)\cup(\xi,\xi+\delta)$ $y=\eta-\epsilon$ $y=\eta+\epsilon$.
«» . , $\lim_{x\to+\infty} f(x) = -\infty$ $M>0$ $N>0$ $y=f(x)$
$(N,+\infty)$ $y=-M$.

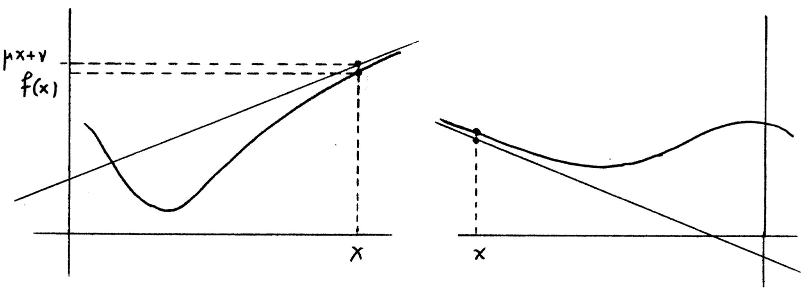


Σχήμα 4.7: $\eta - \epsilon < f(x) < \eta + \epsilon$ x $\xi - \delta < x < \xi$ $\xi < x < \xi + \delta$.

. l $y = \mu x + \nu$ () $+\infty$ $y = f(x)$ $y = f(x)$ $(a, +\infty)$

$$\lim_{x \to +\infty} \big(f(x) - \mu x - \nu\big) = 0.$$

$(x, f(x))$ $y = f(x)$ $(x, \mu x + \nu)$ l . ∵: l $+\infty$.



Σχήμα 4.8: $+\infty$ $-\infty$.

() $-\infty$ $y = f(x)$ $(-\infty, b)$. l $y = \mu x + \nu$

$$\lim_{x \to -\infty} \big(f(x) - \mu x - \nu\big) = 0.$$

, , .

$y = f(x)$ $(a, +\infty)$ $+\infty$ $y = \mu x + \nu$. μ ν. $\lim_{x\to+\infty}(f(x) - \mu x - \nu) = 0$ $\lim_{x\to+\infty} \frac{f(x)-\mu x-\nu}{x} = 0$, , $\mu = \lim_{x\to+\infty} \frac{f(x)}{x}$. , , , $\lim_{x\to+\infty} \frac{f(x)}{x}$ $\pm\infty$, $+\infty$. , , , μ. , , ν $\nu = \lim_{x\to+\infty}(f(x) - \mu x)$, μ . , $\lim_{x\to+\infty}(f(x) - \mu x)$ $\pm\infty$, $+\infty$.

, , $-\infty$.

: $y = x + \frac{1}{x}$ $(-\infty, 0) \cup (0, +\infty)$. : $\mu = \lim_{x\to+\infty} \frac{1}{x}(x + \frac{1}{x}) = 1$ $\nu = \lim_{x\to+\infty}(x+\frac{1}{x}-1x) = 0$. $+\infty$ $y = 1x+0 = x$. : $\mu = \lim_{x\to-\infty} \frac{1}{x}(x+\frac{1}{x}) = 1$ $\nu = \lim_{x\to-\infty}(x + \frac{1}{x} - 1x) = 0$. $-\infty$ $y = 1x + 0 = x$.

$y = f(x)$. , 0 ($\mu = 0$).

.

1. $y = f(x) = \begin{cases} x^2 - 1, & x < 2, \\ \frac{1}{x}, & x \geq 2 \end{cases}$ $\lim_{x\to 2\pm} f(x)$, $\lim_{x\to 0\pm} f(x)$ $\lim_{x\to\pm\infty} f(x)$;

$$y = f(x) = \begin{cases} x^2, & x \geq 2, \\ \frac{1}{x}, & x < 2, x \neq 0. \end{cases}$$

2. :

$$y = ax + b, \quad y = x^n \, (n \,), \quad y = x^a, \quad y = a^x, \quad y = \log_a x,$$

$$y = [x], \quad y = \cos x, \quad y = \sin x, \quad y = \tan x, \quad y = \cot x,$$

$$y = \arccos x, \quad y = \arcsin x, \quad y = \arctan x, \quad y = \operatorname{arc\,cot} x,$$

$$y = \cosh x, \quad y = \sinh x, \quad y = \operatorname{arc\,cosh} x, \quad y = \operatorname{arc\,sinh} x.$$

, ( ) ξ $\pm\infty$ – .

;

3. , , ( ) .

$$y = 2x - 3, \quad y = x^2, \quad y = -5x + \frac{7x+1}{x}, \quad y = \frac{2x^3 + x^2 - 3}{x^2 + 1},$$

$$y = \frac{x^4}{(x-1)(x^2+1)}, \quad y = \frac{1}{x} + \frac{x}{x-1} + \frac{x^2}{x-2}.$$

4. $y = \frac{2x+1}{x-1}$ $x = \frac{y+1}{y-2}$.

, $y = f(x)$ $x = f^{-1}(y)$;

5. $y = 2x - \frac{1}{x}$ $(0, +\infty)$. $(-\infty, +\infty)$ $x = \frac{1}{4}(y + \sqrt{y^2 + 8})$.
.

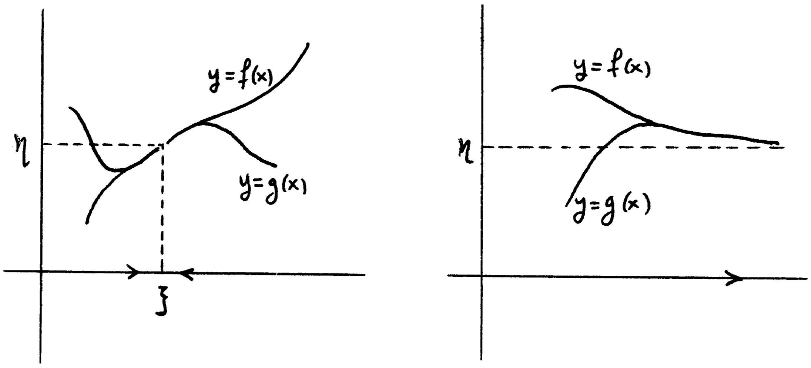
, $y = f(x)$ $x = f^{-1}(y)$; 4.

4.3 .

. .

$y = f(x)$ ξ, , $f(x)$ x ξ $\neq \xi$. , , $y = g(x)$ $(a,\xi)\cup(\xi,b)$, $g(x) = f(x)$ x $(a,\xi)\cup(\xi,b)$. (a,ξ), (ξ,b) , , x ξ ξ. $y = f(x)$ $x \to \xi$, $y = g(x)$.

, $y = g(x)$ $y = f(x)$ $y = f(x)$ − , − $(a,\xi)\cup(\xi,b)$, a b ξ.



Σχήμα 4.9: $\lim_{x\to\xi} f(x) = \lim_{x\to\xi} g(x)$ $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x)$.

, $g(x) = f(x)$ x (ξ,b) $y = f(x)$ $x \to \xi+$, $y = g(x)$. , $g(x) = f(x)$ x (a,ξ) $y = f(x)$ $x \to \xi-$, $y = g(x)$. , $g(x) = f(x)$ x $(a,+\infty)$ $y = f(x)$ $x \to +\infty$, $y = g(x)$. $x \to -\infty$.

:

4.2 $y = f(x)$ $y = g(x)$ $(a,\xi)\cup(\xi,b)$ (ξ,b) (a,ξ) $(a,+\infty)$ $(-\infty,b)$. , , $x \to \xi$ $x \to \xi+$ $x \to \xi-$ $x \to +\infty$ $x \to -\infty$, .

: $y = f(x)$ $y = g(x)$ $(a,\xi)\cup(\xi,b)$ $\lim_{x\to\xi} f(x) = \eta$. .
$\epsilon > 0$. $\lim_{x\to\xi} f(x) = \eta$, $\delta' > 0$ $|f(x)-\eta| < \epsilon$ x $y = f(x)$ $0 < |x-\xi| < \delta'$. $\delta = \min\{\delta', \xi - a, b - \xi\}$, x $0 < |x-\xi| < \delta$ $(a,\xi)\cup(\xi,b)$, , x $g(x) = f(x)$. , $\delta \leq \delta'$, x $0 < |x-\xi| < \delta$ $|g(x)-\eta| = |f(x)-\eta| < \epsilon$. $\lim_{x\to\xi} g(x) = \eta$.

: (1) $\lim_{x\to 0} |x|^{-\frac{1}{2}} = +\infty$. $y = f(x) = \begin{cases} |x|^{-\frac{1}{2}}, & 0 < |x| < \frac{1}{10}, \\ x, & x = 0 \ \ |x| \geq \frac{1}{10}, \end{cases}$ $\lim_{x\to 0} f(x) = +\infty$ $y = |x|^{-\frac{1}{2}}$ $y = f(x)$ $(-\frac{1}{10}, 0)\cup(0,\frac{1}{10})$.

, , $\lim_{x\to 0} f(x)$: $f(x) = |x|^{-\frac{1}{2}}$ $0 < |x| < \frac{1}{10}$ (, , $(-\frac{1}{10}, 0)\cup(0,\frac{1}{10})$), $\lim_{x\to 0} f(x) = \lim_{x\to 0} |x|^{-\frac{1}{2}} = +\infty$.

(2) $y = f(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ x^{-1}, & x > 1 \ \ x < 0. \end{cases}$ $f(x) = x$ $0 < x < 1$, $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} x = 0$. $f(x) = x^{-1}$ $x < 0$, $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x^{-1} = -\infty$.

(3) $y = f(x) = \begin{cases} x^{-1}, & x > 100, \\ x, & x \le 100. \end{cases}$ $f(x) = x^{-1}$ $(100, +\infty)$, $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^{-1} = 0$. $f(x) = x$ $(-\infty, 100)$, $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x = -\infty$.

. .

. lim $\lim_{x\to\xi}$ $\lim_{x\to\xi\pm}$ $\lim_{x\to\pm\infty}$. . , $(a,\xi)\cup(\xi,b)$ $\lim_{x\to\xi}$. .

4.3 . $\lim f(x)$, $\lim(-f(x))$

$$\boxed{\lim(-f(x)) = -\lim f(x).}$$

: $\lim_{x\to\xi} f(x) = \eta$. $\epsilon > 0$, $\delta > 0$ $|f(x) - \eta| < \epsilon$ x $y = f(x)$ $0 < |x - \xi| < \delta$. $|(-f(x)) - (-\eta)| = |\eta - f(x)| = |f(x) - \eta|$. $|(-f(x)) - (-\eta)| < \epsilon$ x $y = f(x)$ $0 < |x - \xi| < \delta$. , $\lim_{x\to\xi}(-f(x)) = -\eta = -\lim_{x\to\xi} f(x)$.

$\lim_{x\to\xi} f(x) = +\infty$. $M > 0$, $\delta > 0$ $f(x) > M$ x $y = f(x)$ $0 < |x - \xi| < \delta$. $-f(x) < -M$ x $y = f(x)$ $0 < |x - \xi| < \delta$. , $\lim_{x\to\xi}(-f(x)) = -\infty = -(+\infty) = -\lim_{x\to\xi} f(x)$.

$\lim_{x\to\xi} f(x) = -\infty$.

$y = f(x)$ $y = g(x)$ $y = f(x) + g(x)$ x .

4.4 . $\lim f(x), \lim g(x)$ $\lim f(x) + \lim g(x)$, $\lim(f(x) + g(x))$

$$\boxed{\lim(f(x) + g(x)) = \lim f(x) + \lim g(x).}$$

: $\lim_{x\to\xi} f(x) = \eta$ $\lim_{x\to\xi} g(x) = \zeta$. $\epsilon > 0$, $\delta' > 0$ $|f(x) - \eta| < \frac{\epsilon}{2}$ x $y = f(x)$ $0 < |x - \xi| < \delta'$ $\delta'' > 0$ $|g(x) - \zeta| < \frac{\epsilon}{2}$ x $y = g(x)$ $0 < |x - \xi| < \delta''$. $\delta = \min\{\delta', \delta''\}$, $\delta \le \delta'$ $\delta \le \delta''$. $|f(x) - \eta| < \frac{\epsilon}{2}$ $|g(x) - \zeta| < \frac{\epsilon}{2}$ x $y = f(x)$ $y = g(x)$ $0 < |x - \xi| < \delta$. $|(f(x) + g(x)) - (\eta + \zeta)| = |(f(x) - \eta) + (g(x) - \zeta)| \le |f(x) - \eta| + |g(x) - \zeta|$. $|(f(x) + g(x)) - (\eta + \zeta)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ x $y = f(x) + g(x)$ $0 < |x - \xi| < \delta$. , $\lim_{x\to\xi}(f(x) + g(x)) = \eta + \zeta = \lim_{x\to\xi} f(x) + \lim_{x\to\xi} g(x)$.

$\lim_{x\to\xi} f(x) = +\infty$ $\lim_{x\to\xi} g(x) = +\infty$. $M > 0$, $\delta' > 0$ $f(x) > \frac{M}{2}$ x $y = f(x)$ $0 < |x - \xi| < \delta'$ $\delta'' > 0$ $g(x) > \frac{M}{2}$ x $y = g(x)$ $0 < |x - \xi| < \delta''$. $\delta = \min\{\delta', \delta''\}$. $\delta \le \delta'$ $\delta \le \delta''$, $f(x) > \frac{M}{2}$ $g(x) > \frac{M}{2}$ x $y = f(x)$ $y = g(x)$ $0 < |x - \xi| < \delta$. , $f(x) + g(x) > \frac{M}{2} + \frac{M}{2} = M$ x $y = f(x) + g(x)$ $0 < |x - \xi| < \delta$. $\lim_{x\to\xi}(f(x) + g(x)) = +\infty = (+\infty) + (+\infty) = \lim_{x\to\xi} f(x) + \lim_{x\to\xi} g(x)$.

$\lim_{x\to\xi} f(x) = +\infty$ $\lim_{x\to\xi} g(x) = \zeta$. $M > 0$, $\delta' > 0$ $f(x) > M - \zeta + 1$ x $y = f(x)$ $0 < |x - \xi| < \delta'$ $\delta'' > 0$ $|g(x) - \zeta| < 1$ x $y = g(x)$ $0 < |x - \xi| < \delta''$. $\delta = \min\{\delta', \delta''\}$. $\delta \le \delta'$ $\delta \le \delta''$, $f(x) > M - \zeta + 1$ $|g(x) - \zeta| < 1$ x $y = f(x)$ $y = g(x)$ $0 < |x - \xi| < \delta$. $|g(x) - \zeta| < 1$ $g(x) > \zeta - 1$, , $f(x) + g(x) > (M - \zeta + 1) + (\zeta - 1) = M$ x $y = f(x) + g(x)$ $0 < |x - \xi| < \delta$. $\lim_{x\to\xi}(f(x) + g(x)) = +\infty = (+\infty) + \zeta = \lim_{x\to\xi} f(x) + \lim_{x\to\xi} g(x)$.

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: (1) $\lim_{x\to0}\left(1 + \frac{1}{|x|}\right) = \lim_{x\to0} 1 + \lim_{x\to0} \frac{1}{|x|} = 1 + (+\infty) = +\infty$.

(2) $\lim_{x\to0+}\left(x + 1 + \frac{1}{x} + \frac{1}{\sqrt{x}}\right) = \lim_{x\to0+} x + \lim_{x\to0+} 1 + \lim_{x\to0+} \frac{1}{x} + \lim_{x\to0+} \frac{1}{\sqrt{x}} = 0 + 1 + (+\infty) + (+\infty) = +\infty$.

(3) $\lim_{x\to+\infty}\left(-1 + \frac{1}{x} + \frac{1}{\sqrt{x}}\right) = \lim_{x\to+\infty}(-1) + \lim_{x\to+\infty} \frac{1}{x} + \lim_{x\to+\infty} \frac{1}{\sqrt{x}} = -1 + 0 + 0 = -1$.

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: (1) $\lim_{x\to 0+}(\frac{1}{x}+3)\to +\infty$, $\lim_{x\to 0+}(-\frac{1}{x})=-\infty$ $\lim_{x\to 0+}\big((\frac{1}{x}+3)+(-\frac{1}{x})\big)=\lim_{x\to 0+}3=3.$
, 3.

(2) $\lim_{x\to+\infty}2x=+\infty$, $\lim_{x\to+\infty}(-x)=-\infty$ $\lim_{x\to+\infty}\big(2x+(-x)\big)=\lim_{x\to+\infty}x=+\infty.$

(3) $\lim_{x\to 0-}(-\frac{1}{x})=+\infty$, $\lim_{x\to 0-}\frac{2}{x}=-\infty$ $\lim_{x\to 0-}(-\frac{1}{x}+\frac{2}{x})=\lim_{x\to 0-}\frac{1}{x}=-\infty.$

(4) $\lim_{x\to 0}(-\frac{2}{x^2})=-\infty.$ ( ) $\lim_{x\to 0}(\frac{2}{x^2}+\frac{1}{x})=+\infty.$, $\lim_{x\to 0}\big((\frac{2}{x^2}+\frac{1}{x})+(-\frac{2}{x^2})\big)=\lim_{x\to 0}\frac{1}{x}$.

$y=f(x)$ $y=g(x)$, $y=f(x)-g(x)$, $f(x)-g(x)=f(x)+(-g(x))$.
:

4.5 . $\lim f(x)$, $\lim g(x)$ $\lim f(x)-\lim g(x)$, $\lim(f(x)-g(x))$

$$\boxed{\lim(f(x)-g(x))=\lim f(x)-\lim g(x).}$$

: (1) $\lim_{x\to+\infty}\big(1-\frac{1}{x}\big)=\lim_{x\to+\infty}1-\lim_{x\to+\infty}\frac{1}{x}=1-0=1.$
(2) $\lim_{x\to 1}\big(x-\frac{1}{\sqrt{|x-1|}}\big)=\lim_{x\to 1}x-\lim_{x\to 1}\frac{1}{\sqrt{|x-1|}}=1-(+\infty)=-\infty.$
(3) $\lim_{x\to 0-}\big(1+\frac{1}{x}-\frac{1}{\sqrt{-x}}\big)=\lim_{x\to 0-}1+\lim_{x\to 0-}\frac{1}{x}-\lim_{x\to 0-}\frac{1}{\sqrt{-x}}=1+(-\infty)-(+\infty)=-\infty.$

$y=f(x)$ $y=g(x)$ $y=f(x)g(x)$ x .

4.6 . $\lim f(x)$, $\lim g(x)$ $\lim f(x)\lim g(x)$, $\lim(f(x)g(x))$

$$\boxed{\lim f(x)g(x)=\lim f(x)\lim g(x).}$$

: $\lim_{x\to\xi}f(x)=\eta$ $\lim_{x\to\xi}g(x)=\zeta$. $\epsilon>0$, $\delta'>0$ $|f(x)-\eta|<\frac{\epsilon}{3|\zeta|+1}$ x $y=f(x)$ $0<|x-\xi|<\delta'$ $\delta''>0$ $|g(x)-\zeta|<\min\{\frac{\epsilon}{3|\eta|+1},\frac{1}{3}\}$ x $y=g(x)$ $0<|x-\xi|<\delta''$. $\delta=\min\{\delta',\delta''\}$, $\delta\le\delta'$ $\delta\le\delta''$. $|f(x)-\eta|<\frac{\epsilon}{3|\zeta|+1}$ $|g(x)-\zeta|<\min\{\frac{\epsilon}{3|\eta|+1},\frac{1}{3}\}$ x $y=f(x)$ $y=g(x)$ $0<|x-\xi|<\delta$. $|f(x)g(x)-\eta\zeta|=|(f(x)-\eta)(g(x)-\zeta)+\eta(g(x)-\zeta)+\zeta(f(x)-\eta)|\le|f(x)-\eta||g(x)-\zeta|+|\eta||g(x)-\zeta|+|\zeta||f(x)-\eta|$. $|f(x)g(x)-\eta\zeta|\le\frac{\epsilon}{3|\zeta|+1}\frac{1}{3}+|\eta|\frac{\epsilon}{3|\eta|+1}+|\zeta|\frac{\epsilon}{3|\zeta|+1}<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon$ x $y=f(x)g(x)$ $0<|x-\xi|<\delta$. $\lim_{x\to\xi}f(x)g(x)=\eta\zeta=\lim_{x\to\xi}f(x)\lim_{x\to\xi}g(x)$.

$\lim_{x\to\xi}f(x)=+\infty$ $\lim_{x\to\xi}g(x)=+\infty$. $M>0$, $\delta'>0$ $f(x)>\sqrt{M}$ x $y=f(x)$ $0<|x-\xi|<\delta'$ $\delta''>0$ $g(x)>\sqrt{M}$ x $y=g(x)$ $0<|x-\xi|<\delta''$. $\delta=\min\{\delta',\delta''\}$, $\delta\le\delta'$ $\delta\le\delta''$. $f(x)>\sqrt{M}$ $g(x)>\sqrt{M}$ x $y=f(x)$ $y=g(x)$ $0<|x-\xi|<\delta$. $f(x)g(x)>\sqrt{M}\sqrt{M}=M$ x $y=f(x)g(x)$ $0<|x-\xi|<\delta$. $\lim_{x\to\xi}f(x)g(x)=+\infty=(+\infty)(+\infty)=\lim_{x\to\xi}f(x)\lim_{x\to\xi}g(x)$.

$\lim_{x\to\xi}f(x)=+\infty$ $\lim_{x\to\xi}g(x)=\zeta>0$. $M>0$, $\delta'>0$ $f(x)>\frac{2M}{\zeta}$ x $y=f(x)$ $0<|x-\xi|<\delta'$ $\delta''>0$ $|g(x)-\zeta|<\frac{\zeta}{2}$ x $y=g(x)$ $0<|x-\xi|<\delta''$. $\delta=\min\{\delta',\delta''\}$, $\delta\le\delta'$ $\delta\le\delta''$. $f(x)>\frac{2M}{\zeta}$ $|g(x)-\zeta|<\frac{\zeta}{2}$ x $y=f(x)$ $y=g(x)$ $0<|x-\xi|<\delta$.

$|g(x)-\zeta| < \frac{\zeta}{2}$ $g(x) > \zeta-\frac{\zeta}{2} = \frac{\zeta}{2}$, , $f(x)g(x) > \frac{2M}{\zeta}\frac{\zeta}{2} = M$ x $y = f(x)g(x)$ $0 < |x-\xi| < \delta$. $\lim_{x\to\xi} f(x)g(x) = +\infty = (+\infty)\zeta = \lim_{x\to\xi} f(x)\lim_{x\to\xi} g(x)$.

.

: (1) . c $\lim f(x)$ $c \cdot \lim f(x)$.

$$\boxed{\lim cf(x) = c\cdot \lim f(x) \qquad (c \neq 0 \quad \lim f(x) = \pm\infty).}$$

$y = f(x)$ $y = c$ c.

(2) $\lim_{x\to 1+} \frac{x+1}{x-1} = \lim_{x\to 1+}(x+1)\lim_{x\to 1+}\frac{1}{x-1} = 2\cdot(+\infty) = +\infty$.

(3) $\lim_{x\to+\infty}(x^2-x) = \lim_{x\to+\infty} x(x-1) = \lim_{x\to+\infty} x \lim_{x\to+\infty}(x-1) = (+\infty)(+\infty-1) = (+\infty)(+\infty) = +\infty$. $\lim_{x\to+\infty}(x^2-x)$.

(4) $\lim_{x\to-1-}\frac{x^2-x+1}{x+1} = \lim_{x\to-1-}(x\cdot x - x + 1)\lim_{x\to-1-}\frac{1}{x+1} = ((-1)(-1)-(-1)+1)(-\infty) = 3(-\infty) = -\infty$.

(5) $\lim_{x\to 0}(\frac{2}{x^2}+\frac{1}{x}) = \lim_{x\to 0}\frac{1}{x^2}\lim_{x\to 0}(2+x) = (+\infty)\cdot 2 = +\infty$. $\lim_{x\to 0}(\frac{2}{x^2}+\frac{1}{x})$ $\lim_{x\to 0}\frac{1}{x}$.

.

: (1) $\lim_{x\to+\infty} x = +\infty$, $\lim_{x\to+\infty}\frac{-2}{x} = 0$ $\lim_{x\to+\infty} x\cdot\frac{-2}{x} = \lim_{x\to+\infty}(-2) = -2$.
-2 .

(2) $\lim_{x\to 0+}\frac{1}{x^2} = +\infty$, $\lim_{x\to 0+} x = 0$ $\lim_{x\to 0+}\frac{1}{x^2}\cdot x = \lim_{x\to 0+}\frac{1}{x} = +\infty$.

(3) $\lim_{x\to 0}\frac{1}{|x|} = +\infty$, $\lim_{x\to 0} x^2 = 0$ $\lim_{x\to 0}\frac{1}{|x|}\cdot x^2 = \lim_{x\to 0}|x| = 0$.

(4) $\lim_{x\to 0}\frac{1}{x^2} = +\infty$, $\lim_{x\to 0} x = 0$ $\lim_{x\to 0}\frac{1}{x^2}\cdot x = \lim_{x\to 0}\frac{1}{x}$.

4.7 $\lim f(x)$ k ,

$$\boxed{\lim f(x)^k = (\lim f(x))^k \qquad (k\ ).}$$

$$\lim f(x)^k = \lim\big(\underbrace{f(x)\cdots f(x)}_{k}\big) = \underbrace{\lim f(x)\cdots\lim f(x)}_{k} = (\lim f(x))^k\,.$$

: (1) $\lim_{x\to+\infty}(1-\frac{1}{x})^3 = \big(\lim_{x\to+\infty}(1-\frac{1}{x})\big)^3 = (1-0)^3 = 1$.

(2) $\lim_{x\to 0-}(\frac{1}{x}-\frac{1}{x^2})^4 = \big(\lim_{x\to 0-}(\frac{1}{x}-\frac{1}{x^2})\big)^4 = \big((-\infty)-(+\infty)\big)^4 = (-\infty)^4 = +\infty$.

(3) k , $\lim_{x\to\xi} x^k = (\lim_{x\to\xi} x)^k = \xi^k$.

$$\boxed{\lim_{x\to\xi} x^k = \xi^k \quad (k\ ).}$$

, , $\xi \neq 0$, $\lim_{x\to\xi}\frac{1}{x^k} = \big(\lim_{x\to\xi}\frac{1}{x}\big)^k = \frac{1}{\xi^k}$.

$$\boxed{\lim_{x\to\xi} x^{-k} = \xi^{-k} \quad (k \quad \xi \neq 0).}$$

(4) k , $\lim_{x\to\xi+} \frac{1}{(x-\xi)^k} = \left(\lim_{x\to\xi+} \frac{1}{x-\xi}\right)^k = (+\infty)^k = +\infty.$

$$\boxed{\lim_{x\to\xi+} (x-\xi)^{-k} = +\infty \quad (k\ ).}$$

k , $\lim_{x\to\xi-} \frac{1}{(x-\xi)^k} = \left(\lim_{x\to\xi-} \frac{1}{x-\xi}\right)^k = (-\infty)^k = +\infty.$ k , $\lim_{x\to\xi-} \frac{1}{(x-\xi)^k} = \left(\lim_{x\to\xi-} \frac{1}{x-\xi}\right)^k = (-\infty)^k = -\infty.$

$$\boxed{\lim_{x\to\xi-} (x-\xi)^{-k} = \begin{cases} +\infty & (k\ ), \\ -\infty & (k\ ). \end{cases}}$$

(5) k , $\lim_{x\to+\infty} x^k = (\lim_{x\to+\infty} x)^k = (+\infty)^k = +\infty.$

$$\boxed{\lim_{x\to+\infty} x^k = +\infty \quad (k\ ).}$$

k , $\lim_{x\to-\infty} x^k = (\lim_{x\to-\infty} x)^k = (-\infty)^k = +\infty.$ k , $\lim_{x\to-\infty} x^k = (\lim_{x\to-\infty} x)^k = (-\infty)^k = -\infty.$

$$\boxed{\lim_{x\to-\infty} x^k = \begin{cases} +\infty & (k\ ), \\ -\infty & (k\ ). \end{cases}}$$

(6) k , $\lim_{x\to\pm\infty} \frac{1}{x^k} = \left(\lim_{x\to\pm\infty} \frac{1}{x}\right)^k = 0^k = 0.$

$$\boxed{\lim_{x\to\pm\infty} x^{-k} = 0 \quad (k\ ).}$$

$y = \frac{1}{f(x)}$ x $y = f(x)$ $f(x) \neq 0.$

4.8 *().* $\lim f(x)$ $\frac{1}{\lim f(x)}$, $\lim f(x) \neq 0,$

$$\boxed{\lim \frac{1}{f(x)} = \frac{1}{\lim f(x)}.}$$

: $\lim_{x\to\xi} f(x) = \eta > 0.$ $\epsilon > 0,$ $\delta > 0$ $|f(x) - \eta| < \min\{\frac{\eta^2\epsilon}{2}, \frac{\eta}{2}\}$ x $y = f(x)$ $0 < |x-\xi| < \delta.$ $|f(x) - \eta| < \frac{\eta^2\epsilon}{2}$ $|f(x) - \eta| < \frac{\eta}{2}$ x $y = f(x)$ $0 < |x-\xi| < \delta.$ $|f(x) - \eta| < \frac{\eta}{2}$ $f(x) > \eta - \frac{\eta}{2} = \frac{\eta}{2}$, , $\left|\frac{1}{f(x)} - \frac{1}{\eta}\right| = \frac{|f(x)-\eta|}{f(x)\eta} < \frac{\frac{\eta^2\epsilon}{2}}{\frac{\eta}{2}\eta} = \epsilon$ x $y = f(x)$ $0 < |x-\xi| < \delta.$ $\lim_{x\to\xi} \frac{1}{f(x)} = \frac{1}{\eta} = \frac{1}{\lim_{x\to\xi} f(x)}.$

$\lim_{x\to\xi} f(x) = +\infty.$ $\epsilon > 0,$ $\delta > 0$ $f(x) > \frac{1}{\epsilon}$ x $y = f(x)$ $0 < |x-\xi| < \delta.$ $0 < \frac{1}{f(x)} < \epsilon$, , $\left|\frac{1}{f(x)} - 0\right| = \frac{1}{f(x)} < \epsilon$ x $y = f(x)$ $0 < |x-\xi| < \delta.$ $\lim_{x\to\xi} \frac{1}{f(x)} = 0 = \frac{1}{+\infty} = \frac{1}{\lim_{x\to\xi} f(x)}.$

$-\infty$.

: (1) $\lim_{x\to1}\frac{1}{x+1}=\frac{1}{\lim_{x\to1}(x+1)}=\frac{1}{2}$.

(2) $\lim_{x\to+\infty}\frac{1}{x^2+x+1}=\frac{1}{\lim_{x\to+\infty}(x^2+x+1)}=\frac{1}{+\infty}=0$.

(3) 0. , $\lim_{x\to0}x=0$ $\lim_{x\to0}\frac{1}{x}$.

. , , , .

$y=f(x)$ – , – ξ, (i) $(a,\xi)\cup(\xi,b)$, ξ, (ii) (ξ,b), ξ ξ, (iii) (a,ξ), ξ ξ. $y=f(x)$ (ξ,b) (ξ), ξ . $y=f(x)$ (a,ξ) (ξ), ξ . , $y=f(x)$ $+\infty$ $(a,+\infty)$ $-\infty$ $(-\infty,b)$.

: (1) $0<\frac{1}{x}<1$ $(1,+\infty)$, $y=\frac{1}{x}$ (, , >0 <1) $+\infty$.

(2) $x(1-x)>0$ $(0,1)$, $y=x(1-x)$ 0 1 . , $y=x(1-x)$ $\frac{1}{4}$, $x(1-x)>0$ $(0,\frac{1}{4})\cup(\frac{1}{4},1)$.

(3) $y=\frac{1}{x(x-1)}$ $(-\infty,0)$ $(0,\frac{1}{2})$ $(\frac{1}{2},1)$ $(1,+\infty)$. , $y=\frac{1}{x(x-1)}$ $-\infty$ 0 1 $+\infty$. : $y=\frac{1}{x(x-1)}$ 0 1!

$\lim f(x)$ $x\to\xi$ $x\to\xi+$ $x\to\xi-$ $x\to+\infty$ $x\to-\infty$ $y=f(x)$ ξ ξ ξ $+\infty$ $-\infty$, , x.

4.9 *()*. (1) $\lim f(x)=0$ $y=f(x)$ x, $\lim\frac{1}{f(x)}=+\infty$.
(2) $\lim f(x)=0$ $y=f(x)$ x, $\lim\frac{1}{f(x)}=-\infty$.

: (1) $\lim_{x\to\xi}f(x)=0$ $f(x)>0$ x $(a,\xi)\cup(\xi,b)$. $M>0$, $\delta'>0$ $|f(x)|=|f(x)-0|<\frac{1}{M}$ x $y=f(x)$ $0<|x-\xi|<\delta'$. $\delta=\min\{\delta',\xi-a,b-\xi\}$, x $0<|x-\xi|<\delta$ $(a,\xi)\cup(\xi,b)$, , $f(x)>0$ x. , $\delta\le\delta'$, $\frac{1}{f(x)}=\frac{1}{|f(x)|}>M$ x $0<|x-\xi|<\delta$. $\lim_{x\to\xi}\frac{1}{f(x)}=+\infty$.
(2) (1).

: $\lim_{x\to+\infty}(\frac{1}{x}-\frac{1}{x^2})=\lim_{x\to+\infty}\frac{1}{x}-\lim_{x\to+\infty}\frac{1}{x^2}=0-0=0$. $\frac{1}{x}-\frac{1}{x^2}>0$ $x>1$. $\lim_{x\to+\infty}\frac{1}{\frac{1}{x}-\frac{1}{x^2}}=+\infty$.

$y=f(x)$ $y=g(x)$, $y=\frac{f(x)}{g(x)}$, .

4.10 . $\lim f(x)$, $\lim g(x)$ $\frac{\lim f(x)}{\lim g(x)}$, $\lim\frac{f(x)}{g(x)}$

$$\boxed{\lim\frac{f(x)}{g(x)}=\frac{\lim f(x)}{\lim g(x)}}.$$

: (1) $\lim_{x\to1}\frac{x^2+x}{x-2}=\frac{\lim_{x\to1}(x^2+x)}{\lim_{x\to1}(x-2)}=\frac{1^2+1}{1-2}=-2$.

(2) $\lim_{x\to+\infty}\frac{x^2-x+1}{x+2}=\lim_{x\to+\infty}x\frac{1-\frac{1}{x}+\frac{1}{x^2}}{1+\frac{2}{x}}=\lim_{x\to+\infty}x\frac{\lim_{x\to+\infty}(1-\frac{1}{x}+\frac{1}{x^2})}{\lim_{x\to+\infty}(1+2\frac{1}{x})}=(+\infty)\frac{1-0+0}{1+2\cdot0}=+\infty$. $\lim_{x\to+\infty}\frac{x^2-x+1}{x+2}$.

$\frac{0}{0}$ $\frac{+\infty}{+\infty}$.

: (1) $\lim_{x\to 0+}(-2x)=0$, $\lim_{x\to 0+} x=0$ $\lim_{x\to 0+}\frac{-2x}{x}=\lim_{x\to 0+}(-2)=-2$. -2 .

(2) $\lim_{x\to 0+} x=0$, $\lim_{x\to 0+} x^2=0$ $\lim_{x\to 0+}\frac{x}{x^2}=\lim_{x\to 0+}\frac{1}{x}=+\infty$.

(3) $\lim_{x\to 0+} x^2=0$, $\lim_{x\to 0+} x=0$ $\lim_{x\to 0+}\frac{x^2}{x}=\lim_{x\to 0+} x=0$.

(4) $\lim_{x\to +\infty} 5x=+\infty$, $\lim_{x\to +\infty} x=+\infty$ $\lim_{x\to +\infty}\frac{5x}{x}=\lim_{x\to +\infty} 5=5$. 5.

(5) $\lim_{x\to +\infty} x^2=+\infty$, $\lim_{x\to +\infty} x=+\infty$ $\lim_{x\to +\infty}\frac{x^2}{x}=\lim_{x\to +\infty} x=+\infty$.

(6) $\lim_{x\to +\infty} x=+\infty$, $\lim_{x\to +\infty} x^2=+\infty$ $\lim_{x\to +\infty}\frac{x}{x^2}=\lim_{x\to +\infty}\frac{1}{x}=0$.

4.11 $\lim f(x)$, $\lim|f(x)|$

$$\boxed{\lim|f(x)|=|\lim f(x)|.}$$

: $\lim_{x\to\xi} f(x)=\eta$. $\epsilon>0$, $\delta>0$ $|f(x)-\eta|<\epsilon$ x $y=f(x)$ $0<|x-\xi|<\delta$. $\big||f(x)|-|\eta|\big|\le|f(x)-\eta|$, $\big||f(x)|-|\eta|\big|<\epsilon$ x $y=f(x)$ $0<|x-\xi|<\delta$. $\lim_{x\to\xi}|f(x)|=|\eta|=|\lim_{x\to\xi} f(x)|$.

$\lim_{x\to\xi} f(x)=+\infty$ $\lim_{x\to\xi} f(x)=-\infty$. $M>0$, $\delta>0$ $f(x)>M$ $f(x)<-M$, , x $y=f(x)$ $0<|x-\xi|<\delta$. $|f(x)|>M$ x $y=f(x)$ $0<|x-\xi|<\delta$. $\lim_{x\to\xi}|f(x)|=+\infty=|\pm\infty|=|\lim_{x\to\xi} f(x)|$.

: (1) $\lim_{x\to 1}|x-2|=|\lim_{x\to 1}(x-2)|=|1-2|=1$.

(2) $\lim_{x\to 0-}|\frac{1}{x}-\frac{1}{x^2}|=|\lim_{x\to 0-}(\frac{1}{x}-\frac{1}{x^2})|=|-\infty-(+\infty)|=|-\infty|=+\infty$.

(3) 4.11. $y=f(x)=\frac{|x|}{x}$ $(-\infty,0)\cup(0,+\infty)$. $\lim_{x\to 0}|f(x)|=\lim_{x\to 0}\left|\frac{|x|}{x}\right|=\lim_{x\to 0} 1=1$. , $\lim_{x\to 0} f(x)$.

. .

$z=g(f(x))$: $y=f(x)$ $z=g(y)$. , , $y=f(x)$ $z=g(y)$. . , $\lim_{x\to\xi} f(x)=\eta$ $\lim_{y\to\eta} g(y)=\zeta$. , , « ». x ξ $\neq\xi$, $y=f(x)$ η , , η. , , $y=f(x)\neq\eta$ ξ, $y=f(x)$ η $\neq\eta$. , $z=g(f(x))=g(y)$ ζ. , , x ξ $\neq\xi$ $z=g(f(x))=g(y)$ ζ. $\lim_{x\to\xi} g(f(x))=\zeta$.

, $z=g(f(x))$, y. , (i) x $y=f(x)$ $f(x)$ y (ii) $\lim_{x\to}$ $\lim_{y\to}$.

.

4.12 . $z=g(f(x))$ $y=f(x)$ $z=g(y)$. $\lim f(x)=\eta$ $y=f(x)$ η x $\lim_{y\to\eta} g(y)$, $\lim g(f(x))=\lim_{y\to\eta} g(y)$.

η $\eta\pm$ $\pm\infty$. :

$$\boxed{\lim g(f(x))=\begin{cases}\lim_{y\to\eta} g(y), & f(x)\to\eta\ \big(f(x)\neq\eta\quad x\big)\\ \lim_{y\to\eta+} g(y), & f(x)\to\eta\ \big(f(x)>\eta\quad x\big)\\ \lim_{y\to\eta-} g(y), & f(x)\to\eta\ \big(f(x)<\eta\quad x\big)\\ \lim_{y\to+\infty} g(y), & f(x)\to+\infty\\ \lim_{y\to-\infty} g(y), & f(x)\to-\infty\end{cases}}$$

: $y = f(x)$ () $(a, \xi) \cup (\xi, b)$ $z = g(x)$ () $(c, \eta) \cup (\eta, d)$ $\lim_{x\to\xi} f(x) = \eta$ $\lim_{y\to\eta} g(y) = \zeta$. $f(x) \neq \eta$ x $(a', \xi)\cup(\xi, b')$, $a \le a' < \xi < b' \le b$. $\lim_{x\to\xi} g(f(x)) = \zeta$.

$\epsilon > 0$, $\delta' > 0$ $|g(y)-\zeta| < \epsilon$ y $z = g(y)$ $0 < |y-\eta| < \delta'$. , $\delta'' > 0$ $|f(x)-\eta| < \delta'$ x $y = f(x)$ $0 < |x-\xi| < \delta''$. $\delta = \min\{\delta'', \xi - a', b' - \xi\}$. x $0 < |x-\xi| < \delta$ $0 < |x-\xi| < \delta''$, , $(a', \xi)\cup(\xi, b')$. x $0 < |x-\xi| < \delta$ $|f(x)-\eta| < \delta'$ $f(x) \neq \eta$, , $0 < |f(x)-\eta| < \delta'$, , $\Big|g(f(x)) - \zeta\Big| < \epsilon$.

: (1) $\lim_{x\to 0} \frac{(\sqrt{x}+1)^4}{(\sqrt{x}+1)^8+(\sqrt{x}+1)^{13}+5}$.

$y = \sqrt{x}+1$ $z = \frac{(\sqrt{x}+1)^4}{(\sqrt{x}+1)^8+(\sqrt{x}+1)^{13}+5}$ $z = \frac{y^4}{y^8+y^{13}+5}$. $\lim_{x\to 0} y = \lim_{x\to 0}(\sqrt{x}+1) = 0+1 = 1$ $y = \sqrt{x}+1 > 1$ x 0.

$\lim_{x\to 0} \frac{(\sqrt{x}+1)^4}{(\sqrt{x}+1)^8+(\sqrt{x}+1)^{13}+5} = \lim_{y\to 1+} \frac{y^4}{y^8+y^{13}+5} = \frac{1}{7}$.

(2) $\lim_{x\to 1+} \big(\frac{1}{\sqrt{x-1}} - \frac{1}{(x-1)^3} + 1\big)$.

$y = \frac{1}{\sqrt{x-1}}$ $z = \frac{1}{\sqrt{x-1}} - \frac{1}{(x-1)^3} + 1$ $z = y - y^6 + 1$. $\lim_{x\to 1+} y = \lim_{x\to 1+} \sqrt{\frac{1}{x-1}} = +\infty$.

$\lim_{x\to 1+} \big(\frac{1}{\sqrt{x-1}} - \frac{1}{(x-1)^3} + 1\big) = \lim_{y\to+\infty}(y - y^6 + 1) = \lim_{y\to+\infty} y^6(-1 + y^{-5} + y^{-6}) = (+\infty)(-1+0+0) = -\infty$.

(3) $\lim_{x\to+\infty} \big((\frac{x-1}{x^2+x+1})^{-2} + (\frac{x-1}{x^2+x+1})^{-4}\big)$.

$y = \frac{x-1}{x^2+x+1}$ $z = (\frac{x-1}{x^2+x+1})^{-2} + (\frac{x-1}{x^2+x+1})^{-4}$ $z = y^{-2}+y^{-4}$. $\lim_{x\to+\infty} y = \lim_{x\to+\infty} \frac{x-1}{x^2+x+1} = \lim_{x\to+\infty} \frac{1}{x} \frac{1-\frac{1}{x}}{1+\frac{1}{x}+\frac{1}{x^2}} = 0 \cdot \frac{1-0}{1+0+0} = 0$ $y = \frac{x-1}{x^2+x+1} > 0$ x $+\infty$ (, , $x > 1$).

$\lim_{x\to+\infty} \big((\frac{x-1}{x^2+x+1})^{-2} + (\frac{x-1}{x^2+x+1})^{-4}\big) = \lim_{y\to 0+}(y^{-2}+y^{-4}) = +\infty$.

(4) , .

$$\lim_{x\to 0+} g\Big(\frac{1}{x}\Big) = \lim_{y\to+\infty} g(y), \qquad \lim_{x\to 0-} g\Big(\frac{1}{x}\Big) = \lim_{y\to-\infty} g(y),$$

$$\lim_{x\to+\infty} g\Big(\frac{1}{x}\Big) = \lim_{y\to 0+} g(y), \qquad \lim_{x\to-\infty} g\Big(\frac{1}{x}\Big) = \lim_{y\to 0-} g(y),$$

$$\lim_{x\to+\infty} g(-x) = \lim_{y\to-\infty} g(y), \qquad \lim_{x\to-\infty} g(-x) = \lim_{y\to+\infty} g(y),$$

$$\lim_{x\to\xi} g(ax+b) = \lim_{y\to a\xi+b} g(y).$$

$y = \frac{1}{x}$, $y = -x$ $y = ax+b$.

. .

, - lim $\lim_{x\to\xi}$ $\lim_{x\to\xi\pm}$ $\lim_{x\to\pm\infty}$. , $(a,\xi)\cup(\xi,b)$ $\lim_{x\to\xi}$.

4.13 $f(x) \le g(x)$ x.
(1) $\lim f(x) = +\infty$, $\lim g(x) = +\infty$.
(2) $\lim g(x) = -\infty$, $\lim f(x) = -\infty$.

: (1) $M > 0$. $\lim_{x\to\xi} f(x) = +\infty$, $\delta' > 0$ $f(x) > M$ x $y = f(x)$ $0 < |x-\xi| < \delta'$. $(a,\xi)\cup(\xi,b)$ $g(x) \geq f(x)$ x . $\delta = \min\{\delta', \xi - a, b - \xi\}$, $\delta \leq \delta'$, , x $0 < |x-\xi| < \delta$ $(a,\xi)\cup(\xi,b)$. $g(x) \geq f(x) > M$ x $0 < |x-\xi| < \delta$. $\lim_{x\to\xi} g(x) = +\infty$.
(2) (1).

: (1) $y = \frac{x^2+x-1}{x}$. $\frac{x^2+x-1}{x} \geq \frac{x^2}{x} = x$ x $(1,+\infty)$ $\lim_{x\to+\infty} x = +\infty$, $\lim_{x\to+\infty} \frac{x^2+x-1}{x} = +\infty$.

(2) $\frac{2}{x^2} + \frac{1}{x} \geq \frac{2}{x^2} - \frac{1}{x^2} = \frac{1}{x^2}$ x $(-1,0)\cup(0,1)$ $\lim_{x\to 0} \frac{1}{x^2} = +\infty$, $\lim_{x\to 0}(\frac{2}{x^2} + \frac{1}{x}) = +\infty$.

4.14 $\lim f(x) = \eta$ $\lim g(x) = \zeta$ $f(x) \leq g(x)$ x, $\eta \leq \zeta$.

: $\lim_{x\to\xi} f(x) = \eta$ $\lim_{x\to\xi} g(x) = \zeta$ $f(x) \leq g(x)$ x $(a,\xi)\cup(\xi,b)$.
( ) $\zeta < \eta$. $\epsilon = \frac{\eta-\zeta}{2} > 0$, $\delta' > 0$ $|f(x)-\eta| < \frac{\eta-\zeta}{2}$ x $y = f(x)$ $0 < |x-\xi| < \delta'$ $\delta'' > 0$ $|g(x)-\zeta| < \frac{\eta-\zeta}{2}$ x $y = g(x)$ $0 < |x-\xi| < \delta''$. $\delta = \min\{\delta', \delta'', \xi - a, b - \xi\}$, x $0 < |x-\xi| < \delta$ $(a,\xi)\cup(\xi,b)$, , $f(x) \leq g(x)$ x. , $\delta \leq \delta'$ $\delta \leq \delta''$, $|f(x)-\eta| < \frac{\eta-\zeta}{2}$ $|g(x)-\zeta| < \frac{\eta-\zeta}{2}$ x $0 < |x-\xi| < \delta$. $f(x) > \eta - \frac{\eta-\zeta}{2} = \frac{\eta+\zeta}{2}$ $g(x) < \zeta + \frac{\eta-\zeta}{2} = \frac{\eta+\zeta}{2}$ x $0 < |x-\xi| < \delta$. $f(x) > \frac{\eta+\zeta}{2} > g(x)$ x $0 < |x-\xi| < \delta$.

: (1) $\lim f(x)$ $f(x) \leq u$ x, $\lim f(x) \leq u$. , $y = u$, $f(x) \leq u$ x $\lim u = u$, $\lim f(x) \leq u$.

(2) $\lim f(x)$ $f(x) \geq l$ x, $\lim f(x) \geq l$. (1).

(3) $\lim f(x)$ x $y = f(x)$ $[l,u]$, $\lim f(x)$ $[l,u]$. (1) (2).

4.15 . $\lim f(x) = \lim h(x) = \rho$ $f(x) \leq g(x) \leq h(x)$ x, $\lim g(x) = \rho$.

: $\lim_{x\to\xi} f(x) = \rho$ $\lim_{x\to\xi} h(x) = \rho$ $f(x) \leq g(x) \leq h(x)$ x $(a,\xi)\cup(\xi,b)$.
$\epsilon > 0$, $\delta' > 0$ $|f(x)-\rho| < \epsilon$ x $y = f(x)$ $0 < |x-\xi| < \delta'$ $\delta'' > 0$ $|h(x)-\rho| < \epsilon$ x $y = h(x)$ $0 < |x-\xi| < \delta''$. $\delta = \min\{\delta', \delta'', \xi - a, b - \xi\}$, x $0 < |x-\xi| < \delta$ $(a,\xi)\cup(\xi,b)$, $f(x) \leq g(x) \leq h(x)$ x. , $\delta \leq \delta'$ $\delta \leq \delta''$, $|f(x)-\rho| < \epsilon$ $|h(x)-\rho| < \epsilon$ x $0 < |x-\xi| < \delta$. $f(x) > \rho - \epsilon$ $h(x) < \rho + \epsilon$ x $0 < |x-\xi| < \delta$. , $\rho - \epsilon < f(x) \leq g(x) \leq h(x) < \rho + \epsilon$, , $|g(x)-\rho| < \epsilon$ x $0 < |x-\xi| < \delta$. $\lim_{x\to\xi} g(x) = \rho$.

: (1) $-\frac{1}{x^2} \leq f(x) < \frac{1}{x}$ $x \geq 3$. $\lim_{x\to+\infty}(-\frac{1}{x^2}) = 0$ $\lim_{x\to+\infty} \frac{1}{x} = 0$, $\lim_{x\to+\infty} f(x) = 0$.

(2) $x - 1 < [x] \leq x$ x. $1 - \frac{1}{x} < \frac{[x]}{x} \leq 1$ $x > 0$. $\lim_{x\to+\infty}(1 - \frac{1}{x}) = 1 - 0 = 1$ $\lim_{x\to+\infty} 1 = 1$, $\lim_{x\to+\infty} \frac{[x]}{x} = 1$.

(3) $-\frac{1}{|x|} \leq \frac{\sin x}{x} \leq \frac{1}{|x|}$ x $(-\infty,0)$ $(0,+\infty)$. $\lim_{x\to+\infty}\big(-\frac{1}{|x|}\big) = \lim_{x\to+\infty} \frac{1}{|x|} = 0$, $\lim_{x\to+\infty} \frac{\sin x}{x} = 0$. $\lim_{x\to-\infty} \frac{\sin x}{x} = 0$.

4.16 (1) $\lim f(x) < u$, $f(x) < u$ x.
(2) $\lim f(x) > l$, $f(x) > l$ x.

: (1) $y = f(x)$ () $(a,\xi)\cup(\xi,b)$ $\lim_{x\to\xi} f(x) = \eta < u$. $\epsilon = u - \eta > 0$. $\delta' > 0$ $|f(x)-\eta| < u - \eta$ x $0 < |x-\xi| < \delta'$. $\delta = \min\{\delta', \xi - a, b - \xi\}$, $\delta \leq \delta'$, $(\xi-\delta,\xi)\cup(\xi,\xi+\delta)$ $(a,\xi)\cup(\xi,b)$. , $|f(x)-\eta| < u - \eta$ x $(\xi-\delta,\xi)\cup(\xi,\xi+\delta)$.

$|f(x)-\eta|<u-\eta$ $f(x)<\eta+(u-\eta)=u$ $f(x)<u$ x $(\xi-\delta,\xi)\cup(\xi,\xi+\delta)$. $f(x)<u$ ξ.

$y=f(x)$ () $(a,\xi)\cup(\xi,b)$ $\lim_{x\to\xi}f(x)=-\infty$. $M>0$ $\geq -u$ (, $M=-u$, $u<0$, $M=1$, $u\geq 0$) M. $\delta'>0$ $f(x)<-M$ x $0<|x-\xi|<\delta'$. $\delta=\min\{\delta',\xi-a,b-\xi\}$, , , $f(x)<-M$ x $(\xi-\delta,\xi)\cup(\xi,\xi+\delta)$. $M\geq -u$, $f(x)<-M$ $f(x)<u$ $f(x)<u$ x $(\xi-\delta,\xi)\cup(\xi,\xi+\delta)$. $f(x)<u$ ξ.

(2) (1).

: (1) $\lim_{x\to 1}\frac{x^7-2x^6+x^4+3x^2-5x+1}{x^8+6x^5-x^4+22x^2+1}=-\frac{1}{29}>-\frac{1}{28}$. $\frac{x^7-2x^6+x^4+3x^2-5x+1}{x^8+6x^5-x^4+22x^2+1}>-\frac{1}{28}$ 1. a,b x $(a,1)\cup(1,b)$.
a,b .

(2) $\lim_{x\to+\infty}\frac{x+\frac{1}{x}+1}{(x+\frac{1}{x})^2+1}=\lim_{t\to+\infty}\frac{t+1}{t^2+1}=0$. $\frac{x+\frac{1}{x}+1}{(x+\frac{1}{x})^2+1}<\frac{1}{5}$ $+\infty$. a $\frac{x+\frac{1}{x}+1}{(x+\frac{1}{x})^2+1}<\frac{1}{5}$ x $(a,+\infty)$.
, , a.

. .

4.17 $\lim f(x)$, $y=f(x)$ x.

: $y=f(x)$ $(a,\xi)\cup(\xi,b)$ $\lim_{x\to\xi}f(x)=\eta$. .
$\epsilon=1$, . $\delta'>0$ $|f(x)-\eta|<1$ x $y=f(x)$ $0<|x-\xi|<\delta'$. $\delta=\min\{\delta',\xi-a,b-\xi\}$, $(\xi-\delta,\xi)\cup(\xi,\xi+\delta)$ $(a,\xi)\cup(\xi,b)$. , $\delta\leq\delta'$, , $|f(x)-\eta|<1$ x $0<|x-\xi|<\delta$, x $(\xi-\delta,\xi)\cup(\xi,\xi+\delta)$. $|f(x)-\eta|<1$ $1-\eta<f(x)<1+\eta$, $y=f(x)$ $(\xi-\delta,\xi)\cup(\xi,\xi+\delta)$, ξ.

: (1) $y=\frac{1}{x}$ $\lim_{x\to+\infty}\frac{1}{x}=0$ $+\infty$. $y=\frac{1}{x}$. , , , $(-\infty,0)\cup(0,+\infty)$ $(0,+\infty)$. , , $(a,+\infty)$ $y=\frac{1}{x}$. , $(1,+\infty)$ $0\leq\frac{1}{x}\leq 1$ x $(1,+\infty)$.

(2) $\lim_{x\to-\infty}\frac{x^5+1}{2x^5-x^3+x^2+8x+1}=\lim_{x\to-\infty}\frac{1+x^{-5}}{2-x^{-2}+x^{-3}+8x^{-4}+x^{-5}}=\frac{1}{2}$. $y=\frac{x^5+1}{2x^5-x^3+x^2+8x+1}$ $-\infty$. , $(-\infty,b)$. , (b) !

4.18 (1) $\lim f(x)=+\infty$, $y=f(x)$ x.
(2) $\lim f(x)=-\infty$, $y=f(x)$ x.

: $y=f(x)$ $(a,\xi)\cup(\xi,b)$ $\lim_{x\to\xi}f(x)=+\infty$. .
$M=1$, . $\delta'>0$ $f(x)>1$ x $0<|x-\xi|<\delta'$. $\delta=\min\{\delta',\xi-a,b-\xi\}$, $(\xi-\delta,\xi)\cup(\xi,\xi+\delta)$ $(a,\xi)\cup(\xi,b)$. , $\delta\leq\delta'$, , $f(x)>1$ x $0<|x-\xi|<\delta$, x $(\xi-\delta,\xi)\cup(\xi,\xi+\delta)$. $y=f(x)$ $(\xi-\delta,\xi)\cup(\xi,\xi+\delta)$, ξ.
, , $y=f(x)$ $(c,\xi)\cup(\xi,d)$. u $f(x)\leq u$ x $(c,\xi)\cup(\xi,d)$. $\lim_{x\to\xi}f(x)\leq u$ $\lim_{x\to\xi}f(x)=+\infty$.

: (1) $\lim_{x\to+\infty}(x-\frac{1}{x})=+\infty$, $y=x-\frac{1}{x}$ $+\infty$. , $y=x-\frac{1}{x}$ $(1,+\infty)$, , $x-\frac{1}{x}\geq 0$ x $(1,+\infty)$.
, $\lim_{x\to 0+}(x-\frac{1}{x})=-\infty$, $y=x-\frac{1}{x}$ 0 . , $(0,1)$, , $x-\frac{1}{x}\leq 0$ x $(0,1)$.

(2) $\lim_{x\to 0-}\frac{(x+\frac{1}{x})^7+1}{3(x+\frac{1}{x})^5+7(x+\frac{1}{x})^4-(x+\frac{1}{x})^3+2}=\lim_{t\to-\infty}\frac{t^7+1}{3t^5+7t^4-t^3+2}=\lim_{t\to-\infty}\frac{t^2(1+t^{-7})}{3+7t^{-1}-t^{-2}+2t^{-5}}=+\infty$. $y=\frac{(x+\frac{1}{x})^7+1}{3(x+\frac{1}{x})^5+7(x+\frac{1}{x})^4-(x+\frac{1}{x})^3+2}$ 0 . a $(a,0)$. , a

$(a,0)$.

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. .

1. $x\to 0\pm$ $x\to\pm\infty$.

$$y=\begin{cases} x, & x\ge 0,\\ \frac{1}{x}, & x<0,\end{cases}\qquad y=\begin{cases} |x-1|^{-\frac{1}{4}}, & |x|\ge 1,\\ \frac{1}{x}, & |x|<1.\end{cases}$$

2. ξ $\lim_{x\to\xi\pm}[x]$. $y=[x]$!
(: $(\xi-1,\xi)$ $(\xi,\xi+1)$, $\cdot$ $y=[x]=\xi-1$ $y=[x]=\xi$.)
ξ.
(: $([\xi],\xi)\cup(\xi,[\xi]+1)$, - $\cdot$ $y=[x]=[\xi]$.)
ξ $\lim_{x\to\xi}[x]$;

3. : $\lim_{x\to\pm\infty}\left[\frac{1}{x}\right]$ $\lim_{x\to\pm\infty} x\left[\frac{1}{x}\right]$.
(: $(-\infty,-1)$ $(1,+\infty)$.)

. .

1. $x\to 1\pm$.

$$y=\frac{1+x}{1+x^3},\ y=\frac{1-x^2}{1-x^3},\ y=\frac{1}{1+(x-1)^{-3}},\ y=\frac{1}{(x-1)^2-|x-1|},$$

$$y=(x-1)^{-1}+|x-1|^{-\frac{1}{2}},\quad y=\left((x-1)^{-1}+|x-1|^{-\frac{1}{2}}\right)^2.$$

2. $x\to\pm\infty$.

$$y=-x+\frac{1}{x}-\frac{2}{x^2},\quad y=\frac{x^3+1}{x^2+1},\quad y=\frac{x^2-4x+3}{x-3},\quad y=\frac{2x^2+3x+1}{x^2+1},$$

$$y=\frac{x^2+x+1}{x^3+1},\quad y=\left(1+\frac{1}{x}\right)^4,\quad y=\frac{(2x+1)^3(3x^2+2)^2(x+4)^{13}}{x^{20}}.$$

3. $f(x)\ne 1$ x $y=f(x)$. $\lim f(x)=2$ $\lim\frac{f(x)+3}{f(x)-1}=5$.

4. $\lim(f(x))^2=1$, $\lim f(x)=1$ $\lim f(x)=-1$; $y=f(x)=\frac{|x|}{x}$ $x\to 0$.

. .

1. ( ) .

$$\lim_{x\to 0}\left(\left(\frac{x^2+1}{x^5+2x^3+2}\right)^8+3\left(\frac{x^2+1}{x^5+2x^3+2}\right)^4+1\right),$$

$$\lim_{x\to 1+}\frac{\sqrt{\frac{x+1}{x-1}}-\sqrt[4]{\frac{x+1}{x-1}}+1}{\sqrt{\frac{x+1}{x-1}}+\sqrt[4]{\frac{x+1}{x-1}}+1}\,,\quad \lim_{x\to 0-}\sqrt[3]{\Big(\sqrt{|x|}+\frac{1}{x}\Big)^3-2\Big(\sqrt{|x|}+\frac{1}{x}\Big)^2+1}\,,$$

$$\lim_{x\to+\infty}\Big(\sqrt[3]{\frac{1}{x^2+x+1}}+\sqrt{\frac{x+1}{\sqrt{x}+1}}\Big).$$

2. .

$$\lim_{x\to\pm\infty} g(x^2)=\lim_{y\to+\infty} g(y),\quad \lim_{x\to 0+} g(\sqrt{x})=\lim_{y\to 0+} g(y),$$

$$\lim_{x\to 0} g\Big(\frac{1}{x^2}\Big)=\lim_{y\to+\infty} g(y).$$

3. $\lim_{x\to+\infty} f(x)$ $f(\sqrt{x})=-3(f(x))^2+1$ $x>0$.

. .

1. 4.13, 4.14 4.15.
2. $\lim_{x\to 1} f(x)$, $x-|x-1|^{\frac{1}{2}}<f(x)\le x+|x-1|^{\frac{1}{2}}$ x $(0,1)\cup(1,\frac{3}{2})$.
3. $\lim_{x\to-\infty} f(x)$, $\frac{x+1}{2x-1}<f(x)<\frac{x-1}{2x+1}$ $x\le -7$.
4. $\lim_{x\to 1\pm} f(x)$, $(x-1)f(x)\ge 1$ x $0<|x-1|<\frac{1}{4}$.
5. B , $[a]\le a<[a]+1$.

$$\lim_{x\to\pm\infty}[x],\quad \lim_{x\to\pm\infty}\frac{[x]}{x}\,,\quad \lim_{x\to\pm\infty}\frac{[x^2]}{x}\,,\quad \lim_{x\to+\infty}\frac{[\sqrt{x}]}{x}\,,\quad \lim_{x\to 1\pm}\Big[\frac{1}{x-1}\Big].$$

6. $\lim f(x)=+\infty$ $y=g(x)$ x. $\lim(f(x)+g(x))=+\infty$.
   $\lim f(x)=-\infty$ $y=g(x)$ x. $\lim(f(x)+g(x))=-\infty$.
7. $\lim f(x)=0$ $y=g(x)$ x. $\lim f(x)g(x)=0$.
8. $\lim f(x)=+\infty$ $-\infty$ $y=g(x)$ x. $\lim f(x)g(x)=+\infty$ $-\infty$, - .
   $\lim f(x)=+\infty$ $-\infty$ $y=g(x)$ x. $\lim f(x)g(x)=-\infty$ $+\infty$, - .
9. 4.16.
10. , :
    (i) $\frac{3x^2+7x+1}{x^2-5x+1}<\frac{301}{100}$ $+\infty$.
    , a x $(a,+\infty)$.
    (ii) $\frac{x^8+1}{4x^4-x^2+2x-1}<\frac{3}{4}$ 1.
    (iii) $(x-\frac{1}{x})^5-(\sqrt{x}-\frac{1}{\sqrt{x}})^3>10^7$ $+\infty$.
    (iv) $-10^{-8}<\frac{x^4+13x^3+25x^2+33}{x^5+2x+1}<10^{-7}$ $-\infty$.

11. $\lim f(x) < \lim g(x)$. $f(x) < g(x)$ x.
(: : $y = g(x) - f(x)$. : ρ $\lim f(x) < \rho < \lim g(x)$ 4.16 $\lim f(x) < \rho$ $\rho < \lim g(x)$. .)

. .

1. 4.17 4.18.

2. , - 0 0 0 .

$$y = \sqrt{x}\,, \quad y = \sqrt{|x|}\,, \quad y = \begin{cases} \frac{1}{x}\,, & x < 0, \\ -\frac{1}{\sqrt{x}}\,, & x > 0, \end{cases}, \quad y = \frac{1}{x^3}\,,$$

$$y = \begin{cases} \frac{1}{x}\,, & x < 0, \\ x\,, & x \geq 0, \end{cases} \quad y = \begin{cases} \frac{1}{|x|}\,, & x < 0, \\ \frac{1}{x-1}\,, & x \geq 0, \end{cases}, \quad y = -\frac{1}{x} + \frac{1}{\sqrt{|x|}}\,.$$

$(a, 0)$ $(0, b)$ $(a, 0) \cup (0, b)$;

3. , - $+\infty$ $-\infty$.

$$y = x^2\,, \quad y = -x^3\,, \quad y = x^{-2}\,, \quad y = |x|^{\frac{1}{2}}\,, \quad y = (-x)^{-\frac{1}{3}}\,, \quad y = -|x|.$$

$(a, +\infty)$ $(-\infty, b)$;

4.4 .

$\lim_{x\to\xi} f(x) = \eta$. (x_n) $y = f(x)$ ξ ξ. , $\lim_{n\to+\infty} x_n = \xi$. $\big(f(x_n)\big)$, . « ». n . x_n ξ $\neq \xi$. $x = x_n$ ξ $\neq \xi$. , $y = f(x_n) = f(x)$ η. , , n , $f(x_n)$ η. $\lim_{n\to+\infty} f(x_n) = \eta$. .

4.19 n x_n $y = f(x)$. $\lim_{x\to\xi} f(x)$, $\lim_{n\to+\infty} x_n = \xi$ $x_n \neq \xi$ n, $\lim_{n\to+\infty} f(x_n) = \lim_{x\to\xi} f(x)$.
ξ $\xi\pm$ $\pm\infty$. :

$$\lim_{n\to+\infty} f(x_n) = \begin{cases} \lim_{x\to\xi} f(x), & x_n \to \xi \ (x_n \neq \xi \quad n) \\ \lim_{x\to\xi+} f(x), & x_n \to \xi \ (x_n > \xi \quad n) \\ \lim_{x\to\xi-} f(x), & x_n \to \xi \ (x_n < \xi \quad n) \\ \lim_{x\to+\infty} f(x), & x_n \to +\infty \\ \lim_{x\to-\infty} f(x), & x_n \to -\infty \end{cases}$$

: $\lim_{x\to\xi} f(x) = \eta$ (x_n) x_n $\neq \xi$, , $\lim_{n\to+\infty} x_n = \xi$. $\lim_{n\to+\infty} f(x_n) = \eta$.
$\epsilon > 0$. $\lim_{x\to\xi} f(x) = \eta$, $\delta > 0$ $|f(x) - \eta| < \epsilon$ x $0 < |x - \xi| < \delta$. $\lim_{n\to+\infty} x_n = \xi$, n_0 $|x_n - \xi| < \delta$ $n \geq n_0$. $x_n \neq \xi$, $0 < |x_n - \xi| < \delta$ $n \geq n_0$. , x_n $y = f(x)$, $|f(x_n) - \eta| < \epsilon$ $n \geq n_0$. $\lim_{n\to+\infty} f(x_n) = \eta$.

¨ .

: (1) $\lim_{x\to+\infty} x^2 = +\infty$ $\lim_{n\to+\infty} n^2 = +\infty$. (x_n) $x_n = n$ $+\infty$

$y = x^2$.

(2) $\lim_{x\to 0}\sqrt{x} = 0$ $\lim_{n\to+\infty}\frac{1}{\sqrt{n}} = 0$. (x_n) $x_n = \frac{1}{n}$ 0, $[0,+\infty)$ $y = \sqrt{x}$ $\neq 0$.

(3) $\big(1+(1+\frac{1}{n})^{2n} - 3(1+\frac{1}{n})^{3n}\big)$ $y = 1 + x^2 - 3x^3$ (x_n) $x_n = (1+\frac{1}{n})^n$. $\lim_{n\to+\infty} x_n = e$ $x_n \neq e$ n. , $\lim_{x\to e}(1 + x^2 - 3x^3) = 1 + e^2 - 3e^3$. $\lim_{n\to+\infty}\big(1+(1+\frac{1}{n})^{2n} - 3(1+\frac{1}{n})^{3n}\big) = \lim_{n\to+\infty}(1+{x_n}^2 - 3{x_n}^3) = 1 + e^2 - 3e^3$.

4.19 . , , , . . $y = f(x)$ x . (x_n) x, $(f(x_n))$. x . , , $(f(x_n))$.

: (1) (x_n) $x_n = \frac{(-1)^{n-1}}{n}$ 0, $\neq 0$ $y = \frac{1}{x}$. (y_n) $y_n = \frac{1}{x_n} = (-1)^{n-1}n$. $\lim_{x\to 0}\frac{1}{x}$.

(2) $\lim_{x\to+\infty}(-1)^{[x]}$ $y = (-1)^{[x]}$. 1 3.4. $[n, n+1)$ $(n \in \mathbf{Z})$ $y = (-1)^{[x]} = (-1)^n$. x , $+1$ -1. , $\lim_{x\to+\infty}(-1)^{[x]}$.
. x $+\infty$, : , , $x_n = n$ $(n \in \mathbf{N})$. $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} n = +\infty$, , $\lim_{n\to+\infty}(-1)^{[x_n]} = \lim_{n\to+\infty}(-1)^{[n]} = \lim_{n\to+\infty}(-1)^n$. $\lim_{x\to+\infty}(-1)^{[x]}$.

.

1. (1 3.4), .

$$\lim_{x\to 0+}(-1)^{[\frac{1}{x}]}, \qquad \lim_{x\to\pm\infty}(x - [x]).$$

($\lim_{x\to\pm\infty}(x-[x])$: $y = x - [x]$ $x - [x] = 0$ $x - [x] = \frac{1}{2}$. $+\infty$ $-\infty$.)

2. $\lim f(x)$, (x_n) x $y = f(x)$ x.

(*i*) $\big(f(x_n)\big)$ $\geq u$, $\lim f(x) \geq u$.

(*ii*) $\big(f(x_n)\big)$ $\leq l$, $\lim f(x) \leq l$.

(x_n) x $y = f(x)$ x. $\big(f(x_n)\big)$ $\geq u$ $\leq l$ $l < u$, $\lim f(x)$.

(: 2.14 4.19.)

4.5 .

$p(x) = a_0 + a_1x + \cdots + a_Nx^N$ $N \geq 1$ $a_N \neq 0$. $\lim_{x\to\xi} x^k = \xi^k$ $\lim_{x\to\xi} a_k = a_k$ $\lim_{x\to\xi}(a_kx^k) = a_k\xi^k$, $\lim_{x\to\xi} p(x) = \lim_{x\to\xi}(a_0 + a_1x + \cdots + a_Nx^N) = a_0 + a_1\xi + \cdots + a_N\xi^N = p(\xi)$.

$$\boxed{\lim_{x\to\xi} p(x) = p(\xi).}$$

$x \to \pm\infty$. $p(x) = a_N x^N (\frac{a_0}{a_N}\frac{1}{x^N} + \cdots + \frac{a_{N-1}}{a_N}\frac{1}{x} + 1)$, $x \to \pm\infty$ 1,

$$\boxed{\lim_{x\to+\infty} p(x) = a_N \cdot (+\infty) = \begin{cases} +\infty\,, & a_N > 0, \\ -\infty\,, & a_N < 0 \end{cases}}$$

$$\boxed{\lim_{x\to-\infty} p(x) = \begin{cases} +\infty\,, & a_N > 0 \;\; N \quad a_N < 0 \;\; N\,, \\ -\infty\,, & a_N < 0 \;\; N \quad a_N > 0 \;\; N\,. \end{cases}}$$

$\lim_{x\to\pm\infty} p(x)$,

$$\boxed{\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_N x^N\,.}$$

: (1) $\lim_{x\to+\infty}(-5x^3 + x^2 - 4x - 12) = \lim_{x\to+\infty}(-5x^3) = -\infty.$

(2) $\lim_{x\to-\infty}(-5x^3 + x^2 - 4x - 12) = \lim_{x\to-\infty}(-5x^3) = +\infty.$

(3) $\lim_{x\to-\infty}(7x^4 + x^3 - x + 5) = \lim_{x\to-\infty} 7x^4 = +\infty.$

. $r(x) = \frac{a_0 + a_1 x + \cdots + a_N x^N}{b_0 + b_1 x + \cdots + b_M x^M}$ $a_N \neq 0,\ b_M \neq 0,$ $r(x) = \frac{a_N}{b_M} x^{N-M} \frac{\frac{a_0}{a_N}\frac{1}{x^N} + \cdots + \frac{a_{N-1}}{a_N}\frac{1}{x} + 1}{\frac{b_0}{b_M}\frac{1}{x^M} + \cdots + \frac{b_{M-1}}{b_M}\frac{1}{x} + 1}$,

$$\lim_{x\to+\infty} r(x) = \begin{cases} \frac{a_N}{b_M} \cdot (+\infty), & N > M, \\ \frac{a_N}{b_M}\,, & N = M, \\ 0, & N < M \end{cases}$$

$$\lim_{x\to-\infty} r(x) = \begin{cases} \frac{a_N}{b_M} \cdot (+\infty), & N - M \quad , \\ \frac{a_N}{b_M} \cdot (-\infty), & N - M \quad , \\ \frac{a_N}{b_M}\,, & N = M, \\ 0, & N < M. \end{cases}$$

$\lim_{x\to\pm\infty} r(x)$,

$$\boxed{\lim_{x\to\pm\infty} r(x) = \lim_{x\to\pm\infty} \frac{a_N x^N}{b_M x^M}\,.}$$

: (1) $\lim_{x\to+\infty} \frac{x^3 - x^2 + 2x + 4}{2x^3 + 1} = \lim_{x\to+\infty} \frac{x^3}{2x^3} = \frac{1}{2}\,.$

(2) $\lim_{x\to+\infty} \frac{-x^3 + x + 4}{2x^2 + 1} = \lim_{x\to+\infty} \frac{-x^3}{2x^2} = (-\frac{1}{2}) \cdot (+\infty) = -\infty.$

(3) $\lim_{x\to+\infty} \frac{3x^2 - x + 2}{x^4 + 1} = \lim_{x\to+\infty} \frac{3x^2}{x^4} = 0.$

(4) $\lim_{x\to-\infty} \frac{3x^2 + 2x + 4}{2x^2 - 1} = \lim_{x\to-\infty} \frac{3x^2}{2x^2} = \frac{3}{2}\,.$

(5) $\lim_{x\to-\infty} \frac{x^3 + 5x^2 - 4}{-3x^4 + 1} = \lim_{x\to-\infty} \frac{x^3}{-3x^4} = 0.$

(6) $\lim_{x\to-\infty} \frac{-x^3 - x + 5}{2x + 1} = \lim_{x\to-\infty} \frac{-x^3}{2x} = (-\frac{1}{2}) \cdot (+\infty) = -\infty.$

(7) $\lim_{x\to-\infty}\frac{3x^4-x^2+4}{2x+1}=\lim_{x\to-\infty}\frac{3x^4}{2x}=\frac{3}{2}\cdot(-\infty)=-\infty.$

x , .

$b_0+b_1\xi+\cdots+b_M\xi^M\neq 0,\ \lim_{x\to\xi}r(x)=\frac{a_0+a_1\xi+\cdots+a_N\xi^N}{b_0+b_1\xi+\cdots+b_M\xi^M}=r(\xi).$,

$$\boxed{\lim_{x\to\xi}r(x)=r(\xi).}$$

: $\lim_{x\to1}\frac{3x-2}{x^4+2x^3-4}=\frac{3\cdot1-2}{1^4+2\cdot1^3-4}=-1.$

$b_0+b_1\xi+\cdots+b_M\xi^M=0,$ $x-\xi$ $b_0+b_1x+\cdots+b_Mx^M.$ $(x-\xi)^m$ ($m\geq1$) $x-\xi$ $b_0+b_1x+\cdots+b_Mx^M,$ $b_0+b_1x+\cdots+b_Mx^M=(x-\xi)^mq(x),$ $q(x)$ $x-\xi$, , $q(\xi)\neq0.$, $x-\xi$ $a_0+a_1\xi+\cdots+a_N\xi^N$, $a_0+a_1\xi+\cdots+a_N\xi^N=(x-\xi)^np(x),$ $n\geq0$ $p(x)$ $x-\xi$, , $p(\xi)\neq0.$, , $r(x)=(x-\xi)^{n-m}\frac{p(x)}{q(x)}$, $\lim_{x\to\xi}\frac{p(x)}{q(x)}=\frac{p(\xi)}{q(\xi)}\neq0,$

$$\lim_{x\to\xi}r(x)=\begin{cases}0\,, & n>m,\\ \frac{p(\xi)}{q(\xi)}\,, & n=m,\end{cases}$$

$$\lim_{x\to\xi}r(x)=\frac{p(\xi)}{q(\xi)}\cdot(+\infty),$$

$m-n$,

$$\lim_{x\to\xi-}r(x)=\frac{p(\xi)}{q(\xi)}\cdot(-\infty),\qquad \lim_{x\to\xi+}r(x)=\frac{p(\xi)}{q(\xi)}\cdot(+\infty),$$

$m-n$.

: (1) $\lim_{x\to1}\frac{x^3-x^2-x+1}{x^4-2x^2+1}$ ' 1 x^4-2x^2+1 , , $x-1.$, , : $x^4-2x^2+1=(x^2-1)^2=(x-1)^2(x+1)^2.$ 1 x^3-x^2-x+1 $x-1$. , : $x^3-x^2-x+1=(x-1)x^2-(x-1)=(x-1)(x^2-1)=(x-1)^2(x+1).$ $\frac{x^3-x^2-x+1}{x^4-2x^2+1}=\frac{(x-1)^2(x+1)}{(x-1)^2(x+1)^2}=\frac{1}{x+1}$ $x\neq1,-1.$, $\lim_{x\to1}\frac{x^3-x^2-x+1}{x^4-2x^2+1}=\lim_{x\to1}\frac{1}{x+1}=\frac{1}{1+1}=\frac{1}{2}.$

(2) $\lim_{x\to1}\frac{x^3+4x^2+x-6}{x^3-x^2-x+1}$ 1 x^3-x^2-x+1 , : $x^3-x^2-x+1=(x-1)^2(x+1).$ 1 x^3+4x^2+x-6 $x-1$: $x^3+4x^2+x-6=x^3-x^2+5x^2-5x+6x-6=(x-1)x^2+(x-1)5x+(x-1)6=(x-1)(x^2+5x+6).$ $x-1$ x^2+5x+6 1 . $\frac{x^3+4x^2+x-6}{x^3-x^2-x+1}=\frac{(x-1)(x^2+5x+6)}{(x-1)^2(x+1)}=\frac{1}{x-1}\frac{x^2+5x+6}{x+1}.$: $\lim_{x\to1+}\frac{x^3+4x^2+x-6}{x^3-x^2-x+1}=(+\infty)\cdot\frac{1^2+5\cdot1+6}{1+1}=+\infty$ $\lim_{x\to1-}\frac{x^3+4x^2+x-6}{x^3-x^2-x+1}=(-\infty)\cdot\frac{1^2+5\cdot1+6}{1+1}=-\infty.$, $\lim_{x\to1}\frac{x^3+4x^2+x-6}{x^3-x^2-x+1}$.

.

1. $x\to\pm\infty.$

$$y=x^4-4x^3\,,\quad y=\frac{x^2+1}{x^3-x^2+1}\,,\quad y=\frac{x^4-x+1}{-3x^4+x^2+1}\,,$$

$$y = \frac{1+x^5-x^8}{1+2x^2}\,, \quad y = \frac{2-2x+x^5}{1-x^2}\,.$$

2. $x \to 1\pm$.

$$y = x^2+2x, \quad y = \frac{x^2-2x+1}{x+1}\,, \quad y = \frac{x^3+2x^2-x-2}{x^4-x^3+x^2-1}\,,$$

$$y = \frac{x^4-x^3-3x^2+5x-2}{x^4+x^3-4x^2+x+1}\,, \quad y = \frac{x+2}{x^4-2x^3+2x^2-2x+1}\,,$$

$$y = \frac{x^3-x^2-x+1}{x^5-3x^4+6x^3-10x^2+9x-3}\,.$$

3. $\lim_{x\to 1\pm} f(x)$

$$f(x) \begin{cases} \leq \frac{x^3-x^2+2x-2}{x^3-x^2-x+1}\,, & 0 \leq x < 1, \\ \geq \frac{x^3-x^2+2x-2}{x^3-x^2-x+1}\,, & x > 1. \end{cases}$$

4. $\lim_{x\to+\infty} f(x)$ $\frac{x^3+x^2-3x-7}{2x^3+8x^2+x+3} \leq f(x) < \frac{x^5+x^4+7}{2x^5-4x^4-x^3-8x-3}$ $x > 5$.

5. ,

(i) $\frac{5}{4} < \frac{x^3-1}{x^2-1} < \frac{3}{2} + 10^{-4}$ 1.

(ii) $\frac{2x^7-14x^6-x^5+3x^4-7x^2-1}{3x^4+x^2+6} < -10^{13}$ $-\infty$.

(iii) $\frac{x^3-x^2+x-1}{x^4-2x^2+1} < -10^9$ 1 .

6. , 0 0 $+\infty$ $-\infty$.

$$y = \frac{x^7+2x^5+5x^2}{-x^6+5x^5+x^4}, \quad y = \frac{3x^5+x^4-5x^3+x^2}{8x^7-x^5-x^4+7x^3}, \quad y = \frac{x^7+x^5+x^3}{-x^7-4x^5+x^3}.$$

4.6 .

$y = x^a$, a , $(0,+\infty)$.

$$\boxed{\lim_{x\to\xi} x^a = \xi^a \quad (\xi > 0).}$$

′ $a > 0$. $\epsilon > 0$ $\delta > 0$ $|x^a - \xi^a| < \epsilon$ $x > 0$ $0 < |x-\xi| < \delta$.
$|x^a - \xi^a| < \epsilon$ $\xi^a - \epsilon < x^a < \xi^a + \epsilon$.

$0 < \epsilon < \xi^a$, $\xi^a - \epsilon < x^a < \xi^a + \epsilon$ $(\xi^a - \epsilon)^{\frac{1}{a}} < x < (\xi^a + \epsilon)^{\frac{1}{a}}$. ξ $(\xi^a - \epsilon)^{\frac{1}{a}}$ $(\xi^a + \epsilon)^{\frac{1}{a}}$, , $\delta = \min\left\{\xi - (\xi^a - \epsilon)^{\frac{1}{a}}, (\xi^a + \epsilon)^{\frac{1}{a}} - \xi\right\}$, x $0 < |x-\xi| < \delta$ $(\xi^a - \epsilon)^{\frac{1}{a}} < x < (\xi^a + \epsilon)^{\frac{1}{a}}$, , $|x^a - \xi^a| < \epsilon$.

$\epsilon \geq \xi^a$, $\xi^a - \epsilon < x^a < \xi^a + \epsilon$ $0 < x^a < \xi^a + \epsilon$ $0 < x < (\xi^a + \epsilon)^{\frac{1}{a}}$. $0 < \xi < (\xi^a + \epsilon)^{\frac{1}{a}}$, $\delta = \min\{\xi - 0, (\xi^a + \epsilon)^{\frac{1}{a}} - \xi\}$, x $0 < |x-\xi| < \delta$ $0 < x < (\xi^a + \epsilon)^{\frac{1}{a}}$, , $|x^a - \xi^a| < \epsilon$.

, , $\delta > 0$ $|x^a - \xi^a| < \epsilon$ $x > 0$ $0 < |x-\xi| < \delta$. $\lim_{x\to\xi} x^a = \xi^a$.

$a<0$ ($-a>0$), $\lim_{x\to\xi} x^a = \lim_{x\to\xi} \frac{1}{x^{-a}} = \frac{1}{\xi^{-a}} = \xi^a$.
, $a=0$, : $\lim_{x\to\xi} x^0 = \lim_{x\to\xi} 1 = 1 = \xi^0$.

$$\boxed{\lim_{x\to 0+} x^a = \begin{cases} 0, & a>0,\\ 1, & a=0,\\ +\infty, & a<0 \end{cases}}$$

$$\boxed{\lim_{x\to +\infty} x^a = \begin{cases} +\infty, & a>0,\\ 1, & a=0,\\ 0, & a<0 \end{cases}}$$

.
$\lim_{x\to\xi} x^a$ $\xi<0$ $\lim_{x\to-\infty} x^a$ $\lim_{x\to 0-} x^a$, $y=x^a$ $(-\infty,0)$, a $a=\frac{m}{n}$ n . $y=x^a=(\sqrt[n]{x})^m$ $y=\sqrt[n]{x}$ n.

$$\boxed{\lim_{x\to\xi} \sqrt[n]{x} = \sqrt[n]{\xi} \quad (n \quad).}$$

($a=\frac{1}{n}$) $\lim_{x\to\xi} x^a = \xi^a$, $\xi>0$. ( ). $\epsilon>0$ $\delta>0$ $|\sqrt[n]{x}-\sqrt[n]{\xi}|<\epsilon$ x $0<|x-\xi|<\delta$. $|\sqrt[n]{x}-\sqrt[n]{\xi}|<\epsilon$ $\sqrt[n]{\xi}-\epsilon<\sqrt[n]{x}<\sqrt[n]{\xi}+\epsilon$ $(\sqrt[n]{\xi}-\epsilon)^n<x<(\sqrt[n]{\xi}+\epsilon)^n$. ξ $(\sqrt[n]{\xi}-\epsilon)^n$ $(\sqrt[n]{\xi}+\epsilon)^n$, , $\delta=\min\left\{\xi-(\sqrt[n]{\xi}-\epsilon)^n, (\sqrt[n]{\xi}+\epsilon)^n-\xi\right\}$, x $0<|x-\xi|<\delta$ $(\sqrt[n]{\xi}-\epsilon)^n<x<(\sqrt[n]{\xi}+\epsilon)^n$, , $|\sqrt[n]{x}-\sqrt[n]{\xi}|<\epsilon$. , , $\lim_{x\to\xi} \sqrt[n]{x} = \sqrt[n]{\xi}$.

$$\boxed{\lim_{x\to-\infty} \sqrt[n]{x} = -\infty, \quad \lim_{x\to+\infty} \sqrt[n]{x} = +\infty \quad (n \quad)}$$

($a=\frac{1}{n}>0$) $\lim_{x\to+\infty} x^a = +\infty$. (M N). $y=-x$: $\lim_{x\to-\infty} \sqrt[n]{x} = \lim_{y\to+\infty} \sqrt[n]{-y} = \lim_{y\to+\infty}(-\sqrt[n]{y}) = -\lim_{y\to+\infty} \sqrt[n]{y} = -(+\infty) = -\infty$.

,

$$\boxed{\lim_{x\to\xi} \sqrt[n]{x} = \sqrt[n]{\xi} \quad (\xi>0 \quad n \quad)}$$

$$\boxed{\lim_{x\to 0+} \sqrt[n]{x} = 0, \quad \lim_{x\to+\infty} \sqrt[n]{x} = +\infty \quad (n \quad).}$$

($a=\frac{1}{n}>0$) $y=x^a$.

: (1) $y=x+1$, $\lim_{x\to+\infty} \sqrt{x+1} = \lim_{y\to+\infty} \sqrt{y} = +\infty$.

(2) $\lim_{x\to+\infty}(\sqrt{x+1}-\sqrt{x}) = 0$.

(1), $(+\infty)-(+\infty)$. $a-b=\frac{a^2-b^2}{a+b}$ $\sqrt{x+1}-\sqrt{x} = \frac{(x+1)-x}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{\sqrt{x+1}+\sqrt{x}}$, , (1), $\lim_{x\to+\infty}(\sqrt{x+1}-\sqrt{x}) = \lim_{x\to+\infty} \frac{1}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{(+\infty)+(+\infty)} =$

0.

(3) $\lim_{x\to 3}\sqrt[5]{\frac{x+1}{x^2+1}} = \lim_{y\to\frac{2}{5}}\sqrt[5]{y} = \sqrt[5]{\frac{2}{5}}$.

(4) $\lim_{x\to 1}\sqrt[4]{(x^2-2x+1)(x^3+x)} = \lim_{y\to 0+}\sqrt[4]{y} = 0$.

(5) $\lim_{x\to 4-}\sqrt[3]{\frac{x+1}{x-4}} = \lim_{y\to-\infty}\sqrt[3]{y} = -\infty$.

(6) $\lim_{x\to+\infty}\frac{\sqrt{x+\frac{1}{x}}+\sqrt[3]{x+\frac{1}{x}}}{x+\frac{1}{x}-\sqrt{x+\frac{1}{x}}} = \lim_{y\to+\infty}\frac{\sqrt{y}+\sqrt[3]{y}}{y-\sqrt{y}} = \lim_{t\to+\infty}\frac{t^3+t^2}{t^6-t^3} = 0$.

.

1. ;

$$\lim_{x\to-\infty} x^{\frac{6}{4}}, \quad \lim_{x\to-1} x^{-\sqrt{2}}, \quad \lim_{x\to 0-} x^{-\frac{1}{2}}, \quad \lim_{x\to 0-} x^{3+\sqrt{3}}.$$

2. $x\to+\infty$.

$$y = \frac{x^{\frac{7}{8}}-3x^{-2}+2x^{\frac{6}{5}}-4}{x^{\frac{6}{5}}-2x^{\frac{9}{8}}+2}, \quad y = \frac{x^{\frac{3}{2}}-2x^{\frac{6}{5}}+1}{x+4x^{\frac{4}{3}}+2}, \quad y = \frac{x^{\frac{7}{4}}-x^{\frac{1}{3}}}{x^2+3x^{\frac{15}{8}}}.$$

3. .

$$\lim_{x\to 3}\left(x^{\frac{1}{3}}-5x^{-\frac{1}{2}}\right), \quad \lim_{x\to 1} x^{\sqrt{2}}, \quad \lim_{x\to-1}\left(2x^{\frac{4}{3}}+x^{-\frac{1}{5}}\right).$$

4. $x\to 0\pm$ $x\to-\infty$.

$$y = x^{\frac{1}{3}}, \quad y = x^{-\frac{1}{3}}, \quad y = x^{\frac{2}{3}}, \quad y = x^{-\frac{2}{3}}, \quad y = 2x^{-\frac{1}{3}}-x^{-\frac{2}{3}}.$$

5. $a\neq 0$. .

$$\lim_{x\to 1\pm}\frac{1}{x^a-1}, \quad \lim_{x\to 1}\frac{1}{(x^a-1)^2}, \quad \lim_{x\to 1}\frac{x^{3a}-1}{x^a-1}.$$

(: : $a<0$ $a>0$.)

6. $x\to+\infty$.

$$y = \sqrt{x^2+1}-x, \quad y = \sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right), \quad y = x\left(\sqrt{x^2+1}-x\right),$$

$$y = \sqrt[3]{x+1}-\sqrt[3]{x}, \quad y = \sqrt[3]{x^2}\left(\sqrt[3]{x+1}-\sqrt[3]{x}\right),$$

$$y = \sqrt{x+1}-2\sqrt{x}+\sqrt{x-1}, \quad y = \sqrt{x^3}\left(\sqrt{x+1}-2\sqrt{x}+\sqrt{x-1}\right).$$

7. .

$$\lim_{x\to 0}\sqrt{x^2+1}, \quad \lim_{x\to\pm\infty}\sqrt{\frac{3x^2-7x+1}{x^2+1}}, \quad \lim_{x\to+\infty}\sqrt{x+\sqrt{x+\sqrt{x}}},$$

$$\lim_{x\to\pm\infty}\sqrt[3]{1+\frac{1}{x}}, \quad \lim_{x\to 0\pm}\sqrt[5]{1+\frac{1}{x}-\frac{1}{x^2}}, \quad \lim_{x\to 1\pm}\left(1-\frac{1}{x-1}\right)^{\frac{2}{3}},$$

$$\lim_{x\to 1+}\frac{\sqrt{\frac{x+1}{x-1}}-3\sqrt[4]{\frac{x+1}{x-1}}+1}{2\sqrt{\frac{x+1}{x-1}}+7\sqrt[4]{\frac{x+1}{x-1}}+3}\,, \qquad \lim_{x\to 1\pm}\frac{\sqrt[3]{\frac{x+1}{x-1}}-3\sqrt[5]{\frac{x+1}{x-1}}+1}{2\sqrt[3]{\frac{x+1}{x-1}}+7\sqrt[7]{\frac{x+1}{x-1}}+3}\,.$$

8. a, b, c $a > 0$. A, B a, b, c

$$\lim_{x\to+\infty}\big(\sqrt{ax^2+bx+c}-Ax-B\big)=0.$$

,

$$\lim_{x\to+\infty}x\big(\sqrt{ax^2+bx+c}-Ax-B\big)=\frac{4ac-b^2}{8a\sqrt{a}}\,.$$

9. (x_n) $a > 0$. 4.19,

(i) $\lim_{n\to+\infty}x_n=+\infty$ $x_n>0$ n, $\lim_{n\to+\infty}{x_n}^a=+\infty$.

(ii) $\lim_{n\to+\infty}x_n=0$ $x_n>0$ n, $\lim_{n\to+\infty}{x_n}^a=0$.

$a < 0$;

.

$$\lim_{n\to+\infty}\Big(\frac{n^3+n+1}{2n^2-1}\Big)^{\sqrt{2}}, \qquad \lim_{n\to+\infty}\sqrt[4]{\frac{n^5+n^3+1}{2n^6+n^2+1}}\,, \qquad \lim_{n\to+\infty}\Big(\frac{2^n}{4^n+1}\Big)^{\frac{3}{4}}.$$

10. (x_n) k. 4.19,

(i) $\lim_{n\to+\infty}x_n=+\infty$, $\lim_{n\to+\infty}\sqrt[k]{x_n}=+\infty$.

(ii) $\lim_{n\to+\infty}x_n=-\infty$, $\lim_{n\to+\infty}\sqrt[k]{x_n}=-\infty$.

(i), (ii) k ;

.

$$\lim_{n\to+\infty}\sqrt[5]{\frac{n^3+n+1}{2n^2-1}}\,, \qquad \lim_{n\to+\infty}\sqrt[5]{\frac{2^n-4^n}{2^n+3^n+1}}\,.$$

4.7 .

$y=a^x$, $a>0$. $y=a^x$ $(-\infty,+\infty)$. ′ :

$$\boxed{\lim_{x\to\xi}a^x=a^\xi\,.}$$

$a>1$. $\epsilon>0$ $\delta>0$ $|a^x-a^\xi|<\epsilon$ x $0<|x-\xi|<\delta$.

$|a^x-a^\xi|<\epsilon$ $a^\xi-\epsilon<a^x<a^\xi+\epsilon$.

$0<\epsilon<a^\xi$, $\log_a(a^\xi-\epsilon)<x<\log_a(a^\xi+\epsilon)$. ξ $\log_a(a^\xi-\epsilon)$ $\log_a(a^\xi+\epsilon)$, $\delta=\min\big\{\xi-\log_a(a^\xi-\epsilon),\log_a(a^\xi+\epsilon)-\xi\big\}$, x $0<|x-\xi|<\delta$ $\log_a(a^\xi-\epsilon)<x<\log_a(a^\xi+\epsilon)$, , $|a^x-a^\xi|<\epsilon$.

$\epsilon\geq a^\xi$, $a^\xi-\epsilon<a^x<a^\xi+\epsilon$ ($0<a^x$) $a^x<a^\xi+\epsilon$ $x<\log_a(a^\xi+\epsilon)$. $\xi<\log_a(a^\xi+\epsilon)$, $\delta=\log_a(a^\xi+\epsilon)-\xi$, x $0<|x-\xi|<\delta$ $x<\log_a(a^\xi+\epsilon)$, , $|a^x-a^\xi|<\epsilon$.

$\lim_{x\to\xi}a^x=a^\xi$.

$0<a<1$ ($\frac{1}{a}>1$), $\lim_{x\to\xi} a^x = \lim_{x\to\xi} \frac{1}{(\frac{1}{a})^x} = \frac{1}{(\frac{1}{a})^\xi} = a^\xi$.

, $a=1$, $\lim_{x\to\xi} 1^x = \lim_{x\to\xi} 1 = 1 = 1^\xi$.

$$\boxed{\lim_{x\to+\infty} a^x = \begin{cases} +\infty\,, & a>1, \\ 1\,, & a=1, \\ 0\,, & 0<a<1. \end{cases}}$$

$a>1$. $M>0$ $N>0$ $a^x>M$ $x>N$. $a^x>M$ $x>\log_a M$. $N=\log_a M>0$, $M>1$, $N=1>0$, $0<M\le 1$, $x>N$ $x>\log_a M$, , $a^x>M$. $\lim_{x\to+\infty} a^x = +\infty$.

$0<a<1$ ($\frac{1}{a}>1$), $\lim_{x\to+\infty} a^x = \lim_{x\to+\infty} \frac{1}{(\frac{1}{a})^x} = \frac{1}{+\infty} = 0$.

, $a=1$, $\lim_{x\to+\infty} 1^x = \lim_{x\to+\infty} 1 = 1$.

:

$$\boxed{\lim_{x\to-\infty} a^x = \begin{cases} 0\,, & a>1, \\ 1\,, & a=1, \\ +\infty\,, & 0<a<1. \end{cases}}$$

, . , , $y=-x$. , $a>1$, $\lim_{x\to-\infty} a^x = \lim_{y\to+\infty} a^{-y} = \lim_{y\to+\infty} \frac{1}{a^y} = \frac{1}{+\infty} = 0$. $a=1$ $0<a<1$.

.

$a>0, a\ne 1$ $y=\log_a x$ $(0,+\infty)$. :

$$\boxed{\lim_{x\to\xi} \log_a x = \log_a \xi \quad (\xi>0).}$$

$a>1$. $\epsilon>0$ $\delta>0$ $|\log_a x - \log_a \xi|<\epsilon$ x $y=\log_a x$ ($x>0$) $0<|x-\xi|<\delta$. $|\log_a x-\log_a\xi|<\epsilon$ $\log_a\xi-\epsilon<\log_a x<\log_a\xi+\epsilon$ $\xi a^{-\epsilon}<x<\xi a^{\epsilon}$. ξ $\xi a^{-\epsilon}$ ξa^{ϵ}, $\delta=\min\{\xi-\xi a^{-\epsilon},\xi a^{\epsilon}-\xi\}$, $x>0$ $0<|x-\xi|<\delta$ $\xi a^{-\epsilon}<x<\xi a^{\epsilon}$, , $|\log_a x-\log_a\xi|<\epsilon$. $\lim_{x\to\xi}\log_a x=\log_a\xi$.

$0<a<1$, $\lim_{x\to\xi}\log_a x = \lim_{x\to\xi}(-\log_{\frac{1}{a}} x) = -\log_{\frac{1}{a}}\xi = \log_a\xi$ $\frac{1}{a}>1$.

$x\to+\infty$:

$$\boxed{\lim_{x\to+\infty} \log_a x = \begin{cases} +\infty\,, & a>1, \\ -\infty\,, & 0<a<1. \end{cases}}$$

$a>1$. $M>0$ $N>0$ $\log_a x>M$ x $y=\log_a x$ ($x>0$) $x>N$. $\log_a x>M$ $x>a^M$, , $N=a^M>0$, $x>N$ $x>a^M$, , $\log_a x>M$. $\lim_{x\to+\infty}\log_a x=+\infty$.

$0<a<1$, $\lim_{x\to+\infty}\log_a x = \lim_{x\to+\infty}(-\log_{\frac{1}{a}} x) = -(+\infty) = -\infty$.

:

$$\boxed{\lim_{x\to 0+} \log_a x = \begin{cases} -\infty\,, & a>1, \\ +\infty\,, & 0<a<1. \end{cases}}$$

$y = \frac{1}{x}$. $a > 1$, $\lim_{x\to 0+} \log_a x = \lim_{y\to+\infty} \log_a \frac{1}{y} = \lim_{y\to+\infty}(-\log_a y) = -\lim_{y\to+\infty} \log_a y = -(+\infty) = -\infty$. $0 < a < 1$.

$a = e$, :

$$\boxed{\lim_{x\to\xi} e^x = e^\xi\,, \qquad \lim_{x\to-\infty} e^x = 0, \qquad \lim_{x\to+\infty} e^x = +\infty}$$

$$\boxed{\lim_{x\to\xi} \log x = \log \xi \quad (\xi > 0), \qquad \lim_{x\to 0+} \log x = -\infty, \qquad \lim_{x\to+\infty} \log x = +\infty.}$$

.

1. $x \to \pm\infty$.

$$y = e^x - e^{2x} + 2, \quad y = \frac{1}{e^x - 1}\,, \quad y = \frac{e^{2x} + e^x + 1}{2e^{2x} - e^x + 2}\,.$$

2. $x \to +\infty$ $x \to 0+$.

$$y = (\log x)^2 - \log x, \quad y = \frac{1}{\log x}\,, \quad y = \frac{1 + 2(\log x)^2}{2 + \log x + (\log x)^3}\,.$$

3. .

$$\lim_{x\to 0\pm} \frac{1}{e^x - 1}\,, \qquad \lim_{x\to 0} \frac{1}{(e^x - 1)^2}\,, \qquad \lim_{x\to 0} \frac{e^{2x} - 1}{e^x - 1}\,,$$

$$\lim_{x\to 0+} \frac{\log(2x)}{\log(3x)}\,, \qquad \lim_{x\to 1\pm} \frac{1}{\log x}\,, \qquad \lim_{x\to 1} \frac{1}{(\log x)^2}\,.$$

4. .

$$\lim_{x\to 2} e^{\frac{1}{x}}\,, \quad \lim_{x\to 0\pm} e^{1-\frac{1}{x}}\,, \quad \lim_{x\to\pm\infty} e^{\frac{1}{x}}\,, \quad \lim_{x\to+\infty} \frac{e^x + e^{\frac{x}{2}} + 1}{2e^x - e^{\frac{x}{3}} + 2}\,,$$

$$\lim_{x\to+\infty} \log(x + 1), \quad \lim_{x\to\pm\infty} \log(x^2 - x + 1), \quad \lim_{x\to 1} \log(x^3 + 1),$$

$$\lim_{x\to\pm\infty} \frac{(\log|x|)^7 - (\log|x|)^4 + 1}{(\log|x|)^5 + (\log|x|)^2 + 1}\,, \quad \lim_{x\to\pm\infty} \log \frac{e^x}{e^{\frac{x}{2}} + 1}\,.$$

5.

$$y = \cosh x, \quad y = \sinh x, \quad y = \tanh x, \quad y = \coth x$$

: $\lim_{x\to\xi}$, $\lim_{x\to\pm\infty}$. $\lim_{x\to 0\pm} \coth x$.

6. 0 0 $+\infty$ $-\infty$;

$$y = e^x\,, \quad y = e^{-|x|}\,, \quad y = \frac{1}{e^x - 1}\,, \quad y = \frac{1}{(e^x - 1)^2}\,,$$

$$y = \log|x|, \quad y = \frac{1}{\log|x|}\,, \quad y = \frac{1}{\log|1 + x|}\,.$$

7. (x_n) $a > 1$. 4.19,

(i) $\lim_{n\to+\infty} x_n = +\infty$, $\lim_{n\to+\infty} a^{x_n} = +\infty$.

(ii) $\lim_{n\to+\infty} x_n = -\infty$, $\lim_{n\to+\infty} a^{x_n} = 0$.

.

$$\lim_{n\to+\infty} 2^n\,, \quad \lim_{n\to+\infty} e^{\frac{n^3+3n-1}{n^2+1}}\,, \quad \lim_{n\to+\infty} e^{-\sqrt{n}}\,, \quad \lim_{n\to+\infty} 2^{2^n}\,, \quad \lim_{n\to+\infty} 2^{\frac{1-\sqrt{n}-n}{1+\sqrt{n}}}\,.$$

8. (x_n) $a > 1$. 4.19,

(i) $\lim_{n\to+\infty} x_n = +\infty$ $x_n > 0$ n, $\lim_{n\to+\infty} \log_a x_n = +\infty$.

(ii) $\lim_{n\to+\infty} x_n = 0$ $x_n > 0$ n, $\lim_{n\to+\infty} \log_a x_n = -\infty$.

.

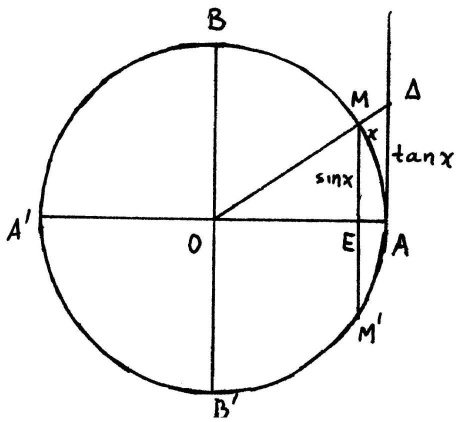
$$\lim_{n\to+\infty} \log\frac{n+1}{2n^2-1}\,, \quad \lim_{n\to+\infty} \log\frac{n^3-n^2+1}{n^2+1}\,, \quad \lim_{n\to+\infty} \log\frac{e^{2n}+1}{e^n+1}\,,$$

$$\lim_{n\to+\infty} \log(3^n-2^n+1), \quad \lim_{n\to+\infty} \frac{(\log\frac{n}{n^2+1})^2 - \log\frac{n}{n^2+1} + 2}{-(\log\frac{n}{n^2+1})^2 + 4\log\frac{n}{n^2+1} - 8}\,.$$

4.8 .

$y = \sin x$.

$$\boxed{|\sin x| \le |x|.}$$



Σχήμα 4.10: < < .

: $0 < x < \frac{\pi}{2}$ x. ′, x $\sin x$. ′ , . ′ $2\sin x$ ′ $2x$. $0 < 2\sin x < 2x$, $0 < \sin x < x$, , $|\sin x| \le |x|$.

$-\frac{\pi}{2} < x < 0$, $0 < -x < \frac{\pi}{2}$, $0 < \sin(-x) < -x$, $x < \sin x < 0$, , $|\sin x| \le |x|$. $x = 0$, $0 = 0$.

, $|x| \ge \frac{\pi}{2}$, $|\sin x| \le 1 < \frac{\pi}{2} \le |x|$.

, $\cos x - \cos\xi = -2\sin\frac{x-\xi}{2}\sin\frac{x+\xi}{2}$ $|\cos x - \cos\xi| = 2\big|\sin\frac{x-\xi}{2}\big|\big|\sin\frac{x+\xi}{2}\big| \le 2\big|\sin\frac{x-\xi}{2}\big| \le 2\big|\frac{x-\xi}{2}\big| = |x-\xi|$. $\epsilon > 0$ $\delta = \epsilon$. x $y = \cos x$ (x) $0 < |x-\xi| < \delta$ $|\cos x - \cos\xi| \le |x-\xi| < \delta = \epsilon$, $|\cos x - \cos\xi| < \epsilon$.

$$\boxed{\lim_{x\to\xi}\cos x = \cos\xi.}$$

$\sin x - \sin\xi = 2\sin\frac{x-\xi}{2}\cos\frac{x+\xi}{2}$ $|\sin x - \sin\xi| \le |x-\xi|$, ,

$$\boxed{\lim_{x\to\xi}\sin x = \sin\xi.}$$

, $\cos\xi \ne 0$, $\xi \ne \frac{\pi}{2} + k\pi$ $(k \in \mathbf{Z})$,

$$\boxed{\lim_{x\to\xi}\tan x = \tan\xi \quad \Big(\xi \ne \frac{\pi}{2} + k\pi,\ k \in \mathbf{Z}\Big).}$$

, $\sin\xi \ne 0$, $\xi \ne k\pi$ $(k \in \mathbf{Z})$,

$$\boxed{\lim_{x\to\xi}\cot x = \cot\xi \quad \big(\xi \ne k\pi,\ k \in \mathbf{Z}\big).}$$

ξ .

$\xi = \frac{\pi}{2} + k2\pi$ $(k \in \mathbf{Z})$, $\lim_{x\to\xi}\sin x = \sin\xi = 1$. , $\lim_{x\to\xi}\cos x = \cos\xi = 0$, , $\cos x > 0$ ξ $\cos x < 0$ ξ . $\lim_{x\to\xi-}\frac{1}{\cos x} = +\infty$ $\lim_{x\to\xi+}\frac{1}{\cos x} = -\infty$. $\xi = -\frac{\pi}{2} + k2\pi$ $(k \in \mathbf{Z})$, :

$$\boxed{\lim_{x\to\xi-}\tan x = +\infty, \quad \lim_{x\to\xi+}\tan x = -\infty \quad \Big(\xi = \frac{\pi}{2} + k\pi,\ k \in \mathbf{Z}\Big).}$$

$$\boxed{\lim_{x\to\xi-}\cot x = -\infty, \quad \lim_{x\to\xi+}\cot x = +\infty \quad \big(\xi = k\pi,\ k \in \mathbf{Z}\big).}$$

$y = \tan x$ $y = \cot x$, , $x = \frac{\pi}{2} + k\pi$ $(k \in \mathbf{Z})$ $y = \tan x$ $x = k\pi$ $(k \in \mathbf{Z})$ $y = \cot x$.

: $\lim_{x\to\pm\infty}\cos x$ $\lim_{x\to\pm\infty}\sin x$.

, $x \to \pm\infty$ $y = \cos x$ $y = \sin x$ «» , -1 1 , , . (πn) $+\infty$ $y = \cos x$ $\big(\cos(\pi n)\big) = \big((-1)^n\big)$. 4.19, $\lim_{x\to+\infty}\cos x$. $\big(\frac{\pi}{2} + \pi n\big)$, $\lim_{x\to+\infty}\sin x$ $x \to -\infty$.

:

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1, \qquad \lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}\,.}$$

, $\frac{0}{0}$.

$$\boxed{|x| \le |\tan x| \quad \Big(|x| < \frac{\pi}{2}\Big).}$$

: $0 < x < \frac{\pi}{2}$, x ′ . x . $\tan x$. $\frac{1}{2}\cdot 1 \cdot \tan x$ $\frac{x}{2}$ (π $\frac{x}{2\pi}\cdot\pi$.) , $0 < x < \tan x$ $|x| < |\tan x|$.
$-\frac{\pi}{2} < x < 0$, $0 < -x < \frac{\pi}{2}$, $0 < -x < \tan(-x)$, $|x| < |\tan x|$.
, $x = 0$, $0 = 0$.

$|\sin x| \le |x|$ $|x| \le |\tan x|$, $\cos x \le \frac{\sin x}{x} \le 1$ x $(-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$. $\lim_{x\to 0} \cos x = \cos 0 = 1$, $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

$\frac{1-\cos x}{x^2} = \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \frac{1-(\cos x)^2}{x^2(1+\cos x)} = \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x}$, $\lim_{x\to 0}\frac{1-\cos x}{x^2} = 1^2 \cdot \frac{1}{1+1} = \frac{1}{2}$.

. $\frac{1-\cos x}{x^2} = \frac{2(\sin\frac{x}{2})^2}{x^2} = \frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$. , $y = \frac{x}{2}$, $\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}} = \lim_{y\to 0}\frac{\sin y}{y} = 1$. $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}\cdot 1^2 = \frac{1}{2}$.

: (1) $\lim_{x\to 0}\frac{\tan x}{x}$ $\frac{\tan x}{x} = \frac{\sin x}{x}\frac{1}{\cos x}$, , $\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\cos x} = 1\cdot 1 = 1$.

(2) $\lim_{x\to 0}\frac{\sin(3x)}{\sin(2x)}$ $\frac{\sin(3x)}{\sin(2x)} = \frac{3}{2}\cdot\frac{\frac{\sin(3x)}{3x}}{\frac{\sin(2x)}{2x}}$. $\lim_{x\to 0}\frac{\sin(3x)}{3x}$ $\lim_{x\to 0}\frac{\sin(2x)}{2x}$.
$y = 3x$, $\lim_{x\to 0}\frac{\sin(3x)}{3x} = \lim_{y\to 0}\frac{\sin y}{y} = 1$. : $\lim_{x\to 0}\frac{\sin(2x)}{2x} = \lim_{y\to 0}\frac{\sin y}{y} = 1$. $\lim_{x\to 0}\frac{\sin(3x)}{\sin(2x)} = \frac{3}{2}\cdot\frac{1}{1} = \frac{3}{2}$.

.

1. .

$$\lim_{x\to 0} x\cot x, \qquad \lim_{x\to 0}\frac{1-\cos x}{x\sin x}, \qquad \lim_{x\to 0}\frac{1-\cos x}{(\sin x)^2}.$$

2. .

$$\lim_{x\to 0}\frac{\sin(3x)}{x}, \quad \lim_{x\to 0}\frac{\tan(3x)}{x}, \quad \lim_{x\to 0}\frac{1-\cos(13x)}{(\sin(7x))^2}, \quad \lim_{x\to\pi}\frac{\sin(3x)}{\sin x},$$

$$\lim_{x\to 0}\frac{\cos(8x)-\cos(15x)}{x^2}, \quad \lim_{x\to\pi}\frac{\sin x}{x-\pi}, \quad \lim_{x\to\frac{\pi}{2}}\frac{\cos x}{x-\frac{\pi}{2}}, \quad \lim_{x\to\pm\infty} x\sin\frac{1}{x},$$

$$\lim_{x\to\pm\infty} x^2\Big(1-\cos\frac{1}{x}\Big), \quad \lim_{x\to 0}\frac{3\sin(7x)-7\sin(3x)}{x^3}, .$$

3. $\lim_{x\to 0+} x^a \sin x = \begin{cases} 0, & a > -1, \\ 1, & a = -1, \\ +\infty, & a < -1. \end{cases}$

4. $a > 0$, $\lim_{x\to 0+} x^a \sin\frac{1}{x} = 0$.

5. $\lim_{x\to+\infty}\sin x$, $\lim_{x\to 0+}\sin\frac{1}{x}$.

6. (x_n). $\lim_{n\to+\infty} x_n = 0$ $x_n \neq 0$ n, $\lim_{n\to+\infty}\frac{\sin x_n}{x_n} = 1$ $\lim_{n\to+\infty}\frac{1-\cos x_n}{{x_n}^2} = \frac{1}{2}$.

.

$$\lim_{n\to+\infty} n\sin\frac{\pi}{n}\,,\quad \lim_{n\to+\infty}\sqrt{n}\sin\frac{\pi}{n}\,,\quad \lim_{n\to+\infty} n^2\sin\frac{\pi}{n}\,,$$

$$\lim_{n\to+\infty} n^2\Big(1-\cos\frac{\pi}{n}\Big),\quad \lim_{n\to+\infty}\frac{\cot\frac{\pi}{2n}}{n}\,.$$

7. .

$$y=\frac{1}{x}\sin x,\quad y=\frac{1}{x^2}\sin x,\quad y=x\sin\frac{1}{x},\quad y=x^2\sin\frac{1}{x},\quad y=\sqrt{x}\sin\frac{1}{x}.$$

(: 6, 7 8 3.10. 3.)

8. 4.19, · (x_n). 6, 7 8 3.10 – – $\sin x=\pm 1$ $\sin\frac{1}{x}=\pm 1$ $(0,+\infty)$.

$$\lim_{x\to+\infty} x\sin x,\quad \lim_{x\to+\infty} x^2\sin x,\quad \lim_{x\to 0+}\sin\frac{1}{x}\,,\quad \lim_{x\to 0+}\frac{1}{x}\sin\frac{1}{x}\,.$$

: $a\leq 0$, $\lim_{x\to 0+} x^a\sin\frac{1}{x}$. 4.

4.9 .

$y=f(x)$ (a,ξ). – – , x (a,ξ) ξ, . $f(x)$, $\lim_{x\to\xi-} f(x)=+\infty$. $f(x)$, u $f(x)\leq u$ x (a,ξ). ′ η $f(x)$, $\lim_{x\to\xi-} f(x)=\eta$, η $\leq u$. $\lim_{x\to\xi-} f(x)=\eta$ $\lim_{x\to\xi-} f(x)=+\infty$.

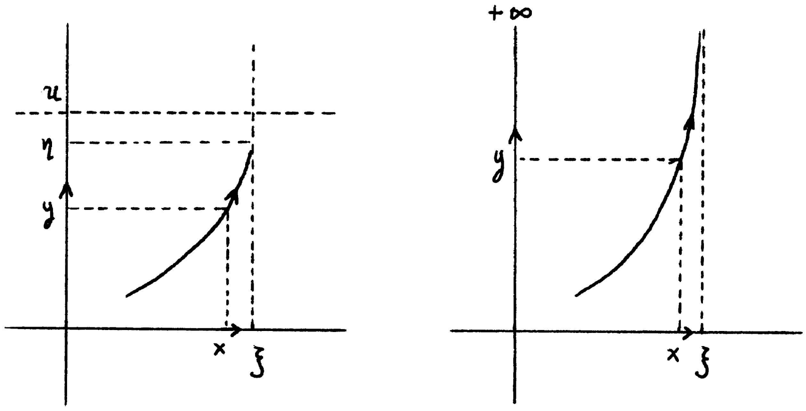
: (i) (a,ξ) $(a,+\infty)$, $y=f(x)$ ξ $+\infty$ (ii) (ξ,b) $(-\infty,b)$, $y=f(x)$ ξ $-\infty$.

4.1 .

4.1 (1) $y=f(x)$. (i) , $y=f(x)$, (ii) $+\infty$, $y=f(x)$. , (i) , $y=f(x)$, (ii) $-\infty$, $y=f(x)$.
(2) $y=f(x)$. (i) , $y=f(x)$, (ii) $-\infty$, $y=f(x)$. , (i) , $y=f(x)$, (ii) $+\infty$, $y=f(x)$.

4.1 · - 2.1 . . , 4.1 – – .

: (1) 4.1 .
, $a>0$, $\lim_{x\to+\infty} x^a=+\infty$ $\lim_{x\to 0+} x^a=0$ $y=x^a$ $(0,+\infty)$. $+\infty$ $-\infty$.
– – η, $\lim_{x\to+\infty} x^a=\eta$. $\lim_{x\to+\infty}(2x)^a=\lim_{y\to+\infty} y^a=\eta$. $\eta=\lim_{x\to+\infty}(2x)^a=\lim_{x\to+\infty}2^a x^a=2^a\lim_{x\to+\infty} x^a=2^a\eta$, , $\eta=2^a\eta$. $\eta=0$ $x\geq 1$ $x^a\geq 1^a=1$, , $\eta=\lim_{x\to+\infty} x^a\geq\lim_{x\to+\infty}1=1$. $+\infty$.



Σχήμα 4.11: .

0. , $x^a > 0$ $x > 0$, $\lim_{x\to 0+} x^a \geq \lim_{x\to 0+} 0 = 0$, , : $\lim_{x\to 0+} x^a = \eta \geq 0$. $\lim_{x\to 0+}(2x)^a = \lim_{y\to 0+} y^a = \eta$. $\eta = \lim_{x\to 0+}(2x)^a = \lim_{x\to 0+} 2^a x^a = 2^a\eta$, , $\eta = 2^a\eta$. $\eta = 0$.

(2) $y = (1+\frac{1}{x})^x$ $(0,+\infty)$.
. (2 6.9). , , , $\lim_{x\to+\infty}(1+\frac{1}{x})^x$ $+\infty$. (n) , $+\infty$, 4.19 $\big((1+\frac{1}{n})^n\big)$. , e , , e. ,

$$\boxed{\lim_{x\to+\infty}\Big(1+\frac{1}{x}\Big)^x = e.}$$

_ , , $\lim_{x\to+\infty}(1+\frac{1}{x})^x = e$: , $y = (1+\frac{1}{x})^x$!!
_ _ .

: (!) $\lim_{x\to+\infty}(1+\frac{1}{x})^x = e$.

$\lim_{n\to+\infty}(1+\frac{1}{n})^n = e$. – , – $\lim_{n\to+\infty}(1+\frac{1}{n+1})^n = \lim_{n\to+\infty}\frac{(1+\frac{1}{n+1})^{n+1}}{1+\frac{1}{n+1}} = \frac{e}{1} = e$ $\lim_{n\to+\infty}(1+\frac{1}{n})^{n+1} = \lim_{n\to+\infty}(1+\frac{1}{n})^n(1+\frac{1}{n}) = e\cdot 1 = e$. $\epsilon > 0$, $n_0{}'$ $e-\epsilon < (1+\frac{1}{n+1})^n < e+\epsilon$ $n \geq n_0{}'$ $n_0{}''$ $e-\epsilon < (1+\frac{1}{n})^{n+1} < e+\epsilon$ $n_0{}''$. () $N = \max\{n_0{}', n_0{}''\}$, $N \geq n_0{}'$ $N \geq n_0{}''$. $e-\epsilon < (1+\frac{1}{n+1})^n < e+\epsilon$ $e-\epsilon < (1+\frac{1}{n})^{n+1} < e+\epsilon$ $n \geq N$. (′) $x > N$ $[x] \geq N$, $e-\epsilon < \Big(1+\frac{1}{[x]+1}\Big)^{[x]} \leq (1+\frac{1}{x})^x \leq \Big(1+\frac{1}{[x]}\Big)^{[x]+1} < e+\epsilon$. , , $\epsilon > 0$ $N > 0$ $e-\epsilon < (1+\frac{1}{x})^x < e+\epsilon$, , $\Big|(1+\frac{1}{x})^x - e\Big| < \epsilon$ $x > N$. $\lim_{x\to+\infty}(1+\frac{1}{x})^x = e$.

.

1.
$$\lim_{x\to+\infty}\Big(1-\frac{1}{x}\Big)^x = \frac{1}{e}\,.$$

(: $(1-\frac{1}{x})^x = (\frac{x-1}{x})^x = \frac{1}{(\frac{x}{x-1})^x} = \frac{1}{(1+\frac{1}{x-1})^{x-1}(1+\frac{1}{x-1})}$.)

$$\boxed{\lim_{x\to+\infty}\left(1+\frac{t}{x}\right)^x = e^t}$$

t, : $t>0$, $t=0$, $t<0$.

(: $t>0$, $\lim_{x\to+\infty}\big(1+\frac{1}{x}\big)^x = e$. $t<0$, $\lim_{x\to+\infty}\big(1-\frac{1}{x}\big)^x = \frac{1}{e}$.)

$\lim_{x\to+\infty}\big(1+\frac{1}{x}\big)^x = e$ $\lim_{x\to+\infty}\Big(1-\frac{1}{x}\Big)^x = \frac{1}{e}$ $\lim_{x\to+\infty}\Big(1+\frac{t}{x}\Big)^x = e^t$.)

2. 4.19, .

$$\lim_{n\to+\infty}\left(1+\frac{3}{2n}\right)^{4n}, \quad \lim_{n\to+\infty}\left(1-\frac{1}{4n}\right)^{n}, \quad \lim_{n\to+\infty}\left(1-\frac{1}{2n}\right)^{3n},$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{\frac{n}{5}}, \quad \lim_{n\to+\infty}\left(1+\frac{1}{n^2}\right)^{n^2}, \quad \lim_{n\to+\infty}\left(1+\frac{3}{2\sqrt{n}}\right)^{\frac{\sqrt{n}}{4}}.$$

3. $a>1$. $\log_a(ax) = 1+\log_a x$ $y=\log_a x$, $\lim_{x\to+\infty}\log_a x$ $\lim_{x\to 0+}\log_a x$.

4. $a>1$. $a^{x+1} = aa^x$ $y=a^x$, $\lim_{x\to\pm\infty} a^x$.

5. $y=f(x)$ $[1,+\infty)$ $f(\sqrt{n})\geq \log n$ n. $\lim_{x\to+\infty} f(x)$, , ;
(: 4.19.)

6. $y=f(x)$ $(0,2)$ $f(\frac{1}{n}) = 1-\frac{1}{\sqrt{n}}$ n. $\lim_{x\to 0+} f(x)$, , ;

Κεφάλαιο 5

.

: « ϵ δ». , , ... , - . Bolzano . . . . .

5.1 , .

$y=f(x)$ ξ ξ

$$\lim_{x\to\xi} f(x)=f(\xi).$$

, ξ, $[\xi,b)$, (a,ξ), $(a,\xi]$, (ξ,b), (a,b), $a<\xi<b$. , $y=f(x)$ ξ ξ $(a,\xi)\cup(\xi,b)$, $y=f(x)$ ξ.
$y=f(x)$ ξ ¨ ξ , $f(\xi)$.
(i) $y=f(x)$ $[\xi,b)$ (a,ξ), $\lim_{x\to\xi} f(x)=f(\xi)$, , $\lim_{x\to\xi+} f(x)=f(\xi)$. (ii) $y=f(x)$ $(a,\xi]$ (ξ,b), $\lim_{x\to\xi-} f(x)=f(\xi)$. (iii) $y=f(x)$ (a,b) $a<\xi<b$, $\lim_{x\to\xi-} f(x)=\lim_{x\to\xi+} f(x)=f(\xi)$.
, $y=f(x)$ $[\xi,b)$ $\lim_{x\to\xi+} f(x)=f(\xi)$, ξ . , $y=f(x)$ $(a,\xi]$ $\lim_{x\to\xi-} f(x)=f(\xi)$, ξ . : (i) $y=f(x)$ $[\xi,b)$ (a,ξ), ξ ξ . (ii) $y=f(x)$ $(a,\xi]$ (ξ,b), ξ ξ . (iii) $y=f(x)$ (a,b) $a<\xi<b$, ξ ξ .

: (1) $y=x^2$ 3, $\lim_{x\to3} x^2=9=3^2$.

(2) $y=\sqrt{x}$ 0, $\lim_{x\to0+}\sqrt{x}=0=\sqrt{0}$.

(3) $y=[x]$ $y=0$ $(0,1)$, $\lim_{x\to1-}[x]=\lim_{x\to1-}0=0\neq1=[1]$. , $y=[x]$ $y=1$ $(1,2)$, $\lim_{x\to1+}[x]=\lim_{x\to1+}1=1=[1]$. $y=[x]$ 1 , , 1. $y=[x]$ $y=0$ $(0,\frac{1}{2})\cup(\frac{1}{2},1)$, $\lim_{x\to\frac{1}{2}}[x]=\lim_{x\to\frac{1}{2}}0=0=[\frac{1}{2}]$. $y=[x]$ $\frac{1}{2}$.

(4) $\sqrt{-x^2(x+1)}$ $(-\infty,-1]\cup\{0\}$. , 0 0, 0.

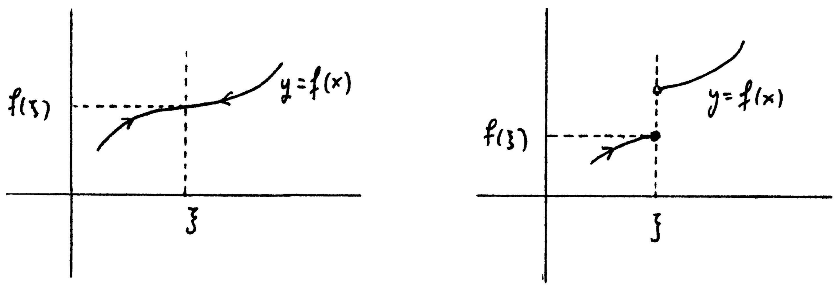
(5) $y=c$ ξ. , $\lim_{x\to\xi} c=c$, ξ ξ.

, , , . , A , A, A, A.

: $y = f(x) = \begin{cases} x\,, & x < 0, \\ x+1\,, & x \geq 0, \end{cases}$ $(-\infty, +\infty)$. $y = f(x)$ 0. : $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x = 0 \neq 1 = f(0)$, 0 , $\lim_{x\to 0+} f(x) = \lim_{x\to 0+}(x+1) = 1 = f(0)$, 0 . $[0, +\infty)$, $y = f(x) = x + 1$ x $[0, +\infty)$. , ξ $[0, +\infty)$ $\lim_{x\to\xi} f(x) = \lim_{x\to\xi}(x + 1) = \xi + 1 = f(\xi)$. $[0, +\infty)$ 0 0 . , , $y = f(x)$ $[0, +\infty)$.

. $y = f(x)$ ξ ξ, $\lim_{x\to\xi} f(x)$. $\lim_{x\to\xi} f(x) = f(\xi)$, ξ, $|f(x) - f(\xi)|$ $|x - \xi|$ $\neq 0$. , , , $|x - \xi|$ 0 , , x ξ, $f(x)$ $f(\xi)$, , $|f(x) - f(\xi)|$ 0 , , $|f(x) - f(\xi)|$. 0 $|x - \xi|$. , $y = f(x)$ ξ $|f(x) - f(\xi)|$ $|x - \xi|$.
ξ , , ξ. , ′ , $|x - \xi|$ x ξ , , $f(x)$ $f(\xi)$, , $|f(x) - f(\xi)|$ 0 , , $|f(x) - f(\xi)|$.
, . $y = f(x)$ ξ ξ $|f(x) - f(\xi)|$ $|x - \xi|$, , $\epsilon > 0$ $\delta > 0$ $|f(x) - f(\xi)| < \epsilon$ x $|x - \xi| < \delta$. $0 < |x - \xi|$ ξ.
. $y = f(x)$ ξ, ξ , x ξ, $(x, y) = (x, f(x))$ $(\xi, f(\xi))$. ¨( ) : ξ «» $(\xi, f(\xi))$, , «» . (, , 8 .) , . «» $y = f(x)$. $y = f(x)$ ξ $\epsilon > 0$ $\delta > 0$ x $(\xi - \delta, \xi + \delta)$ $f(x)$ $(x, f(x))$ $f(\xi) - \epsilon$ $f(\xi) + \epsilon$, , $(\xi - \delta, \xi + \delta)$ $y = f(\xi) - \epsilon$ $y = f(\xi) + \epsilon$.

: , , .



Σχήμα 5.1: ξ ξ .

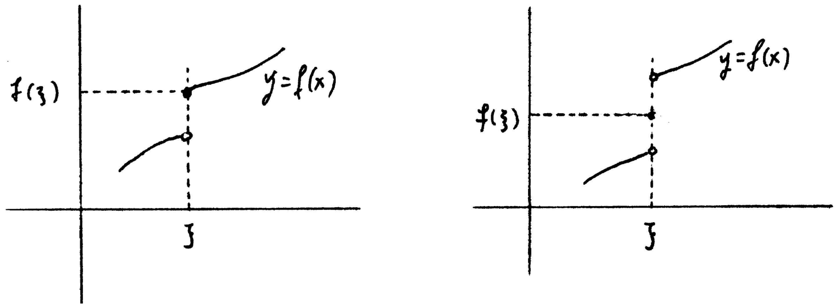
(1) $y = p(x) = a_0 + a_1x + \cdots + a_Nx^N$, ξ $\lim_{x\to\xi} p(x) = p(\xi)$.

(2) $y = r(x) = \frac{a_0+a_1x+\cdots+a_Nx^N}{b_0+b_1x+\cdots+b_Mx^M}$. , ξ , ξ , $\lim_{x\to\xi} r(x) = r(\xi)$.

(3) $y = \cos x$ $y = \sin x$, ξ $\lim_{x\to\xi} \cos x = \cos\xi$ $\lim_{x\to\xi} \sin x = \sin\xi$.
, $y = \tan x$ $y = \cot x$ · ξ , $\xi \neq \frac{\pi}{2} + k\pi$ $(k \in \mathbf{Z})$ $\xi \neq k\pi$ $(k \in \mathbf{Z})$, $\lim_{x\to\xi} \tan x = \tan\xi$ $\lim_{x\to\xi} \cot x = \cot\xi$.

(4) $y = x^a$ · ξ , (i) ξ, a ($\xi = 0$, $a \leq 0$) (ii) $\xi \geq 0$, a ($\xi = 0$, $a \leq 0$), $\lim_{x\to\xi} x^a = \xi^a$.

(5) $a > 0$, $y = a^x$, ξ $\lim_{x\to\xi} a^x = a^\xi$.



Σχήμα 5.2: ξ ξ.

(6) $a > 0$, $a \neq 1$, $y = \log_a x$. , $\xi > 0$ $\lim_{x \to \xi} \log_a x = \log_a \xi$.

.

1. 1. « ϵ δ».

$$y = x, \qquad y = 2x - 3, \qquad y = x^2\,, \qquad y = \frac{1}{x}\,, \qquad y = \sqrt{x}\,.$$

2. 0;

$$y = \begin{cases} \frac{\sin x}{x}\,, & x \neq 0, \\ 1\,, & x = 0, \end{cases} \qquad y = \begin{cases} \frac{1 - \cos x}{x^2}\,, & x \neq 0, \\ \frac{1}{2}\,, & x = 0. \end{cases}$$

3. ;

$$y = \begin{cases} x\,, & x \leq 0, \\ \frac{1}{x}\,, & x > 0, \end{cases} \quad y = \begin{cases} 0\,, & x = 0, \\ \frac{1}{|x|}\,, & x \neq 0, \end{cases} \quad y = \begin{cases} x^2\,, & x \neq 0, \\ 1\,, & x = 0, \end{cases}$$

$$y = \begin{cases} x^2 + 1, & x < 0, \\ \sqrt{x + 1}\,, & x \geq 0, \end{cases} \quad y = \begin{cases} x^2\,, & x \leq -\pi \quad x > \pi, \\ \sin x\,, & -\pi < x \leq \pi. \end{cases}$$

4. ; 2 4.3.

$$y = [x], \quad y = [2x], \quad y = x - [x], \quad y = x - [x] - \frac{1}{2}\,, \quad y = \left|x - [x] - \frac{1}{2}\right|.$$

5. $\lim_{h \to 0} \big(f(\xi + h) - f(\xi - h)\big) = 0$ $y = f(x)$ ξ.

: $y = f(x) = \begin{cases} 1\,, & x = 0, \\ 0\,, & x \neq 0 \end{cases}$ $\xi = 0$.

6. $y = f(x)$ ξ ξ, M $|f(x)| \leq M$ x $(a, \xi) \cup (\xi, b)$. « ϵ δ» ($\epsilon > 0$ $\delta > 0$) $y = g(x) = (x - \xi) f(x)$ ξ.

7. $M \geq 0$ $\rho > 0$ $y = f(x)$ (a,b), $a < \xi < b$. $|f(x) - f(\xi)| \leq M|x-\xi|^{\rho}$ x (a,b), « ϵ δ» ($\epsilon > 0$ $\delta > 0$) ξ.

$y = f(x)$, **Hölder-** ξ **Hölder-** ρ. $\rho = 1$, $y = f(x)$ **Lipschitz-** ξ.

$y = x$, $y = |x|$, $y = \cos x$, $y = \sin x$, $y = \sqrt{|x|}$ $y = x\sqrt{|x|}$ Hölder- 0 Hölder-.

(*) Hölder- ξ $\xi \neq 0$ Hölder-. Hölder- $\xi = 0$ $\xi \neq 0$.

8. $y = f(x)$ ξ «» $(\xi, f(\xi))$, , «» . , , . .

$$y = \begin{cases} x(-1)^{[\frac{1}{x}]}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$. 1 3.4.

0.

(: .)

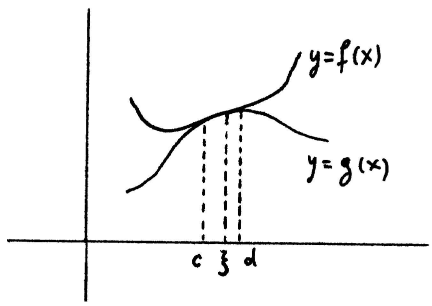
0.

5.2 .

. 5.1 4.2.

5.1 $y = f(x)$ $y = g(x)$ (a,b) $a < \xi < b$ $[\xi, b)$ $(a, \xi]$. ξ ξ ξ , , .

: $y = f(x)$ $y = g(x)$ (a,b) $a < \xi < b$, $(a,\xi) \cup (\xi, b)$ $f(\xi) = g(\xi)$. $y = f(x)$ ξ, $\lim_{x\to\xi} f(x) = f(\xi)$. $y = f(x)$ $y = g(x)$ $(a,\xi)\cup(\xi,b)$, $\lim_{x\to\xi} g(x) = \lim_{x\to\xi} f(x) = f(\xi)$. , $f(\xi) = g(\xi)$, $\lim_{x\to\xi} g(x) = g(\xi)$, , $y = g(x)$ ξ.



Σχήμα 5.3: ξ.

$[\xi, b)$ $(a, \xi]$.

: (1) $y = \begin{cases} x+1, & x \leq 1, \\ x-1, & 1 < x \end{cases}$ $y = x+1$ $(-\infty, 1]$. 1 , , 1 . 1 .

(2) $y = x^2$ $y = \begin{cases} 1+x, & |x| \geq 10^{-10}, \\ x^2, & |x| < 10^{-10}, \end{cases}$ $(-10^{-10}, 10^{-10})$. 0, 0.

. 5.2 , , , .

5.2 $y=f(x)$ $y=g(x)$ ξ ξ ξ , $y=f(x)+g(x)$, $y=f(x)-g(x)$, $y=f(x)g(x)$ $y=|f(x)|$ ξ ξ ξ , . $\frac{f(x)}{g(x)}$, $g(\xi)\neq 0$.

: $\lim_{x\to\xi} f(x)=f(\xi)$ $\lim_{x\to\xi} g(x)=g(\xi)$

$$\lim_{x\to\xi}(f(x)+g(x))=\lim_{x\to\xi} f(x)+\lim_{x\to\xi} g(x)=f(\xi)+g(\xi)$$

, , $y=f(x)+g(x)$ ξ. ξ , , .

: (1) $y=\frac{\sqrt{x}+e^x}{(x-2x^2)\log x}$ ξ , $y=\sqrt{x}$, $y=e^x$, $y=\log x$ $y=x-2x^2$. $(0,+\infty)$. ξ $(0,\frac{1}{2})\cup(\frac{1}{2},1)\cup(1,+\infty)$.

(2) $y=\frac{x^2+\sqrt{x}}{\sin x+\cos x}$ $\xi\geq 0$, $\xi\geq 0$ $\neq -\frac{\pi}{4}+k\pi$ $(k\in\mathbf{Z})$.

. .

5.3 $z=g\big(f(x)\big)$ $y=f(x)$ $z=g(y)$. $y=f(x)$ ξ $z=g(y)$ $\eta=f(\xi)$, $z=g\big(f(x)\big)$ ξ.

: $\epsilon>0$. $z=g(y)$ η, $\delta'>0$ $|g(y)-g(\eta)|<\epsilon$ y $z=g(y)$ $|y-\eta|<\delta'$. $y=f(x)$ ξ, $\delta>0$ $|f(x)-\eta|=|f(x)-f(\xi)|<\delta'$ x $y=f(x)$ $|x-\xi|<\delta$. , x $y=f(x)$ $|x-\xi|<\delta$ $|f(x)-\eta|<\delta'$, $f(x)$ $z=g(y)$, $\big|g\big(f(x)\big)-g(\eta)\big|<\epsilon$. $\big|g\big(f(x)\big)-g\big(f(\xi)\big)\big|<\epsilon$ x $y=f(x)$ – , x $z=g\big(f(x)\big)$ – $|x-\xi|<\delta$. $z=g\big(f(x)\big)$ ξ.

5.3 « »: x ξ, $y=f(x)$, , $f(\xi)=\eta$, $g\big(f(x)\big)=g(y)$ $g(\eta)=g\big(f(\xi)\big)$. , x ξ, $g\big(f(x)\big)$ $g\big(f(\xi)\big)$. , , , 4.12.

: (1) $z=\sin\sqrt{x}$ $\xi\geq 0$. , $y=\sqrt{x}$ $\xi\geq 0$ $z=\sin y$ $\eta=\sqrt{\xi}$ – .

(2) $z=\sqrt{\sin x}$ $[k2\pi,\pi+k2\pi]$ $(k\in\mathbf{Z})$ ′ $\sin x\geq 0$. $y=\sin x$ ξ ′ – ξ – $z=\sqrt{y}$ $\eta=\sin\xi$ ≥ 0. $z=\sqrt{\sin x}$ ξ .

5.3 – – 4.12 $z=g\big(f(x)\big)$, $y=f(x)$ $z=g(y)$, $\lim f(x)$ η $z=g(y)$ η. , $\lim_{x\to\xi} f(x)=\eta$ $z=g(y)$ η. « »: x ξ $\neq\xi$, $y=f(x)$, , η, $g\big(f(x)\big)=g(y)$ $g(\eta)$. , x ξ $\neq\xi$, $g\big(f(x)\big)$ $g(\eta)$. , $\lim_{x\to\xi} g\big(f(x)\big)=g(\eta)$.

5.4 **.** $z=g\big(f(x)\big)$ $y=f(x)$ $z=g(y)$. (i) $\lim f(x)=\eta$ $y=g(x)$ η (ii) $\lim f(x)=\eta$ $f(x)\geq\eta$ x $y=g(x)$ η (iii) $\lim f(x)=\eta$ $f(x)\leq\eta$ x $y=g(x)$ η , $\lim g\big(f(x)\big)=g(\eta)$.

: $\lim_{x\to\xi} f(x)=\eta$ $z=g(y)$ η. $\lim_{x\to\xi} g(f(x))=g(\eta)$.

$\epsilon>0$, $\delta'>0$ $|g(y)-g(\eta)|<\epsilon$ y $z=g(y)$ $|y-\eta|<\delta'$. , $\delta>0$ $|f(x)-\eta|<\delta'$ x $y=f(x)$ $0<|x-\xi|<\delta$. x $y=f(x)$ $0<|x-\xi|<\delta$ $|f(x)-\eta|<\delta'$, $f(x)$ $z=g(y)$, $|g(f(x))-g(\eta)|<\epsilon$. , x $y=f(x)$ – , x $z=g(f(x))$ – $0<|x-\xi|<\delta$ $|g(f(x))-g(\eta)|<\epsilon$. $\lim g(f(x))=g(\eta)$.

: (1) $\lim_{x\to 0}\frac{(\sqrt{x}+1)^4}{(\sqrt{x}+1)^8+(\sqrt{x}+1)^{13}+5}$.

$y=\sqrt{x}+1$, $z=\frac{(\sqrt{x}+1)^4}{(\sqrt{x}+1)^8+(\sqrt{x}+1)^{13}+5}$ $z=\frac{y^4}{y^8+y^{13}+5}$. $\lim_{x\to 0} y = \lim_{x\to 0}(\sqrt{x}+1)=1$ $z=\frac{y^4}{y^8+y^{13}+5}$ 1.

$\lim_{x\to 0}\frac{(\sqrt{x}+1)^4}{(\sqrt{x}+1)^8+(\sqrt{x}+1)^{13}+5}=\frac{1^4}{1^8+1^{13}+5}=\frac{1}{7}$.

(2) $\lim_{x\to+\infty}\frac{\sqrt{\frac{x-1}{x^2+x+1}}+\left(\frac{x-1}{x^2+x+1}\right)^2+\frac{x-1}{x^2+x+1}+1}{3\left(\frac{x-1}{x^2+x+1}\right)^4+2\sqrt{\frac{x-1}{x^2+x+1}}+1}$.

$y=\frac{x-1}{x^2+x+1}$ $z=\frac{\sqrt{\frac{x-1}{x^2+x+1}}+\left(\frac{x-1}{x^2+x+1}\right)^2+\frac{x-1}{x^2+x+1}+1}{3\left(\frac{x-1}{x^2+x+1}\right)^4+2\sqrt{\frac{x-1}{x^2+x+1}}+1}$ $z=\frac{\sqrt{y}+y^2+y+1}{3y^4+2\sqrt{y}+1}$. $\lim_{x\to+\infty} y=\lim_{x\to+\infty}\frac{x-1}{x^2+x+1}=0$ $z=\frac{\sqrt{y}+y^2+y+1}{3y^4+2\sqrt{y}+1}$ 0.

$\lim_{x\to+\infty}\frac{\sqrt{\frac{x-1}{x^2+x+1}}+\left(\frac{x-1}{x^2+x+1}\right)^2+\frac{x-1}{x^2+x+1}+1}{3\left(\frac{x-1}{x^2+x+1}\right)^4+2\sqrt{\frac{x-1}{x^2+x+1}}+1}=\frac{\sqrt{0}+0^2+0+1}{3\cdot 0^4+2\sqrt{0}+1}=1.$

(3) $\lim_{x\to+\infty}\left((\frac{\sin x}{x})^3+(\frac{\sin x}{x})^2+3\right)$.

$y=\frac{\sin x}{x}$ $z=(\frac{\sin x}{x})^3+(\frac{\sin x}{x})^2+3$ $z=y^3+y^2+3$. $\lim_{x\to+\infty} y = \lim_{x\to+\infty}\frac{\sin x}{x}=0$ $z=y^3+y^2+3$ 0.

$\lim_{x\to+\infty}\left((\frac{\sin x}{x})^3+(\frac{\sin x}{x})^2+3\right)=0^3+0^2+3=3.$

4.12 5.4. ′ 4.12 $y=f(x)$ $\pm\infty$ 5.4 $y=f(x)$. , 5.4 4.12. , $\lim f(x)=\eta$, 4.12 $f(x)\neq\eta$ x. 5.4 $z=g(y)$ η. 5.4 4.12 4.12 5.4 (a $\frac{\sin x}{x}\neq 0$ $x>a$).

. .

5.5 $f(x)\le g(x)$ ξ ξ $y=f(x)$ $y=g(x)$ ξ ξ , , $f(\xi)\le g(\xi)$.

: 4.14 $\lim_{x\to\xi} f(x)=f(\xi)$ $\lim_{x\to\xi} g(x)=g(\xi)$ $\lim_{x\to\xi\pm} f(x)=f(\xi)$ $\lim_{x\to\xi\pm} g(x)=g(\xi)$.

: (1) $f(x)\le\sin x$ x $(0,\frac{\pi}{4})$ $y=f(x)$ 0 , , $y=\sin x$ 0 , $f(0)\le\sin 0=0$.

(2) $l\le f(x)\le u$ ξ $y=f(x)$ ξ, $l\le f(\xi)\le u$. 5.5 $y=f(x)$ $y=l$ $y=u$.

5.6 $y=f(x)$ ξ ξ .
(1) $f(\xi)<u$, $f(x)<u$ ξ ξ , .
(2) $f(\xi)>l$, $f(x)>l$ ξ ξ , .

: 4.16 $\lim_{x\to\xi} f(x)=f(\xi)$ $\lim_{x\to\xi} g(x)=g(\xi)$ $\lim_{x\to\xi\pm} f(x)=f(\xi)$ $\lim_{x\to\xi\pm} g(x)=g(\xi)$.

: $y=\frac{\sin x+\cos x}{\sin x-\cos x}$ $\frac{\pi}{3}$ $\frac{\pi}{3}$ $\frac{\frac{\sqrt{3}}{2}+\frac{1}{2}}{\frac{\sqrt{3}}{2}-\frac{1}{2}}=2+\sqrt{3}$. $3<2+\sqrt{3}<4$, (a,b) $a<\frac{\pi}{3}<b$ $3<\frac{\sin x+\cos x}{\sin x-\cos x}<4$ x . 5.6 $l=3$ $u=4$.

. 5.7 4.17.

5.7 $y = f(x)$ ξ ξ , ξ ξ , .

: $y = \frac{1}{x}$ 1. 5.7 1 ( ). , (a, b) $a < 1 < b$ $y = \frac{1}{x}$. , $(\frac{1}{2}, \frac{3}{2})$, 1, $\frac{2}{3} < \frac{1}{x} < 2$ x .

.

1. ;

$$y = \frac{x^2 \log x + xe^x}{(\sin x - \cos x)^2}, \qquad y = x^{-\frac{3}{4}}(\log x)^2 \frac{\tan x - \cot x}{(\sin x)^2 - 2\sin x + 1}.$$

2. − − .

$$y = \sin(x^2), \quad y = \log(x^2 + 2), \quad y = e^{x^3 - 2x}, \quad y = 2^{2^x}, \quad y = \frac{1}{\sqrt{e^x - 1}},$$

$$y = \sin(\log x), \quad y = \sqrt{1 - \cos x}, \quad y = e^{\frac{1}{\sin x}}, \quad y = [x^2], \quad y = [\sqrt{x}],$$

$$y = (x^2 - 5x + 6)^{\sqrt{2}}, \quad y = \log(x^2 - 5x + 6), \quad y = \log(\log x),$$

$$y = \log(\sin x), \quad y = \log(1 - \cos x), \quad y = \tan(\sin x - \cos x).$$

3. $f(x) > 0$ x $y = f(x)$. $y = f(x)$ $y = g(x)$ ξ, $y = f(x)^{g(x)}$ ξ. (: $f(x)^{g(x)} = e^{g(x)\log(f(x))}$.)

.

(*i*) $y = x^x$ $(0, +\infty)$.

(*ii*) $y = (x^2 - 3)^{\frac{x-2}{x+2}}$ $(-\infty, -2) \cup (-2, -\sqrt{3}) \cup (\sqrt{3}, +\infty)$.

(*iii*) $y = (2 - x^2)^{\log x}$ $(0, \sqrt{2})$.

(*iv*) $y = (\log x)^{\log x}$ $(1, +\infty)$.

4. .

$$\lim_{x \to +\infty} \sin \frac{1}{\sqrt{x}}, \quad \lim_{x \to -\infty} \cos\left(\frac{\sin x}{x}\right), \quad \lim_{x \to 1} e^{(x-1)\sin\frac{1}{x-1}}.$$

4.12;

5. 5.5 5.6.

6. $[0, b)$ x $[0, b)$ $\frac{1}{2} < \frac{\log(1+x) + \cos\sqrt{x}}{e^x + \sin\sqrt{x}} < \frac{3}{2}$.

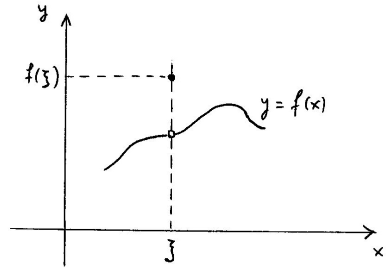
7. (a, b) − − 1 x (a, b) $\frac{1}{2} < \frac{x^8 - x^5 + 3}{4x^4 - 1} < \frac{3}{2}$ $\frac{1}{6} < \frac{e^x - 2^x}{7x - 3} < \frac{1}{4}$. (: ′ . (a, b) ;)

8. 5.7.

5.3 .

$y = f(x)$ ξ, ξ . , ξ $y = f(x)$, ξ .
ξ $y = f(x)$ ξ, ξ ( ) ξ. , , .

1. $\lim_{x\to\xi} f(x)$ $\neq f(\xi)$. ξ ξ.



Σχήμα 5.4: ξ.

$y = f(x)$ $\xi -$ $\xi -$ ξ. , $y = g(x)$ $g(x) = f(x)$ $x \neq \xi$ $y = f(x)$ $g(\xi) = \lim_{x\to\xi} f(x)$. $y = g(x)$ $y = f(x)$ $y = f(x)$ ξ. , , $y = g(x)$ ξ, $\lim_{x\to\xi} g(x) = \lim_{x\to\xi} f(x) = g(\xi)$. ($g(x) = f(x)$ $x \neq \xi$ $g(\xi)$.) $y = g(x)$ ξ.

: (1) $y = f(x) = \begin{cases} x+1, & x \neq 0, \\ 0, & x = 0, \end{cases}$ 0, $\lim_{x\to 0} f(x) = \lim_{x\to 0}(x+1) = 1$ $f(0) = 0$.

, 0 0, 1 (0), $y = g(x) = \begin{cases} x+1, & x \neq 0, \\ 1, & x = 0, \end{cases}$ $y = x + 1$, 0.

(2) $y = f(x) = \begin{cases} \sqrt{x}, & x > 0, \\ 1, & x = 0, \end{cases}$ 0, $\lim_{x\to 0} f(x) = \lim_{x\to 0} \sqrt{x} = 0$ $\neq f(0) = 1$.

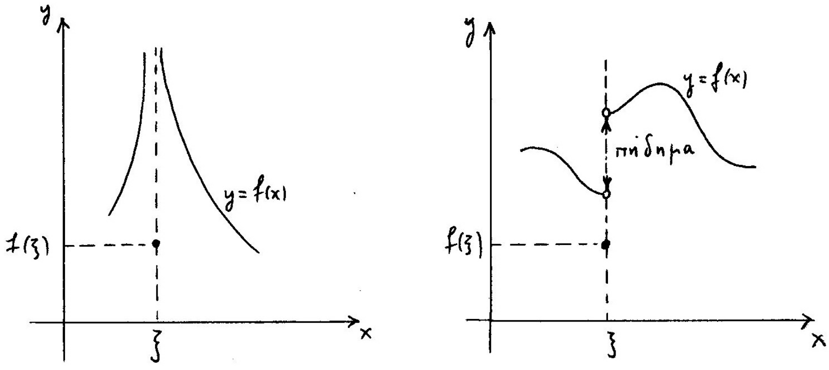
0 0, 0 (1), $y = g(x) = \begin{cases} \sqrt{x}, & x > 0, \\ 0, & x = 0, \end{cases}$ $= \sqrt{x}$ 0.

2. (*i*) $\lim_{x\to\xi} f(x)$, $+\infty$ $-\infty$, (*ii*) $\lim_{x\to\xi+} f(x)$ $\lim_{x\to\xi-} f(x)$. ξ ξ. (*ii*) $\lim_{x\to\xi+} f(x) - \lim_{x\to\xi-} f(x)$, $\neq 0$, $y = f(x)$ ξ.

: (1) $y = f(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0, \\ 0, & x = 0 \end{cases}$ 0, $\lim_{x\to 0} f(x) = +\infty$.

$y = f(x) = \begin{cases} \frac{1}{\sqrt{-x}}, & x < 0, \\ 0, & x = 0. \end{cases}$

(2) $y = f(x) = \begin{cases} \frac{1}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases}$ 0, $\lim_{x\to 0+} f(x) = +\infty$ $\lim_{x\to 0-} f(x) = -\infty$, $\lim_{x\to 0+} f(x) \neq \lim_{x\to 0-} f(x)$. 0 $+\infty - (-\infty) = +\infty$.

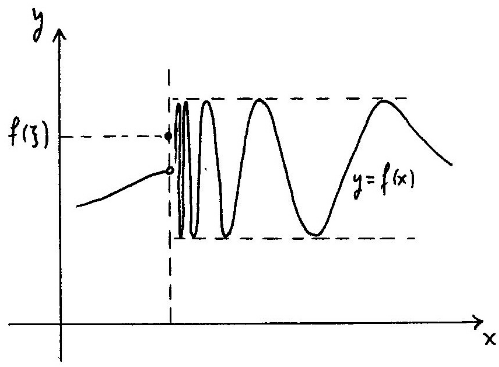


Σχήμα 5.5: ξ: $= +\infty$.

(3) $y = f(x) = \begin{cases} x+1, & x \geq 0, \\ x, & x < 0 \end{cases}$ 0, $\lim_{x\to 0+} f(x) = 1$ $\lim_{x\to 0-} f(x) = 0$, $\lim_{x\to 0+} f(x) \neq \lim_{x\to 0-} f(x)$. 0 $1 - 0 = 1$.

0 , 0 .

3. , $\lim_{x\to \xi+} f(x)$ $\lim_{x\to \xi-} f(x)$, ξ ξ.



Σχήμα 5.6: ξ: .

: (1) $y = f(x) = \begin{cases} \sin\frac{1}{x}, & x > 0, \\ x, & x \leq 0 \end{cases}$ 0, 0. , $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \sin\frac{1}{x} = \lim_{t\to +\infty} \sin t$.

0 , $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x = 0 = f(0)$.

(2) $y = f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0, \\ 1, & x = 0 \end{cases}$ 0, 0. , $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \sin\frac{1}{x} = \lim_{t\to+\infty} \sin t$ ( ) $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \sin\frac{1}{x} = \lim_{t\to-\infty} \sin t$, , .

, ξ $y = f(x)$, $f(\xi)$ ξ. ξ .

$y = f(x)$ (a, b) $a < \xi < b$. , , (a, ξ) , $f(x) \leq f(\xi)$ x (a, ξ). , 4.1, $\lim_{x\to\xi-} f(x)$ $\lim_{x\to\xi-} f(x) \leq f(\xi)$. , (ξ, b), $f(x) \geq f(\xi)$ x (ξ, b). $\lim_{x\to\xi+} f(x)$ $\geq f(\xi)$. , , $\lim_{x\to\xi-} f(x)$ $\lim_{x\to\xi+} f(x)$ $\lim_{x\to\xi-} f(x) \leq f(\xi) \leq \lim_{x\to\xi+} f(x)$. . $\lim_{x\to\xi-} f(x) = \lim_{x\to\xi+} f(x)$, $\lim_{x\to\xi-} f(x) = f(\xi) = \lim_{x\to\xi+} f(x)$, , ξ. $\lim_{x\to\xi-} f(x) < \lim_{x\to\xi+} f(x)$, ξ $\lim_{x\to\xi+} f(x) - \lim_{x\to\xi-} f(x) > 0$.

$y = f(x)$ (a, b) $a < \xi < b$· , $\lim_{x\to\xi-} f(x) \geq f(\xi) \geq \lim_{x\to\xi+} f(x)$.

:

$y = f(x)$ (a, b) $a < \xi < b$, (i) ξ (ii) -
ξ , ,
, .

, , .

.

1. 0 . 0 0. .

$$y = \begin{cases} \frac{|x|}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases} \quad y = \begin{cases} x^2, & x \neq 0, \\ 1, & x = 0, \end{cases} \quad y = \begin{cases} \frac{1}{|x|}, & x \neq 0, \\ -1, & x = 0, \end{cases}$$

$$y = \begin{cases} x, & x \leq 0, \\ \frac{1}{x}, & x > 0, \end{cases} \quad y = \begin{cases} \sin\frac{1}{x}, & x > 0, \\ 1, & x \leq 0, \end{cases} \quad y = \begin{cases} \frac{\tan x}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

2. 4 5.1.

3. $y = \log[x]$ $[1, +\infty)$.

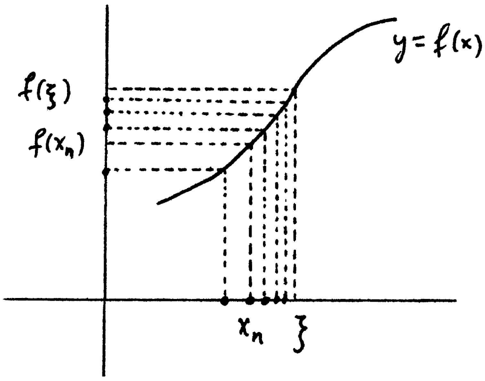
4. $y = f(x)$, − − (i) (ii) .

5.4 .

$y = f(x)$ ξ. (x_n) $y = f(x)$ ξ. $\lim_{n\to+\infty} x_n = \xi$. « »: n , x_n , , ξ, $f(x_n)$ $f(\xi)$. , , .

5.8 $y = f(x)$ ξ (x_n) . $\lim_{n\to+\infty} x_n = \xi$, $\lim_{n\to+\infty} f(x_n) = f(\xi)$.

: $\epsilon > 0$, , $y = f(x)$ ξ, $\delta > 0$ $|f(x) - f(\xi)| < \epsilon$ x $y = f(x)$ $|x - \xi| < \delta$. $\lim_{n\to+\infty} x_n = \xi$, n_0 $|x_n - \xi| < \delta$ $n \geq n_0$. $|f(x_n) - f(\xi)| < \epsilon$ $n \geq n_0$, , $\lim_{n\to+\infty} f(x_n) = f(\xi)$.



Σχήμα 5.7: $\lim x_n = \xi$ $\lim f(x_n) = f(\xi)$.

: (1) $y = p(x)$ $\lim_{n\to+\infty} x_n = \xi$, $\lim_{n\to+\infty} p(x_n) = p(\xi)$.

(2) $y = r(x)$, (x_n) ξ $\lim_{n\to+\infty} x_n = \xi$, $\lim_{n\to+\infty} r(x_n) = r(\xi)$.

(3) $\lim_{n\to+\infty} x_n = \xi$, $\lim_{n\to+\infty} \cos x_n = \cos\xi$ $\lim_{n\to+\infty} \sin x_n = \sin\xi$.

(4) (x_n) , ξ $\lim_{n\to+\infty} x_n = \xi$, $\lim_{n\to+\infty} {x_n}^a = \xi^a$.

(5) $a > 0$ $\lim_{n\to+\infty} x_n = \xi$, $\lim_{n\to+\infty} a^{x_n} = a^\xi$.
, $x_n = \frac{1}{n}$ n, .

$$\boxed{\lim_{n\to+\infty} \sqrt[n]{a} = 1 \qquad (a > 0).}$$

(6) (x_n) , ξ $\lim_{n\to+\infty} x_n = \xi$, $\lim_{n\to+\infty} \log_a x_n = \log_a \xi$.

$\lim_{n\to+\infty} x_n = \xi$ $\lim_{n\to+\infty} f(x_n) = f(\xi)$:

$$\boxed{\lim_{n\to+\infty} f(x_n) = f\big(\lim_{n\to+\infty} x_n\big).}$$

, «» $\lim_{n\to+\infty}$ f ¨ $y = f(x)$ ξ, .
5.8 – – 4.19 – . , 5.8 ( ) 4.19 , , $\neq \xi$.

: $\lim_{n\to+\infty} \sin\big(\frac{1+(-1)^{n-1}}{n}\big)$ $\lim_{n\to+\infty} \frac{1+(-1)^{n-1}}{n} = 0$ $y = \sin x$ 0.
$\lim_{n\to+\infty} \sin\big(\frac{1+(-1)^{n-1}}{n}\big) = \sin\big(\lim_{n\to+\infty} \frac{1+(-1)^{n-1}}{n}\big) = \sin 0 = 0$.
4.19, $\frac{1+(-1)^{n-1}}{n} = 0$ n.

.

1. .

$$\Big(\Big(1+\frac{1}{n}\Big)^8 + 4\Big(1+\frac{1}{n}\Big)^5 + 7\Big), \quad \Big(e^{\frac{1+(-1)^n}{n}}\Big), \quad \Big(\log\Big(1+\frac{1}{n}\Big)\Big), \quad \Big(\tan\frac{1}{2^n}\Big),$$

$$\left(2^{\frac{3n^4+n-4}{n^4+n^3+4}}\sin\left(\frac{\pi}{2}-\frac{1}{n^2}\right)\right),\ \left(n\log\left(1+\frac{1}{n}\right)\right),\ \left(\left(\frac{n^2+3}{4n^2-3}\right)^{\frac{3}{2}}\log\left(\cos\frac{1}{n}\right)\right).$$

5.5 .

– – .

5.1 . $y=f(x)$ $[a,b]$. .

5.1 . $y=f(x)$ $(a,f(a))$ $(b,f(b))$ x $[a,b]$ $(x,f(x))$. – – $(x,f(x))$, $f(x)$, . 5.1 , . , , , , .

: (1) $y=\begin{cases}\frac{1}{x}, & -1\le x<0 \ \ 0<x\le 1,\\ 0, & x=0,\end{cases}$ $[-1,1]$, 0 $[-1,1]$.

(2) $y=\begin{cases}x, & -1\le x<0 \ \ 0<x\le 1,\\ 1, & x=0,\end{cases}$ $[-1,1]$, 0 $[-1,1]$.

(3) H $y=\frac{1}{x(x-1)}$ $(0,1)$ $(0,1)$.

(4) $y=x$ $(-1,1)$ $(-1,1)$.

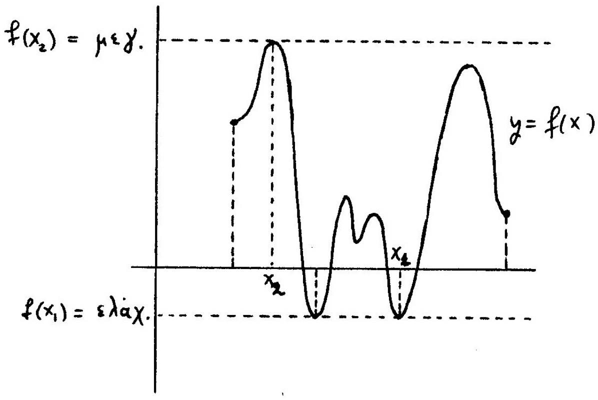
(5) $y=x$ $(-\infty,+\infty)$ $(-\infty,+\infty)$.

(6) $y=\frac{1}{x^2+1}$ $(-\infty,+\infty)$ $(-\infty,+\infty)$.

5.2 **-** . $y=f(x)$ $[a,b]$. x_1,x_2 $[a,b]$

$$f(x_1)\le f(x)\le f(x_2)$$

x $[a,b]$.



Σχήμα 5.8: - .

5.1, $y = f(x)$ $(a, f(a))$ $(b, f(b))$. , , – – , $(x_1, f(x_1))$, , «», , $(x_2, f(x_2))$, , «». , , x_1 x_2 . x_1, x_2 . x_1 () x_2 () .

5.2 x_1, x_2 . .

5.2 , . , , , , .

: (1) $y = \begin{cases} x+1\,, & -1 \le x < 0, \\ 0\,, & x = 0, \\ x-1\,, & 0 < x \le 1, \end{cases}$ $[-1, 1]$, 0 .

(2) $y = \begin{cases} 0\,, & -1 \le x < 0, \\ 1\,, & 0 \le x \le 1, \end{cases}$ $[-1, 1]$, 0 .

(3) H $y = x$ $(-1, 1)$.

(4) $y = \begin{cases} x+2\,, & -2 < x < -1, \\ -x\,, & -1 \le x \le 1, \\ x-2\,, & 1 < x < 2, \end{cases}$ $(-2, 2)$.

(5) $y = x$ $(-\infty, +\infty)$.

(6) $y = \begin{cases} \frac{1}{x}\,, & |x| > 1, \\ x\,, & |x| \le 1, \end{cases}$ $(-\infty, +\infty)$.

5.3 . $y = f(x)$ $[a, b]$. c, $f(a)$ $f(b)$, . c $f(a) \le c \le f(b)$ $f(b) \le c \le f(a)$ ξ $[a, b]$

$$f(\xi) = c$$

, , c $f(x) = c$ ( ) ξ $[a, b]$.

$f(a) = f(b)$, $c = f(a) = f(b)$, $f(x) = c$ · $\xi = a$ $\xi = b$. , $f(a) \ne f(b)$ $c = f(a)$ $c = f(b)$, $f(x) = c$ · $\xi = a$ $\xi = b$, . , $f(a) < c < f(b)$ $f(b) < c < f(a)$, c $f(a)$ $f(b)$, 5.3 . , ′ a b $f(x) = c$, (a, b).

, , 5.3. $y = c$ c . $(a, f(a))$ $(b, f(b))$ $f(a)$ $f(b)$, , , , $y = c$. $y = f(x)$, $(a, f(a))$ $(b, f(b))$, $y = c$. (ξ, η), $\eta = f(\xi)$, , $\eta = c$ $y = c$. $f(\xi) = c$.

5.3 $f(x) = c$, ξ. , , ξ . ξ c.

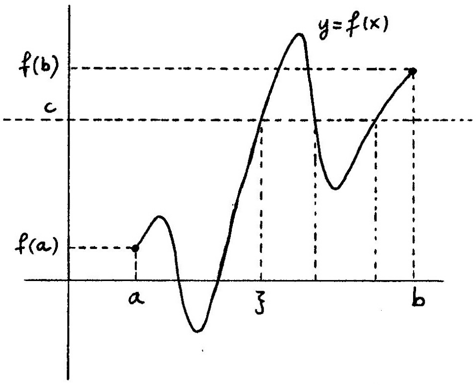
5.3 , . , , , .

: (1) $y = f(x) = \begin{cases} 1\,, & 0 < x \le 1, \\ 0\,, & x = 0, \end{cases}$ $[0, 1]$, 0 $\frac{1}{2}$ – c $f(0) = 0$ $f(1) = 1$ – .

(2) $y = f(x) = \begin{cases} x\,, & 0 \le x < \frac{1}{2}, \\ x - \frac{1}{2}, & \frac{1}{2} \le x \le 1, \end{cases}$ $[0, 1]$, $\frac{1}{2}$ c $f(0) = 0$ $f(1) = \frac{1}{2}$.

.

: (1) $\cos x = x$ $[0, \frac{\pi}{2}]$.



Σχήμα 5.9: .

$y = \cos x - x$ $[0, \frac{\pi}{2}]$. . $\cos 0 - 0 = 1$ $\cos\frac{\pi}{2} - \frac{\pi}{2} = -\frac{\pi}{2}$ 0 . ξ $[0, \frac{\pi}{2}]$ $\cos\xi - \xi = 0$, , $\cos\xi = \xi$. , , ξ $(0, \frac{\pi}{2})$.

(2) $x^3 - 5x^2 - 18x + 7 = 0$.

, . a b $a < b$ 0 $y = x^3 - 5x^2 - 18x + 7$ a b , , $a^3 - 5a^2 - 18a + 7$ $b^3 - 5b^2 - 18b + 7$. - : $a = 0$, $a^3 - 5a^2 - 18a + 7 = 7$, $b = 1$, $b^3 - 5b^2 - 18b + 7 = -15$. ξ $(0, 1)$ $\xi^3 - 5\xi^2 - 18\xi + 7 = 0$.

, , . , , . . $\lim_{x\to-\infty}(x^3 - 5x^2 - 18x + 7) = -\infty$, a ( ) $a^3 - 5a^2 - 18a + 7 < 0$. , $\lim_{x\to+\infty}(x^3 - 5x^2 - 18x + 7) = +\infty$, b $b^3 - 5b^2 - 18b + 7 > 0$. , ξ $[a, b]$ $\xi^3 - 5\xi^2 - 18\xi + 7 = 0$.

5.13 .

.

5.9 *Bolzano.* $y = f(x)$ $[a, b]$. $f(a)f(b) < 0$, ξ (a, b) $f(\xi) = 0$.

, $f(a)f(b) < 0$ $f(a) < 0 < f(b)$ $f(b) < 0 < f(a)$.

5.10 **.** $y = f(x)$ I *( ).* $f(x) \neq 0$ x I, $f(x) > 0$ x I $f(x) < 0$ x I.

I. , a I $f(a) < 0$ b I $f(b) > 0$. I , $[a, b]$ $[b, a]$ a b I , , $y = f(x)$. ξ $[a, b]$ $[b, a]$, , I $f(\xi) = 0$, . $f(x) > 0$ x I $f(x) < 0$ x I.

x-, x- x-.

.

. - .

1. $(0,1)$;

$$y=x^2\,,\quad y=x^2-x+1,\quad y=\sin(\pi x),\quad y=\cot(\pi x),\quad y=\sin(2\pi x).$$

2. $y=\sin\frac{1}{x}$ $(0,+\infty)$. ;
$y=x\sin x$ $y=\frac{1}{x}\sin\frac{1}{x}$ $(0,+\infty)$.
(: 6, 7 8 3.10.)

3. $y=\frac{1}{1+x}\sin\frac{1}{x}$ $(0,+\infty)$.
(: $y=\frac{1}{1+x}$ $y=-\frac{1}{1+x}$ $(0,+\infty)$ 6, 7 8 3.10.)

4. () $t=a$ $t=b$ $(a<b)$. $t=a$ $t=b$.
¨ ;

5. $y=f(x)$ $[a,b]$ $f(x)>l$ x $[a,b]$. $\rho>l$ $f(x)\geq\rho$ x $[a,b]$.
;

6. $y=f(x)$ $y=g(x)$ $[a,b]$ $f(x)>g(x)$ x $[a,b]$. ρ $f(x)\geq g(x)+\rho$ x $[a,b]$.
(: $y=f(x)-g(x)$ $[a,b]$.)
;

. .

1. $x^7-3x^6+5x^5+13x^4-x^3-12x^2-5x+1=0$ $[0,1]$.

2. $e^x=x+2$.

3. $\frac{3}{x}+\frac{2}{x-1}+\frac{1}{x-2}+\frac{5}{x-3}=0$ $(0,1)$, $(1,2)$ $(2,3)$.
(: , .)

4. $\tan x=x$ $(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi)$ $(k\in\mathbf{Z})$.

5. $y=f(x)$ $[0,1]$ $0\leq f(x)\leq 1$ x $[0,1]$. ξ $[0,1]$ $f(\xi)=\xi^2$.
(: $y=f(x)-x^2$.)

6. $y=f(x)$ $y=g(x)$ $[a,b]$. $f(a)<g(a)$ $f(b)>g(b)$, ξ (a,b) $f(\xi)=g(\xi)$.
(: $y=f(x)-g(x)$.)
;

7. () . $t=a$ $t=b$ . .
¨ ;

8. (*) $y=f(x)$ -- I. $y=f(x)$ I.
(: , x_1,x_2 x_3 I $x_1<x_2<x_3$ $f(x_1),f(x_3)<f(x_2)$ $f(x_1),f(x_3)>f(x_2)$. c $\max\{f(x_1),f(x_3)\}<c<f(x_2)$.)

9. $y = f(x)$ $y = g(x)$ I. $f(x) \neq g(x)$ x I, $f(x) < g(x)$ x I $f(x) > g(x)$ x I.

(: $y = f(x) - g(x)$.)

(*) , , $y = h(x)$ I. x I $h(x) = f(x)$ $h(x) = g(x)$, $h(x) = f(x)$ x I $h(x) = g(x)$ x I.

(: $y = h(x)$ $y = \frac{f(x)+g(x)}{2}$.)

;

10. (*) $y = f(x)$ $y = g(x)$ I $g(x)^2 = f(x)^2$ $f(x) \neq 0$ x I. $g(x) = f(x)$ x I $g(x) = -f(x)$ x I.

(: ′ $g(x) \neq 0$ x I. : $y = f(x)$ $y = g(x)$. : $y = h(x) = \frac{g(x)}{f(x)}$. : . : .)

;

$y = f(x)$ I $[0, +\infty)$ $(-\infty, 0]$ $f(x)^2 = |x|$ x I. $f(x) = \sqrt{|x|}$ x I $f(x) = -\sqrt{|x|}$ x I.

$y = f(x)$ $(-\infty, +\infty)$ $f(x)^2 = x^2$ x;

$y = f(x)$ I $[-1, 1]$ $x^2 + f(x)^2 = 1$ x I. $f(x) = \sqrt{1 - x^2}$ x I $f(x) = -\sqrt{1 - x^2}$ x I.

11. (**) 9 .

$y = f_1(x), y = f_2(x), \dots, y = f_n(x)$ I x I n . ;

, , $y = h(x)$ I x I (x) n , , $(h(x) - f_1(x)) \cdots (h(x) - f_n(x)) = 0$ x I. $y = h(x)$ $y = f_1(x), y = f_2(x), \dots, y = f_n(x)$;

12. (*) $y = f(x)$ I $[1, +\infty)$ $[0, 1]$ $(f(x) - x)(f(x) - x^2)(f(x) - x^3) = 0$ x I. $f(x) = x$ x I $f(x) = x^2$ x I $f(x) = x^3$ x I.

(: .)

$I = [0, +\infty)$, – – $y = f(x)$;

5.6 .

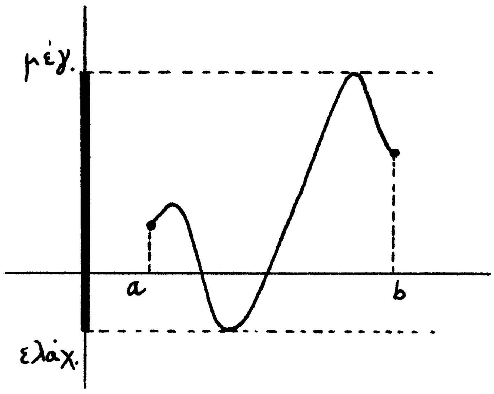
$y = f(x)$, , c $f(x) = c$. – – .

5.11 - I, (I) I.

: $y = f(x)$ $[a, b]$, x_1, x_2 $[a, b]$ $f(x_1) \leq f(x) \leq f(x_2)$ x $[a, b]$. $m_1 = f(x_1), m_2 = f(x_2)$, , , $[m_1, m_2]$. $[m_1, m_2]$. , c $[m_1, m_2]$. $[x_1, x_2]$ $[x_2, x_1]$, $[a, b]$. $y = f(x)$ $[x_1, x_2]$ $[x_2, x_1]$ c $m_1 = f(x_1)$ $m_2 = f(x_2)$, ξ x_1 x_2 , , $[a, b]$ $c = f(\xi)$. c $[m_1, m_2]$, $[m_1, m_2]$. $[m_1, m_2]$.

. . 6 , , . , .

: (1) $y = x^2$ $[1, 4]$, $1^2 = 1$ $4^2 = 16$. $[1, 4]$ $[1, 16]$.



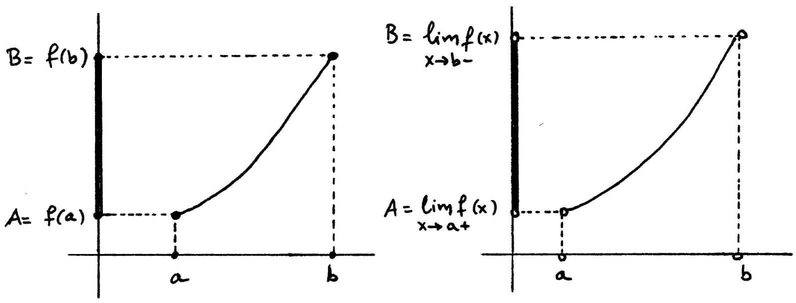
Σχήμα 5.10: $= [\ ,\ ]$.

(2) $y = \frac{1}{x}$ $[\frac{1}{2}, 3]$, $\frac{1}{3}$ $\frac{1}{\frac{1}{2}} = 2$. $[\frac{1}{2}, 3]$ $[\frac{1}{3}, 2]$.

(3) $y = x^2 - 6x + 5$ $[-1, 6]$. $y = (x-3)^2 - 4$, $[-1, 3]$ $[3, 6]$. $[-1, 6]$ $3^2 - 6 \cdot 3 + 5 = -4$ $(-1)^2 - 6(-1) + 5 = 12$ $6^2 - 6 \cdot 6 + 5 = 5$, 12. $[-1, 6]$ $[-4, 12]$. : $[-1, 3]$ $[-4, 12]$ $[3, 6]$ $[-4, 5]$.

. . , 6, .
5.12 , 4.1, , · .

5.12 (1) $y = f(x)$ $I = [a, b]$. $J = [A, B]$, $A = f(a)$, $B = f(b)$.
(2) $y = f(x)$ I. J I ( ).
(3) $y = f(x)$ I . J I I ( ).
(1), (2) (3) I. J . , (1) $J = [B, A]$ $J = [A, B]$.



Σχήμα 5.11: .

: (1) 5.11, $y = f(x)$ $[a, b]$ $B = f(b)$ $A = f(a)$.
(2) $y = f(x)$ (a, b) $\lim_{x \to a+} f(x) = A$ $\lim_{x \to b-} f(x) = B$. a $-\infty$ b $+\infty$. A B, .

x (a,b). x' $a < x' < x$, x'' $a < x'' < x' < x$ $f(x'') < f(x') < f(x)$, $A = \lim_{x''\to a+} f(x'') \le f(x')$, , $A < f(x)$. , x' $x < x' < b$, x'' $x < x' < x'' < b$ $f(x) < f(x') < f(x'')$, $f(x') \le \lim_{x''\to b-} f(x'') = B$, , $f(x) < B$. $y = f(x)$ (A,B), (A,B).

, c (A,B), $A < c < B$. 4.16 $f(x) < c$ a $c < f(x)$ b. a' (a,b) a $f(a') < c$ b' (a,b) b $c < f(b')$. , $y = f(x)$ $[a', b']$, , , c $y = f(x)$ $[a', b']$, , (a,b). (A,B) $y = f(x)$.

$y = f(x)$ (A,B).

(3) $y = f(x)$ $[a,b)$ $f(a) = A$ $\lim_{x\to b-} f(x) = B$.

x $[a,b)$, $a \le x < b$, $A \le f(x) < B$, , $y = f(x)$ $[A,B)$.

, c $[A,B)$, $f(a) = A \le c < B$, b' $[a,b)$ b $c < f(b')$. $f(a) \le c < f(b')$ c $[a,b']$, , $[a,b)$. $[A,B)$ $y = f(x)$.

$y = f(x)$ $[A,B)$.

$y = f(x)$ $(a,b]$. , .

: (1) $y = 2x^2+1$ $(1,3)$. $(1,3)$ $\lim_{x\to 1+}(2x^2+1) = 3$ $\lim_{x\to 3-}(2x^2+1) = 19$. , $y = 2x^2+1$ $(1,3)$ $(3,19)$.

(2) $y = \frac{x+1}{x-1}$ $(1,+\infty)$. $(1,+\infty)$ $\lim_{x\to 1+}\frac{x+1}{x-1} = +\infty$ $\lim_{x\to+\infty}\frac{x+1}{x-1} = 1$. $y = \frac{x+1}{x-1}$ $(1,+\infty)$ $(1,+\infty)$.

(3) $y = \log\frac{1}{x}$ $[1,+\infty)$. $\log\frac{1}{1} = 0$ $\lim_{x\to+\infty}\log\frac{1}{x} = -\infty$. $[1,+\infty)$, $[1,+\infty)$ $(-\infty,0]$.

(4) $y = \tanh x = \frac{e^x - e^{-x}}{e^x+e^{-x}}$ $(-\infty,+\infty)$. $\lim_{x\to-\infty}\tanh x = -1$ $\lim_{x\to+\infty}\tanh x = 1$, , $(-1,1)$.

(5) n**-** . n $b \ge 0$ $a \ge 0$ $a^n = b$.

. $y = x^n$ $[0,+\infty)$ $0^n = 0$ $\lim_{x\to+\infty} x^n = +\infty$. $y = x^n$ $[0,+\infty)$ $[0,+\infty)$. , $b \ge 0$ $[0,+\infty)$, $a \ge 0$ $a^n = b$.

, , 1.2, 1. , , – – . , *1.2!*

, , .

5.13 (1) $y = p(x) = a_0 + a_1x + \cdots + a_{2n-1}x^{2n-1}$ *(* $a_{2n-1} \ne 0$*).* $(-\infty,+\infty)$.

(2) $y = p(x) = a_0 + a_1x + \cdots + a_{2n}x^{2n}$ *(* $a_{2n} \ne 0$*).* $a_{2n} > 0$, , m, $[m,+\infty)$. $a_{2n} < 0$, , m, $(-\infty,m]$.

: (1) $a_{2n-1} > 0$, $\lim_{x\to-\infty} p(x) = -\infty$ $\lim_{x\to+\infty} p(x) = +\infty$. c 4.16 $p(x) < c$ $-\infty$ $c < p(x)$ $+\infty$. a b $p(a) < c < p(b)$. c . $(-\infty,+\infty)$ $y = p(x)$, , $(-\infty,+\infty)$.

$a_{2n-1} < 0$, $\lim_{x\to-\infty} p(x) = +\infty$ $\lim_{x\to+\infty} p(x) = -\infty$.

(2) $a_{2n} > 0$. $y = p(x)$, $p(0) = a_0$. $\lim_{x\to-\infty} p(x) = +\infty$ $\lim_{x\to+\infty} p(x) = +\infty$, 4.16 a $p(x) > a_0$ x $(-\infty,a)$ b $p(x) > a_0$ $(b,+\infty)$. $y = p(x)$ $[a,b]$, , m, . , 0 $[a,b]$, $m \le p(0) = a_0$, , $m \le p(x)$ x $(-\infty,a)$ x $(b,+\infty)$. m $y = p(x)$ $(-\infty,+\infty)$ $[a,b]$.

, , $y = p(x)$ $[m,+\infty)$. , c $[m,+\infty)$, $m \le c < +\infty$. m $y = p(x)$, x_0 $p(x_0) = m$. 4.16, $c < \lim_{x\to+\infty} p(x)$, b' $c < p(b')$. c $p(x_0)$ $p(b')$, c . $[m,+\infty)$ $y = p(x)$ $[m,+\infty)$.

$a_{2n} < 0$, .

: (1) $y = -2x^5 + 4x^4 - 3x^3 - x^2 + 7x - 1$ $(-\infty,+\infty)$.

(2) $y = x^4 - 4x^3 + 4x^2 - 7$ $y = x^2(x-2)^2 - 7$. $x^4 - 4x^3 + 4x^2 - 7 \geq -7$ x , , -7 $x = 0$ $x = 2$. , -7 $y = x^4 - 4x^3 + 4x^2 - 7$, , $[-7, +\infty)$.

6 .

.

1. ;

$$y = -2x^3 + x^2 - 5x + 6, \qquad y = x^4 - 2x^2 + 7, \qquad y = x^6 - 3x^4 + 3x^2 - 1.$$

2. $y = a_0 + a_1 x + \cdots + a_N x^N$. $a_0 a_N < 0$, ξ $p(\xi) = 0$.

3. . ( ) , , .
   (i) $y = \sin x$ $y = \cos(5x)$ $[-\frac{\pi}{4}, \frac{\pi}{2}]$.
   (ii) $y = x + \frac{1}{x}$ $(-\infty, -1]$, $[-1, 0)$, $(0, 1]$ $[1, +\infty)$.
   (iii) $y = e^x + x$ $y = \frac{1}{1+e^{2x}}$ $(-\infty, +\infty)$.

4. $y = \frac{3}{x} + \frac{2}{x-1} + \frac{1}{x-2} + \frac{5}{x-3}$ $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, $(2, 3)$ $(3, +\infty)$.
   .
   $\frac{3}{x} + \frac{2}{x-1} + \frac{1}{x-2} + \frac{5}{x-3} = c$, c ;
   (: : $c < 0$, $c > 0$ $c = 0$.)
   3 .

5.7 .

, $y = f(x)$ I, , , -- I $x = f^{-1}(y)$. , , J $y = f(x)$ I, $x = f^{-1}(y)$ J I. 5.12 $y = f(x)$, , I. : J , , I. 5.14 5.12, $x = f^{-1}(y)$ J.

5.14 (1) $y = f(x)$ $I = [a, b]$. $J = [A, B]$, $A = f(a)$, $B = f(b)$. , $x = f^{-1}(y)$ $[A, B]$ $[a, b]$.
(2) $y = f(x)$ I. J I ( ). , $x = f^{-1}(y)$ J I.
(3) $y = f(x)$ I . J I I ( ). , $x = f^{-1}(y)$ J I.

(1), (2) (3) I. J . , (1) $J = [B, A]$ $J = [A, B]$.

: (1) 5.12 $y = f(x)$ $[A, B]$ $A = f(a)$ $B = f(b)$, , $x = f^{-1}(y)$ $[A, B]$ $[a, b]$. $x = f^{-1}(y)$ η $[A, B]$.

η $[A, B]$ $\xi = f^{-1}(\eta)$. $A < \eta < B$, $\epsilon > 0$ x_1, x_2 $[a, b]$ $\xi - \epsilon \leq x_1 < \xi < x_2 \leq \xi + \epsilon$. $y_1 = f(x_1)$, $y_2 = f(x_2)$ $[A, B]$, $y_1 < \eta < y_2$. $\delta = \min\{\eta - y_1, y_2 - \eta\}$. T y $|y - \eta| < \delta$ $y_1 \leq \eta - \delta < y < \eta + \delta \leq y_2$, $\xi - \epsilon \leq x_1 = f^{-1}(y_1) < f^{-1}(y) < f^{-1}(y_2) = x_2 \leq \xi + \epsilon$, , $|f^{-1}(y) - f^{-1}(\eta)| = |f^{-1}(y) - \xi| < \epsilon$. $x = f^{-1}(y)$ η.

$\eta = A$, $\xi = f^{-1}(\eta) = a$, $\epsilon > 0$ x_2 $[a, b]$ $a < x_2 \leq a + \epsilon$. $y_2 = f(x_2)$ $[A, B]$, $A < y_2$. $\delta = y_2 - A$. T y $A \leq y < A + \delta = y_2$ $a = f^{-1}(A) \leq f^{-1}(y) < f^{-1}(y_2) = x_2 \leq a + \epsilon$, , $|f^{-1}(y) - f^{-1}(A)| = |f^{-1}(y) - a| < \epsilon$. $x = f^{-1}(y)$ $\eta = A$.

$\eta = B$, $x = f^{-1}(y)$ η. $x = f^{-1}(y)$ $[A, B]$.
(2) − (3) (1), $x = f^{-1}(y)$ η J (1).

: (1) $y = x^3 + x$ $(-\infty, +\infty)$.
$\lim_{x\to-\infty}(x^3 + x) = -\infty$ $\lim_{x\to+\infty}(x^3 + x) = +\infty$, $(-\infty, +\infty)$ − $y = x^3 + x$. $(-\infty, +\infty)$ $(-\infty, +\infty)$.
.

(2) $y = -xe^x + 1$ $[0, +\infty)$.
$-0e^0 + 1 = 1$ $\lim_{x\to+\infty}(-xe^x + 1) = -\infty$, $(-\infty, 1]$ $(-\infty, 1]$ $[0, +\infty)$.
, . , , $-xe^x + 1 = y$ x.

.

: (1) . $y = e^x$ $(-\infty, +\infty)$.
$\lim_{x\to-\infty} e^x = 0$ $\lim_{x\to+\infty} e^x = +\infty$, $(0, +\infty)$, $x = \log y$, $(0, +\infty)$ $(-\infty, +\infty)$.
$x = \log y$ $(0, +\infty)$. , , $x = \log y$ *5.14.* , $\lim_{y\to0+} \log y = -\infty$ $\lim_{y\to+\infty} \log y = +\infty$. , , 5.14. , $x = \log y$ $(0, +\infty)$, $\lim_{y\to0+} \log y$ $\lim_{y\to+\infty} \log y$. , , $(-\infty, +\infty)$, $\lim_{y\to0+} \log y = -\infty$ $\lim_{y\to+\infty} \log y = +\infty$.

(2) n- . (*i*) n. $y = x^n$ $(-\infty, +\infty)$.
$\lim_{x\to-\infty} x^n = -\infty$ $\lim_{x\to+\infty} x^n = +\infty$, $(-\infty, +\infty)$, $x = \sqrt[n]{y}$, $(-\infty, +\infty)$ $(-\infty, +\infty)$.
$x = \sqrt[n]{y}$ $(-\infty, +\infty)$. , , $x = \sqrt[n]{y}$ *5.14.* , $x = \sqrt[n]{y}$ $(-\infty, +\infty)$, $\lim_{y\to-\infty} \sqrt[n]{y}$ $\lim_{y\to+\infty} \sqrt[n]{y}$. , , $(-\infty, +\infty)$, $\lim_{y\to-\infty} \sqrt[n]{y} = -\infty$ $\lim_{y\to+\infty} \sqrt[n]{y} = +\infty$.
(*ii*) n. $y = x^n$ $[0, +\infty)$.
$0^n = 0$ $\lim_{x\to+\infty} x^n = +\infty$, $[0, +\infty)$, $x = \sqrt[n]{y}$, $[0, +\infty)$ $[0, +\infty)$. , $x = \sqrt[n]{y}$ $[0, +\infty)$, $\sqrt[n]{0} = 0$ $\lim_{y\to+\infty} \sqrt[n]{y}$. , , $[0, +\infty)$, $\lim_{y\to+\infty} \sqrt[n]{y} = +\infty$.

(3) . , -, -, - -. , , . 5.14 .
(*i*) H $y = \cos x$ $[0, \pi]$. $\cos 0 = 1$ $\cos \pi = -1$, $[0, \pi]$ $[-1, 1]$. , , -, $x = \arccos y$ $[-1, 1]$ $[0, \pi]$.
(*ii*) H $y = \sin x$ $[-\frac{\pi}{2}, \frac{\pi}{2}]$. $\sin(-\frac{\pi}{2}) = -1$ $\sin\frac{\pi}{2} = 1$, $[-\frac{\pi}{2}, \frac{\pi}{2}]$ $[-1, 1]$. -, $x = \arcsin y$ $[-1, 1]$ $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
(*iii*) H $y = \tan x$ $(-\frac{\pi}{2}, \frac{\pi}{2})$. $\lim_{x\to-\frac{\pi}{2}+} \tan x = -\infty$ $\lim_{x\to\frac{\pi}{2}-} \tan x = +\infty$, $(-\frac{\pi}{2}, \frac{\pi}{2})$ $(-\infty, +\infty)$. -, $x = \arctan y$ $(-\infty, +\infty)$ $(-\frac{\pi}{2}, \frac{\pi}{2})$. , , : $\lim_{y\to-\infty} \arctan y = -\frac{\pi}{2}$ $\lim_{y\to+\infty} \arctan y = \frac{\pi}{2}$.
(*iv*) H $y = \cot x$ $(0, \pi)$. $\lim_{x\to0+} \cot x = +\infty$ $\lim_{x\to\pi-} \cot x = -\infty$, $(0, \pi)$ $(-\infty, +\infty)$. , , -, $x = \operatorname{arccot} y$ $(-\infty, +\infty)$ $(0, \pi)$. : $\lim_{y\to-\infty} \operatorname{arccot} y = \pi$ $\lim_{y\to+\infty} \operatorname{arccot} y = 0$.

(4) . , - - , . , - - . 5.14 . , , .
(*i*) $y = \cosh x = \frac{e^x + e^{-x}}{2}$ $[0, +\infty)$. $\cosh 0 = 1$ $\lim_{x\to+\infty} \cosh x = +\infty$, $[0, +\infty)$ $[1, +\infty)$. - , $x = \operatorname{arccosh}\, y$, $[1, +\infty)$ $[0, +\infty)$.

(ii) $y = \sinh x = \frac{e^x - e^{-x}}{2}$ $(-\infty, +\infty)$. $\lim_{x\to-\infty} \sinh x = -\infty$ $\lim_{x\to+\infty} \sinh x = +\infty$, $(-\infty, +\infty)$ $(-\infty, +\infty)$. - , $x = \text{arcsinh } y$, $(-\infty, +\infty)$ $(-\infty, +\infty)$.

$x = \text{arccosh } y = \log(x + \sqrt{x^2 - 1})$ $x = \text{arcsinh } y = \log(x + \sqrt{x^2 + 1})$

.

3.11.

(iii) $y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $(-\infty, +\infty)$. $\lim_{x\to-\infty} \tanh x = -1$ $\lim_{x\to+\infty} \tanh x = 1$, $(-\infty, +\infty)$ $(-1, 1)$. - , $x = \text{arctanh } y$, $(-1, 1)$ $(-\infty, +\infty)$.

(iv) $y = \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ $(0, +\infty)$. $\lim_{x\to 0+} \coth x = +\infty$ $\lim_{x\to+\infty} \coth x = 1$, $(0, +\infty)$ $(1, +\infty)$.

, $y = \coth x$ $(-\infty, 0)$. $\lim_{x\to-\infty} \coth x = -1$ $\lim_{x\to 0-} \coth x = -\infty$, $(-\infty, 0)$ $(-\infty, -1)$.

$y = \coth x$ $(-\infty, 0) \cup (0, +\infty)$ -- $(-\infty, -1) \cup (1, +\infty)$. , - , $x = \text{arccoth } y$, $(-\infty, -1) \cup (1, +\infty)$ $(-\infty, 0) \cup (0, +\infty)$. $x = \text{arccoth } y$ $(-\infty, -1) \cup (1, +\infty)$.

, $x = \text{arccoth } y$ $(-\infty, -1)$ $(-\infty, 0)$ $(1, +\infty)$ $(0, +\infty)$.

: $x = \text{arccoth } y$ $(-\infty, -1) \cup (1, +\infty)$ $(-\infty, -1)$ $(1, +\infty)$.

.

1. , . , .

   (i) $y = x^2 + 2x$ $[0, 1]$.

   (ii) $y = \frac{1}{x}$ $(0, 1]$.

   (iii) $y = \frac{1}{x^2+1}$ $[0, +\infty)$.

2.

   (i) $\text{arctanh } y = \frac{1}{2} \log \frac{1+y}{1-y}$ y $(-1, 1)$.

   (ii) $\text{arccoth } y = \frac{1}{2} \log \frac{y+1}{y-1}$ y $(-\infty, -1) \cup (1, +\infty)$.

   $x = \text{arctanh } y$ $x = \text{arccoth } y$

$$x = \frac{1}{2} \log \left| \frac{y+1}{y-1} \right|$$

   $(-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$ $(-\infty, +\infty)$. .

3. (*) $y = f(x) = \frac{1}{2}(x - \frac{1}{x})$ $(0, +\infty)$.

   $y = f(x)$, $(-\infty, +\infty)$ $x = f^{-1}(y)$. $f^{-1}(y) - \frac{1}{f^{-1}(y)} = 2y$ y $(-\infty, +\infty)$.

   $x = h(y)$ $(-\infty, +\infty)$, , $h(y) - \frac{1}{h(y)} = 2y$ y $(-\infty, +\infty)$, $h(y) = f^{-1}(y)$ y $(-\infty, +\infty)$.

   $y = f(x) = \frac{1}{2}(x - \frac{1}{x})$ $(-\infty, 0)$.

   .

4. (*) $y = f(x) = \frac{1}{2}(x + \frac{1}{x})$ $(0, +\infty)$.

$f(\frac{1}{x}) = f(x)$ x $(0, +\infty)$, , --.

$y = f(x)$ $[1, +\infty)$, $[1, +\infty)$ $x = g_1(y)$ $[1, +\infty)$ $[1, +\infty)$.

$y = f(x)$ $(0, 1]$, $[1, +\infty)$ $x = g_2(y)$ $[1, +\infty)$ $(0, 1]$.

$g_1(y) + \frac{1}{g_1(y)} = 2y = g_2(y) + \frac{1}{g_2(y)}$ y $[1, +\infty)$.

$x = h(y)$ $[1, +\infty)$ $h(y) + \frac{1}{h(y)} = 2y$ y $[1, +\infty)$, $h(y) = g_1(y)$ y $[1, +\infty)$ $h(y) = g_2(y)$ y $[1, +\infty)$.

$y = f(x) = \frac{1}{2}(x + \frac{1}{x})$ $(-\infty, 0)$.

.

5. (*) $y = f(x) = x^3 - 3x$.

$y = f(x)$ $(-\infty, -1]$, $[-1, 1]$ $[1, +\infty)$.

$y = f_1(x)$, $y = f_2(x)$ $y = f_3(x)$ $y = f(x)$ $(-\infty, -1]$, $[-1, 1]$ $[1, +\infty)$, .

$y = f_1(x)$, $y = f_2(x)$ $y = f_3(x)$: , , ;

$x = g_1(y)$, $x = g_2(y)$ $x = g_3(y)$ $y = f_1(x)$, $y = f_2(x)$ $y = f_3(x)$, , : , , ;

, $x = g(y)$ $x = g_1(y)$, $x = g_2(y)$ $x = g_3(y)$, $g(y)^3 - 3g(y) = y$ y .

I $x = g(y)$ I : $g(y)^3 - 3g(y) = y$ y I. $I = [-2, +\infty)$, $x = g(y)$ $x = g_3(y)$. $I = (-\infty, 2]$, $x = g(y)$ $x = g_1(y)$. , , $I = [-2, 2]$, $x = g(y)$ I $x = g_1(y)$ $x = g_2(y)$ $x = g_3(y)$.

$x = g_1(y)$, $x = g_2(y)$ $x = g_3(y)$;

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Κεφάλαιο 6

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. . . . , . . . Fermat, Rolle (Lagrange Cauchy). : , . . : , , , , . l' Hopitâl. – : , . .

6.1 .

. .

( ) . – – $y = f(x)$ (a,b) $(\xi, f(\xi))$ ξ (a,b). – l – $(\xi, f(\xi))$. $(x, f(x))$ $x \neq \xi$. x ξ – l_x – $(\xi, f(\xi))$ $(x, f(x))$, $(\xi, f(\xi))$ $(x, f(x))$ l. , l_x l. , l_x

$$\frac{f(x) - f(\xi)}{x - \xi} .$$

$$\boxed{l = \lim_{x \to \xi} \; l_x = \lim_{x \to \xi} \frac{f(x) - f(\xi)}{x - \xi} .}$$

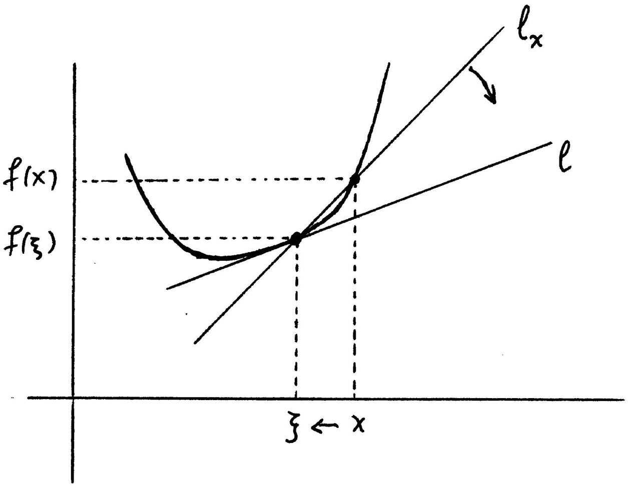
. .

. – – . . $s(t_1)$ $s(t_2)$ t_1 t_2,

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1} .$$

, , τ; , τ τ t. , τ t τ t τ. ,

$$\boxed{\tau = \lim_{t \to \tau} \frac{s(t) - s(\tau)}{t - \tau} .}$$



Σχήμα 6.1: l_x l.

.

1.

(i) () .

(ii) () () .

6.2 .

. , , : − $f(x)$ x $s(t)$ t . .
, , .

$y = f(x)$ ξ ξ, (a, b), $a < \xi < b$, $(a, \xi]$ (ξ, b) $[\xi, b)$ (a, ξ). $\lim_{x\to\xi} \frac{f(x)-f(\xi)}{x-\xi}$, $y = f(x)$ ξ, ξ $y = f(x)$

$$\boxed{f'(\xi) \qquad Df(\xi) \qquad \frac{d\,f(x)}{dx}\Big|_{x=\xi} \qquad \frac{d\,y}{dx}\Big|_{x=\xi} = \lim_{x\to\xi} \frac{f(x)-f(\xi)}{x-\xi}\,.}$$

$f'(\xi)$ $\pm\infty$, ξ.
$y = f(x)$ ξ , , .

: (1) $y = x^2$ 1 1 $\frac{d\,y}{dx}\big|_{x=1} = \frac{d\,x^2}{dx}\big|_{x=1} = \lim_{x\to 1} \frac{x^2-1^2}{x-1} = \lim_{x\to 1}(x+1) = 2$.

(2) $y = \sqrt[3]{x}$ $(-\infty, +\infty)$ 0 $\frac{d\,y}{dx}\big|_{x=0} = \frac{d\,\sqrt[3]{x}}{dx}\big|_{x=0} = \lim_{x\to 0} \frac{\sqrt[3]{x}-\sqrt[3]{0}}{x-0} = \lim_{x\to 0} \frac{1}{(\sqrt[3]{x})^2} = +\infty$. , 0.

$\lim_{x\to\xi}\frac{f(x)-f(\xi)}{x-\xi}$, , . $h = x-\xi$,

$$\lim_{x\to\xi}\frac{f(x)-f(\xi)}{x-\xi} = \lim_{h\to 0}\frac{f(\xi+h)-f(\xi)}{h}.$$

$\lim_{x\to\xi}\frac{f(x)-f(\xi)}{x-\xi}$ 0. , $y = f(x)$ ξ, 0 $\frac{0}{0}$. $\frac{dy}{dx}$. $\Delta x = x-\xi$, , $\Delta y = y-\eta = f(x)-f(\xi)$, , $\frac{f(x)-f(\xi)}{x-\xi}$ $\frac{\Delta y}{\Delta x}$. ′ x ξ $\neq \xi$. − − ′ , dx. Δx dx , $y = f(x)$ ξ, $\Delta y = f(x)-f(\xi)$ dy. , $\frac{\Delta y}{\Delta x}$ $\frac{dy}{dx}$. $\frac{dy}{dx}$ 0 . , , $\frac{dy}{dx}$.

.

$y = f(x)$ $[\xi, b)$. $\lim_{x\to\xi+}\frac{f(x)-f(\xi)}{x-\xi}$, $y = f(x)$ ξ ξ

$$f'_+(\xi) \quad D_+f(\xi) \quad \frac{d\,f(x)}{dx}\Big|_{x=\xi+} \quad \frac{d\,y}{dx}\Big|_{x=\xi+} = \lim_{x\to\xi+}\frac{f(x)-f(\xi)}{x-\xi}.$$

, $y = f(x)$ $(a, \xi]$. $\lim_{x\to\xi-}\frac{f(x)-f(\xi)}{x-\xi}$, $y = f(x)$ ξ ξ

$$f'_-(\xi) \quad D_-f(\xi) \quad \frac{d\,f(x)}{dx}\Big|_{x=\xi-} \quad \frac{d\,y}{dx}\Big|_{x=\xi-} = \lim_{x\to\xi-}\frac{f(x)-f(\xi)}{x-\xi}.$$

$y = f(x)$ (a, b) $a < \xi < b$. , $y = f(x)$ ξ, ξ ξ ξ, $f'_-(\xi) = f'_+(\xi) = f'(\xi)$. , $y = f(x)$ ξ , $y = f(x)$ ξ .

: (1) $y = |x|$ $(-\infty, +\infty)$. 0: $\frac{d\,|x|}{dx}\Big|_{x=0+} = \lim_{x\to 0+}\frac{|x|-|0|}{x-0} = \lim_{x\to 0+}\frac{x}{x} = \lim_{x\to 0+}1 = 1$, $\frac{d\,|x|}{dx}\Big|_{x=0-} = \lim_{x\to 0-}\frac{|x|-|0|}{x-0} = \lim_{x\to 0-}\frac{-x}{x} = \lim_{x\to 0-}(-1) = -1$. 0.

(2) $y = \sqrt{|x|}$ $(-\infty, +\infty)$. : $\frac{d\sqrt{|x|}}{dx}\Big|_{x=0+} = \lim_{x\to 0+}\frac{\sqrt{|x|}-\sqrt{|0|}}{x-0} = \lim_{x\to 0+}\frac{1}{\sqrt{x}} = +\infty$, $\frac{d\sqrt{|x|}}{dx}\Big|_{x=0-} = \lim_{x\to 0-}\frac{\sqrt{|x|}-\sqrt{|0|}}{x-0} = \lim_{x\to 0-}\frac{1}{-\sqrt{-x}} = -\infty$. 0.

(3) $y = \begin{cases}\sqrt{x}, & x>0,\\ 0, & x=0,\\ -\sqrt{-x}, & x<0,\end{cases}$ $(-\infty, +\infty)$: $\frac{d\,y}{dx}\Big|_{x=0+} = \lim_{x\to 0+}\frac{\sqrt{x}-0}{x-0} = \lim_{x\to 0+}\frac{1}{\sqrt{x}} = +\infty$ $\frac{dy}{dx}\Big|_{x=0-} = \lim_{x\to 0-}\frac{-\sqrt{-x}-0}{x-0} = \lim_{x\to 0-}\frac{1}{\sqrt{-x}} = +\infty$. 0 $\frac{dy}{dx}\Big|_{x=0} = +\infty$.

(4) $y = \begin{cases}x\sin\frac{1}{x}, & x\neq 0,\\ 0, & x=0,\end{cases}$ $(-\infty, +\infty)$. $\frac{d\,y}{dx}\Big|_{x=0+} = \lim_{x\to 0+}\frac{x\sin\frac{1}{x}-0}{x-0} = \lim_{x\to 0+}\sin\frac{1}{x} = \lim_{t\to+\infty}\sin t$, , $\frac{dy}{dx}\Big|_{x=0-} = \lim_{x\to 0-}\frac{x\sin\frac{1}{x}-0}{x-0} = \lim_{x\to 0-}\sin\frac{1}{x} = \lim_{t\to-\infty}\sin t$, , . 0.

$y = f(x)$ $[\xi, b)$ (a, ξ), $\lim_{x\to\xi}\frac{f(x)-f(\xi)}{x-\xi}$, , $\lim_{x\to\xi+}\frac{f(x)-f(\xi)}{x-\xi}$ ξ . , ξ ξ. , $y = f(x)$ $(a, \xi]$ (ξ, b), $\lim_{x\to\xi}\frac{f(x)-f(\xi)}{x-\xi}$

$\lim_{x\to\xi-}\frac{f(x)-f(\xi)}{x-\xi}$ ξ ξ.

: $y=\sqrt{x}$ $[0,+\infty)$ 0 $\frac{d\sqrt{x}}{dx}\Big|_{x=0}=\frac{d\sqrt{x}}{dx}\Big|_{x=0+}=\lim_{x\to 0+}\frac{\sqrt{x}-\sqrt{0}}{x-0}=\lim_{x\to 0+}\frac{1}{\sqrt{x}}=+\infty$.

ξ $y=f(x)$, ξ $f'(\xi)$, $y=f(x)$,

$$\boxed{f'(x) \qquad Df(x) \qquad \frac{d\,f(x)}{dx} \qquad \frac{d\,y}{dx}\,.}$$

.

1. ( ) 0 .

$$y=2,\quad y=x,\quad y=3x^2-5x+3,\quad y=\sqrt[4]{x}\,,\quad y=\sqrt[5]{x}\,,$$

$$y=\sin x,\quad y=\cos x,\quad y=\tan x,\quad y=\begin{cases}2x\,, & x\le 0,\\ -3x\,, & x\ge 0,\end{cases}$$

$$y=\begin{cases}2x^2\,, & x\le 0,\\ -3x^2\,, & x\ge 0,\end{cases}\quad y=\begin{cases}-2\sqrt{-x}\,, & x\le 0,\\ \sqrt[3]{x}\,, & x\ge 0,\end{cases}$$

$$y=\begin{cases}0, & x\ne 0,\\ 1, & x=0,\end{cases}\quad y=\begin{cases}\sqrt{x}\,, & x\ge 0,\\ -1, & x<0,\end{cases}\quad y=\begin{cases}1-x, & x>0,\\ 0, & x=0,\\ -1-x, & x<0.\end{cases}$$

2. $y=\begin{cases}2x^2+x+1, & x\le 0,\\ ax+b, & x>0.\end{cases}$ a, b 0.

$$y=\begin{cases}2x^2+x+1, & x<0,\\ ax+b, & x\ge 0.\end{cases}$$

3. $y=g(x)$ $(a,\xi]$ $y=h(x)$ $[\xi,b)$ $g(\xi)=h(\xi)$ $g'_-(\xi)=h'_+(\xi)$. $y=f(x)=\begin{cases}g(x)\,, & a<x\le\xi,\\ h(x)\,, & \xi\le x<b,\end{cases}$ ξ $f'(\xi)=g'_-(\xi)=h'_+(\xi)$.

4. 1 6.1

   (i) () .

   (ii) () () .

6.3 , .

. $y=c$, c x, . , ξ

$$\frac{d\,c}{dx}\Big|_{x=\xi}=\lim_{x\to\xi}\frac{c-c}{x-\xi}=\lim_{x\to\xi}0=0.$$

,

$$\boxed{\frac{dc}{dx} = 0.}$$

. $y = x$ $\left.\frac{d\,x}{dx}\right|_{x=\xi} = \lim_{x\to\xi} \frac{x-\xi}{x-\xi} = \lim_{x\to\xi} 1 = 1.$ $y = x$

$$\boxed{\frac{d\,x}{dx} = 1.}$$

. n ≥ 2, $y = x^n$

$$\boxed{\frac{d\,x^n}{dx} = nx^{n-1} \quad (n \quad \geq 2).}$$

, ξ

$$\left.\frac{d\,x^n}{dx}\right|_{x=\xi} = \lim_{x\to\xi} \frac{x^n - \xi^n}{x-\xi} = \lim_{x\to\xi}(x^{n-1} + x^{n-2}\xi + \cdots + x\xi^{n-2} + \xi^{n-1})$$
$$= n\xi^{n-1},$$

$\lim_{x\to\xi} x^{n-k}\xi^{k-1} = \xi^{n-k}\xi^{k-1} = \xi^{n-1}$ n.
$\frac{d\,x^n}{dx} = nx^{n-1}$, $n \geq 2$, $n = 1$: $\frac{d\,x^1}{dx} = 1x^0$. , $\frac{d\,x}{dx} = 1$, $x \neq 0$, 0^0.

. $y = x^n$ n 0 . , , $\xi = 0$ 0 $y = x^n$ $n \leq 0$. , $n = 0$, $y = x^0$ $y = 1$ $(-\infty, 0) \cup (0, +\infty)$, , $0x^{0-1}$ $(-\infty, 0) \cup (0, +\infty)$. $n < 0$, $m = -n$, m $\xi \neq 0$

$$\left.\frac{d\,x^n}{dx}\right|_{x=\xi} = \lim_{x\to\xi} \frac{x^n - \xi^n}{x-\xi} = \lim_{x\to\xi} \frac{x^{-m} - \xi^{-m}}{x-\xi} = \lim_{x\to\xi} \frac{\xi^m - x^m}{\xi^m x^m (x-\xi)}$$
$$= -\lim_{x\to\xi} \frac{x^{m-1} + x^{m-2}\xi + \cdots + x\xi^{m-2} + \xi^{m-1}}{\xi^m x^m}$$
$$= -\frac{m\xi^{m-1}}{\xi^{2m}} = -m\xi^{-m-1} = n\xi^{n-1}.$$

$$\boxed{\frac{d\,x^n}{dx} = nx^{n-1} \quad (n \quad \leq 0, x \neq 0).}$$

. n ≥ 3, $y = \sqrt[n]{x}$ $(-\infty, +\infty)$. $(-\infty, 0) \cup (0, +\infty)$

$$\boxed{\frac{d\,\sqrt[n]{x}}{dx} = \frac{1}{n}\,\frac{\sqrt[n]{x}}{x} \quad (n \quad \geq 3, x \neq 0).}$$

0 $\left.\frac{d\,\sqrt[n]{x}}{dx}\right|_{x=0} = \lim_{x\to 0} \frac{\sqrt[n]{x} - \sqrt[n]{0}}{x-0} = \lim_{x\to 0} \frac{1}{\sqrt[n]{x^{n-1}}} = +\infty$ $n-1$, $\sqrt[n]{x^{n-1}} > 0$ $x \neq 0$. 0 .

$\xi \neq 0$ $y = \sqrt[n]{x}$. $\eta = \sqrt[n]{\xi}$

$$\frac{d\,\sqrt[n]{x}}{dx}\Big|_{x=\xi} = \lim_{x\to\xi}\frac{\sqrt[n]{x}-\sqrt[n]{\xi}}{x-\xi} = \lim_{y\to\eta}\frac{y-\eta}{y^n-\eta^n}$$
$$= \lim_{y\to\eta}\frac{1}{y^{n-1}+y^{n-2}\eta+\cdots+y\eta^{n-2}+\eta^{n-1}}$$
$$= \frac{1}{n\eta^{n-1}} = \frac{1}{n}\frac{\eta}{\eta^n} = \frac{1}{n}\frac{\sqrt[n]{\xi}}{\xi}.$$

n , $y = \sqrt[n]{x}$ $[0,+\infty)$ $(0,+\infty)$. ,

$$\boxed{\frac{d\,\sqrt[n]{x}}{dx} = \frac{1}{n}\frac{\sqrt[n]{x}}{x} \quad (n \quad , x > 0).}$$

0 $+\infty$. .

. $y = x^a$, a . $y = x^a$ $(-\infty,+\infty)$ (0, $a \le 0$), a , $[0,+\infty)$ (0, $a \le 0$), a .

$$\boxed{\frac{d\,x^a}{dx} = ax^{a-1}\begin{pmatrix} a & >1 & , \\ a & \le 1 & , x \neq 0, \\ a & >1 & , x \ge 0, \\ a & \le 1 & , x > 0. \end{pmatrix}}$$

, $a = \frac{m}{n}$, m n . $t = \sqrt[n]{x}$. $y = t^m$, , $\eta = \xi^a$ $\tau = \sqrt[n]{\xi}$. :

$$\frac{d\,x^a}{dx}\Big|_{x=\xi} = \lim_{x\to\xi}\frac{y-\eta}{x-\xi} = \lim_{t\to\tau}\frac{t^m-\tau^m}{t^n-\tau^n}$$
$$= \lim_{t\to\tau}\frac{t^{m-1}+t^{m-2}\tau+\cdots+t\tau^{m-2}+\tau^{m-1}}{t^{n-1}+t^{n-2}\tau+\cdots+t\tau^{n-2}+\tau^{n-1}}$$
$$= \frac{m\tau^{m-1}}{n\tau^{n-1}} = \frac{m}{n}\tau^{m-n} = \frac{m}{n}\left(\sqrt[n]{\xi}\right)^{m-n} = a\xi^{a-1}.$$

. ,

$$\boxed{\frac{d\,\cos x}{dx} = -\sin x, \qquad \frac{d\,\sin x}{dx} = \cos x.}$$

$$\frac{d\,\cos x}{dx}\Big|_{x=\xi} = \lim_{x\to\xi}\frac{\cos x - \cos\xi}{x-\xi} = \lim_{x\to\xi}\frac{-2\sin\frac{x-\xi}{2}\sin\frac{x+\xi}{2}}{x-\xi}$$
$$= -\lim_{x\to\xi}\frac{\sin\frac{x-\xi}{2}}{\frac{x-\xi}{2}}\sin\frac{x+\xi}{2} = -1\cdot\sin\frac{\xi+\xi}{2} = -\sin\xi$$

, ,

$$\frac{d\,\sin x}{dx}\Big|_{x=\xi} = \lim_{x\to\xi}\frac{\sin x - \sin\xi}{x-\xi} = \lim_{x\to\xi}\frac{2\sin\frac{x-\xi}{2}\cos\frac{x+\xi}{2}}{x-\xi}$$
$$= \lim_{x\to\xi}\frac{\sin\frac{x-\xi}{2}}{\frac{x-\xi}{2}}\cos\frac{x+\xi}{2} = 1\cdot\cos\frac{\xi+\xi}{2} = \cos\xi.$$

.

1. , , .

$$\left.\frac{d\,x^3}{dx}\right|_{x=1}, \quad \left.\frac{d\,\sqrt[5]{x}}{dx}\right|_{x=-1}, \quad \left.\frac{d\,\sqrt[3]{x}}{dx}\right|_{x=0}, \quad \left.\frac{d\,\sqrt{x}}{dx}\right|_{x=-1}, \quad \left.\frac{d\,\sqrt[4]{x}}{dx}\right|_{x=1},$$

$$\left.\frac{d\,\sqrt[6]{x}}{dx}\right|_{x=0}, \quad \left.\frac{d\,x^{-1}}{dx}\right|_{x=-2}, \quad \left.\frac{d\,x^{\frac{2}{3}}}{dx}\right|_{x=-1}, \quad \left.\frac{d\,x^{\frac{4}{5}}}{dx}\right|_{x=0}, \quad \left.\frac{d\,x^{\frac{5}{2}}}{dx}\right|_{x=0},$$

$$\left.\frac{d\,x^{\frac{5}{2}}}{dx}\right|_{x=-3}, \quad \left.\frac{d\,x^{-\frac{1}{2}}}{dx}\right|_{x=-3}, \quad \left.\frac{d\,x^{-\frac{3}{2}}}{dx}\right|_{x=0}.$$

2. 0; -1;

$$\frac{d\,x^3}{dx}, \quad \frac{d\,x^{-3}}{dx}, \quad \frac{d\,x^0}{dx}, \quad \frac{d\,\sqrt{x}}{dx}, \quad \frac{d\,\sqrt[5]{x}}{dx},$$

$$\frac{d\,x^{\frac{4}{3}}}{dx}, \quad \frac{d\,x^{\frac{2}{3}}}{dx}, \quad \frac{d\,x^{\frac{3}{2}}}{dx}, \quad \frac{d\,x^{-\frac{3}{2}}}{dx}, \quad \frac{d\,x^{-\frac{4}{5}}}{dx}.$$

3. .

$$\frac{d\,\sin x}{dx} = -1, \quad \frac{d\,\sin x}{dx} + \sin x = \sqrt{2}, \quad \cos x\frac{d\,\sin x}{dx} - \sin x\frac{d\,\cos x}{dx} = 1.$$

4. .

$$y = (\sin x)^2, \quad y = (\cos x)^3, \quad y = \sin(2x), \quad y = \cos(7x).$$

5. .

.

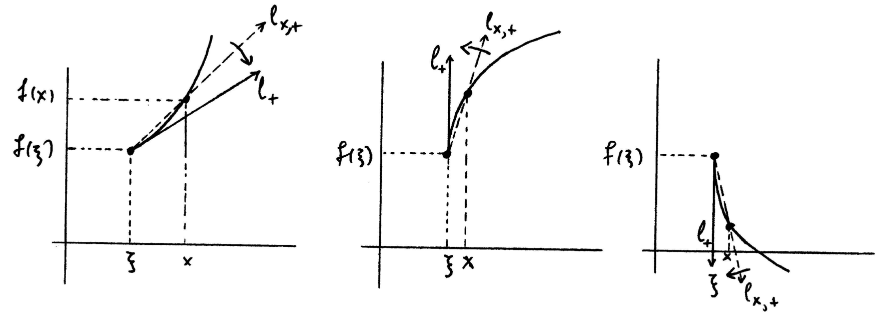
.

6.4 .

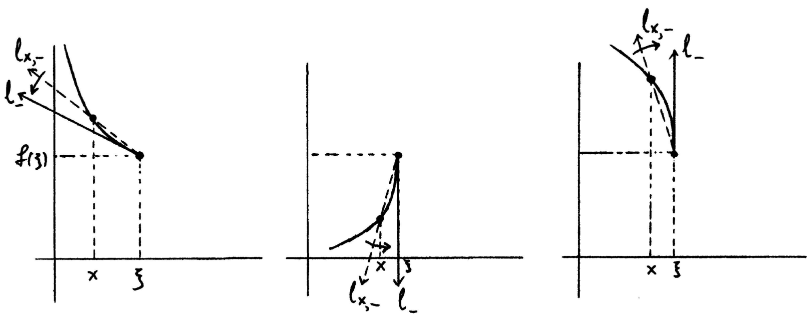
.

$y = f(x)$ $[\xi, b)$. x (ξ, b), $l_{x,+}$ $(\xi, f(\xi))$ $(x, f(x))$. , , x ξ. $(x, f(x))$ $(\xi, f(\xi))$ ($y = f(x)$ ξ), $l_{x,+}$ l_+ $(\xi, f(\xi))$ $y = f(x)$ $(\xi, f(\xi))$. $\frac{f(x)-f(\xi)}{x-\xi}$. (i) $f'_+(\xi) = \lim_{x\to\xi+}\frac{f(x)-f(\xi)}{x-\xi}$, l_+ , : l_+ , , , , l_+ , 0. (ii) $f'_+(\xi) = +\infty$, $l_{x,+}$, l_+ . (iii) $f'_+(\xi) = -\infty$, $l_{x,+}$, l_+ . , $\lim_{x\to\xi+}\frac{f(x)-f(\xi)}{x-\xi}$, .

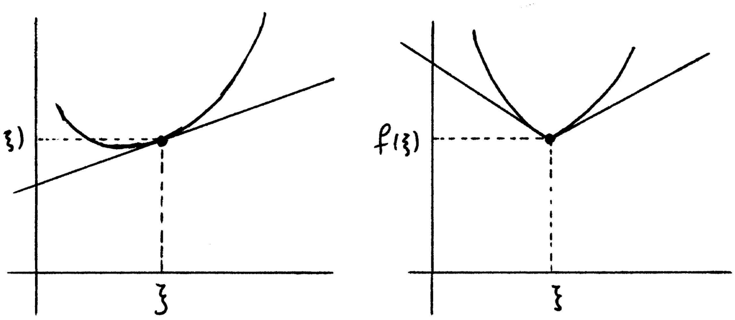
«» . $y = f(x)$ $(a, \xi]$. x (a, ξ), $l_{x,-}$ $(\xi, f(\xi))$ $(x, f(x))$. x ξ. $(x, f(x))$ $(\xi, f(\xi))$ ($y = f(x)$ ξ), $l_{x,-}$ l_-



Σχήμα 6.2: .



Σχήμα 6.3: .



Σχήμα 6.4: . .

$(\xi, f(\xi))$ $y = f(x)$ $(\xi, f(\xi))$. $l_{x,-}$ $\frac{f(x)-f(\xi)}{x-\xi}$. (i) $f'_-(\xi) = \lim_{x\to\xi-} \frac{f(x)-f(\xi)}{x-\xi}$, l_- , : l_- , , , , l_- ,

0. (ii) $f'_{-}(\xi) = +\infty$, $l_{x,-}$, l_{-} . (iii) $f'_{-}(\xi) = -\infty$, $l_{x,-}$, l_{-} . $\lim_{x\to\xi-} \frac{f(x)-f(\xi)}{x-\xi}$, .

$y = f(x)$ (a, b), $a < \xi < b$. , , $y = f(x)$ ξ , $(\xi, f(\xi))$, , , l $(\xi, f(\xi))$. , , , . , , .

: (1) $y = |x|$ $(0, 0)$. $(0, 0)$, $\frac{d\,|x|}{dx}\big|_{x=0+} = 1$. $(0, 0)$, $\frac{d\,|x|}{dx}\big|_{x=0-} = -1$. , $(0, 0)$.

(2) $y = \sqrt{|x|}$ $(0, 0)$. $\frac{d\sqrt{|x|}}{dx}\big|_{x=0+} = +\infty$, $(0, 0)$. , $\frac{d\sqrt{|x|}}{dx}\big|_{x=0-} = -\infty$, $(0, 0)$. $(0, 0)$.

(3) $y = \begin{cases} \sqrt{x}\,, & 0 \leq x, \\ -\sqrt{-x}\,, & x \leq 0, \end{cases}$ $(0, 0)$. $\frac{dy}{dx}\big|_{x=0+} = +\infty$, $(0, 0)$. , $\frac{dy}{dx}\big|_{x=0-} = +\infty$, $(0, 0)$. , $(0, 0)$, , , · $x = 0$.

$y = f(x)$ ξ ξ , $(\xi, f(\xi))$.

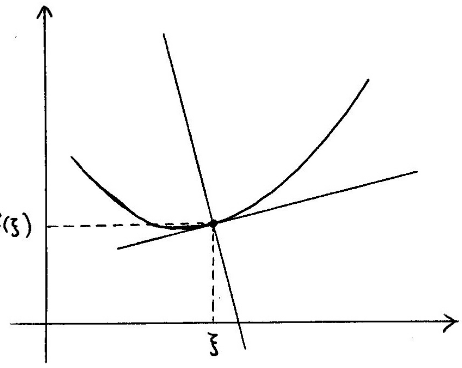
: $y = \sqrt{x}$ $(0, 0)$. $\frac{d\sqrt{x}}{dx}\big|_{x=0+} = +\infty$, $(0, 0)$.

$y = f(x)$ (a, b), $a < \xi < b$ $y = f(x)$ ξ. , l $(\xi, f(\xi))$ $f'(\xi)$, :

: $y = f'(\xi)(x - \xi) + f(\xi) \quad (f'(\xi) \neq \pm\infty)$.

, , $(\xi, f(\xi))$. $y = f(x)$ $(\xi, f(\xi))$. $f'(\xi) \neq 0$, $-\frac{1}{f'(\xi)}$, ,

$$y = -\frac{1}{f'(\xi)}(x - \xi) + f(\xi).$$



Σχήμα 6.5: .

$f'(\xi)=0$, ,

$$x=\xi.$$

, $y=f(x)$ ξ $f'(\xi)=+\infty$ $f'(\xi)=-\infty$, $(\xi,f(\xi))$

: $x=\xi$ $(f'(\xi)=\pm\infty)$,

$$y=f(\xi).$$

: (1) (ξ,ξ^2) $y=x^2$ $y=2\xi(x-\xi)+\xi^2$. (ξ,ξ^2) $y=-\frac{1}{2\xi}(x-\xi)+\xi^2$, $\xi\neq 0$, $x=0$, $\xi=0$.

(2) $(\xi,\xi^{\frac{1}{3}})$ $y=x^{\frac{1}{3}}$ $y=\frac{1}{3}\xi^{-\frac{2}{3}}(x-\xi)+\xi^{\frac{1}{3}}$, $\xi\neq 0$, $x=0$, $\xi=0$. $(\xi,\xi^{\frac{1}{3}})$ $y=-3\xi^{\frac{2}{3}}(x-\xi)+\xi^{\frac{1}{3}}$, $\xi\neq 0$, $y=0$, $\xi=0$.
$(\xi,\xi^{\frac{1}{3}})$ $y=x$, , $\frac{1}{3}\xi^{-\frac{2}{3}}=1$, $\xi=\pm\frac{1}{\sqrt{27}}$.

.

1. ( ) , 0 1 6.2.

2. . ;

   (i) $y=x^2$ . .

   (ii) $y=x^3$.

3. 6.3.

4. b c $y=x^2+bx+c$ $y=x$ $(1,1)$.

5. $y=x^{\frac{1}{3}}$ $y=-\frac{4}{3}x+\frac{2}{3}$; $y=\frac{4}{3}x+\frac{2}{3}$; $x=4$; $y=1$;

6. $y=f(x)=\begin{cases} x\sin\frac{1}{x}, & x\neq 0,\\ 0, & x=0.\end{cases}$

   0. 0.

   $l_{x,+}$ $(0,0)$ $(x,f(x))$ $x\to 0+$. $l_{x,+}$ $(0,0)$; $l_{x,-}$ $(0,0)$ $(x,f(x))$ $x\to 0-$.

7. $y=\begin{cases} x^2\sin\frac{1}{x}, & x\neq 0,\\ 0, & x=0.\end{cases}$

   0 0 0.

   , $l_{x,+}$ $l_{x,-}$ $(0,0)$ $(x,f(x))$ $x\to 0+$ $x\to 0-$, .

8. $y^4=x^3$. (i) y x . (ii) x , $y=x^{\frac{3}{4}}$ $y=-x^{\frac{3}{4}}$ $[0,+\infty)$ $(0,0)$.

   $y^2=x^3$. , , $(0,0)$;

9. , $xy=a\,(a>0)$, x- y- ( ). x,y .

10. , $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a\,(a>0)$, x- y- $a^{\frac{3}{2}}$.

    . ;

6.5 .

. .

$y = f(x)$ $y = g(x)$ (a,b) $a < \xi < b$. $\frac{f(x)-f(\xi)}{x-\xi}$ $\frac{g(x)-g(\xi)}{x-\xi}$ x $(a,\xi)\cup(\xi,b)$, , , . , $y = f(x)$ $y = g(x)$ $(a,\xi]$ $[\xi,b)$. :

6.1 $y = f(x)$ $y = g(x)$ (a,b) $a < \xi < b$ $(a,\xi]$ $[\xi,b)$. ξ, , .

: (1) $y = \begin{cases} \frac{|x|}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$ $(-\infty,+\infty)$ $y = 1$ $(-\infty,+\infty)$ $(0,+\infty)$. $(0,+\infty)$ 0. , $y = -1$ $(-\infty,+\infty)$ $(-\infty,0)$. $(-\infty,0)$ 0. : (a,b) $a < 0 < b$ $y = 0$. 0 0. , 0.

(2) $y = \sqrt{|x|}$ $(-\infty,+\infty)$ $[0,+\infty)$ $y = \sqrt{x}$ $[0,+\infty)$. $y = \sqrt{|x|}$ $(0,+\infty)$ $y = \sqrt{x}$, $\frac{d\sqrt{|x|}}{dx} = \frac{d\sqrt{x}}{dx} = \frac{1}{2}\frac{\sqrt{x}}{x}$ x $(0,+\infty)$. , $y = \sqrt{|x|}$ 0 $y = \sqrt{x}$ 0 , $\frac{d\sqrt{|x|}}{dx}\Big|_{x=0+} = \frac{d\sqrt{x}}{dx}\Big|_{x=0+} = +\infty$.

. .

6.2 $y = f(x)$ ξ ξ ξ , ξ ξ ξ , .

: $y = f(x)$ ξ, $f'(\xi)$. $f(x) = \frac{f(x)-f(\xi)}{x-\xi}(x-\xi) + f(\xi)$

$$\lim_{x\to\xi} f(x) = \lim_{x\to\xi} \frac{f(x)-f(\xi)}{x-\xi} \lim_{x\to\xi}(x-\xi) + f(\xi) = f'(\xi)\cdot 0 + f(\xi) = f(\xi).$$

$y = f(x)$ ξ.

, , .

: (1) 6.2 . $y = |x|$ 0 0.

(2) 6.2 ξ . $+\infty$ $-\infty$, $y = f(x)$ ′ ξ. $y = \begin{cases} \frac{|x|}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$ $(-\infty,+\infty)$. 0 , $\lim_{x\to0+}\frac{|x|}{x} = \lim_{x\to0+} 1 = 1$ $\lim_{x\to0-}\frac{|x|}{x} = \lim_{x\to0-}(-1) = -1$ 0 0. , $\frac{dy}{dx}\Big|_{x=0+} = \lim_{x\to0+}\frac{1-0}{x-0} = +\infty$ $\frac{dy}{dx}\Big|_{x=0-} = \lim_{x\to0-}\frac{-1-0}{x-0} = +\infty$, 0 $\frac{dy}{dx}\Big|_{x=0} = +\infty$.

, $y = f(x)$ ξ ξ, $(\xi, f(\xi))$. 6.2 , , ξ. , , $\pm\infty$, ξ. $(0,0)$.

. .

6.3 $y = f(x)$ $y = g(x)$ ξ. , , , $g(\xi) \neq 0$, ξ *:*

$$\boxed{\begin{array}{l} (f+g)'(\xi) = f'(\xi) + g'(\xi), \quad (f-g)'(\xi) = f'(\xi) - g'(\xi), \\ (fg)'(\xi) = g(\xi)f'(\xi) + f(\xi)g'(\xi), \quad \left(\frac{f}{g}\right)'(\xi) = \frac{g(\xi)f'(\xi)-f(\xi)g'(\xi)}{g(\xi)^2}. \end{array}}$$

.

: .

$$\begin{aligned}(f+g)'(\xi) &= \lim_{x\to\xi}\frac{(f(x)+g(x))-(f(\xi)+g(\xi))}{x-\xi}\\ &= \lim_{x\to\xi}\left(\frac{f(x)-f(\xi)}{x-\xi}+\frac{g(x)-g(\xi)}{x-\xi}\right)\\ &= \lim_{x\to\xi}\frac{f(x)-f(\xi)}{x-\xi}+\lim_{x\to\xi}\frac{g(x)-g(\xi)}{x-\xi}\\ &= f'(\xi)+g'(\xi)\end{aligned}$$

. :

$$\begin{aligned}(fg)'(\xi) &= \lim_{x\to\xi}\frac{f(x)g(x)-f(\xi)g(\xi)}{x-\xi}\\ &= \lim_{x\to\xi}\left(g(x)\frac{f(x)-f(\xi)}{x-\xi}+f(\xi)\frac{g(x)-g(\xi)}{x-\xi}\right)\\ &= \lim_{x\to\xi}g(x)\lim_{x\to\xi}\frac{f(x)-f(\xi)}{x-\xi}+f(\xi)\lim_{x\to\xi}\frac{g(x)-g(\xi)}{x-\xi}\\ &= g(\xi)f'(\xi)+f(\xi)g'(\xi).\end{aligned}$$

, :

$$\begin{aligned}\left(\frac{f}{g}\right)'(\xi) &= \lim_{x\to\xi}\frac{\frac{f(x)}{g(x)}-\frac{f(\xi)}{g(\xi)}}{x-\xi}\\ &= \lim_{x\to\xi}\left(\frac{1}{g(x)}\frac{f(x)-f(\xi)}{x-\xi}-\frac{f(\xi)}{g(x)g(\xi)}\frac{g(x)-g(\xi)}{x-\xi}\right)\\ &= \lim_{x\to\xi}\frac{1}{g(x)}\lim_{x\to\xi}\frac{f(x)-f(\xi)}{x-\xi}\\ &\quad -\frac{f(\xi)}{g(\xi)}\lim_{x\to\xi}\frac{1}{g(x)}\lim_{x\to\xi}\frac{g(x)-g(\xi)}{x-\xi}\\ &= \frac{1}{g(\xi)}f'(\xi)-\frac{f(\xi)}{g(\xi)}\frac{1}{g(\xi)}g'(\xi)=\frac{g(\xi)f'(\xi)-f(\xi)g'(\xi)}{g(\xi)^2}.\end{aligned}$$

(, , ξ)

$$D(f\pm g)=Df\pm Dg,\qquad D(fg)=gDf+fDg,\qquad D\left(\frac{f}{g}\right)=\frac{gDf-fDg}{g^2}$$

$$\frac{d(f(x)\pm g(x))}{dx}=\frac{d\,f(x))}{dx}\pm\frac{d\,g(x)}{dx},\quad \frac{d(f(x)g(x))}{dx}=g(x)\frac{d\,f(x)}{dx}+f(x)\frac{d\,g(x)}{dx},$$

$$\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx}=\frac{g(x)\frac{d\,f(x)}{dx}-f(x)\frac{d\,g(x)}{dx}}{g(x)^2}$$

$$\frac{d(y\pm z)}{dx}=\frac{dy}{dx}\pm\frac{dz}{dx},\quad \frac{d(yz)}{dx}=z\frac{dy}{dx}+y\frac{dz}{dx},\quad \frac{d\left(\frac{y}{z}\right)}{dx}=\frac{z\frac{dy}{dx}-y\frac{dz}{dx}}{z^2},$$

$z=g(x)$ $y=g(x)$ $y=f(x)$.

: (1) $y=f(x)$ ξ c , $y=cf(x)$ ξ

$$\boxed{(cf)'(\xi)=cf'(\xi).}$$

, $y = c$, $(cf)'(\xi) = 0 \cdot f(\xi) + cf'(\xi) = cf'(\xi)$.

(2) $y = a_0 + a_1 x + \cdots + a_N x^N$. $\frac{d\,(a_k x^k)}{dx} = a_k \frac{d\,x^k}{dx} = k a_k x^{k-1}$, ,

$$\boxed{\frac{d}{dx}\big(a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N\big) = a_1 + 2a_2 x + \cdots + N a_N x^{N-1}\,.}$$

(3)

$$\frac{d\left(\frac{x^2+x-1}{x^3+2}\right)}{dx} = \frac{(x^3+2)\frac{d\,(x^2+x-1)}{dx} - (x^2+x-1)\frac{d\,(x^3+2)}{dx}}{(x^3+2)^2}$$

$$= \frac{(x^3+2)(2x+1) - (x^2+x-1)3x^2}{(x^3+2)^2} = \frac{-x^4 - 2x^3 + 3x^2 + 4x + 2}{(x^3+2)^2}\,.$$

.

(4)

$$\boxed{\frac{d\,\tan x}{dx} = \frac{1}{(\cos x)^2}\,, \qquad \frac{d\,\cot x}{dx} = -\frac{1}{(\sin x)^2}\,.}$$

:

$$\frac{d\,\tan x}{dx} = \frac{\cos x \frac{d\,\sin x}{dx} - \sin x \frac{d\,\cos x}{dx}}{(\cos x)^2} = \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2}\,.$$

.

. .

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6.4 . $z = (g\circ f)(x) = g\big(f(x)\big)$ $y = f(x)$ $z = g(y)$. $y = f(x)$ ξ $z = g(y)$ $\eta = f(\xi)$, $z = (g\circ f)(x) = g\big(f(x)\big)$ ξ

$$\boxed{(g\circ f)'(\xi) = g'(\eta) f'(\xi) = g'\big(f(\xi)\big) f'(\xi).}$$

:

$$z = G(y) = \begin{cases} \frac{g(y)-g(\eta)}{y-\eta}\,, & y \quad z = g(y) \quad y \neq \eta, \\ g'(\eta)\,, & y = \eta. \end{cases}$$

$z = G(y)$ $z = g(y)$. $z = G(y)$ η $\lim_{y\to\eta} G(y) = \lim_{y\to\eta} \frac{g(y)-g(\eta)}{y-\eta} = g'(\eta) = G(\eta)$.
, $G(y)$ $g(y) - g(\eta) = G(y)(y-\eta)$ y $z = g(y)$, $y = \eta$. , ,

$$\begin{aligned} \lim_{x\to\xi} \frac{g(f(x)) - g(f(\xi))}{x-\xi} &= \lim_{x\to\xi} \frac{G(f(x))(f(x) - f(\xi))}{x-\xi} \\ &= \lim_{x\to\xi} G(f(x)) \lim_{x\to\xi} \frac{f(x)-f(\xi)}{x-\xi} \\ &= G(f(\xi)) f'(\xi) = G(\eta) f'(\xi) = g'(\eta) f'(\xi), \end{aligned}$$

$\lim_{x\to\xi} G(f(x)) = G(f(\xi))$, $z = G(f(x))$ ξ.

:

$$D(g\circ f)(\xi) = Dg(\eta) Df(\xi) = Dg\big(f(\xi)\big) Df(\xi)$$

$$\frac{d\,g(f(x))}{dx}\Big|_{x=\xi} = \frac{d\,g(y)}{dy}\Big|_{y=\eta}\frac{d\,f(x)}{dx}\Big|_{x=\xi} = \frac{d\,g(y)}{dy}\Big|_{y=f(\xi)}\frac{d\,f(x)}{dx}\Big|_{x=\xi}$$

$$\frac{dz}{dx}\Big|_{x=\xi} = \frac{dz}{dy}\Big|_{y=\eta}\frac{dy}{dx}\Big|_{x=\xi} = \frac{dz}{dy}\Big|_{y=f(\xi)}\frac{dy}{dx}\Big|_{x=\xi}.$$

, x (ξ), ,

$$D(g\circ f)(x) = Dg\big(f(x)\big)Df(x)$$

$$\frac{d\,g(f(x))}{dx} = \frac{d\,g(y)}{dy}\Big|_{y=f(x)}\frac{d\,f(x)}{dx}$$

$$\frac{dz}{dx} = \frac{dz}{dy}\Big|_{y=f(x)}\frac{dy}{dx}.$$

$\big|_{y=f(x)}$, $\frac{d\,g(f(x))}{dx} = \frac{d\,g(y)}{dy}\frac{d\,f(x)}{dx}$ $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$. , $\frac{d\,g(y)}{dy}$ $\frac{dz}{dy}$ $z = g'(y)$, y. , x (x) , , y $f(x)$ x. , ,

$$\frac{d\,g(f(x))}{dx} = \frac{d\,g(y)}{dy}\frac{d\,f(x)}{dx}, \qquad \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx},$$

, , y $f(x)$. . , , $\frac{dz}{dx}$, $\frac{dz}{dy}$ $\frac{dy}{dx}$.

: (1) $z = \sin(x^2+3)$ 2. $y = x^2+3$ $z = \sin y$. $y = x^2+3$ 2 $\frac{dy}{dx}\big|_{x=2} = \frac{d\,(x^2+3)}{dx}\big|_{x=2} = 2x|_{x=2} = 4$ $z = \sin y$ $2^2+3 = 7$ $\frac{dz}{dy}\big|_{y=7} = \frac{d\,\sin y}{dy}\big|_{y=7} = \cos y|_{y=7} = \cos 7$. $\frac{d\,\sin(x^2+3)}{dx}\big|_{x=2} = \frac{dz}{dx}\big|_{x=2} = \frac{dz}{dy}\big|_{y=7}\frac{dy}{dx}\big|_{x=2} = 4\cos 7$.

(2) $z = \sin(x^2+3)$. $y = x^2+3$ $z = \sin y$. $y = x^2+3$ $\frac{dy}{dx} = \frac{d\,(x^2+3)}{dx} = 2x$ $z = \sin y$ $\frac{dz}{dy} = \frac{d\,\sin y}{dy} = \cos y$. $\frac{d\,\sin(x^2+3)}{dx} = \frac{dz}{dx} = \frac{dz}{dy}\big|_{y=x^2+3}\frac{dy}{dx} = \cos y\big|_{y=x^2+3}2x = 2x\cos(x^2+3)$.

(3) $z = (\sin x)^n$. $y = \sin x$ $z = y^n$ $\frac{d\,(\sin x)^n}{dx} = \frac{dz}{dx} = \frac{dz}{dy}\big|_{y=\sin x}\frac{dy}{dx} = ny^{n-1}\big|_{y=\sin x}\cos x = n(\sin x)^{n-1}\cos x$.

. .

. $y = f(x)$ (a,b) ξ (a,b). , $\frac{f(x)-f(\xi)}{x-\xi} \geq 0$ $x \neq \xi$ (a,b) , , $f'(\xi) = \lim_{x\to\xi}\frac{f(x)-f(\xi)}{x-\xi}$ ≥ 0 $+\infty$. , $y = f(x)$, $f'(\xi)$ ≤ 0 $-\infty$.

– – $[\xi, b)$ $(a,\xi]$, $f'_+(\xi)$ $f'_-(\xi)$.

6.5 . $y = f(x)$ I ξ. $y = f(x)$, , J $\eta = f(\xi)$ $x = f^{-1}(y)$ J. $y = f(x)$ ξ, $x = f^{-1}(y)$ η

$$\boxed{\big(f^{-1}\big)'(\eta) = \begin{cases} \frac{1}{f'(\xi)}, & f'(\xi) \quad > 0, \\ 0, & f'(\xi) = +\infty, \\ +\infty, & f'(\xi) = 0. \end{cases}}$$

$y = f(x)$, : <0 >0 $-\infty$ $+\infty$.

:

$$\begin{aligned}\left(f^{-1}\right)'(\eta) &= \lim_{y\to\eta} \frac{f^{-1}(y)-f^{-1}(\eta)}{y-\eta} \\ &= \lim_{x\to\xi} \frac{x-\xi}{f(x)-f(\xi)} = \lim_{x\to\xi} \frac{1}{\frac{f(x)-f(\xi)}{x-\xi}}\,.\end{aligned}$$

$\frac{1}{f'(\xi)}$, $f'(\xi)$ >0, 0 $f'(\xi)$ $+\infty$. $f'(\xi)=0$, $\frac{f(x)-f(\xi)}{x-\xi}$ 0 , , $\frac{1}{\frac{f(x)-f(\xi)}{x-\xi}}$ $+\infty$.

:

$$D(f^{-1})(\eta) = \frac{1}{Df(\xi)}\,, \quad \frac{d\,f^{-1}(y)}{dy}\Big|_{y=\eta} = \frac{1}{\frac{d\,f(x)}{dx}\Big|_{x=\xi}}\,, \quad \frac{dx}{dy}\Big|_{y=\eta} = \frac{1}{\frac{dy}{dx}\Big|_{x=\xi}}\,.$$

$$\left(f^{-1}\right)'(y) = \frac{1}{f'\big(f^{-1}(y)\big)}\,, \qquad D(f^{-1})(y) = \frac{1}{Df\big(f^{-1}(y)\big)}\,,$$

$$\frac{d\,f^{-1}(y)}{dy} = \frac{1}{\frac{d\,f(x)}{dx}\Big|_{x=f^{-1}(y)}}\,, \qquad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}\Big|_{x=f^{-1}(y)}}\,.$$

, $\frac{d\,f^{-1}(y)}{dy} = \frac{1}{\frac{d\,f(x)}{dx}}$ $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ $\frac{d\,f^{-1}(y)}{dy} = \frac{dx}{dy}$, $x = f^{-1}(y)$, y $\frac{d\,f(x)}{dx} = \frac{dy}{dx}$ x. , ,

$$\frac{d\,f^{-1}(y)}{dy} = \frac{1}{\frac{d\,f(x)}{dx}}\,, \qquad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}\,,$$

x $f^{-1}(y)$ y. $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, . . , , $y = x$. . . : $y = x$.

: (1) $\frac{d\,\sqrt[n]{y}}{dy} = \frac{1}{n}\frac{\sqrt[n]{y}}{y}$ $x = \sqrt[n]{y}$. . $x = \sqrt[n]{y}$ $y = x^n$,

$$\frac{d\,\sqrt[n]{y}}{dy} = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}\Big|_{x=\sqrt[n]{y}}} = \frac{1}{nx^{n-1}\Big|_{x=\sqrt[n]{y}}} = \frac{1}{n}\frac{\sqrt[n]{y}}{y}\,.$$

, , , $\frac{d\,x^a}{dx} = ax^{a-1}$ $y = x^a$ a . , $a = \frac{m}{n}$, m n, $u = \sqrt[n]{x}$ $y = x^a = u^m$

$$\begin{aligned}\frac{d\,x^a}{dx} &= \frac{dy}{dx} = \frac{dy}{du}\Big|_{u=\sqrt[n]{x}}\frac{du}{dx} = mu^{m-1}\Big|_{u=\sqrt[n]{x}} \cdot \frac{1}{n}\frac{\sqrt[n]{x}}{x} \\ &= \frac{m}{n}(\sqrt[n]{x})^{m-1}\frac{\sqrt[n]{x}}{x} = \frac{m}{n}\frac{(\sqrt[n]{x})^m}{x} = ax^{a-1}\,.\end{aligned}$$

(2) $y = \sin x$ $[-\frac{\pi}{2}, \frac{\pi}{2}]$ $[-1, 1]$. , , $x = \arcsin y$, $[-1, 1]$ $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$x = \arcsin y$. $y = \sin x$ $\frac{d \sin x}{dx} = \cos x = \sqrt{1-(\sin x)^2}$ $+\infty$.
$\cos x = \pm\sqrt{1-(\sin x)^2}$ $+$ $\cos x \geq 0$ x $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
$\sqrt{1-y^2} = \sqrt{1-(\sin x)^2} > 0$,

$$\frac{d \arcsin y}{dy} = \frac{1}{\frac{d \sin x}{dx}\big|_{x=\arcsin y}} = \frac{1}{\sqrt{1-(\sin x)^2}\big|_{x=\arcsin y}} = \frac{1}{\sqrt{1-y^2}}.$$

$\sqrt{1-y^2} = \sqrt{1-(\sin x)^2} = 0$, $\frac{d \arcsin y}{dy} = +\infty$.

$$\boxed{\frac{d \arcsin y}{dy} = \begin{cases} \frac{1}{\sqrt{1-y^2}}, & -1 < y < 1, \\ +\infty, & y = \pm 1. \end{cases}}$$

$x = \arccos y$, $[-1, 1]$ $[0, \pi]$.

$$\boxed{\frac{d \arccos y}{dy} = \begin{cases} -\frac{1}{\sqrt{1-y^2}}, & -1 < y < 1, \\ -\infty, & y = \pm 1. \end{cases}}$$

(3) $y = \tan x$ $(-\frac{\pi}{2}, \frac{\pi}{2})$ $(-\infty, +\infty)$. , , $x = \arctan y$ $(-\infty, +\infty)$ $(-\frac{\pi}{2}, \frac{\pi}{2})$. $x = \arctan y$. $\frac{d \tan x}{dx} = \frac{1}{(\cos x)^2}$ 0 $+\infty$,

$$\frac{d \arctan y}{dy} = \frac{1}{\frac{d \tan x}{dx}\big|_{x=\arctan y}} = \frac{1}{\frac{1}{(\cos x)^2}\big|_{x=\arctan y}} = (\cos x)^2\big|_{x=\arctan y}$$

$$= \frac{1}{1+(\tan x)^2}\bigg|_{x=\arctan y} = \frac{1}{1+y^2}.$$

$$\boxed{\frac{d \arctan y}{dy} = \frac{1}{1+y^2}.}$$

$x = \operatorname{arccot} y$, $(-\infty, +\infty)$ $(0, \pi)$.

$$\boxed{\frac{d \operatorname{arc\,cot} y}{dy} = -\frac{1}{1+y^2}.}$$

•

1. .

$$y = x^2 - 3x + 1 - \frac{x^3+2}{x^2+1}, \quad y = \frac{x^3 - x + 4\sin x}{x^2 + \sin x + 2}, \quad y = \sin x + \tan x.$$

2. .

$$y = \sin(x^n), \quad y = (\tan x)^n, \quad y = \tan(x^n), \quad y = \sqrt[n]{1+\cos x},$$

$$y = \frac{(\sin x)^3 - 3(\sin x)^2 + 1}{(\sin x)^2 + 4\sin x + 4}, \quad y = \sin(\arccos x), \quad y = \arcsin(\cos x),$$

$$y = \arctan(\tan x), \quad y = \tan(\arctan x).$$

3. $y = \begin{cases} x\sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

0. $(-\infty, 0) \cup (0, +\infty)$.

4. $y = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

$(-\infty, +\infty)$. $(-\infty, +\infty)$;

5. . a $y = \begin{cases} x^a\sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

a $(-\infty, +\infty)$; $(-\infty, +\infty)$; $(-\infty, +\infty)$;

6. $x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1}-1}{x-1} - 1$ $x + 2x^2 + 3x^3 + \cdots + nx^n$ $x + 4x^2 + 9x^3 + \cdots + n^2x^n$.

7. $y = f_1(x), \ldots, y = f_n(x)$ ξ 0 ξ. $g(x) = f_1(x) \cdots f_n(x)$,

$$\frac{g'(\xi)}{g(\xi)} = \frac{f_1{}'(\xi)}{f_1(\xi)} + \cdots + \frac{f_n{}'(\xi)}{f_n(\xi)}.$$

8. $y = r(x)$ $\lim_{x\to+\infty} \frac{xr'(x)}{r(x)} = 0$;

9. (a, b) $y = r(x)$ $r'(x) = \frac{1}{x}$ x (a, b).
(: . .)

10. .

11. .

12. $y = f(x)$ $f(x)^2 + 4f(x) = x^3 - 5x^2 - 5x + 21$ x .
$(2f(x) + 4)f'(x) = 3x^2 - 10x - 5$ x .
, .

13. $y = f(x) = x^3 + 3x^2 + 3x + 7$ $(-\infty, +\infty)$.
$y = f(x)$ $(-\infty, +\infty)$. ;
$x = f^{-1}(y)$, $\left(f^{-1}\right)'(y) = \begin{cases} \frac{1}{3(f^{-1}(y)+1)^2}, & y \neq 6, \\ +\infty, & y = 6. \end{cases}$
$x = f^{-1}(y)$, .
(: $y = (x+1)^3 + 6$.)

14. κ μ. .

15. κ μ. ;

16. l . v ( ), ( ). , , .

17. $y^2 = 4x^3$. ( ) ( ); x, y.
(: $x = x(t), y = y(t)$ $y^2 = 4x^3$.)

6.6 , .

.

. $y = \log_a x$, $a > 0$, $a \neq 1$, $(0, +\infty)$ $(-\infty, +\infty)$.

$$\boxed{\frac{d \log_a x}{dx} = \frac{1}{\log a}\frac{1}{x} \qquad (x > 0).}$$

$\lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^t = e$, $\lim_{t\to+\infty}\left(1-\frac{1}{t}\right)^t = \lim_{t\to+\infty}\frac{1}{(1+\frac{1}{t-1})^t} = \lim_{t\to+\infty}\frac{1}{(1+\frac{1}{t-1})^{t-1}(1+\frac{1}{t-1})} = \frac{1}{e\cdot 1} = \frac{1}{e}$.

$y = \log_a x$ $\lim_{t\to+\infty}\log_a\left(1+\frac{1}{t}\right)^t = \log_a e = \frac{1}{\log a}$ $\lim_{t\to+\infty}\log_a\left(1-\frac{1}{t}\right)^t = \log_a\frac{1}{e} = -\log_a e = -\frac{1}{\log a}$.

$t = \frac{x}{h}$.

$$\lim_{h\to 0+}\frac{\log_a(x+h)-\log_a x}{h} = \lim_{h\to 0+}\frac{1}{h}\log_a\left(\frac{x+h}{x}\right) = \frac{1}{x}\lim_{h\to 0+}\frac{x}{h}\log_a\left(1+\frac{h}{x}\right)$$

$$= \frac{1}{x}\lim_{t\to+\infty} t\log_a\left(1+\frac{1}{t}\right) = \frac{1}{x}\lim_{t\to+\infty}\log_a\left(1+\frac{1}{t}\right)^t = \frac{1}{\log a}\frac{1}{x}.$$

, $t = -\frac{x}{h}$.

$$\lim_{h\to 0-}\frac{\log_a(x+h)-\log_a x}{h} = \lim_{h\to 0-}\frac{1}{h}\log_a\left(\frac{x+h}{x}\right) = \frac{1}{x}\lim_{h\to 0-}\frac{x}{h}\log_a\left(1+\frac{h}{x}\right)$$

$$= -\frac{1}{x}\lim_{t\to+\infty} t\log_a\left(1-\frac{1}{t}\right) = -\frac{1}{x}\lim_{t\to+\infty}\log_a\left(1-\frac{1}{t}\right)^t = \frac{1}{\log a}\frac{1}{x}.$$

$\frac{d\ \log_a x}{dx} = \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h} = \frac{1}{\log a}\frac{1}{x}$.

$$\frac{d\ \log x}{dx} = \frac{1}{x} \qquad (x > 0).$$

:

$$\frac{d\ \log|x|}{dx} = \frac{1}{x} \qquad (x \neq 0).$$

$(0, +\infty)$ $\frac{d\ \log|x|}{dx} = \frac{d\ \log x}{dx} = \frac{1}{x}$ $(-\infty, 0)$ $\frac{d\ \log|x|}{dx} = \frac{d\ \log(-x)}{dx} = \frac{1}{-x}(-1) = \frac{1}{x}$

.

. $y = a^x$, $a > 0$, $(-\infty, +\infty)$ $(0, +\infty)$. :

$$\boxed{\frac{d\,a^x}{dx} = a^x \log a.}$$

$a > 0,\ a \neq 1,$. $y = a^x$ $x = \log_a y$, ,

$$\frac{d\,a^x}{dx} = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}\big|_{y=a^x}} = \frac{1}{\frac{1}{\log a}\frac{1}{y}\big|_{y=a^x}} = a^x \log a.$$

$a = 1,\ y = 1^x = 1$. $1^x \log 1$, , ′ .
$y = e^x$:

$$\frac{d\,e^x}{dx} = e^x\,.$$

. , $y = x^a$ a . ′ $y = x^a$ $[0, +\infty),\ a > 0,$ $(0, +\infty),\ a < 0.$

$$\boxed{\frac{d\,x^a}{dx} = ax^{a-1} \qquad \begin{pmatrix} a\ > 1,\ x \geq 0 \\ a\ < 1,\ x > 0. \end{pmatrix}}$$

$x > 0.$ $y = x^a = e^{\log(x^a)} = e^{a\log x}$. $z = a\log x,$ $y = e^z$

$$\frac{d\,x^a}{dx} = \frac{dy}{dx} = \frac{dy}{dz}\Big|_{z=a\log x}\frac{dz}{dx} = e^z\Big|_{z=a\log x}\frac{a}{x} = e^{a\log x}\frac{a}{x} = x^a\frac{a}{x} = ax^{a-1}\,.$$

a $< 0,$ $y = x^a$ $0.$ a $0 < a < 1,$ $\frac{d\,x^a}{dx}\big|_{x=0} = \lim_{x\to 0+}\frac{x^a - 0^a}{x-0} = \lim_{x\to 0+} x^{a-1} = +\infty.$, a $a > 1,$ $\frac{d\,x^a}{dx}\big|_{x=0} = \lim_{x\to 0+}\frac{x^a - 0^a}{x-0} = \lim_{x\to 0+} x^{a-1} = 0 = a0^{a-1}\,.$

.

1. .

$$y = x\log x, \quad y = \log\big|\log|x|\big|, \quad y = \log\left(e^{3x^2+4} + \sin\left(x^{-\frac{5}{4}}\right)\right),$$

$$y = 2^{x^2+1}\log_3(\sin x), \quad y = 3^{-\sin(\log x)}\,, \quad y = \sin\left(e^{\sqrt{\log(x^2+1)}}\right).$$

2. .

$$\lim_{x\to 1}\frac{\log x}{x-1}\,, \qquad \lim_{x\to 0}\frac{e^x - 1}{x}\,, \qquad \lim_{x\to 1}\frac{x^a - 1}{x-1}\,.$$

$$\lim_{x\to 1}\frac{\log x}{x^a - 1} = \frac{1}{a}\,, \quad \lim_{x\to 1}\frac{x^a - x^b}{x-1} = a - b, \quad \lim_{x\to 1}\frac{x^a - x^b}{\log x} = a - b,$$

$$\lim_{x\to 0}\frac{a^x - b^x}{x} = \log\frac{a}{b}\,, \quad \lim_{x\to 0}\frac{e^{ax} - e^{bx}}{x} = a - b, \quad \lim_{x\to 0}\frac{\tanh(ax)}{x} = a.$$

3. $y = f(x)$ $y = g(x)$ $(a, b),$ $a < \xi < b$ $f(x) > 0$ x $(a, b).$
$\xi,$ $y = f(x)^{g(x)}$ ξ $\xi.$
.

$$y = x^x\,, \quad y = (x^2+1)^{\sin x}\,, \quad y = |x-1|^{x-2}|x-2|^{x-1}\,.$$

4.
$$\frac{d\cosh x}{dx} = \sinh x, \qquad \frac{d\sinh x}{dx} = \cosh x,$$
$$\frac{d\tanh x}{dx} = \frac{1}{(\cosh x)^2}, \qquad \frac{d\coth x}{dx} = -\frac{1}{(\sinh x)^2} \quad (x \neq 0).$$

5.
$$\frac{d\,\mathrm{arccosh}\ y}{dy} = \begin{cases} \frac{1}{\sqrt{y^2-1}}, & y > 1, \\ +\infty, & y = 1, \end{cases} \qquad \frac{d\,\mathrm{arcsinh}\ y}{dy} = \frac{1}{\sqrt{y^2+1}},$$
$$\frac{d\,\mathrm{arctanh}\ y}{dy} = \frac{1}{1-y^2} \quad (|y| < 1), \qquad \frac{d\,\mathrm{arccoth}\ y}{dy} = \frac{1}{1-y^2} \quad (|y| > 1).$$

6.7 .

$x = x(t)$ $y = y(t)$ I t. C $(x, y) = (x(t), y(t))$, t I, . I C .

t , $(x, y) = (x(u), y(u))$, $(x, y) = (x(s), y(s))$.

$x = x(t)$ $y = y(t)$ C , , t .
′ , .

: (1) $x = x(t) = \kappa t + \lambda$ $y = y(t) = \mu t + \nu$ t $(-\infty, +\infty)$, $\kappa, \mu \neq 0$. – – C. , , , , , .
, t, C $\mu x - \kappa y = \mu\lambda - \kappa\nu$. , (x, y) $\mu x - \kappa y = \mu\lambda - \kappa\nu$, , $\kappa \neq 0$, $t = \frac{1}{\kappa}x - \frac{\lambda}{\kappa}$ t $x = \kappa t + \lambda$ $y = \mu t + \nu$. , C (x, y) $\mu x - \kappa y = \mu\lambda - \kappa\nu$. $\mu \neq 0$. , , $\mu x - \kappa y = \mu\lambda - \kappa\nu$ l : $\kappa \neq 0$, $y = y(x) = \frac{\mu}{\kappa}x + \frac{\kappa\nu - \mu\lambda}{\kappa}$, $\mu \neq 0$, $x = x(y) = \frac{\kappa}{\mu}y + \frac{\mu\lambda - \kappa\nu}{\mu}$. , $\kappa \neq 0$ $\mu \neq 0$, , . $\kappa = 0$, , $x = \lambda$, , . $\mu = 0$, , $y = \nu$, , . , , C l.
, l
$$ax + by = c,$$
$a, b \neq 0$. $\kappa = -b$, $\mu = a$ λ, ν $\mu\lambda - \kappa\nu = c$. , C $x = x(t) = \kappa t + \lambda$ $y = y(t) = \mu t + \nu$ l.
$t = 0$ $t = 1$ $A_0 = (\lambda, \nu)$ $A_1 = (\kappa + \lambda, \mu + \nu)$ $\overrightarrow{A_0A_1} = (\kappa, \mu)$ – t.
$x = x(t) = \kappa t + \lambda$ $y = y(t) = \mu t + \nu$, $(x, y) = (\kappa t + \lambda, \mu t + \nu) = (\kappa, \mu)t + (\lambda, \nu)$. (κ, μ) (λ, ν) , , (κ, μ) ($t = 0$) (λ, ν). t $(-\infty, +\infty)$, $(x, y) = (\kappa t + \lambda, \mu t + \nu)$.

(2) $x = x(t) = r_0 \cos t + x_0$ $y = y(t) = r_0 \sin t + y_0$ t $I = [t_0, t_0 + 2\pi]$ 2π, $r_0 > 0$, (x_0, y_0) r_0 . , $(x, y) = (r_0 \cos t + x_0, r_0 \sin t + y_0)$
$$(x - x_0)^2 + (y - y_0)^2 = {r_0}^2$$
, , (x, y) $(x - x_0)^2 + (y - y_0)^2 = {r_0}^2$ $(x, y) = (r_0 \cos t + x_0, r_0 \sin t + y_0)$ t I. , , $x = x(t) = r_0 \cos t + x_0$ $y = y(t) = r_0 \sin t + y_0$ $(x - x_0)^2 + (y - y_0)^2 = {r_0}^2$. , t I , $(x, y) = (r_0 \cos t + x_0, r_0 \sin t + y_0)$. t $< 2\pi$, $(x, y) = (r_0 \cos t + x_0, r_0 \sin t + y_0)$.

(3) $x = x(t) = \kappa_0 \cos t + x_0$ $y = y(t) = \mu_0 \sin t + y_0$ t $I = [t_0, t_0 + 2\pi]$ 2π, $\kappa_0, \mu_0 > 0$, (x_0, y_0) $2\kappa_0$ $2\mu_0$. $(x, y) = (\kappa_0 \cos t + x_0, \mu_0 \sin t + y_0)$

$$\Big(\frac{x - x_0}{\kappa_0}\Big)^2 + \Big(\frac{y - y_0}{\mu_0}\Big)^2 = 1$$

. , $x = x(t) = \kappa_0 \cos t + x_0$ $y = \mu_0 \sin t + y_0$ $(\frac{x-x_0}{\kappa_0})^2 + (\frac{y-y_0}{\mu_0})^2 = 1$. t I , $(x, y) = (\kappa_0 \cos t + x_0, \mu_0 \sin t + y_0)$. t $< 2\pi$, $(x, y) = (\kappa_0 \cos t + x_0, \mu_0 \sin t + y_0)$.

(4) $y = f(x)$ I, $(x, y) = (x, f(x))$ x I, . , $x = t$ $y = f(t)$ t I.

, $x = g(y)$ I , $x = g(t)$ $y = t$ t I.

. $x = x(t)$, $y = y(t)$ $z = z(t)$ I t, $(x, y, z) = (x(t), y(t), z(t))$, t I, .

: (1) $x = x(t) = \kappa t + \lambda$, $y = y(t) = \mu t + \nu$ $z = z(t) = \rho t + \sigma$ t $(-\infty, +\infty)$, $\kappa, \mu, \rho \neq 0$, . , $(x, y, z) = (\kappa, \mu, \rho)t + (\lambda, \nu, \sigma)$, (κ, μ, ρ) ($t = 0$) (λ, ν, σ).

(2) $x = x(t) = r_0 \cos t + x_0$, $y = y(t) = r_0 \sin t + y_0$ $z = z(t) = \frac{h_0}{2\pi} t + z_0$ t $(-\infty, +\infty)$, $r_0 > 0$ $h_0 > 0$, «» , . t 2π , $(x, y) = (r_0 \cos t + x_0, r_0 \sin t + y_0)$ xy- (x_0, y_0) r_0 , , z h_0 − − . , (x, y, z) , , r_0 l xy- $(x_0, y_0, 0)$.

, , .

, , $x = x(t)$ $y = y(t)$ I t. , , $(x(\tau), y(\tau))$ τ I. $\tau + h$ I, h , , $(x(\tau + h), y(\tau + h))$.

$$x = x(t) = \frac{x(\tau + h) - x(\tau)}{h}(t - \tau) + x(\tau),$$

$$y = y(t) = \frac{y(\tau + h) - y(\tau)}{h}(t - \tau) + y(\tau).$$

h 0, $(x(\tau), y(\tau))$, , ,

:	$x = x(t) = x'(\tau)(t - \tau) + x(\tau)$, $y = y(t) = y'(\tau)(t - \tau) + y(\tau)$.

t

:	$x'(\tau)\big(y - y(\tau)\big) = y'(\tau)\big(x - x(\tau)\big)$.

$(x'(\tau), y'(\tau))$.

$(x(t), y(t))$ $t = \tau$, , . $x = x(t) = x'(\tau)(t - \tau) + x(\tau)$ $y = y(t) = y'(\tau)(t - \tau) + y(\tau)$ $x'(\tau), y'(\tau)$ $\neq 0$. , $x = x(\tau), y = y(\tau)$. τ I, t $\geq \tau$, τ I, $\leq \tau$, τ .

, $x = x(t)$, $y = y(t)$ $z = z(t)$ I t, $(x(\tau), y(\tau), z(\tau))$

$$\boxed{:\quad \begin{aligned} x &= x(t) = x'(\tau)(t-\tau) + x(\tau), \\ y &= y(t) = y'(\tau)(t-\tau) + y(\tau), \\ z &= z(t) = z'(\tau)(t-\tau) + z(\tau). \end{aligned}}$$

$$(x'(\tau), y'(\tau), z'(\tau))$$

$(x(t), y(t), z(t))$ $t = \tau$. , $x'(\tau), y'(\tau), z'(\tau)$ $\neq 0$.

: (1) $x = x(t) = t\cos t,\ y = y(t) = \sin t$ t $(-\infty, +\infty)$. $(x(0), y(0)) = (0, 0)$ $x = x'(0)(t-0) + x(0) = t,\ y = y'(0)(t-0) + y(0) = t$. (t) $y = x$. $(x'(0), y'(0)) = (1, 1)$.

(2) $x = x(t) = t,\ y = y(t) = t^2,\ z = z(t) = t^3$ t $(-\infty, +\infty)$. $(x(1), y(1), z(1)) = (1, 1, 1)$ $x = x'(1)(t-1) + x(1) = t,\ y = y'(1)(t-1) + y(1) = 2t - 1,\ z = z'(1)(t-1) + z(1) = 3t - 2$. $(x'(1), y'(1), z'(1)) = (1, 2, 3)$.

(3) $y = f(x)\ (x\ \ I)$ $x = t$ $y = f(t)$ t I. $(\xi, f(\xi))$ $y = f'(\xi)(x-\xi) + f(\xi)$ x $(-\infty, +\infty)$. $x = 1\cdot(t-\xi) + \xi$ $y = f'(\xi)(t-\xi) + f(\xi)$ t $(-\infty, +\infty)$. $x = t$ $y = f'(\xi)(t-\xi) + f(\xi)$ t $(-\infty, +\infty)$, t, $y = f'(\xi)(x-\xi) + f(\xi)$ x $(-\infty, +\infty)$. , .

.

1. : .

2. .

3. () $x = x(t) = r_0\cos t + x_0,\ y = y(t) = r_0\sin t + y_0$ $z = z(t) = \frac{h_0}{2\pi}t + z_0$ t $(-\infty, +\infty)$, $r_0 > 0$ $h_0 > 0$.

4. () $x = x(t) = r_0 t\cos t + x_0,\ y = y(t) = r_0 t\sin t + y_0$ $z = z(t) = \frac{h_0}{2\pi}t + z_0$ t $(-\infty, +\infty)$, $r_0 > 0$ $h_0 > 0$.

5. $a > 0$. (x, y)
$$x^2 - y^2 = a^2\,.$$
$x = \sqrt{t^2 + a^2}$ $y = t$ t $(-\infty, +\infty)$ $x = -\sqrt{t^2 + a^2}$ $y = t$ t $(-\infty, +\infty)$.
$x = x(t) = a\cosh t$ $y = y(t) = a\sinh t$ t $(-\infty, +\infty)$ $x = x(t) = -a\cosh t$ $y = y(t) = a\sinh t$ t $(-\infty, +\infty)$.
, . : , .

6. (x, y)
$$xy = b,$$
$b \neq 0$, $x = x(t) = t$ $y = y(t) = \frac{b}{t}$ $x = x(t) = \frac{b}{t}$ $y = t$ t $(0, +\infty)$ t $(-\infty, 0)$.
.

7. $x^2 - y^2 = a^2\,(a > 0)$ $xy = b\,(b \neq 0)$, .

8. $x = x(t) = t^3$ $y = y(t) = t^4$ t $(-\infty, +\infty)$, , C (t^3, t^4).

C $(0,0)$.

, $x = x(s) = s$ $y = y(s) = s^{\frac{4}{3}}$ s $(-\infty, +\infty)$.

C $(0,0)$;

6.8 .

$y = f(x)$ (a,b) ξ (a,b). ξ $y = f(x)$ (c,d), , , ξ, $f(x) \le f(\xi)$ x (c,d). $y = f(x)$ $[\xi, b)$ (a,ξ), ξ $y = f(x)$ $[\xi, d)$ $f(x) \le f(\xi)$ x $[\xi, d)$. , $y = f(x)$ $(a,\xi]$ (ξ, b), ξ $(c,\xi]$ $f(x) \le f(\xi)$ x $(c,\xi]$.

ξ $y = f(x)$ $f(x) \ge f(\xi)$ $f(x) \le f(\xi)$.

, ξ $y = f(x)$ ξ , , ξ.

ξ $y = f(x)$.

, , **()** , . , , **()** , .

$y = f(x)$ ξ : ( ) $(\xi, f(\xi))$, , $(\xi, f(\xi))$.

: (1) 0 $y = 1 + x^2(x+1)$, 0 1 $1 + x^2(x+1) \ge 1$ x $(-1, +\infty)$ 0. 0 () , < 1. , -2 $-3 < 1$, , $\lim_{x\to-\infty}(1 + x^2(x+1)) = -\infty$.

(2) 0 $y = x - \sqrt{x}$ $[0, +\infty)$, 0 0 $x - \sqrt{x} \le 0$ $[0,1)$. 0 () , > 0.

(3) , , . , ,

$$y = \begin{cases} x \sin \frac{1}{x}, & x > 0, \\ 0, & x = 0, \end{cases}$$

$[0, +\infty)$.

. , 6, 7 8 3.10, 7 8 4.8, 2 3 5.5, 6.2, 6 7 6.4 3, 4 5 6.5.

0, $\lim_{x\to0+} x \sin\frac{1}{x} = 0$. $x = \frac{1}{\frac{\pi}{2} + n2\pi}$ $(n \in \mathbf{Z}, n \ge 0)$ $x \sin \frac{1}{x} = x > 0$ $x = \frac{1}{\frac{3\pi}{2} + n2\pi}$ $(n \in \mathbf{Z}, n \ge 0)$ $x \sin\frac{1}{x} = -x < 0$. n , 0. , $[0, d)$, 0. 0 .

.

6.1 ***Fermat.*** $y = f(x)$ (a,b) ξ (a,b). ξ $y = f(x)$,

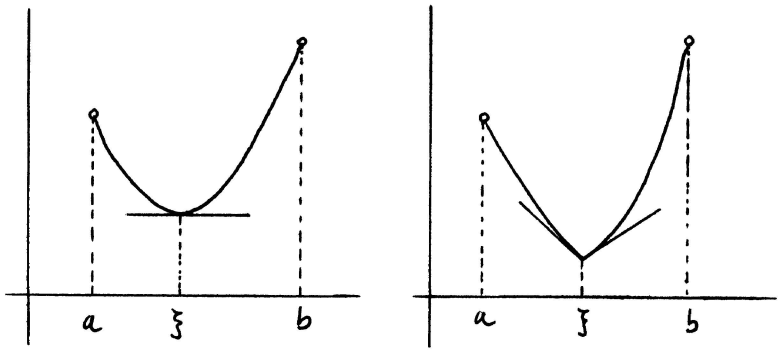
(i) $y = f(x)$ ξ,

(ii) $y = f(x)$ ξ $f'(\xi) = 0$.

: ξ $y = f(x)$. (c,d) (a,b) ξ $f(x) \le f(\xi)$ x (c,d). (i), $f'(\xi)$. $f'_+(\xi)$ $f'_-(\xi)$ $f'(\xi)$ – ′ $\pm\infty$.

x (ξ, d) $\frac{f(x)-f(\xi)}{x-\xi} \le 0$. $f'(\xi) = f'_+(\xi) = \lim_{x\to\xi+} \frac{f(x)-f(\xi)}{x-\xi} \le 0$. , x (c,ξ) $\frac{f(x)-f(\xi)}{x-\xi} \ge 0$. $f'(\xi) = f'_-(\xi) = \lim_{x\to\xi-} \frac{f(x)-f(\xi)}{x-\xi} \ge 0$. $f'(\xi) = 0$, , (ii).

ξ , , ≥ 0 ≤ 0 .



Σχήμα 6.6: Fermat.

Fermat . $y = f(x)$ ξ ξ $y = f(x)$, $(\xi, f(\xi))$ 0, .

: (1) 0 () $y = |x|$, $(-\infty, +\infty)$, 0.

(2) 0 () $y = x^2$, $(-\infty, +\infty)$, 0 , , $\frac{dx^2}{dx}\big|_{x=0} = 2x\big|_{x=0} = 0$.

Fermat .

,

: () ,

0. –

.

: $y = 2x^3 - 9x^2 + 12x + 5$ $[0, 4]$. $y = \frac{d(2x^3 - 9x^2 + 12x + 5)}{dx} = 6x^2 - 18x + 12 = 6(x-1)(x-2)$, $[0, 4]$ 0 4 1 2 . 5, 37, 10 9, . . $[0, 4]$, . , , 0 (5) 4 (37). 1 2 . $[0, 2]$, . 0, 1 2 – . 5, 10 9, 1 $[0, 2]$, , $[0, 4]$.

, 1, 2 4 10, 9 37, , 2 $[1, 4]$, , $[0, 4]$.

.

Fermat .

6.6 (1) $y = f(x)$ $[\xi, b)$ ξ $y = f(x)$.
(i) $f'_+(\xi)$,
(ii) $f'_+(\xi)$ $f'_+(\xi) \le 0$ $f'_+(\xi) \ge 0$, .
(2) $y = f(x)$ $(a, \xi]$ ξ $y = f(x)$.
(i) $f'_-(\xi)$,

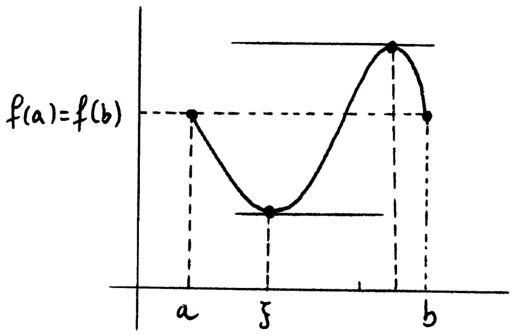
(ii) $f'_-(\xi)$ $f'_-(\xi) \geq 0$ $f'_-(\xi) \leq 0$, .

: (1) ξ $y = f(x)$. $[\xi, d)$ $f(x) \leq f(\xi)$ x $[\xi, d)$. (i), $f'_+(\xi)$. $\frac{f(x)-f(\xi)}{x-\xi} \leq 0$ x (ξ, d) , , $f'_+(\xi) = \lim_{x \to \xi+} \frac{f(x)-f(\xi)}{x-\xi} \leq 0$.
(1) (2) .

, $f'_+(\xi) \leq 0$ $f'_+(\xi) \geq 0$, $f'_+(\xi) = -\infty$ $f'_+(\xi) = +\infty$, . $f'_-(\xi)$.

: (1) $y = x$ $[0, 2]$ 0 0 , , $1 \geq 0$. 2 2 $1 \geq 0$.
(2) $y = \sqrt{x}$ $[0, 2]$ 0 0 , , $+\infty \geq 0$.

6.2 ***Rolle***. $y = f(x)$ $[a, b]$ (a, b). $f(a) = f(b)$, ξ (a, b) $f'(\xi) = 0$.



Σχήμα 6.7: Rolle.

: .
(i) $y = f(x)$ $[a, b]$, $f(a) = f(b)$, 0 (a, b), .
$y = f(x)$ $[a, b]$, (ii) $> f(a) = f(b)$ (iii) $< f(a) = f(b)$. .
(ii) - ξ $[a, b]$, $y = f(x)$. $f(a) = f(b)$, $f(\xi) > f(a) = f(b)$, , ξ (a, b). , $y = f(x)$ ξ , Fermat, $f'(\xi) = 0$.
(iii) - ξ $[a, b]$, $y = f(x)$. $< f(a) = f(b)$, $f(\xi) < f(a) = f(b)$, , ξ (a, b). , $y = f(x)$ ξ , Fermat, $f'(\xi) = 0$.
ξ (a, b) $f'(\xi) = 0$.

: $y = x^3 + 2x^2 - 3x - 5$ $[-2, \sqrt{3}]$, $(-2, \sqrt{3})$ 1. ξ $(-2, \sqrt{3})$, $y = 3x^2 + 4x - 3$ 0. ξ $3x^2 + 4x - 3 = 0$. $\frac{-2 \pm \sqrt{13}}{3}$ $(-2, \sqrt{3})$.

Rolle ξ $f'(\xi) = 0$.
$y = f(x)$ Rolle (a, b), ξ (a, b) $f'(\xi) = 0$.

: $y = |x|$ $[-1, 1]$ 1. , ξ $(-1, 1)$ 0.

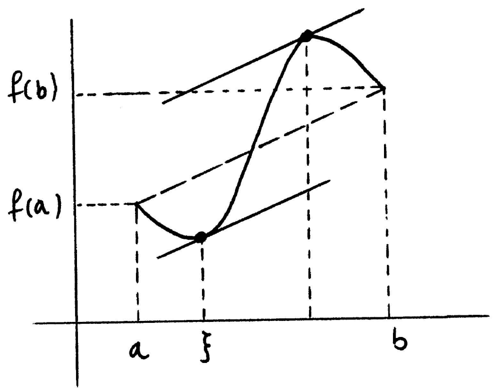
Rolle $+\infty$ $-\infty$ (a,b). .

: $y = x^{\frac{1}{3}} - x^{\frac{7}{3}}$ $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} - \frac{7}{3}x^{\frac{4}{3}}$, $x \neq 0$, $\frac{dy}{dx} = +\infty$, $x = 0$. 0 $[-1,1]$. 0 $\pm\frac{1}{\sqrt{7}}$.

.

6.3 ***(Lagrange)***. $y = f(x)$ $[a,b]$ (a,b). ξ (a,b)

$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(\xi).}$$



Σχήμα 6.8: .

: $y = h(x) = (b-a)f(x) - (f(b)-f(a))x$ $[a,b]$ (a,b). , $h(a) = h(b)$. ξ (a,b) $h'(\xi) = 0$. $(b-a)f'(\xi) - (f(b)-f(a)) = 0$, , $\frac{f(b)-f(a)}{b-a} = f'(\xi)$.

Rolle 6.3. , $f(a) = f(b)$, 6.3 ξ (a,b) $f'(\xi) = \frac{f(b)-f(a)}{b-a} = 0$. , 6.3 Rolle. 6.3 Rolle. .
$y = f(x)$ 6.3, $\frac{f(b)-f(a)}{b-a}$ $(a, f(a))$ $(b, f(b))$. , 6.3 , $y = f(x)$ $[a,b]$ (a,b), $(\xi, f(\xi))$, , $(a, f(a))$ $(b, f(b))$.

: $y = x^{\frac{1}{3}}$ $[-1,1]$ $(-1,1)$. 0 $+\infty$ $x \neq 0$ $\frac{1}{3}x^{-\frac{2}{3}}$. ξ $(-1,1)$ $\frac{1}{3}\xi^{-\frac{2}{3}} = \frac{1^{\frac{1}{3}} - (-1)^{\frac{1}{3}}}{1-(-1)} = 1$. , $\xi = \pm\frac{1}{3\sqrt{3}}$.

.

6.4 ***(Cauchy)***. $y = f(x)$ $y = g(x)$ $[a,b]$ (a,b) (i) $g(a) \neq g(b)$ (ii) x (a,b) $f'(x) = g'(x) = 0$. ξ (a,b)

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}.}$$

: $y = h(x) = (g(b)-g(a))f(x)-(f(b)-f(a))g(x)$ $[a,b]$ (a,b). , $h(a) = h(b)$. ξ (a,b) $h'(\xi) = 0$. $(g(b)-g(a))f'(\xi)-(f(b)-f(a))g'(\xi) = 0$, , $(g(b)-g(a))f'(\xi) = (f(b)-f(a))g'(\xi)$. $g(a) \neq g(b)$, $\frac{f(b)-f(a)}{g(b)-g(a)}g'(\xi) = f'(\xi)$. $g'(\xi) = 0$, $f'(\xi) = 0$, . $g'(\xi) \neq 0$, $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$.

: $0 < a < b$, m, n ξ $a < \xi < b$ $\frac{b^m-a^m}{b^n-a^n} = \frac{m\xi^{m-1}}{n\xi^{n-1}} = \frac{m}{n}\xi^{m-n}$.

6.4 Rolle. , 6.3 6.4. , $y = g(x) = x$ 6.4, 6.3. , Rolle 6.3. , , .

.

1. 0 ;

$$y = \begin{cases} x\sin\frac{1}{x}, & x < 0, \\ 0, & x = 0, \end{cases} \quad y = \begin{cases} x^2\sin\frac{1}{x}, & x > 0, \\ 0, & x = 0, \end{cases}$$

$$y = \begin{cases} x\big(1+\sin\frac{1}{x}\big), & x > 0, \\ 0, & x = 0, \end{cases} \quad y = \begin{cases} x^2\big(-1+\sin\frac{1}{x}\big), & x > 0, \\ 0, & x = 0. \end{cases}$$

0.

(: .)

.

2. $a < b$. ξ

(i) $a < \xi < b$ $\frac{\sin b - \sin a}{b-a} = \cos\xi$.

(ii) $a < \xi < b$ $\frac{\sin b - \sin a}{e^b - e^a} = e^{-\xi}\cos\xi$.

(iii) $a < \xi < b$ $\frac{e^b - e^a}{\arctan b - \arctan a} = (1+\xi^2)e^{\xi}$.

3. $x^3 - 12x = c$ $[-2,2]$; $(-\infty,-2]$; $[2,+\infty)$;

4. $y = 2 - x^{\frac{2}{5}}$ 1 1 -1. ξ $(-1,1)$;

5. Rolle , n- n .

6. $y = (x+4)(x+1)(x-2)(x-3)$.

, $a_1 < a_2 < \ldots < a_n$ $y = (x-a_1)(x-a_2)\cdots(x-a_n)$ $n-1$. $a_1, \ldots, a_n$.

(: .)

7. $x^2 = x\sin x + \cos x$. 0.

8. $e^x = 1$; $e^x = 1+x$; $e^x = 1+x+\frac{x^2}{2}$; : $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}$;

(: .)

9. $y = f(x)$ $[1,4]$, $f(1) = -7$ $f'(x) \geq 3$ x $(1,4)$.

$f(4) = 1$; $f(4) \geq 2$.

$\mu \geq 2$ $y = f(x)$ $f(4) = \mu$.

(: $y = f(x) = ax + b$.)

10. $y = f(x)$ I I. y_1 y_2 d, x_1 x_2 I $f(x_1) = y_1$ $f(x_2) = y_2$.

(*i*) $|f'(x)| \geq m > 0$ x I, x_1 x_2 $\frac{d}{m}$.

(*ii*) $|f'(x)| \leq m < +\infty$ x I, x_1 x_2 $\frac{d}{m}$.

11. $y = f(x)$ I $f'(x) \neq 0$ x I.

$y = f(x)$ -- I.

12. (*) $y = f(x)$ $y = g(x)$ I $f(x)g'(x) - f'(x)g(x) \neq 0$ x I. $f(x) = 0$ I $g(x) = 0$.

(: a, b I $a < b$ $f(a) = f(b) = 0$. $g(x) \neq 0$ x (a,b). $g(a) = 0$ $g(b) = 0$; , $y = \frac{f(x)}{g(x)}$ $[a,b]$.)

$y = \cos x$ $y = \sin x$ $(-\infty, +\infty)$;

13. $y = f(x)$ $[a,b]$ a, b.

$f'_+(a) < 0 < f'_-(b)$, $y = f(x)$ $[a,b]$ (-) (a,b).

$f'_+(a) > 0 > f'_-(b)$;

14. (**) **Darboux**. $y = f(x)$ $[a,b]$ $f'_+(a) < c < f'_-(b)$ $f'_+(a) > c > f'_-(b)$. ξ (a,b) $f'(\xi) = c$.

(: $y = g(x) = f(x) - cx$.)

« » $[a,b]$, , ¨ .

15. $y = f(x)$ $[\xi - h, \xi + h]$ $(\xi - h, \xi) \cup (\xi, \xi + h)$. :

(*i*) ζ $(0,h)$ $\frac{f(\xi+h)-f(\xi-h)}{h} = f'(\xi+\zeta) + f'(\xi-\zeta)$.

(*ii*) ζ $(0,h)$ $\frac{f(\xi+h)-2f(\xi)+f(\xi-h)}{h} = f'(\xi+\zeta) - f'(\xi-\zeta)$.

16. (*) $y = f(x)$ $(0,+\infty)$ $|f'(x)| \leq \frac{1}{x}$ $x > 0$, $\lim_{x\to+\infty}(f(x+\sqrt{x}) - f(x)) = 0$.

(: $x > 0$ $[x, x+\sqrt{x}]$.)

17. (**) $y = f(x)$ $(0,+\infty)$ $\lim_{x\to+\infty} f'(x) = 0$, $\lim_{x\to+\infty}(f(x+1)-f(x)) = 0$.

(: $\epsilon > 0$ $\lim_{x\to+\infty} f'(x) = 0$. , x $[x, x+1]$ $|f(x+1)-f(x)| < \epsilon$.)

18. (**) $y = f(x)$ $[\xi, b)$ (ξ, b). $\lim_{x\to\xi+} f'(x)$, $f'_+(\xi)$.

(: $\epsilon > 0$ $\lim_{x\to\xi+} f'(x) = \eta$. x ξ $[\xi, x]$ $\left|\frac{f(x)-f(\xi)}{x-\xi} - \eta\right| < \epsilon$.)

$y = f(x)$ $(a, \xi]$ (a, ξ). $\lim_{x\to\xi-} f'(x)$, $f'_-(\xi)$.

6.9 .

. .

6.7 $y = f(x)$ I.
(1) $y = f(x)$ I $f'(x) = 0$ x I.
(2) $y = f(x)$ I $f'(x) \geq 0$ x .
(3) $y = f(x)$ I $f'(x) \leq 0$ x I.

: (1) $y = f(x)$, . , $f'(x) = 0$ x I. x_1 x_2 I $x_1 < x_2$ (I). $y = f(x)$ $[x_1, x_2]$ (x_1, x_2). ξ (x_1, x_2) , , ξ I, $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(\xi) = 0$. $f(x_1) = f(x_2)$. $y = f(x)$, $y = f(x)$ I.
(2) $y = f(x)$ I, , 6.5, $f'(x) \geq 0$ x . , $f'(x) \geq 0$ x . (1), x_1 x_2 I $x_1 < x_2$ ξ (x_1, x_2) , , ξ I, $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(\xi) \geq 0$. $f(x_1) \leq f(x_2)$, , $y = f(x)$ I.
(3) (2).

6.7. . , . , , .

6.7.

6.8 $y = f(x)$ I.
(1) $f'(x) > 0$ x , $y = f(x)$ I.
(2) $f'(x) < 0$ x , $y = f(x)$ I.

: 6.7.

6.7 6.8 $f'(x) \geq 0$ $f'(x) > 0$ $f'(x) = +\infty$. , $f'(x) \leq 0$ $f'(x) < 0$ $f'(x) = -\infty$. 6.9 .
(1) (2) 6.8. $y = f(x)$, , $f'(x) \geq 0$ x I. $y = f(x)$.

: $y = x^3$ $(-\infty, +\infty)$ $f'(x) > 0$ x $(-\infty, +\infty)$. : $\frac{d\,x^3}{dx} = 3x^2$ x > 0 $x \neq 0$ 0 $x = 0$.

6.7 6.8 . , .

: (1) $y = \frac{|x|}{x}$, $(-\infty, 0) \cup (0, +\infty)$, $(-\infty, 0) \cup (0, +\infty)$. -1 $(-\infty, 0)$ 1 $(0, +\infty)$.

(2) $y = \frac{1}{x}$ $-\frac{1}{x^2} < 0$, $(-\infty, 0) \cup (0, +\infty)$, $(-\infty, 0) \cup (0, +\infty)$. $(-\infty, 0)$ $(0, +\infty)$.

.

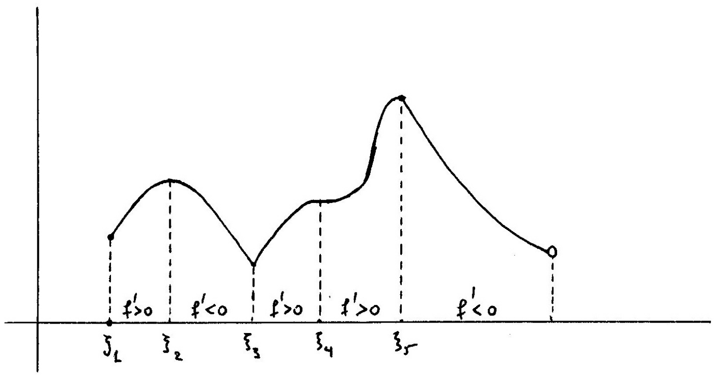
6.9 $y = f(x)$ (a, b), (c, d) (a, b) ξ (c, d).
(1) $f'(x) \geq 0$ x (c, ξ) $f'(x) \leq 0$ x (ξ, d), ξ $y = f(x)$.
(2) $f'(x) \leq 0$ x (c, ξ) $f'(x) \geq 0$ x (ξ, d), ξ $y = f(x)$.

: (1) $y = f(x)$ $(c, \xi]$ $[\xi, d)$, $f(\xi)$ (c, d).
(2) .

6.9.

$y = f(x)$
( ) $\xi_1, \xi_2, \ldots, \xi_n$
()

. (i) () -
, (ii) ξ_k -
-
(iii) ξ_k -
-
.



Σχήμα 6.9: .

$y = f(x)$ $\xi_1, \xi_2, \ldots, \xi_n$ · .

: (1) 6.8. $y = 2x^3 - 9x^2 + 12x + 5$ $[0, 4]$ $y = 6x^2 - 18x + 12 = 6(x-1)(x-2)$ $(0, 4)$. $(0, 1)$ $(2, 4)$ $(1, 2)$. $[0, 1]$ $[2, 4]$ $[1, 2]$, , 0 2 1 4 .

(2) $y = x^4(x-1)^4$ $(-\infty, +\infty)$ $\frac{d\,(x^4(x-1)^4)}{dx} = 4x^3(x-1)^4 + 4x^4(x-1)^3 = 8x^3(x-1)^3(x-\frac{1}{2})$. $(0, \frac{1}{2})$ $(1, +\infty)$ $(-\infty, 0)$ $(\frac{1}{2}, 1)$. $[0, \frac{1}{2}]$ $[1, +\infty)$ $(-\infty, 0]$ $[\frac{1}{2}, 1]$. , 0 1 $\frac{1}{2}$. 0 1 0 , $0 \le x^4(x-1)^4$ x, 0 1 . $\frac{1}{2}$, $\frac{1}{256}$, $\lim_{x \to \pm\infty} x^4(x-1)^4 = +\infty$.

(3) $y = x + \frac{1}{x}$ $(-\infty, 0)$ $(0, +\infty)$. $\frac{d}{dx}(x + \frac{1}{x}) = 1 - \frac{1}{x^2}$ $(-\infty, -1)$ $(1, +\infty)$ $(-1, 0)$ $(0, 1)$. $(-\infty, -1]$ $[1, +\infty)$ $[-1, 0)$ $(0, 1]$, , -1 1 . , -1 $(-\infty, 0)$ 1 $(0, +\infty)$.

(4) $y = |\sin x|$ $[0, 2\pi]$ $y = \sin x$ $(0, \pi)$ $y = -\sin x$ $(\pi, 2\pi)$. , $\frac{d\,\sin x}{dx} = \cos x$ $(0, \pi)$ $\frac{d\,(-\sin x)}{dx} = -\cos x$ $(\pi, 2\pi)$. $(0, \frac{\pi}{2})$ $(\pi, \frac{3\pi}{2})$ $(\frac{\pi}{2}, \pi)$ $(\frac{3\pi}{2}, 2\pi)$. $[0, \frac{\pi}{2}]$ $[\pi, \frac{3\pi}{2}]$ $[\frac{\pi}{2}, \pi]$ $[\frac{3\pi}{2}, 2\pi]$. , 0, π 2π $\frac{\pi}{2}$ $\frac{3\pi}{2}$. 0, , 1, .

, 0, π 2π. 0: $\frac{d\,|\sin x|}{dx}\Big|_{x=0} = \lim_{x\to 0+}\frac{|\sin x|-|\sin 0|}{x-0} = \lim_{x\to 0+}\frac{\sin x-\sin 0}{x-0} = \cos 0 = 1$. 2π: $\frac{d\,|\sin x|}{dx}\Big|_{x=2\pi} = \lim_{x\to 2\pi-}\frac{|\sin x|-|\sin(2\pi)|}{x-2\pi} = \lim_{x\to 2\pi-}\frac{-\sin x+\sin(2\pi)}{x-2\pi} = -\lim_{x\to 2\pi-}\frac{\sin x-\sin(2\pi)}{x-2\pi} = -\cos(2\pi) = -1$. , π . : $\frac{d\,|\sin x|}{dx}\Big|_{x=\pi+} = \lim_{x\to\pi+}\frac{|\sin x|-|\sin\pi|}{x-\pi} = \lim_{x\to\pi+}\frac{-\sin x+\sin\pi}{x-\pi} = -\lim_{x\to\pi+}\frac{\sin x-\sin\pi}{x-\pi} = -\cos\pi = 1$ $\frac{d\,|\sin x|}{dx}\Big|_{x=\pi-} = \lim_{x\to\pi-}\frac{|\sin x|-|\sin\pi|}{x-\pi} = \lim_{x\to\pi-}\frac{\sin x-\sin\pi}{x-\pi} = \cos\pi = -1$.

. , .

6.7 6.8.

: (1) $\frac{d\,\cos x}{dx} = -\sin x$ $\frac{d\,\sin x}{dx} = \cos x$ $(\cos x)^2 + (\sin x)^2 = 1$ x. $\frac{d\,((\cos x)^2+(\sin x)^2)}{dx} = -2\cos x\sin x + 2\sin x\cos x = 0$ x $(-\infty,+\infty)$. 6.7 $y = (\cos x)^2 + (\sin x)^2$ $(-\infty,+\infty)$, c $(\cos x)^2 + (\sin x)^2 = c$ x $(-\infty,+\infty)$. c, $x\cdot$: $c = (\cos 0)^2 + (\sin 0)^2 = 1$.

(2)

$$\boxed{e^x \geq x+1}$$

x ( ).
. $y = f(x) = e^x - x - 1$ $(-\infty,+\infty)$. $f(0) = 0$ $'$ $x > 0$. ξ $(0,x)$ $\frac{e^x-x-1}{x} = \frac{f(x)-f(0)}{x-0} = f'(\xi) = e^\xi - 1$. $\xi > 0$, $e^\xi - 1 > 0$, , $e^x - x - 1 > 0$.
, , $x < 0$, ξ $(x,0)$ $\frac{e^x-x-1}{x} = \frac{f(x)-f(0)}{x-0} = f'(\xi) = e^\xi - 1$. $\xi < 0$, $e^\xi - 1 < 0$, , $e^x - x - 1 > 0$.
, , $x \neq 0$ $e^x > x+1$ $x = 0$, , $e^0 = 0+1$.
. . $y = f(x) = e^x - x - 1$ $f'(x) = e^x - 1$ > 0 $x > 0$ < 0 $x < 0$. $[0,+\infty)$ $(-\infty,0]$. $f(x) > f(0) = 0$ $x \neq 0$.

(3) $\cos x \geq 1 - \frac{x^2}{2}$ x.
. $y = f(x) = \cos x - 1 + \frac{x^2}{2}$ $(-\infty,+\infty)$. $f(0) = 0$ $x \neq 0$. ξ $(0,x)$ $(x,0)$ $\frac{\cos x-1+\frac{x^2}{2}}{x} = \frac{f(x)-f(0)}{x-0} = f'(\xi) = \xi - \sin\xi$. $x > 0$, $\xi > 0$, , $\xi > \sin\xi$.
, $x < 0$, $\xi < 0$, , $\xi < \sin\xi$. $\cos x > 1 - \frac{x^2}{2}$.
. $y = f(x) = \cos x - 1 + \frac{x^2}{2}$ $f'(x) = x - \sin x$ > 0 $(0,+\infty)$ < 0 $(-\infty,0)$. $[0,+\infty)$ $(-\infty,0]$, , $f(x) > f(0) = 0$ $x \neq 0$.

.

. .

1. .

$$y = x^2 - x - 1,\quad y = x^3 - 15x^2 + 72x + 7,\quad y = \frac{x^2-2x+3}{x^2+2x+3},\quad y = \frac{\sqrt{x}}{x+4},$$

$$y = x^2e^{-x},\quad y = \sin x - \cos x,\quad y = \frac{\sin(3x)}{3} - \cos x,\quad y = x + \sin x,$$

$$y = x + |\sin x|, \quad y = \frac{\log x}{x}, \quad y = |x|e^{-|x-1|}, \quad y = \arctan x - \log(1+x^2)\,.$$

2. (*) $y = \left(1 + \frac{1}{x}\right)^x$ $(0, +\infty)$.

3. (**) $y = \begin{cases} x - x^2 \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

$\frac{dy}{dx}\big|_{x=0} = 1 > 0.$

$a > 0$, , $(-a, a)$.

4. .

(i) $y = (x-1)|x|$ $[-1, 3]$.

(ii) $y = |x^2 - 3x + 2|$ $[-3, 10]$.

(iii) $y = \frac{(\log x)^2}{x}$ $[1, 3]$.

(iv) $y = x + \frac{1}{x}$ $[\frac{1}{3}, 3]$.

(v) $y = e^x \sin x$ $[0, 2\pi]$.

5. $a > 0$ $y = a \log x + 2a - x$ $(0, +\infty)$.

6. $a_1 < a_2 < \cdots < a_{n-1} < a_n$ . .

$$y = (x - a_1)^2 + \cdots + (x - a_n)^2\,, \qquad y = |x - a_1| + \cdots + |x - a_n|.$$

7. $y = f(x)$ Hölder- ξ Hölder- > 1 (7 5.1). $f'(\xi) = 0$.

8. (*) $|f(x_2) - f(x_1)| \leq M|x_2 - x_1|^\rho$ x_1, x_2 I. $y = f(x)$ **Hölder-** I **Hölder-** ρ. $\rho = 1$, $y = f(x)$ **Lipschitz-** I.

, $\rho > 1$, $y = f(x)$ I.

(: .)

9. $y = f(x)$ (a, b), $a < \xi < b$ $f'(\xi) > 0$. $f(x) > f(\xi)$ ξ $f(x) < f(\xi)$ ξ .

(: $f'(\xi)$ 4.16.)

3.

ξ $y = f(x)$;

$f'(\xi) < 0$;

10. (*) $y = f(x)$ I $f'(x) \neq 0$ x I. 11 $y = f(x)$ -- I.

(1) $y = f(x)$ I.

(: : 8 5.5. : Fermat , x_1, x_3 I $x_1 < x_3$ $y = f(x)$ $[x_1, x_3]$ x_1, x_3 . x_1, x_2, x_3 I $x_1 < x_2 < x_3$ $f(x_1) < f(x_2) < f(x_3)$ $f(x_1) > f(x_2) > f(x_3)$. : (2) .)

(2) $f'(x) > 0$ x I $f'(x) < 0$ x I.

(: : (1) . : a, b I $a < b$ $f'(a) < 0 < f'(b)$ $f'(a) > 0 > f'(b)$, 13 14 .)

11. . l $M = (x_0, y_0)$. M l M l $d(M, l)$.

(i) l $x = \kappa$,

$$d(M, l) = |\kappa - x_0|.$$

(ii) l $y = \mu x + \nu$,

$$d(M, l) = \frac{|\mu x_0 + \nu - y_0|}{\sqrt{1 + \mu^2}}.$$

$ax + by = c$, $a, b \neq 0$,

$$d(M, l) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}.$$

12. l . ;

13. A_1 d_1 A_2 d_2 . () v_1 v_2 . A_1 A_2 .

14. h r.

;

;

15. $y = f(x)$ $y = g(x)$ $[0, b]$ $(0, b)$, $f(0) = g(0) = 0$ $f'(x) > 0$ $g'(x) > 0$ x $(0, b)$.

(i) $y = f'(x)$ $(0, b)$, $y = \frac{f(x)}{x}$ $(0, b]$.

(*) (ii) $y = \frac{f'(x)}{g'(x)}$ $(0, b)$, $y = \frac{f(x)}{g(x)}$ $(0, b]$.

$$y = \frac{x}{\sin x}, \quad y = \frac{\frac{1}{2}x^2}{1 - \cos x}, \quad y = \frac{\frac{1}{6}x^3}{x - \sin x}, \quad \cdots$$

$(0, \frac{\pi}{2})$.

. , .

1. $\frac{d\,(\arccos y + \arcsin y)}{dy} = 0$ y $(-1, 1)$.

2 1.4, $\arccos y + \arcsin y = \frac{\pi}{2}$ y $[-1, 1]$.

.

2. $\frac{d\,(\arctan x + \arctan \frac{1}{x})}{dx} = 0$ $x \neq 0$.

$\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$ $x > 0$ $\arctan x + \arctan \frac{1}{x} = -\frac{\pi}{2}$ $x < 0$.

6.7 $y = \arctan x + \arctan \frac{1}{x}$;

3. $y = f(x)$ $y = g(x)$ (a,b) $a < 0 < b$. $f'(x) = g(x)$, $g'(x) = -f(x)$ x (a,b) $f(0) = 0$ $g(0) = 1$.

;

$f(x)^2 + g(x)^2 = 1$ x (a,b).

$y = F(x)$ $y = G(x)$, $F(x) = f(x)$ $G(x) = g(x)$ x (a,b).

(: $y = (F(x) - f(x))^2 + (G(x) - g(x))^2$.)

4. $\frac{2}{\pi}x < \sin x < x$ x $(0, \frac{\pi}{2})$.

5. $\log \frac{1+x}{1-x} > 2x + \frac{2x^3}{3}$ x $(0,1)$.

6. $\log \frac{1+x}{1-x} < 2x + \frac{2x^3}{3} + \frac{x^4}{2}$ x $(0, \frac{1}{2}]$.

7. $e^{\frac{x}{x+1}} < 1 + x$ $x > -1$.

8. $x > \arctan x > x - \frac{x^3}{3}$ $x > 0$.

9. $y = f(x)$ $[a,b]$ (a,b).

(i) $f'(x) \geq \mu$ x (a,b), $f(a) + \mu(x-a) \leq f(x) \leq f(b) + \mu(x-b)$ x $[a,b]$.

(ii) $f'(x) \leq \mu$ x (a,b), $f(b) + \mu(x-b) \leq f(x) \leq f(a) + \mu(x-a)$ x $[a,b]$.

10. $0 < x < y < +\infty$. :

(i) $a > 1$, $ax^{a-1} < \frac{y^a - x^a}{y-x} < ay^{a-1}$.

(ii) $0 < a < 1$, $ay^{a-1} < \frac{y^a - x^a}{y-x} < ax^{a-1}$.

11. $x < y$ $a > 0$, $a \neq 1$. $a^x \log a < \frac{a^y - a^x}{y-x} < a^y \log a$.

12. $0 < x < y$. $\frac{1}{y} < \frac{1}{y-x} \log(\frac{y}{x}) < \frac{1}{x}$.

13. $0 \leq x < y$. $\frac{y-x}{1+y^2} < \arctan y - \arctan x < \frac{y-x}{1+x^2}$.

14.

$$e^x \geq 1 + \frac{x}{1!}\,, \quad e^x \geq 1 + \frac{x}{1!} + \frac{x^2}{2!}\,, \quad e^x \geq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$x \geq 0$.

;

, $x \leq 0$, , , .

15.

$$\sin x \leq \frac{x}{1!}\,, \quad \sin x \geq \frac{x}{1!} - \frac{x^3}{3!}\,, \quad \sin x \leq \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$x \geq 0$ $x \leq 0$.

, ,

$$\cos x \geq 1 - \frac{x^2}{2!}\,, \quad \cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\,, \quad \cos x \geq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

x.

;

16. (**) $a_1, \dots, a_n > 0$.

$y = \frac{a_1+\cdots+a_n+x}{(n+1)\sqrt[n+1]{a_1\cdots a_n x}}$ $(0,+\infty)$.

Cauchy.

$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}\,.$$

(: .)

: $\sqrt[n]{a_1 \cdots a_n} = \frac{a_1+\cdots+a_n}{n}$ $a_1 = \ldots = a_n$.

17. $a_1, \dots, a_n, b_1, \dots, b_n, \mu_1, \dots, \mu_n > 0$ $\mu_1 + \cdots + \mu_n = 1$. , $0 < t < 1$ $a, b > 0$.

Young.

$$a^{1-t}b^t \leq (1-t)a + tb$$

$a = b$.

(: $x = \frac{b}{a}$ $y = x^t - tx + t$ $(0,+\infty)$.)

(*) **Hölder.**

$$a_1{}^{1-t}b_1{}^t + \cdots + a_n{}^{1-t}b_n{}^t \leq \big(a_1 + \cdots + a_n\big)^{1-t}\big(b_1 + \cdots + b_n\big)^t$$

$\frac{a_k}{A} = \frac{b_k}{B}$ k, $A = a_1 + \cdots + a_n$ $B = b_1 + \cdots + b_n$.

(: Young $\frac{a_k}{A}$, $\frac{b_k}{B}$.)

(*) $y = (\mu_1 a_1{}^x + \cdots + \mu_n a_n{}^x)^{\frac{1}{x}}$ $(-\infty, 0) \cup (0, +\infty)$.

(: $0 < x < x'$, Hölder μ_k, $\mu_k a_k{}^{x'}$ $t = \frac{x}{x'}$. $x < x' < 0$ $x < 0 < x'$.)

(*) $\lim_{x\to 0}(\mu_1 a_1{}^x + \cdots + \mu_n a_n{}^x)^{\frac{1}{x}} = a_1{}^{\mu_1} \cdots a_n{}^{\mu_n}$.

(: $\log(\mu_1 a_1{}^x + \cdots + \mu_n a_n{}^x)^{\frac{1}{x}} = \frac{\log\left(\mu_1 a_1{}^x + \cdots + \mu_n a_n{}^x\right)}{\mu_1 a_1{}^x + \cdots + \mu_n a_n{}^x - 1}\,\frac{\mu_1 a_1{}^x + \cdots + \mu_n a_n{}^x - 1}{x}$.)

(*)

$$(\mu_1 a_1{}^x + \cdots + \mu_n a_n{}^x)^{\frac{1}{x}} \leq a_1{}^{\mu_1} \cdots a_n{}^{\mu_n} \leq (\mu_1 a_1{}^{x'} + \cdots + \mu_n a_n{}^{x'})^{\frac{1}{x'}}$$

x, x' $x < 0 < x'$. Cauchy .

6.10 .

$y = f(x)$ ξ $y = f'(x)$, , ξ, $\lim_{x\to\xi} \frac{f'(x)-f'(\xi)}{x-\xi}$. $y = f(x)$ ξ

$$f''(\xi) \quad D^2 f(\xi) \quad \frac{d^2 f(x)}{dx^2}\Big|_{x=\xi} \quad \frac{d^2 y}{dx^2}\Big|_{x=\xi} = \lim_{x\to\xi} \frac{f'(x)-f'(\xi)}{x-\xi}.$$

ξ $\pm\infty$ ξ. $y = f(x)$ $[\xi, b)$ $(a, \xi]$, , , ξ. , , , n- $(n-1)$- . $f^{(1)}$ $f^{(2)}$. f''' $f^{(3)}$, n- -

$$f^{(n)}(\xi) \quad D^n f(\xi) \quad \frac{d^n f(x)}{dx^n}\Big|_{x=\xi} \quad \frac{d^n y}{dx^n}\Big|_{x=\xi}.$$

, , n- $y = f(x)$ ξ

$$f^{(n)}(\xi) = \lim_{x\to\xi} \frac{f^{(n-1)}(x) - f^{(n-1)}(\xi)}{x-\xi}$$

$(n-1)$- $\pm\infty$ ξ.
, $y = f^{(0)}(x)$ $y = f(x)$.

: (1) n , $y = x^n$ $\frac{d\,x^n}{dx} = nx^{n-1}$, $\frac{d^2\,x^n}{dx^2} = n(n-1)x^{n-2}$, $\frac{d^3\,x^n}{dx^3} = n(n-1)(n-2)x^{n-3}$, … , $\frac{d^{n-1}\,x^n}{dx^{n-1}} = n(n-1)\cdots 2x$ $\frac{d^n\,x^n}{dx^n} = n(n-1)\cdots 2\cdot 1$. n- , 0, $\frac{d^m\,x^n}{dx^m} = 0$ $m > n$.

(2) a 0, $y = x^a$ $\frac{d\,x^a}{dx} = ax^{a-1}$, $\frac{d^2\,x^a}{dx^2} = a(a-1)x^{a-2}$, $\frac{d^3\,x^a}{dx^3} = a(a-1)(a-2)x^{a-3}$, , m $\frac{d^m\,x^a}{dx^m} = a(a-1)\cdots(a-m+1)x^{a-m}$. x 0 , , 0.

(3) $a > 0$, $a \neq 1$, $y = a^x$ $\frac{d\,a^x}{dx} = a^x \log a$, $\frac{d^2\,a^x}{dx^2} = a^x(\log a)^2$, , $\frac{d^m\,a^x}{dx^m} = a^x(\log a)^m$ m.
$y = e^x$ $\frac{d^m\,e^x}{dx^m} = e^x$ m.

(4) $y = \sin x$ $\frac{d\,\sin x}{dx} = \cos x$, $\frac{d^2\,\sin x}{dx^2} = -\sin x$, $\frac{d^3\,\sin x}{dx^3} = -\cos x$, $\frac{d^4\,\sin x}{dx^4} = \sin x$. «»: $\sin x$, $\cos x$, $-\sin x$, $-\cos x$. $\frac{d^{2k}\,\sin x}{dx^{2k}} = (-1)^k \sin x$ $\frac{d^{2k-1}\,\sin x}{dx^{2k-1}} = (-1)^{k-1}\cos x$ k.
, $y = \cos x$ $\frac{d\,\cos x}{dx} = -\sin x$, $\frac{d^2\,\cos x}{dx^2} = -\cos x$, $\frac{d^3\,\cos x}{dx^3} = \sin x$, $\frac{d^4\,\cos x}{dx^4} = \cos x$. «»: $\cos x$, $-\sin x$, $-\cos x$, $\sin x$. $\frac{d^{2k}\,\cos x}{dx^{2k}} = (-1)^k \cos x$ $\frac{d^{2k-1}\,\cos x}{dx^{2k-1}} = (-1)^k \sin x$ k.

.

. .

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6.10 . $y = f(x)$ (a, b), ξ (a, b) $y = f(x)$ ξ.
(1) $f'(\xi) = 0$ $f''(\xi) > 0$, ξ $y = f(x)$.
(2) $f'(\xi) = 0$ $f''(\xi) < 0$, ξ $y = f(x)$.

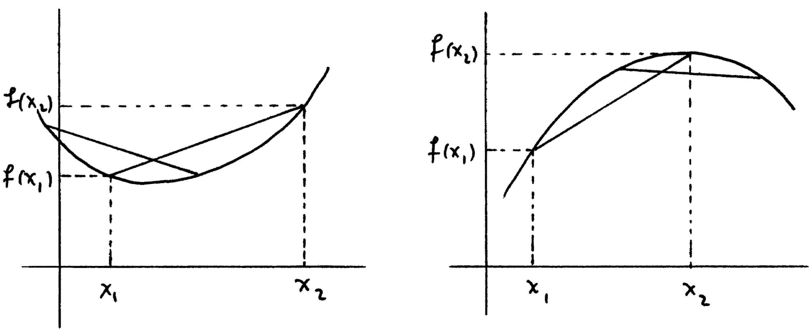
: (1) $f'(\xi)=0$ $f''(\xi)>0$. $f''(\xi)=\lim_{x\to\xi}\frac{f'(x)-f'(\xi)}{x-\xi}$, $(a,\xi)\cup(\xi,b)$ $\frac{f'(x)-f'(\xi)}{x-\xi}>0$ x . , $f'(x)>f'(\xi)=0$ x (ξ,b) $f'(x)<f'(\xi)=0$ x (a,ξ). $y=f(x)$ $[\xi,b)$ $(a,\xi]$, $(a,\xi]$ $[\xi,b)$, , ξ .
(2) .

: (1) $y=x^2$ $\frac{d\,x^2}{dx}=2x$ $\frac{d^2\,x^2}{dx^2}=\frac{d(2x)}{dx}=2$, $\frac{d\,x^2}{dx}\big|_{x=0}=0$ $\frac{d^2\,x^2}{dx^2}\big|_{x=0}=2>0$. 0 .

(2) . $y=x^4$ $\frac{d\,x^4}{dx}=4x^3$ $\frac{d^2\,x^4}{dx^2}=\frac{d(4x^3)}{dx}=12x^2$, $\frac{d\,x^4}{dx}\big|_{x=0}=0$ $\frac{d^2\,x^4}{dx^2}\big|_{x=0}=0$. , 0 .

. .

$y=f(x)$ I x_1 x_2 I $x_1<x_2$ $[x_1,x_2]$ $(x_1,f(x_1))$ $(x_2,f(x_2))$. , $y=f(x)$ I x_1 x_2 I $x_1<x_2$ $[x_1,x_2]$ $(x_1,f(x_1))$ $(x_2,f(x_2))$.



Σχήμα 6.10: .

. l $(x_1,f(x_1))$ $(x_2,f(x_2))$

$$y=\frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1)+f(x_1).$$

x $[x_1,x_2]$, $(x,f(x))$ $y=f(x)$ $(x_1,f(x_1))$ $(x_2,f(x_2))$ $(x,f(x))$ (x,y) l,

$$f(x)\leq\frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1)+f(x_1).$$

, , $y=f(x)$ I x_1 x_2 I $x_1<x_2$ x $[x_1,x_2]$.
, $y=f(x)$ I x_1 x_2 I $x_1<x_2$ x $[x_1,x_2]$:

$$f(x)\geq\frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1)+f(x_1).$$

. t $[0,1]$ $x=(1-t)x_1+tx_2$ x_1 x_2. , x x_1 x_2 t $[0,1]$ $x=(1-t)x_1+tx_2$. , t, $t=\frac{x-x_1}{x_2-x_1}$, $[0,1]$. , ,

$f\big((1-t)x_1+tx_2\big) \le \big(f(x_2)-f(x_1)\big)t+f(x_1) = (1-t)f(x_1)+tf(x_2)$.
$y=f(x)$ I x_1 x_2 $x_1<x_2$

$$f\big((1-t)x_1+tx_2\big) \le (1-t)f(x_1)+tf(x_2) \qquad (0\le t\le 1).$$

:

$$f\big((1-t)x_1+tx_2\big) \ge (1-t)f(x_1)+tf(x_2)$$

$f\big((1-t)x_1+tx_2\big) \le (1-t)f(x_1)+tf(x_2)$. , $x_1=x_2$, , , .

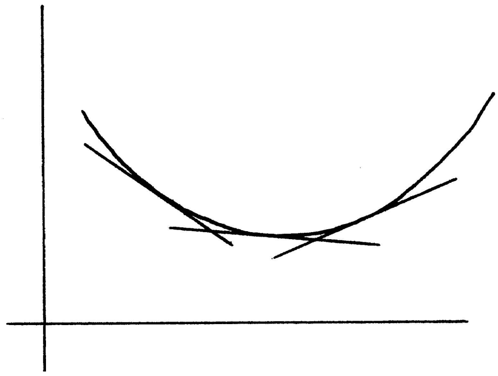
: (1) $y=\mu x+\nu$ $(-\infty,+\infty)$. : $\mu((1-t)x_1+tx_2)+\nu = (1-t)(\mu x_1+\nu)+t(\mu x_2+\nu)$.

(2) $y=x^2$ $(-\infty,+\infty)$. , $2x_1x_2 \le x_1^2+x_2^2$, : $((1-t)x_1+tx_2)^2 = (1-t)^2x_1^2+2t(1-t)x_1x_2+t^2x_2^2 \le (1-t)^2x_1^2+t(1-t)(x_1^2+x_2^2)+t^2x_2^2 = (1-t)x_1^2+tx_2^2$.

(3) $y=|x|$ $(-\infty,+\infty)$. : $|(1-t)x_1+tx_2| \le |(1-t)x_1|+|tx_2| = (1-t)|x_1|+t|x_2|$.

.

6.11 $y=f(x)$ I I.
(1) $y=f(x)$ I .
(2) $y=f(x)$ I .



Σχήμα 6.11: : .

: (1) $y=f(x)$ I. x_1 x_2 I $x_1<x_2$ $f'(x_1)\le f'(x_2)$. x x_1 x_2 $f(x) \le \frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1)+f(x_1)$. $\frac{f(x)-f(x_1)}{x-x_1} \le \frac{f(x_2)-f(x_1)}{x_2-x_1}$, , $x\to x_1+$, $f'(x_1) \le \frac{f(x_2)-f(x_1)}{x_2-x_1}$. , $\frac{f(x_2)-f(x_1)}{x_2-x_1} \le \frac{f(x)-f(x_2)}{x-x_2}$, , $x\to x_2-$, $\frac{f(x_2)-f(x_1)}{x_2-x_1} \le f'(x_2)$. , $f'(x_1)\le f'(x_2)$.

, . x_1 x_2 I ( ) $x_1<x_2$ $x\in(x_1,x_2)$. $[x_1,x]$ (x_1,x), $\xi_1\in(x_1,x)$ $\frac{f(x)-f(x_1)}{x-x_1} = f'(\xi_1)$. $\xi_2\in(x,x_2)$ $\frac{f(x_2)-f(x)}{x_2-x} = f'(\xi_2)$. , $f'(\xi_1)\le f'(\xi_2)$, $\frac{f(x)-f(x_1)}{x-x_1} \le \frac{f(x_2)-f(x)}{x_2-x}$. $f(x) \le \frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1)+f(x_1)$ x (x_1,x_2) $x=x_1$ $x=x_2$, I.

(2) (1).

: $y = \begin{cases} 2x^2, & x \le 0, \\ x^2, & 0 \le x, \end{cases}$ $\frac{dy}{dx} = \begin{cases} 4x, & x \le 0, \\ 2x, & 0 \le x. \end{cases}$ $(-\infty,+\infty)$, , $(-\infty,+\infty)$.
, 6.12, 0.

6.11 . $y=f(x)$ l_x $(x, f(x))$. l_x $f'(x)$. 6.11 $y=f(x)$, x , l_x . , $y=f(x)$, x , l_x .
, .

6.12 $y=f(x)$ I I.
(1) $y=f(x)$ I $f''(x) \ge 0$ x I.
(2) $y=f(x)$ I $f''(x) \le 0$ x I.

: (1) $y = x(x-1)(x-2)$ $\frac{dy}{dx} = 3x^2-6x+2$ $\frac{d^2y}{dx^2} = 6x-6$. $(-\infty,1)$ $\frac{d^2y}{dx^2} \le 0$, , $(-\infty,1]$. $(1,+\infty)$ $\frac{d^2y}{dx^2} \ge 0$, $[1,+\infty)$.

(2) n , $y=x^n$ $(-\infty,+\infty)$. n , $y=x^n$ $(-\infty,0]$ $[0,+\infty)$.
, n , $\frac{d^2\,x^n}{dx^2} = n(n-1)x^{n-2} \ge 0$ x , n , $\frac{d^2\,x^n}{dx^2} = n(n-1)x^{n-2} \ge 0$ $x>0$ $\frac{d^2\,x^n}{dx^2} = n(n-1)x^{n-2} \le 0$ $x<0$.

(3) $y=x^a$ $(0,+\infty)$, $a \le 0$ $a \ge 1$, $(0,+\infty)$, $0 \le a \le 1$. $\frac{d^2\,x^a}{dx^2} = a(a-1)x^{a-2}$ $a(a-1)$.

(4) $y=a^x$ $(-\infty,+\infty)$ $a>0$ $\frac{d^2\,a^x}{dx^2} = a^x(\log a)^2 \ge 0$ x.

(5) $y=\log_a x$ $(0,+\infty)$, $a>1$, $(0,+\infty)$, $0<a<1$. $\frac{d^2\,\log_a x}{dx^2} = -\frac{1}{x^2}\,\frac{1}{\log a}$ $-\frac{1}{\log a}$.

. .

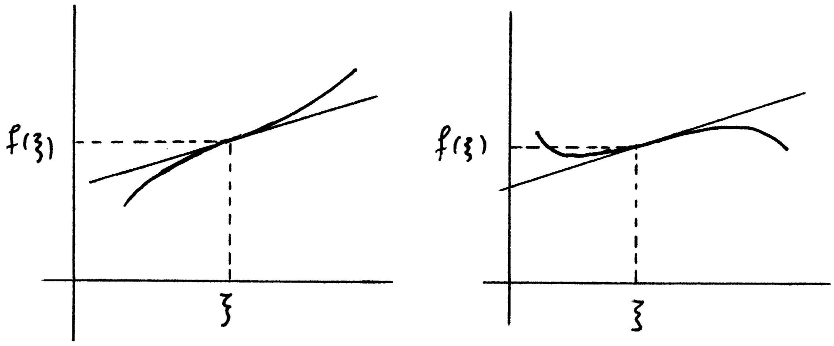
$y=f(x)$ (a,b) ξ (a,b). $y=f(x)$ ξ. $y=f(x)$ ξ, $(\xi, f(\xi))$ $y = f'(\xi)(x-\xi)+f(\xi)$, $f(x) \ge f'(\xi)(x-\xi)+f(\xi)$ x $(c,\xi]$ $f(x) \le f'(\xi)(x-\xi)+f(\xi)$ x $[\xi,d)$, , $f(x) \le f'(\xi)(x-\xi)+f(\xi)$ x $(c,\xi]$ $f(x) \ge f'(\xi)(x-\xi)+f(\xi)$ x $[\xi,d)$, ξ $y=f(x)$. , $f'(\xi)$ $+\infty$ $-\infty$ ξ $y=f(x)$.
ξ $y=f(x)$ $(\xi,f(\xi))$ $(\xi,f(\xi))$ $(\xi,f(\xi))$.
6.13 ξ $y=f(x)$ ξ, $f'(\xi)$.

6.13 $y=f(x)$ (a,b), ξ (a,b) $y=f(x)$ ξ. $y=f(x)$ $(c,\xi]$ $[\xi,d)$, , $(c,\xi]$ $[\xi,d)$, ξ .

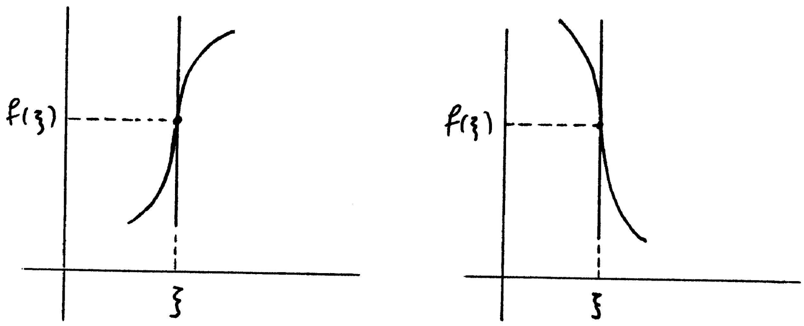
: , , $y=f(x)$ $(c,\xi]$ $[\xi,d)$. 6.11 , $(c,\xi]$, $\frac{f(x)-f(\xi)}{x-\xi} \le f'(\xi)$ x $(c,\xi]$, $[\xi,d)$, $\frac{f(x)-f(\xi)}{x-\xi} \le f'(\xi)$ x $[\xi,d)$. $f(x) \ge f'(\xi)(x-\xi)+f(\xi)$ x $(c,\xi]$ $f(x) \le f'(\xi)(x-\xi)+f(\xi)$ x $[\xi,d)$.

, , . :

6.14 $y=f(x)$ (a,b), ξ (a,b) $y=f(x)$ ξ. $f''(x) \ge 0$ x (c,ξ) $f''(x) \le 0$ x (ξ,d) , , $f''(x) \le 0$ x (c,ξ) $f''(x) \ge 0$ x (ξ,d), ξ .



Σχήμα 6.12: : $f'(\xi)$.



Σχήμα 6.13: : $f'(\xi) = +\infty$ $-\infty$.

: $y = x^3$ $\frac{dy}{dx} = 3x^2$ $\frac{d^2y}{dx^2} = 6x$. $\frac{d^2y}{dx^2} \leq 0$ $(-\infty, 0)$ $\frac{d^2y}{dx^2} \geq 0$ $(0, +\infty)$, 0 .

. .

l $y = f(x)$ $(\xi, f(\xi))$ l l. , l $y = \mu x + \nu$, $f(\xi) = \mu\xi + \nu$ $f(x) \geq \mu x + \nu$ x $y = f(x)$.

, l $y = f(x)$ $(\xi, f(\xi))$ l l. , l $y = \mu x + \nu$, $f(\xi) = \mu\xi + \nu$ $f(x) \leq \mu x + \nu$ x $y = f(x)$.

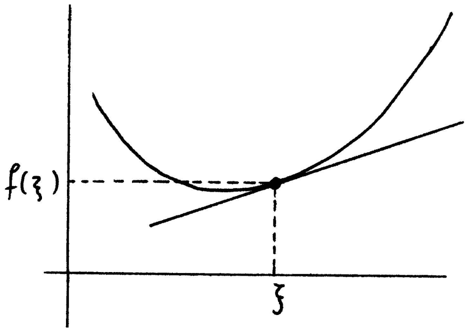
$f(\xi) = \mu\xi + \nu$ l $y = \mu x + f(\xi) - \mu\xi$, , $y = \mu(x - \xi) + f(\xi)$. , , l $y = f(x)$ $(\xi, f(\xi))$ $y = \mu(x - \xi) + f(\xi)$

$$f(x) \geq \mu(x - \xi) + f(\xi)$$

x $y = f(x)$. , , :

$$f(x) \leq \mu(x - \xi) + f(\xi)$$

$f(x) \geq \mu(x - \xi) + f(\xi)$. , μ.



Σχήμα 6.14: .

: (1) $y=|x|$ $(0,0)$ $y=\mu x$. $|x|\geq \mu x$ x $(-\infty,+\infty)$. $x=1$ $\mu\leq 1$ $x=-1$ $-1\leq\mu$, , $-1\leq\mu\leq 1$. , $-1\leq\mu\leq 1$, $\mu x\leq|\mu x|=|\mu||x|\leq|x|$ x. $y=|x|$ $(0,0)$ $y=\mu x$ $(-1\leq\mu\leq 1)$.

(2) $y=(x^2-1)^2$ $(0,1)$ $y=\mu x+1$. $(x^2-1)^2\geq\mu x+1$ x $(-\infty,+\infty)$. , $x=1$ $0\geq\mu+1$ $x=-1$ $0\geq-\mu+1$ . .

6.15 .

6.15 $y=f(x)$ I, ξ I, $y=f(x)$ ξ l $y=f(x)$ $(\xi,f(\xi))$. $y=f(x)$ $(\xi,f(\xi))$, l.

: $y=\mu(x-\xi)+f(\xi)$, $f(x)\geq\mu(x-\xi)+f(\xi)$ x I. μ. $x>\xi$ I $\frac{f(x)-f(\xi)}{x-\xi}\geq\mu$, $x\to\xi+$, $f'(\xi)\geq\mu$. , $x<\xi$ I $\frac{f(x)-f(\xi)}{x-\xi}\leq\mu$, $x\to\xi-$, $f'(\xi)\leq\mu$. $\mu=f'(\xi)$, $y=f'(\xi)(x-\xi)+f(\xi)$ l.

.

: (1) $y=x^2$ $(3,9)$. , , $y=6(x-3)+9$. $x^2\geq 6(x-3)+9$ x. $(x-3)^2\geq 0$ x , , .

(2) , $y=x^3$ $(3,27)$. , $y=27(x-3)+27$ $x^3\geq 27(x-3)+27$ x. $x^3-27x+54\geq 0$ x . $\lim_{x\to-\infty}(x^3-27x+54)=-\infty$. : $(-10)^3-27(-10)+54=-676<0$. $y=x^3$ $(3,27)$.

6.16 $y=f(x)$ I, ξ I, $y=f(x)$ ξ l $y=f(x)$ $(\xi,f(\xi))$.
(1) $y=f(x)$ I, l $(\xi,f(\xi))$.
(2) $y=f(x)$ I, l $(\xi,f(\xi))$.

: (1) l $y=f'(\xi)(x-\xi)+f(\xi)$. x I $x>\xi$ $f'(\xi)\leq\frac{f(x)-f(\xi)}{x-\xi}$, , $f(x)-f(\xi)\geq f'(\xi)(x-\xi)$, $f(x)\geq f'(\xi)(x-\xi)+f(\xi)$. , x I $x<\xi$ $f'(\xi)\geq\frac{f(x)-f(\xi)}{x-\xi}$, , $f(x)-f(\xi)\geq f'(\xi)(x-\xi)$, $f(x)\geq f'(\xi)(x-\xi)+f(\xi)$. , , $f(x)\geq f'(\xi)(x-\xi)+f(\xi)$ x I , , l . l 6.15.
(2) (1).

: $y=e^x$ $(-\infty,+\infty)$. , $y=e^x$ $(0,e^0)=(0,1)$, $y=x+1$.

$e^x \geq x+1$ x $(-\infty,+\infty)$.

. .

. , , ( ) .

: (1) $e^{\frac{x_1+x_2}{2}} \leq \frac{e^{x_1}+e^{x_2}}{2}$ x_1, x_2 .
$y = e^x$. , $y = e^x$ $(-\infty,+\infty)$, $e^{(1-t)x_1+tx_2} \leq (1-t)e^{x_1} + te^{x_2}$ x_1 x_2 , $x_1 \neq x_2$, t $[0,1]$. $t = \frac{1}{2}$, $x_1 \neq x_2$. $x_1 = x_2$, , .

(2) $(x_1+x_2)\log\frac{x_1+x_2}{2} \leq x_1\log x_1 + x_2\log x_2$ $x_1, x_2 > 0$.
, $y = x\log x$ $(0,+\infty)$, $\frac{d^2\,(x\log x)}{dx^2} = \frac{1}{x} \geq 0$ x $(0,+\infty)$. , , $t = \frac{1}{2}$, $\frac{x_1+x_2}{2}\log\frac{x_1+x_2}{2} \leq \frac{x_1\log x_1 + x_2\log x_2}{2}$ x_1, x_2 $(0,+\infty)$, $x_1 \neq x_2$. $x_1 \neq x_2$, $x_1 = x_2$, , .

(3) $x^{\frac{3}{4}} \leq \frac{3}{4}(x-1)+1$ $x \geq 0$.
$-$; $y = x^{\frac{3}{4}}$ $[0,+\infty)$. $y = x^{\frac{3}{4}}$ $(1,1)$ «» .

.

. .

1. $y = \begin{cases} x^2\,, & x \geq 0, \\ -x^2\,, & x \leq 0, \end{cases}$ $(-\infty,+\infty)$, $(-\infty,0)\cup(0,+\infty)$ 0.
   , k $y = \begin{cases} x^k\,, & x \geq 0, \\ -x^k\,, & x \leq 0, \end{cases}$ ( ) $\frac{d^n y}{dx^n}$ n.
2. $y = g(x)$ $(a,\xi]$ $y = h(x)$ $[\xi,b)$. $g(\xi) = h(\xi)$, $g'_-(\xi) = h'_+(\xi)$ $g''_-(\xi) = h''_+(\xi)$, $y = f(x) = \begin{cases} g(x)\,, & a < x \leq \xi, \\ h(x)\,, & \xi \leq x < b, \end{cases}$ ξ $f''(\xi) = g''_-(\xi) = h''_+(\xi)$.
3. $y = p(x)$ $n \geq 1$, $y = p'(x)$ $n-1$. . (: !)
   $N \geq 0$. $f^{(N+1)}(x) = 0$ x ( ) I $y = f(x)$ $\leq N$.
4. (*) $y = f(x)$ (a,b) $f(x)f''(x) \geq 0$ x (a,b). (a,b) $f(x)f'(x) = 0$, $y = f(x)$.
   (: $\big(f(x)f'(x)\big)'$.)
5. $y = p(x) = a_0 + a_1x + \cdots + a_Nx^N$.

$$p^{(n)}(0) = n!a_n, \qquad a_n = \frac{p^{(n)}(0)}{n!}$$

   $n = 0,1,\ldots,N$.
   , $p^{(n)}(0) = 0$ $n \geq N+1$.
   $y_0, y_1, \ldots, y_N$. $y = p(x)$ $\leq N$ $p^{(n)}(0) = y_n$ $n = 0,1,\ldots,N$. ;

6. $p(x)$ $k \geq 2$.

, $p(x)$ $(x-\xi)^k$, , $p'(x)$, $(x-\xi)^{k-1}$.

, , $p'(x)$ $(x-\xi)^{k-1}$ $p(\xi)=0$, $p(x)$ $(x-\xi)^k$.

(: $p(x)=q(x)(x-\xi)^k+r(x)$ $q(x), r(x)$, $r(x)$ k.)

7. $y=(x^2-1)^n$. n $\frac{d^n y}{dx^n}=0$ n $(-1,1)$.

(: .)

8. **Leibniz**:

$$(fg)^{(n)}(x)=\sum_{k=0}^{n}\binom{n}{k} f^{(k)}(x)g^{(n-k)}(x).$$

(: Newton.)

9. $y=f(x)=e^{-\frac{1}{x}}$ $(0,+\infty)$.

n $f^{(n)}(x)=x^{-2n}p_n(x)e^{-\frac{1}{x}}$ x $(0,+\infty)$, $p_n(x)$ $n-1$. : $p_1(x)=1$, $p_2(x)=1-2x$, $p_3(x)=1-6x+6x^2$.

$p_{n+1}(x)=x^2{p_n}'(x)+(1-2nx)p_n(x)$ $(0,+\infty)$.

(: $f^{(n)}(x)=x^{-2n}p_n(x)e^{-\frac{1}{x}}$.)

$p_{n+2}(x)=(1-2(n+1)x)p_{n+1}(x)-n(n+1)x^2p_n(x)$ $(0,+\infty)$.

(: $x^2f'(x)=f(x)$ $(0,+\infty)$ n Leibniz .)

x^{n-1} $p_n(x)$ $(-1)^{n-1}n!$.

$x^2{p_n}''(x)-(2nx-2x-1){p_n}'(x)+n(n-1)p_n(x)=0$ $(0,+\infty)$.

10. $y=f(x)=e^{\frac{x^2}{2}}$ $(-\infty,+\infty)$.

n $f^{(n)}(x)=p_n(x)e^{\frac{x^2}{2}}$ x $(-\infty,+\infty)$, $p_n(x)$ n. : $p_1(x)=x$, $p_2(x)=1+x^2$, $p_3(x)=3x+x^3$.

$p_{n+1}(x)={p_n}'(x)+xp_n(x)$ $(-\infty,+\infty)$.

$p_{n+1}(x)=xp_n(x)+np_{n-1}(x)$ $(-\infty,+\infty)$.

(: $f'(x)=xf(x)$ $(-\infty,+\infty)$ n Leibniz .)

x^n $p_n(x)$ 1.

${p_n}''(x)+x{p_n}'(x)-np_n(x)=0$ $(-\infty,+\infty)$.

11. (x,y), (x',y') (x'',y'') xy- . R

$$\frac{\sqrt{(x'-x)^2+(y'-y)^2}\sqrt{(x''-x)^2+(y''-y)^2}\sqrt{(x'-x'')^2+(y'-y'')^2}}{2\big|(x'-x)(y''-y)-(x''-x)(y'-y)\big|}.$$

$x=x(t)$ $y=y(t)$ $(a<t<b)$ xy-. t (a,b) $h>0$ $(x(t),y(t))$, $(x(t+h),y(t+h))$ $(x(t-h),y(t-h))$. $R_{t,h}$ R_t $\lim_{h\to 0+}R_{t,h}$, R_t $(x(t),y(t))$.

$$R_t = \frac{\big((x'(t))^2 + (y'(t))^2\big)^{\frac{3}{2}}}{\big|x''(t)y'(t) - x'(t)y''(t)\big|}\,.$$

$x = x(t) = x_0 + r_0 \cos t$ $y = y(t) = y_0 + r_0 \sin t$. ; ;

$x = x(t) = x_0 + \kappa_0 \cos t$ $y = y(t) = y_0 + \mu_0 \sin t$. ; ;

$y = f(x)$ $(x, f(x))$, x , $\frac{(1+(f'(x))^2)^{\frac{3}{2}}}{|f''(x)|}$.

$y = x^2$ $y = \frac{1}{x}$.

; « »;

. .

1. .

$$y = x^3 - 4x^2 + x + 3, \qquad y = xe^x\,, \qquad y = x\log x.$$

. .

1. .

$$y = x^3-3x^2+6x, \quad y = x^2(x-1)^2\,, \quad y = \frac{x}{x+1}\,, \quad y = \frac{1}{\log x}\,, \quad y = \sin x.$$

2. $y = f(x)$ $y = -f(x)$.
$y = f(x)$ ≤ 1 .

3. (*) $y = f(x)$ I x_1, x_0, x_2 I $x_1 < x_0 < x_2$. $(x_0, f(x_0))$ $(x_1, f(x_1))$ $(x_2, f(x_2))$, , x $x_1 < x < x_2$ $(x, f(x))$, , $[x_1, x_2]$ $(x_1, f(x_1))$ $(x_2, f(x_2))$.
(: $x_1 < x < x_0$. x_1, x_0, x_2, x_1, x, x_0 x, x_0, x_2.)

4. (*) $y = f(x)$ $(-\infty, +\infty)$, $(-\infty, +\infty)$.
(: $f(x) \leq u$ x. x_1, x_2 $x_1 < x_2$. $x > x_2$ x_1, x_2, x $f(x) \leq u$, $x \to +\infty$ $f(x_2) \leq f(x_1)$. , $x < x_1$ $f(x_2) \geq f(x_1)$.)

5. (*) $y = f(x)$ I, I.
(: ξ I. x_1, x_2 I $x_1 < \xi < x_2$. $\xi < x < x_2$, x_1, ξ, x ξ, x, x_2 $\lim_{x\to\xi+} f(x)$.)

. .

1. .

$$y = x^3-3x^2+6x, \quad y = x^2(x-1)^2\,, \quad y = \frac{x}{x+1}\,, \quad y = \frac{1}{\log x}\,, \quad y = \sin x.$$

2. 0 $y = \begin{cases} x^2\left(\frac{|x|}{x} + \sin\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$

6.13 6.14;

3. $y = f(x)$ (a,b) ξ (a,b). $f'(x) \geq f'(\xi)$ x (a,b) $f'(x) \leq f'(\xi)$ x (a,b), ξ $y = f(x)$.

. .

1. − − .

$$y = x,\ y = |x|,\ y = x^2\,,\ y = x^3\,,\ y = e^{-2x}\,,\ y = \frac{1}{x^2+1}\,,\ y = x\log x.$$

2. (*) $y = f(x)$ I.

(1) x I $y = f(x)$ $(x, f(x))$, I.

(: x_1, x, x_2 I $x_1 < x < x_2$ $f(x_1) \geq \mu(x_1 - x) + f(x)$ $f(x_2) \geq \mu(x_2 - x) + f(x)$.)

(2) (1). $y = f(x)$ I, x I $y = f(x)$ $(x, f(x))$.

(: x_0 I. $\frac{f(x)-f(x_0)}{x-x_0}$ x $I \setminus \{x_0\}$. $\lim_{x\to x_0\pm} \frac{f(x)-f(x_0)}{x-x_0}$, $\lim_{x\to x_0-} \frac{f(x)-f(x_0)}{x-x_0} \leq \lim_{x\to x_0+} \frac{f(x)-f(x_0)}{x-x_0}$. μ $y = \mu(x-x_0)+f(x_0)$ $y = f(x)$ $(x_0, f(x_0))$.)

. .

1. $a \geq 1$ $a \leq 0$.

$$\big((1-t)x_1 + tx_2\big)^a \leq (1-t){x_1}^a + t{x_2}^a \qquad (x_1, x_2 > 0, 0 \leq t \leq 1),$$

$$x^a \geq a\xi^{a-1}(x - \xi) + \xi^a \qquad (x, \xi > 0).$$

$0 \leq a \leq 1$.

2. $a > 0$.

$$a^{(1-t)x_1 + tx_2} \leq (1-t)a^{x_1} + ta^{x_2} \qquad (0 \leq t \leq 1),$$

$$a^x \geq a^\xi \log a\,(x - \xi) + a^\xi\,.$$

3.

$$\log\big((1-t)x_1 + tx_2\big) \geq (1-t)\log x_1 + t\log x_2 \qquad (x_1, x_2 > 0, 0 \leq t \leq 1),$$

$$\log x \leq \frac{1}{\xi}(x - \xi) + \log\xi \qquad (x, \xi > 0).$$

4. (*) Young 17 6.9, $a^{1-t}b^t \leq (1-t)a + tb$ $(a, b > 0, 0 < t < 1)$: (i) $y = \log x$ $(0, +\infty)$ (ii) x^t $(0, +\infty)$ $0 < t < 1$.

(: (i) $a^{1-t}b^t \leq (1-t)a + tb$ $(1-t)\log a + t\log b \leq \log\big((1-t)a + tb\big)$.
(ii) $y = x^t$ $(1,1)$ $x = \frac{b}{a}$.)

5. (*) $y=f(x)$ I. $x_1,\dots,x_n$ I $\mu_1,\dots,\mu_n>0$ $\mu_1+\dots+\mu_n=1$.

$$f\big(\mu_1x_1+\dots+\mu_nx_n\big)\le\mu_1f(x_1)+\dots+\mu_nf(x_n).$$

(: n. n $n+1$ $t=\mu_{n+1}$, $x_1{}'=\frac{\mu_1}{1-\mu_{n+1}}x_1+\dots+\frac{\mu_n}{1-\mu_{n+1}}x_n$ $x_2{}'=x_{n+1}$.)

Hölder 17 6.9 $y=x^t$ $(0,+\infty)$ $0<t<1$.

(: $x_1=\frac{b_1}{a_1}$, $\dots$, $x_n=\frac{b_n}{a_n}$ $\mu_1=\frac{a_1}{A}$, $\dots$, $\mu_n=\frac{a_n}{A}$, $A=a_1+\dots+a_n$.)

$a_1{}^{\mu_1}\cdots a_n{}^{\mu_n}\le(\mu_1a_1{}^x+\dots+\mu_na_n{}^x)^{\frac{1}{x}}$ 17 6.9 ( ) $y=-\log x$ $(0,+\infty)$.

(: $x_1=a_1{}^x$, $\dots$, $x_n=a_n{}^x$.)

6.11 .

$\frac{0}{0}$ $\frac{\pm\infty}{\pm\infty}$. l' Hopitâl.

. .

l' Hopitâl $\frac{0}{0}$.

6.17 ***l' Hopitâl***. $y=f(x)$ $y=g(x)$ (ξ,b) $g(x)\ne0$ $g'(x)\ne0$ x (ξ,b). , , $\lim_{x\to\xi+}f(x)=\lim_{x\to\xi+}g(x)=0$. $\lim_{x\to\xi+}\frac{f'(x)}{g'(x)}$, $\lim_{x\to\xi+}\frac{f(x)}{g(x)}$.

$$\lim_{x\to\xi+}\frac{f(x)}{g(x)}=\lim_{x\to\xi+}\frac{f'(x)}{g'(x)}.$$

( ) *:* $x\to\xi-$, $x\to\xi$, $x\to+\infty$ $x\to-\infty$.

: $y=f(x)$ $y=g(x)$ $'$ ξ, ξ $f(\xi)=0$ $g(\xi)=0$. $\lim_{x\to\xi+}f(x)=\lim_{x\to\xi+}g(x)=0$, $[\xi,b)$.

$\lim_{x\to\xi+}\frac{f'(x)}{g'(x)}=\eta$. $\epsilon>0$, $\delta>0$ $\left|\frac{f'(x)}{g'(x)}-\eta\right|<\epsilon$ x (ξ,b) $\xi<x<\xi+\delta$. 6.4 x (ξ,b) ζ (ξ,x) $\frac{f(x)}{g(x)}=\frac{f(x)-f(\xi)}{g(x)-g(\xi)}=\frac{f'(\zeta)}{g'(\zeta)}$. , x (ξ,b) $\xi<x<\xi+\delta$ ζ (ξ,b) $\xi<\zeta<\xi+\delta$, $\left|\frac{f'(\zeta)}{g'(\zeta)}-\eta\right|<\epsilon$, , $\left|\frac{f(x)}{g(x)}-\eta\right|<\epsilon$. , , $\left|\frac{f(x)}{g(x)}-\eta\right|<\epsilon$ x (ξ,b) $\xi<x<\xi+\delta$. $\lim_{x\to\xi+}\frac{f(x)}{g(x)}=\eta$.

: $\lim_{x\to\xi+}\frac{f'(x)}{g'(x)}=\pm\infty$. , $x\to\xi-$ $x\to\xi$.

$x\to+\infty$ $x\to0+$.

$y=f(x)$ $y=g(x)$ $(a,+\infty)$ $a>0$, $g(x)\ne0$ $g'(x)\ne0$ x $(a,+\infty)$ $\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}g(x)=0$. $\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}$ $\lim_{x\to+\infty}\frac{f(x)}{g(x)}$. $t=\frac{1}{x}$, $F(t)=f(\frac{1}{t})=f(x)$ $G(t)=g(\frac{1}{t})=g(x)$ $(0,\frac{1}{a})$ $G(t)=g(x)\ne0$ $G'(t)=-\frac{1}{t^2}g'(\frac{1}{t})=-x^2g'(x)\ne0$ t $(0,\frac{1}{a})$. , $\lim_{t\to0+}\frac{F'(t)}{G'(t)}=\lim_{x\to+\infty}\frac{-x^2f'(x)}{-x^2g'(x)}=\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}$, , $\lim_{t\to0+}\frac{F'(t)}{G'(t)}$. $\lim_{t\to0+}\frac{F(t)}{G(t)}$. $\lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{t\to0+}\frac{F(t)}{G(t)}$, $\lim_{x\to+\infty}\frac{f(x)}{g(x)}$ $\lim_{x\to+\infty}\frac{f'(x)}{g'(x)}$.

$x\to-\infty$ $x\to0-$.

: $\lim_{x\to0}\frac{e^x-1}{\sin x}$. $(-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2})$ $\sin x\ne0$ $\frac{d\sin x}{dx}=\cos x\ne0$. , $\lim_{x\to0}(e^x-1)=\lim_{x\to0}\sin x=0$. : $\lim_{x\to0}\frac{e^x}{\cos x}=\frac{1}{1}=1$. , $\lim_{x\to0}\frac{e^x-1}{\sin x}=1$.

l' Hopitâl $\frac{\pm\infty}{\pm\infty}$, , .

6.18 ***l' Hopitâl***. $y = f(x)$, $y = g(x)$ (ξ, b) $g(x) \neq 0$ $g'(x) \neq 0$ x (ξ, b). , , $\lim_{x\to\xi+} g(x) = +\infty$ $-\infty$. $\lim_{x\to\xi+} \frac{f'(x)}{g'(x)}$, $\lim_{x\to\xi+} \frac{f(x)}{g(x)}$.

$$\lim_{x\to\xi+} \frac{f(x)}{g(x)} = \lim_{x\to\xi+} \frac{f'(x)}{g'(x)}.$$

( ) *:* $x \to \xi-$, $x \to \xi$, $x \to +\infty$ $x \to -\infty$.

: $\lim_{x\to\xi+} \frac{f'(x)}{g'(x)} = \eta$. $\epsilon > 0$, $\delta' > 0$ $\left|\frac{f'(x)}{g'(x)} - \eta\right| < \frac{\epsilon}{6}$ x (ξ, b) $\xi < x < \xi + \delta'$. x_0 (ξ, b) $\xi < x_0 < \xi + \delta'$. , $\lim_{x\to\xi+} |g(x)| = +\infty$, $\delta'' > 0$ $|g(x)| > \max\left\{|g(x_0)|, \frac{3}{\epsilon}|f(x_0)|, \frac{3|\eta|}{\epsilon}|g(x_0)|\right\}$ x (ξ, b) $\xi < x < \xi + \delta''$. $\delta = \min\{x_0 - \xi, \delta''\}$. x (ξ, b) $\xi < x < \xi + \delta$ $\xi < x < x_0 < \xi + \delta'$ $\xi < x < \xi + \delta''$. 6.4 x ζ (x, x_0) $\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(\zeta)}{g'(\zeta)}$. ζ (ξ, b) $\xi < \zeta < \xi + \delta'$, $\left|\frac{f(x)-f(x_0)}{g(x)-g(x_0)} - \eta\right| = \left|\frac{f'(\zeta)}{g'(\zeta)} - \eta\right| < \frac{\epsilon}{6}$. $|f(x) - f(x_0) - \eta(g(x) - g(x_0))| < \frac{\epsilon}{6}|g(x) - g(x_0)|$, $|f(x) - \eta g(x)| < \frac{\epsilon}{6}(|g(x)| + |g(x_0)|) + |f(x_0)| + |\eta||g(x_0)|$, , $\left|\frac{f(x)}{g(x)} - \eta\right| < \frac{\epsilon}{6}\left(1 + \frac{|g(x_0)|}{|g(x)|}\right) + \frac{|f(x_0)|}{|g(x)|} + |\eta|\frac{|g(x_0)|}{|g(x)|} < \frac{\epsilon}{6}(1+1) + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. , , $\delta > 0$ $\left|\frac{f(x)}{g(x)} - \eta\right| < \epsilon$ x (ξ, b) $\xi < x < \xi + \delta$. $\lim_{x\to\xi+} \frac{f(x)}{g(x)} = \eta$.

$\lim_{x\to\xi+} \frac{f'(x)}{g'(x)} = \pm\infty$ $x \to \xi-$ $x \to \xi$. $x \to \pm\infty$ $x \to 0\pm$ 6.17.

l' Hopitâl $\lim_{x\to\xi+} f(x)$ ( ). , $\frac{\pm\infty}{\pm\infty}$ l' Hopitâl, .

: (1)

$$\boxed{\lim_{x\to+\infty} \frac{x^b}{a^x} = 0 \quad (b > 0, a > 1).}$$

$b = 1$, $\lim_{x\to+\infty} \frac{x}{a^x} = 0$ $a > 1$. $\lim_{x\to+\infty} x = +\infty$ $\lim_{x\to+\infty} a^x = +\infty$, $\frac{+\infty}{+\infty}$. $(-\infty, +\infty)$ $a^x \neq 0$ $\frac{d\,a^x}{dx} = a^x \log a \neq 0$. $\frac{1}{a^x \log a}$ $\lim_{x\to+\infty} \frac{1}{a^x \log a} = 0$. $\lim_{x\to+\infty} \frac{x}{a^x} = 0$.

. $a^{\frac{1}{b}} > 1$, : $\lim_{x\to+\infty} \frac{x^b}{a^x} = \lim_{x\to+\infty} \left(\frac{x}{(a^{\frac{1}{b}})^x}\right)^b = \left(\lim_{x\to+\infty} \frac{x}{(a^{\frac{1}{b}})^x}\right)^b = 0^b = 0$.

(2)

$$\boxed{\lim_{x\to+\infty} \frac{(\log x)^b}{x^a} = 0 \quad (b > 0, a > 0).}$$

$\lim_{x\to+\infty} \frac{\log x}{x^a} = 0$ $a > 0$. $\lim_{x\to+\infty} \log x = +\infty$ $\lim_{x\to+\infty} x^a = +\infty$, $\frac{+\infty}{+\infty}$. $(0, +\infty)$ $x^a \neq 0$ $\frac{d\,x^a}{dx} = ax^{a-1} \neq 0$. $\frac{\frac{1}{x}}{ax^{a-1}} = \frac{1}{ax^a}$ $\lim_{x\to+\infty} \frac{1}{ax^a} = 0$. $\lim_{x\to+\infty} \frac{\log x}{x^a} = 0$.

: $\lim_{x\to+\infty} \frac{(\log x)^b}{x^a} = \lim_{x\to+\infty} \left(\frac{\log x}{x^{\frac{a}{b}}}\right)^b = 0^b = 0$ $\frac{a}{b} > 0$.

(3) $\lim_{x\to+\infty} \frac{x - \cos x}{x}$ $\frac{+\infty}{+\infty}$. , $\lim_{x\to+\infty} x = +\infty$. , $x - \cos x \geq x - 1$ $\lim_{x\to+\infty}(x - \cos x) = +\infty$.

: $\left|\frac{\cos x}{x}\right| \leq \frac{1}{x}$ $x > 0$, $\lim_{x\to+\infty} \frac{\cos x}{x} = 0$, , $\lim_{x\to+\infty} \frac{x-\cos x}{x} = \lim_{x\to+\infty} \left(1 - \frac{\cos x}{x}\right) = 1 - 0 = 1$.

! $\frac{1+\sin x}{1} = 1 + \sin x$, , $\lim_{x\to+\infty} \sin x$.

, , l' Hopitâl. , .

, , $\frac{0}{0}$ $\frac{\pm\infty}{\pm\infty}$. l' Hopitâl. , , .

(1) $\lim f(x) = 0$ $\lim g(x) = \pm\infty$ $\lim f(x)g(x)$, $0(\pm\infty)$. $\lim \frac{f(x)}{\frac{1}{g(x)}}$, $\frac{0}{0}$. $\lim \frac{g(x)}{\frac{1}{f(x)}}$, $\frac{\pm\infty}{\pm\infty}$.
(2) $\lim f(x) = +\infty$ $\lim g(x) = -\infty$ $\lim(f(x) + g(x))$, $(+\infty) + (-\infty)$. $\lim \left(\frac{1}{g(x)} + \frac{1}{f(x)}\right) f(x)g(x)$, $0(-\infty)$. .
(3) $\lim f(x) = 0$, $y = f(x)$, $\lim g(x) = 0$ $\lim f(x)^{g(x)}$, 0^0. $\lim e^{g(x)\log f(x)}$, $0(-\infty)$.
(4) $\lim f(x) = +\infty$ $\lim g(x) = 0$ $\lim f(x)^{g(x)}$, $(+\infty)^0$. $\lim e^{g(x)\log f(x)}$, $0(+\infty)$.
(5) , $\lim f(x) = 1$ $\lim g(x) = \pm\infty$ $\lim f(x)^{g(x)}$, $1^{\pm\infty}$. $\lim e^{g(x)\log f(x)}$, $(\pm\infty)0$.

: (1) $\lim_{x\to0+} x\log x$, $0(-\infty)$. $x\log x = \frac{\log x}{\frac{1}{x}}$, $\lim_{x\to0+} \frac{1}{x} = +\infty$. : $\frac{1}{x} \neq 0$ $\frac{d\left(\frac{1}{x}\right)}{dx} = -\frac{1}{x^2} \neq 0$ x $(0, +\infty)$. , $\lim_{x\to0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to0+} x = 0$. $\lim_{x\to0+} x\log x = 0$.

(2) $\lim_{x\to0+} x^x$ 0^0. $x^x = e^{x\log x}$, , $\lim_{x\to0+} x^x = \lim_{x\to0+} e^{x\log x} = e^0 = 1$ $y = e^x$ 0.

(3) $\lim_{x\to0+} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ $(+\infty) - (+\infty)$. $\frac{1}{\sin x} - \frac{1}{x} = \frac{x-\sin x}{x\sin x}$ $\lim_{x\to0+}(x - \sin x) = 0$ $\lim_{x\to0+} x\sin x = 0$. : $x\sin x \neq 0$ $\frac{d\,(x\sin x)}{dx} = \sin x + x\cos x \neq 0$ x $(0, \frac{\pi}{2})$ ($\sin x > 0$, $x > 0$ $\cos x > 0$). , $\lim_{x\to0+} \frac{1-\cos x}{\sin x + x\cos x}$ $\frac{0}{0}$ $\lim_{x\to0+}(1 - \cos x) = 0$ $\lim_{x\to0+}(\sin x + x\cos x) = 0$. ( ) , $\sin x + x\cos x \neq 0$ x $(0, \frac{\pi}{2})$ $\frac{d}{dx}(\sin x + x\cos x) = 2\cos x - x\sin x \neq 0$ x $(0, \frac{\pi}{4})$. $\lim_{x\to0+} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$. $\lim_{x\to0+} \frac{1-\cos x}{\sin x + x\cos x} = 0$, , $\lim_{x\to0+} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = 0$.

(4) $\lim_{x\to+\infty} \left(1 + \frac{x}{1+x^2}\right)^x$ $1^{+\infty}$. $\left(1 + \frac{x}{1+x^2}\right)^x = e^{x\log\left(1+\frac{x}{1+x^2}\right)}$, $\lim_{x\to+\infty} x\log\left(1 + \frac{x}{1+x^2}\right)$, $(+\infty)0$. , , $x\log\left(1 + \frac{x}{1+x^2}\right) = \frac{\log\left(1+\frac{x}{1+x^2}\right)}{\frac{1}{x}}$ $\frac{0}{0}$. : $\frac{1}{x} \neq 0$ $\frac{d\left(\frac{1}{x}\right)}{dx} = -\frac{1}{x^2} \neq 0$ x $(0, +\infty)$. $\frac{d\,\log\left(1+\frac{x}{1+x^2}\right)}{dx} = \frac{\frac{1-x^2}{(1+x^2)^2}}{1+\frac{x}{1+x^2}} = \frac{1-x^2}{(1+x^2)(1+x+x^2)}$ $\lim_{x\to+\infty} \frac{x^2(x^2-1)}{(1+x^2)(1+x+x^2)} = 1$. $\lim_{x\to+\infty} x\log\left(1+\frac{x}{1+x^2}\right) = 1$, , $\lim_{x\to+\infty} \left(1 + \frac{x}{1+x^2}\right)^x = \lim_{x\to+\infty} e^{x\log\left(1+\frac{x}{1+x^2}\right)} = e^1 = e$ $y = e^x$ 1.

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l' Hopitâl .

. (a_n) (b_n) $b_n \neq 0$ n $\lim_{n\to+\infty} a_n = 0$ $\lim_{n\to+\infty} b_n = 0$. $\lim_{n\to+\infty} \frac{a_n}{b_n}$, .

l' Hopitâl, $y = f(x)$ $y = g(x)$, $[1, +\infty)$, :

$$f(n) = a_n \qquad g(n) = b_n \qquad (n\ ).$$

, $[1, +\infty)$, $g(x) \neq 0$ $g'(x) \neq 0$ x $[1, +\infty)$, , $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = 0$. , , $\lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$, $\lim_{x\to+\infty} \frac{f(x)}{g(x)}$. , 4.19 $y = \frac{f(x)}{g(x)}$ (x_n) $x_n = n$, $\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{f(n)}{g(n)}$.

: $\lim_{n\to+\infty} \arctan\frac{1}{n}\tan(\frac{\pi}{2} - \frac{1}{n})$.
$\lim_{n\to+\infty} \arctan\frac{1}{n} = 0$ $\lim_{n\to+\infty} \tan(\frac{\pi}{2} - \frac{1}{n}) = +\infty$, $0 \cdot (+\infty)$.
$\arctan\frac{1}{n}\tan(\frac{\pi}{2} - \frac{1}{n}) = \arctan\frac{1}{n}\cot\frac{1}{n} = \frac{\arctan\frac{1}{n}}{\tan\frac{1}{n}}$ $\frac{0}{0}$. $y = f(x) = \arctan\frac{1}{x}$ $y = g(x) = \tan\frac{1}{x}$ $(\frac{2}{\pi}, +\infty)$. , $\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to+\infty} \frac{x^2(\cos\frac{1}{x})^2}{1+x^2} = 1$. (n) $+\infty$, $\lim_{n\to+\infty} \frac{\arctan\frac{1}{n}}{\tan\frac{1}{n}} = \lim_{x\to+\infty} \frac{\arctan\frac{1}{x}}{\tan\frac{1}{x}} = 1$.

. (a_n) (b_n) $b_n \neq 0$ n $\lim_{n\to+\infty} b_n = +\infty$ $-\infty$. , , $\lim_{n\to+\infty} \frac{a_n}{b_n}$, . l' Hopitâl, $y = f(x)$ $y = g(x)$, $[1, +\infty)$, :

$$f(n) = a_n \qquad g(n) = b_n \qquad (n\ ).$$

$[1, +\infty)$, $g(x) \neq 0$ $g'(x) \neq 0$ x $[1, +\infty)$ $\lim_{x\to+\infty} g(x) = +\infty$ $-\infty$. , , $\lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$, $\lim_{x\to+\infty} \frac{f(x)}{g(x)}$, $\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{f(n)}{g(n)}$.

: (1)

$$\boxed{\lim_{n\to+\infty} \frac{n^b}{a^n} = 0 \quad (b > 0, a > 1) \qquad \lim_{n\to+\infty} \frac{(\log n)^b}{n^a} = 0 \quad (b > 0, a > 0).}$$

$y = x^b$ $y = a^x$, $\lim_{x\to+\infty} \frac{x^b}{a^x}$. l' Hopitâl, $\lim_{x\to+\infty} \frac{x^b}{a^x} = 0$ 4.19 (x_n) $x_n = n$. .

(2) .

$$\boxed{\lim_{n\to+\infty} \sqrt[n]{n} = 1.}$$

$\sqrt[n]{n} = n^{\frac{1}{n}} = e^{\frac{\log n}{n}}$ $(+\infty)^0$ $\frac{+\infty}{+\infty}$. $\lim_{n\to+\infty} \frac{\log n}{n} = 0$. $\lim_{n\to+\infty} \sqrt[n]{n} = \lim_{n\to+\infty} e^{\frac{\log n}{n}} = e^0 = 1$.

.

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1. ′ , .

$$\lim_{x\to+\infty}\frac{e^{\frac{1}{4}x}}{x^{13}}\,,\qquad \lim_{x\to+\infty}\frac{\sqrt[7]{x}}{(\log x)^5}\,,\qquad \lim_{x\to+\infty}\frac{e^{\frac{x}{2}}-(\log x)^4}{x^{100}-e^{\frac{x}{4}}}\,,\qquad \lim_{x\to0}\frac{e^{-\frac{1}{x^2}}}{x^5}\,,$$

$$\lim_{x\to+\infty}\left(e^x-x^{10}\right),\qquad \lim_{x\to0+}\left(\frac{1}{x^5}-\frac{2}{x^2}+(\log x)^7\right).$$

2. l' Hopitâl, .

$$\lim_{x\to1}\frac{1-x+\log x}{1-\sqrt{2-x}}\,,\qquad \lim_{x\to0}\frac{\log(1+x)}{e^{2x}-1}\,,\qquad \lim_{x\to2}\frac{x^2-3x+2}{x^2-5x+6}\,,$$

$$\lim_{x\to0}\frac{\sin x}{\arctan x}\,,\quad \lim_{x\to1}\frac{x^a-1}{x^b-1}\,,\quad \lim_{x\to0}\frac{a^x-1}{b^x-1}\quad (a,b>0, b\neq1),$$

$$\lim_{x\to0}\frac{1-\cos(1-\cos x)}{x^4}\,,\qquad \lim_{x\to0}\frac{x\sin(\sin x)-(\sin x)^2}{x^6}\,,$$

$$\lim_{x\to0}\frac{e-(1+x)^{\frac{1}{x}}}{x}\,,\qquad \lim_{x\to+\infty}\frac{\log(\log x)}{\log x}\,,\qquad \lim_{x\to+\infty}\frac{\log\big(\log(\log x)\big)}{\log(\log x)}\,,$$

$$\lim_{x\to0+}\sin x\log x,\qquad \lim_{x\to1+}\log x\log(x-1),\qquad \lim_{x\to+\infty}\frac{x(x^{\frac{1}{x}}-1)}{\log x}\,,$$

$$\lim_{x\to0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right),\ \lim_{x\to0}\left(\frac{1}{\log(1+x)}-\frac{1}{x}\right),\ \lim_{x\to0}\left(\cot x-\frac{1}{x}\right),$$

$$\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan(2x)}\,,\qquad \lim_{x\to0+}x^{x^x-1}\,,\qquad \lim_{x\to0}(x+e^x)^{\frac{1}{x}}\,.$$

3. $\lim_{x\to+\infty}\frac{\cosh x}{e^x}$ l' Hopitâl ?
, l' Hopitâl ?

4. a, b $\lim_{x\to0}\left(\frac{1-\cos x}{x^4}+ax^{-2}+b\right)=0$.

5. $y=\begin{cases}\frac{1}{\log|x|}, & x\neq0,\\ 0, & x=0,\end{cases}$ Hölder- 0 (7 5.1).
(: .)

6. , , , () () , ( ).

$$y=xe^{-x}\,,\quad y=xe^{-x^2}\,,\quad y=x\log x,\quad y=\frac{\log x}{x}\,,\quad y=x^{\frac{1}{x}}\,,\quad y=x^x\,.$$

7. . , l' Hopitâl .

$$\lim_{x\to0}\frac{e^x-1}{x}\,,\qquad \lim_{x\to0}\frac{e^x-1-\frac{1}{1!}x}{x^2}\,,$$

$$\lim_{x\to0}\frac{e^x-1-\frac{1}{1!}x-\frac{1}{2!}x^2}{x^3}\,,\qquad \lim_{x\to0}\frac{e^x-1-\frac{1}{1!}x-\frac{1}{2!}x^2-\frac{1}{3!}x^3}{x^4}\,.$$

.

8. l' Hopitâl $\lim_{x\to 0}\frac{\sin x}{x}=1$;

9. .

$$\lim_{x\to 0}\frac{\sin x-\frac{1}{1!}x}{x^3},\quad \lim_{x\to 0}\frac{\cos x-1+\frac{1}{2!}x^2}{x^4},$$

$$\lim_{x\to 0}\frac{\sin x-\frac{1}{1!}x+\frac{1}{3!}x^3}{x^5},\quad \lim_{x\to 0}\frac{\cos x-1+\frac{1}{2!}x^2-\frac{1}{4!}x^4}{x^6}.$$

.

10. . , l' Hopitâl .

$$\lim_{x\to 0}\frac{\log(1+x)}{x},\quad \lim_{x\to 0}\frac{\log(1+x)-x}{x^2},$$

$$\lim_{x\to 0}\frac{\log(1+x)-x+\frac{1}{2}x^2}{x^3},\quad \lim_{x\to 0}\frac{\log(1+x)-x+\frac{1}{2}x^2-\frac{1}{3}x^3}{x^4}.$$

.

11. $y=f(x)$ $n-1$ (a,b) $a<\xi<b$. $f^{(n)}(\xi)$,

$$\lim_{x\to\xi}\frac{f(x)-f(\xi)-\frac{f^{(1)}(\xi)}{1!}(x-\xi)^1-\cdots-\frac{f^{(n-1)}(\xi)}{(n-1)!}(x-\xi)^{n-1}}{(x-\xi)^n}=\frac{f^{(n)}(\xi)}{n!}.$$

: l' Hopitâl?

7, 9 10 .

12. (*) . $y=f(x)$ $2m-1$ (a,b), $a<\xi<b$ $f^{(2m)}(\xi)$. $f^{(1)}(\xi)=\ldots=f^{(2m-1)}(\xi)=0$, :

(i) $f^{(2m)}(\xi)>0$, ξ $y=f(x)$.

(ii) $f^{(2m)}(\xi)<0$, ξ $y=f(x)$.

(: .)

13. (*) $y=f(x)$ $2m$ (a,b), $a<\xi<b$ $f^{(2m+1)}(\xi)$. $f^{(1)}(\xi)=\ldots=f^{(2m)}(\xi)=0$ $f^{(2m+1)}(\xi)\neq 0$, ξ $y=f(x)$.

(: .)

14. (*) $y=f(x)$ (a,b) $a<\xi<b$.

$f'(\xi)$, $\lim_{h\to 0+}\frac{f(\xi+h)-f(\xi-h)}{2h}=f'(\xi)$. l' Hopitâl;

(: $\frac{f(\xi+h)-f(\xi-h)}{2h}=\frac{1}{2}\frac{f(\xi+h)-f(\xi)}{h}+\frac{1}{2}\frac{f(\xi-h)-f(\xi)}{-h}$.)

$f''(\xi)$, $\lim_{h\to 0+}\frac{f(\xi+h)-2f(\xi)+f(\xi-h)}{h^2}=f''(\xi)$.

(: $f''(\xi)$ ¨ (c,d) $c<\xi<d$. l' Hopitâl.)

15. $y=f(x)$ $(0,+\infty)$ $\lim_{x\to+\infty}f'(x)=\eta$. $\lim_{x\to+\infty}\frac{f(x)}{x}=\eta$.

16. 18 6.8 l' Hopitâl.

17. $^{(*)}$ $y=f(x)$ $(0,+\infty)$ $\eta=\lim_{x\to+\infty}(f(x)+f'(x))$. $\lim_{x\to+\infty}f(x)=\eta$ $\lim_{x\to+\infty}f'(x)=0$.

(: $f(x)=\frac{f(x)e^x}{e^x}$.)

18. $\lim_{x\to0+}x^{-m}e^{-\frac{1}{x}}=0$ m.

$^{(*)}$ $y=h(x)=\begin{cases}e^{-\frac{1}{x}}\,, & x>0,\\ 0\,, & x\le 0.\end{cases}$ $y=h(x)$ $(-\infty,+\infty)$, , $h^{(n)}(0)=0$ n.

(: n $y=h(x)$ n $(-\infty,+\infty)$, , $h^{(n)}(0)=0$. 9 6.10.)

19. $^{(*)}$ $y=f(x)=\begin{cases}e^{-\frac{2}{1-x^2}}\,, & -1<x<1,\\ 0\,, & |x|\ge 1.\end{cases}$ $y=f(x)$ $(-\infty,+\infty)$.

(: .)

20. $^{(*)}$ $y=e^x$ $(-\infty,+\infty)$.

(: $p_n(x)e^{nx}+\cdots+p_1(x)e^x+p_0(x)=0$ $(-\infty,+\infty)$, $p_n(x),\ldots,p_1(x),p_0(x)$. $p_n(x)e^{nx}$ $x\to+\infty$.)

. .

1. .

$$\Big(\frac{(\log n)^{13}}{n^2}\Big),\quad \Big(\frac{\sqrt{n}}{(\log n)^{95}}\Big),\quad (ne^{-n}),\quad (n^3e^{-n}),\quad \Big(\frac{e^{\frac{n}{5}}}{n^{100}}\Big).$$

2. $\lim_{n\to+\infty}\sqrt[n]{n}=1$.

$$\lim_{n\to+\infty}\sqrt[n]{n+1}=1,\quad \lim_{n\to+\infty}\sqrt[n]{n^2}=1,\quad \lim_{n\to+\infty}\sqrt[n]{n^3+3n^2+n+2}=1.$$

(: $1\le\sqrt[n]{n+1}\le\sqrt[n]{2n}=\sqrt[n]{2}\,\sqrt[n]{n}$ $\sqrt[n]{n^2}=\big(\sqrt[n]{n}\big)^2$.)

3. .

$$\Big(\frac{\log(\log n)}{\log n}\Big),\quad \Big(\frac{\log(\log(\log n))}{\log(\log n)}\Big),\quad \Big(n-\cot\frac{1}{n}\Big).$$

6.12 , .

. .

$y=f(x)$ $y=g(x)$ $f(x),g(x)\neq0$ ξ, x $(a,\xi)\cup(\xi,b)$. $\lim_{x\to\xi}\big|\frac{f(x)}{g(x)}\big|=0$, $y=f(x)$ $y=g(x)$ ξ $y=g(x)$ $y=f(x)$ ξ. , , $\lim_{x\to\xi}\big|\frac{f(x)}{g(x)}\big|=0$ $\lim_{x\to\xi}\big|\frac{g(x)}{f(x)}\big|=+\infty$. , l u $0<l\le\big|\frac{f(x)}{g(x)}\big|\le u<+\infty$ ξ, $y=f(x)$ $y=g(x)$ ξ.

$\rho=\lim_{x\to\xi}\big|\frac{f(x)}{g(x)}\big|$, $l<\rho$ (, $l=\frac{\rho}{2}$) $u>\rho$ (, $u=2\rho$), , 4.16, $l<\big|\frac{f(x)}{g(x)}\big|<u$ ξ , , $y=f(x)$ $y=g(x)$ ξ.

, , : $x \to \xi+$, $x \to \xi-$, $x \to +\infty$ $x \to -\infty$.

: (1) $y = x^b$ $(b > 0)$ $y = a^x$ $(a > 1)$ $+\infty$, , , $\lim_{x\to+\infty} \frac{x^b}{a^x} = 0$.

(2) $y = (\log x)^c$ $(c > 0)$ $y = x^b$ $(b > 0)$ $+\infty$, $\lim_{x\to+\infty} \frac{(\log x)^c}{x^b} = 0$.

(3) $y = a_0 + a_1x + \cdots + a_Nx^N$ $y = b_0 + b_1x + \cdots + b_Nx^N$ ($a_N, b_N \neq 0$) $+\infty$, $\lim_{x\to+\infty} \left|\frac{a_0+a_1x+\cdots+a_Nx^N}{b_0+b_1x+\cdots+b_Nx^N}\right| = \left|\frac{a_N}{b_N}\right|$.
, $y = a_0 + a_1x + \cdots + a_Nx^N$ ($a_N \neq 0$) $y = b_0 + b_1x + \cdots + b_Mx^M$ ($b_M \neq 0$), $N < M$, $+\infty$. , $\lim_{x\to+\infty} \frac{a_0+a_1x+\cdots+a_Nx^N}{b_0+b_1x+\cdots+b_Mx^M} = 0$.

(4) $y = a^{-\frac{1}{x}}$ $(a > 1)$ $y = x^b$ $(b > 0)$ 0 , $\lim_{x\to0+} \frac{a^{-\frac{1}{x}}}{x^b} = \lim_{t\to+\infty} \frac{t^b}{a^t} = 0$.

(5) H $y = 1 - \cos x$ $y = x^2$ 0, $\lim_{x\to0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

(6) H $y = \sin x$ $y = x$ 0, $\lim_{x\to0} \frac{\sin x}{x} = 1$.

(7) $y = \left(\log \frac{1}{x}\right)^c$ $(c > 0)$ $y = \frac{1}{x^b}$ $(b > 0)$ 0 , $\lim_{x\to0+} \frac{\left(\log \frac{1}{x}\right)^c}{\frac{1}{x^b}} = \lim_{t\to+\infty} \frac{(\log t)^c}{t^b} = 0$.

(8) $\lim_{x\to+\infty} \frac{2x+x\sin x}{x} = \lim_{x\to+\infty}(2 + \sin x)$. , $y = 2x + x\sin x$ $y = x$ $+\infty$. , $\frac{2x+x\sin x}{x} = 2 + \sin x$ $1 \leq \frac{2x+x\sin x}{x} \leq 3$ x $(0, +\infty)$.

, , $x \to +\infty$: , .
(3) , N , N $+\infty$. $y = a_0 + a_1x + \cdots + a_Nx^N$ ($a_N \neq 0$) , , N $+\infty$. , , (3), «» $+\infty$: . , N $y = x^N$. , , « N» N $y = x^N$ $+\infty$.

: $\frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_mx^m}$ $(a_n, b_m \neq 0)$, $n > m$, $N = n-m$ $+\infty$. $\lim_{x\to+\infty} \left|\frac{\frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_mx^m}}{x^{n-m}}\right| = \left|\frac{a_n}{b_m}\right| > 0$.

« » $y = x^b$ $(b > 0)$ b , ′ . , , $y = x^b$ $(b > 0)$ $+\infty$. , b $+\infty$ $y = x^b$ $(b > 0)$ $+\infty$. «» , $\lim_{x\to+\infty} \frac{x^{b_1}}{x^{b_2}} = \lim_{x\to+\infty} \frac{1}{x^{b_2-b_1}} = 0$ $0 < b_1 < b_2$. , , $+\infty$ $+\infty$ $+\infty$.
, $y = f(x)$ $y = x^b$ $(b > 0)$, $u, l > 0$ $l \leq \left|\frac{f(x)}{x^b}\right| \leq u$, , $|f(x)| \geq lx^b$ $+\infty$. $\lim_{x\to+\infty}(lx^b) = +\infty$, $\lim_{x\to+\infty} |f(x)| = +\infty$.
, $a > 1$, $y = a^x$ $+\infty$. $y = a^x$ $(a > 1)$ $+\infty$. «» a. , $\lim_{x\to+\infty} \frac{{a_1}^x}{{a_2}^x} = \lim_{x\to+\infty} \left(\frac{a_1}{a_2}\right)^x = 0$ $1 < a_1 < a_2$.
, $c > 0$, $y = (\log x)^c$ $+\infty$. $y = (\log x)^c$ $(c > 0)$ $+\infty$. «» c, $\lim_{x\to+\infty} \frac{(\log x)^{c_1}}{(\log x)^{c_2}} = \lim_{x\to+\infty} \frac{1}{(\log x)^{c_2-c_1}} = 0$ $0 < c_1 < c_2$.
, $+\infty$ $+\infty$ $+\infty$. .
(1) (2)

$+\infty$

.

. . .

$y = f(x)$ $y = g(x)$ $g(x) \neq 0$ x $(a,\xi) \cup (\xi, b)$. $\lim_{x\to\xi} \frac{f(x)}{g(x)} = 0$,

$$f(x) = \big(g(x)\big) \quad \xi$$

« $y = f(x)$ $y = g(x)$» ξ. $y = \frac{f(x)}{g(x)}$ ξ, u $|f(x)| \le u|g(x)|$ ξ,

$$f(x) = \big(g(x)\big) \quad \xi$$

« $y = f(x)$ $y = g(x)$» ξ. : $x \to \xi+$, $x \to \xi-$, $x \to +\infty$ $x \to -\infty$.

: (1) $y = f(x)$ $y = g(x)$ x, $f(x) = \big(g(x)\big)$ x.
, , , , $f(x) \neq 0$ x.
, $(\log x)^c = \big(x^b\big)$ $x^b = \big(a^x\big)$ $+\infty$ $a > 1$, $b > 0$ $c > 0$. , $x^{b_1} = \big(x^{b_2}\big)$ $+\infty$ $x^{b_2} = \big(x^{b_1}\big)$ 0 $0 < b_1 < b_2$.

(2) $y = f(x)$ $y = g(x)$ x, $f(x) = \big(g(x)\big)$ x.
, $\sin x = (x)$ $1 - \cos x = (x^2)$ 0.

$y = f(x)$ $y = g(x)$ $g(x) \neq 0$ x $(a,\xi) \cup (\xi, b)$. $\lim_{x\to\xi} \frac{f(x)}{g(x)} = 1$,

$$f(x) \sim g(x) \quad \xi$$

$y = f(x)$ $y = g(x)$ ξ. : $x \to \xi+$, $x \to \xi-$, $x \to +\infty$ $x \to -\infty$.
$\lim_{x\to\xi} \frac{f(x)}{g(x)} = 1$ $\frac{f(x)}{g(x)} \neq 0$ ξ , , $f(x) \neq 0$ ξ. , $f(x) \sim g(x)$ ξ, $f(x), g(x) \neq 0$ ξ.

: (1) $\sin x \sim x$ $1 - \cos x \sim \frac{1}{2}x^2$ 0.

(2) $e^x - 1 \sim x$ 0, $\lim_{x\to 0} \frac{e^x - 1}{x} = \frac{d\, e^x}{dx}\Big|_{x=0} = 1$.

(3) $\log(1+x) \sim x$ 0, $\lim_{x\to 0} \frac{\log(1+x)}{x} = \frac{d \log x}{dx}\Big|_{x=1} = 1$.

(4) $\tan x \sim \frac{1}{\frac{\pi}{2} - x}$ $\frac{\pi}{2}$, $\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\frac{1}{\frac{\pi}{2}-x}} = \lim_{t\to 0} t \cot t = \lim_{t\to 0} \frac{t}{\sin t} \cos t = 1$.

.

$f(x) - g(x) = \big(cg(x)\big)$, c $\neq 0$, $f(x) \sim g(x)$.

, $\lim \frac{f(x)-g(x)}{cg(x)} = 0$, $\lim \frac{f(x)-g(x)}{g(x)} = 0$, $\lim \big(\frac{f(x)}{g(x)} - 1\big) = 0$, $\lim \frac{f(x)}{g(x)} = 1$ $f(x) \sim g(x)$.

: .

(1) $\sin x - x = (x)$ $\cos x - 1 + \frac{1}{2}x^2 = (x^2)$ 0.

(2) $e^x - 1 - x = (x)$ 0.

(3) $\log(1+x) - x = (x)$ 0.

(4) $\tan x - \frac{1}{\frac{\pi}{2}-x} = \left(\frac{1}{\frac{\pi}{2}-x}\right)$ $\frac{\pi}{2}$.

$y = g_1(x), \dots, y = g_n(x)$ $y = g(x)$ x, $y = g_1(x) + \cdots + g_n(x)$ $y = g(x)$ x. : $\lim \frac{g_1(x)+\cdots+g_n(x)}{g(x)} = \lim \frac{g_1(x)}{g(x)} + \cdots + \frac{g_n(x)}{g(x)} = 0 + \cdots + 0 = 0.$

$y = f(x) = g(x) + g_1(x) + \cdots + g_n(x)$ $y = g_1(x), \dots, y = g_n(x)$ $y = g(x)$ x, $y = g(x)$ x $f(x) \sim g(x)$ x. , $\lim \frac{f(x)}{g(x)} = \lim \left(1 + \frac{g_1(x)+\cdots+g_n(x)}{g(x)}\right) = 1.$

. , $y = f(x) = g(x) + g_1(x) + \cdots + g_n(x)$ $y = g(x)$ x , $y = f(x)$. , , $f(x) \sim g(x)$ x , , $\lim f(x) = \lim \left(\frac{f(x)}{g(x)} g(x)\right) = 1 \cdot \lim g(x).$

: $y = g(x) + g_1(x) + \cdots + g_n(x)$ $y = h(x) + h_1(x) + \cdots + h_m(x)$ $y = g(x)$ $y = h(x)$, , x, $\frac{g(x)+g_1(x)+\cdots+g_n(x)}{h(x)+h_1(x)+\cdots+h_m(x)} \sim \frac{g(x)}{h(x)}$ x , , $y = \frac{g(x)}{h(x)}$ x , $\frac{g(x)+g_1(x)+\cdots+g_n(x)}{h(x)+h_1(x)+\cdots+h_m(x)}$.

: (1) « » . $x^n = (x^N)$ $+\infty$ $n < N$, $a_0 + a_1 x + \cdots + a_N x^N$ $(a_N \neq 0)$ $a_N x^N$ $+\infty$, $a_0 + a_1 x + \cdots + a_N x^N \sim a_N x^N$, , $\lim_{x\to+\infty}(a_0 + a_1 x + \cdots + a_N x^N) = \lim_{x\to+\infty} a_N x^N$.

, $\frac{a_0+a_1x+\cdots+a_Nx^N}{b_0+b_1x+\cdots+b_Mx^M} \sim \frac{a_Nx^N}{b_Mx^M}$ $+\infty$, , $\lim_{x\to+\infty} \frac{a_0+a_1x+\cdots+a_Nx^N}{b_0+b_1x+\cdots+b_Mx^M} = \lim_{x\to+\infty} \frac{a_Nx^N}{b_Mx^M}$.

(2) $xe^{2x} - x^2 + 3e^x - x^2 e^{\frac{x}{2}}$ $2e^{2x} + \log x - x^2 e^x$ $+\infty$. $\frac{xe^{2x}-x^2+3e^x-x^2e^{\frac{x}{2}}}{2e^{2x}+\log x - x^2e^x} \sim \frac{xe^{2x}}{2e^{2x}} = \frac{x}{2}$ $+\infty$, $\lim_{x\to+\infty} \frac{xe^{2x}-x^2+3e^x-x^2e^{\frac{x}{2}}}{2e^{2x}+\log x-x^2e^x} = \lim_{x\to+\infty} \frac{x}{2} = +\infty.$

.

. .

1. $a > 1$. $+\infty$ $y = a^x$, $y = a^{a^x}$ $y = a^{a^{a^x}}$.

   .

2. $+\infty$ $y = \log x$, $y = \log(\log x)$ $y = \log\big(\log(\log x)\big)$.

   .

3. $y = x$, $y = \log(e^x + x\log x)$ $y = e^{(1+\frac{1}{x})\log x}$ $+\infty$.

   $y = \log\frac{1}{x}$ $y = \frac{x^2 + x\log\frac{1}{x}}{\sin x + x^2}$ 0.

4. $+\infty$, .

$$y = \frac{x^3e^x - x^5e^{\frac{x}{2}}}{xe^x + \sin x}, \quad y = e^{\frac{x}{5}} + x^3e^{\frac{x}{6}} - x, \quad y = e^{3\log(2+\log x)}.$$

5. ∞ $y = \frac{1}{x^b}$ $(b > 0)$. , $+\infty$ $y = \frac{1}{a^x}$ $(a > 0)$ $y = \frac{1}{(\log x)^c}$ $(c > 0)$, , $+\infty$.

   $+\infty$ 0 $+\infty$.

   $+\infty$.

.

$y = \frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_mx^m}$ $(a_n, b_m \neq 0)$, $n < m$, $+\infty$.

$+\infty$, .

$$y = e^{-x} + 2e^{-x^2}, \quad y = \frac{1}{\log(x+\log x)}, \quad y = \log\left(e^{\frac{1}{x}} + \frac{1}{x^2}\right).$$

. . .

1. 0.

$$\frac{1}{1-x} - 1 = (1), \quad \frac{1}{1-x} - (1+x) = (x), \quad \frac{1}{1-x} - (1+x+x^2) = (x^2).$$

.

2. 0.

$$e^x - 1 = (1), \quad e^x - \left(1 + \frac{x}{1!}\right) = (x), \quad e^x - \left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right) = (x^2).$$

.

3. 0.

$$\sin x - \left(\frac{x}{1!} - \frac{x^3}{3!}\right) = (x^3), \qquad \cos x - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right) = (x^4),$$

$$\sin x - \left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!}\right) = (x^5), \qquad \cos x - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right) = (x^6).$$

.

4. 0.

$$\log(1+x) - x = (x), \qquad \log(1+x) - \left(x - \frac{x^2}{2}\right) = (x^2),$$

$$\log(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) = (x^3).$$

.

5. 0.

$$\arctan x - x = (x) \qquad \arctan x - \left(x - \frac{x^3}{3}\right) = (x^3),$$

$$\arctan x - \left(x - \frac{x^3}{3} + \frac{x^5}{5}\right) = (x^5).$$

.

6. $f(x)-(a+bx+cx^2+dx^3)=(x^3)$ 0. $f(x)-(a+bx+cx^2)=(x^2)$, $f(x)-(a+bx)=(x)$ $f(x)-a=(1)$ 0.

a,b,c,d

$$\frac{x}{e^x-1}-(a+bx+cx^2+dx^3)=(x^3)$$

0.

7. .11 .

$y=f(x)$ $n-1$ (a,b) $a<\xi<b$. $f^{(n)}(\xi)$,

$$f(x)-\Big(f(\xi)+\frac{f^{(1)}(\xi)}{1!}(x-\xi)^1+\cdots+\frac{f^{(n)}(\xi)}{n!}(x-\xi)^n\Big)=\ \big((x-\xi)^n\big)$$

ξ.

1, 2, 3, 4 5 .

8. $+\infty$;

$$e^{2x}\log x-x^5e^x\,,\quad x\log x-\frac{x^2}{\log x}+x\sqrt{x}\log(\log x),\quad x^2\log x-x^2+3x\sin x.$$

9. 0;

$$\frac{2}{x}+\frac{1}{x\sqrt{x}}-\frac{3}{\sqrt{x}}\,,\quad 1+2x-x\sqrt{x}\,,\quad \frac{1}{(\sin x)^2}+\frac{3}{x}-\frac{1}{x\sqrt{x}}\,.$$

10. $y=f(x)$ $y=g(x)$, $y=f(x)$ $y=g(x)$. :

$$\big(g(x)\big)=\big(g(x)\big).$$

: $\big(g(x)\big)=\big(g(x)\big)$.

, $y=f_1(x)$ $y=f_2(x)$ $y=g(x)$, $y=f_1(x)+f_2(x)$ $y=g(x)$. :

$$\big(g(x)\big)+\big(g(x)\big)=\big(g(x)\big).$$

:

$$\big(g(x)\big)+\big(g(x)\big)=\big(g(x)\big),$$

$$\big(g_1(x)\big)\big(g_2(x)\big)=\big(g_1(x)g_2(x)\big),\ \ \big(g_1(x)\big)\big(g_2(x)\big)=\big(g_1(x)g_2(x)\big).$$

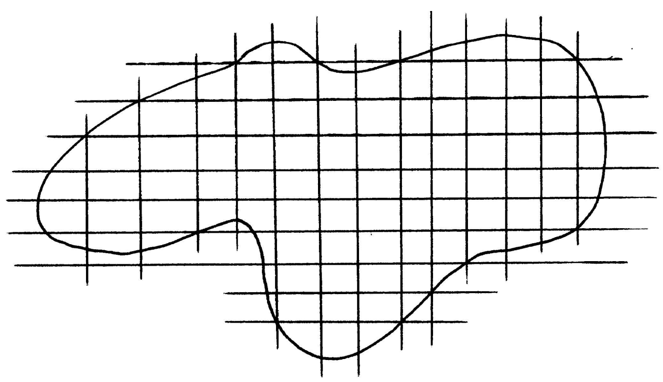
Κεφάλαιο 7

Riemann.

. , , Riemann. Riemann: Riemann, . Riemann . Riemann . Riemann . Riemann . , . Riemann : , , , , , , , .

7.1 .

. . . , , , « ».



Σχήμα 7.1: .

. : , , , , . . . , , « » : , . . , , « » , , « ». « » . , , « » $[a,b]$ x-, $y = f(x)$ $a \leq x \leq b$. $(a,0)$ $(a,f(a))$. $(b,0)$ $(b,f(b))$. , $f(x) \geq 0$ x $[a,b]$. « », $[a,b]$ $y = f(x)$. $y = f(x)$ $[a,b]$. A E.

. $[a,b]$ $x_0 = a, x_1, \ldots, x_{n-1}, x_n = b$. a b. , , $[x_0,x_1] = [a,x_1]$,

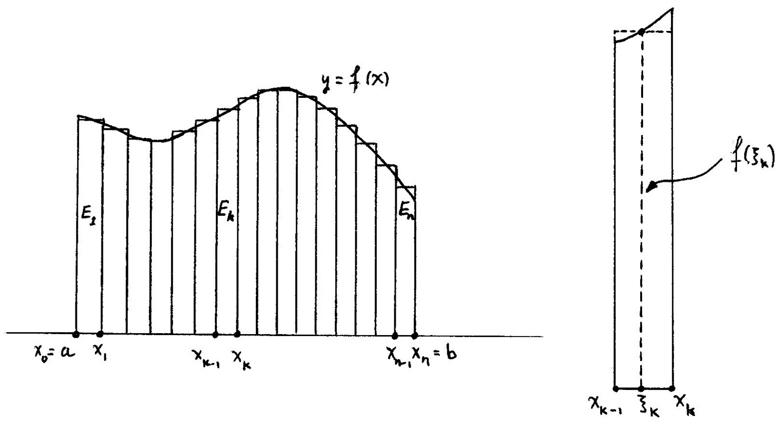
$[x_1, x_2]$, $\ldots$, $[x_{k-1}, x_k]$, $\ldots$, $[x_{n-2}, x_{n-1}]$ $[x_{n-1}, x_n] = [x_{n-1}, b]$. , , · , , δ. , $b - a$ $[a, b]$, δ n. $b - a < n\delta$, $n > \frac{b-a}{\delta}$, , δ n . $x_0 = a$, x_1, $\ldots$, x_{n-1}, $x_n = b$ $\Delta = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ $[a, b]$. $[x_0, x_1]$, n- $[x_{n-1}, x_n]$, , k- $(1 \leq k \leq n)$ $[x_{k-1}, x_k]$. , $n = 1$, $[x_0, x_1] = [a, b]$. , $[a, b]$. ′ n , , $n \geq 2$ $x_1, \ldots, x_{n-1}$ (, $a < x_1 < \ldots < x_{n-1} < b$).

: $[a, b]$ $\frac{b-a}{n}$. $x_0 = a, x_1 = a + \frac{b-a}{n}$, $x_2 = a + 2\frac{b-a}{n}$, $\ldots$, $x_k = a + k\frac{b-a}{n}$, $\ldots$, $x_{n-1} = a + (n-1)\frac{b-a}{n}$ $x_n = a + n\frac{b-a}{n} = b$.

Δ ,

$$(\Delta) = \max\big\{x_k - x_{k-1} \,:\, 1 \leq k \leq n\big\}.$$

, Δ .



Σχήμα 7.2: .

Δ $[a, b]$, « » n « », k- $[x_{k-1}, x_k]$, $y = f(x)$ $x_{k-1} \leq x \leq x_k$ ¨ . $(x_{k-1}, 0)$ $(x_{k-1}, f(x_{k-1}))$. $(x_k, 0)$ $(x_k, f(x_k))$. A_k E_k. , ,

$$E = E_1 + \cdots + E_n\,.$$

$A_1, \ldots, A_n$, , , . , : Δ , $[x_{k-1}, x_k]$, x , $f(x)$, , , , . $y = f(x)$ $[a, b]$. , Δ . , ξ_k $[x_{k-1}, x_k]$, $f(x)$ $[x_{k-1}, x_k]$ $f(\xi_k)$, A_k $\widetilde{A_k}$ $[x_{k-1}, x_k]$ $f(\xi_k)$. , E_k A_k $\widetilde{E_k} = f(\xi_k)(x_k - x_{k-1})$ $\widetilde{A_k}$, ,

$$E_k \approx \widetilde{E_k} = f(\xi_k)(x_k - x_{k-1})\,.$$

, , , $\widetilde{E_k}$ E_k , , $\big|E_k - \widetilde{E_k}\big|$, . Δ $\big|E_k - \widetilde{E_k}\big|$. , , $k = 1, \ldots, n$ $\widetilde{E_k}$ $\widetilde{E} = \widetilde{E_1} + \cdots + \widetilde{E_n}$, $\widetilde{A}$ $\widetilde{A_1}, \ldots, \widetilde{A_n}$. E_k $\widetilde{E_k}$, $E = E_1 + \cdots + E_n$

$\widetilde{E} = \widetilde{E_1} + \cdots + \widetilde{E_n} = f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_n)(x_n - x_{n-1})$, ,

$$E \approx \widetilde{E_1} + \cdots + \widetilde{E_n} = f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_n)(x_n - x_{n-1}).$$

, :

$$\begin{aligned} \left|E - \widetilde{E}\right| &= \left|(E_1 + \cdots + E_n) - \left(\widetilde{E_1} + \cdots + \widetilde{E_n}\right)\right| \\ &\le \left|E_1 - \widetilde{E_1}\right| + \cdots + \left|E_n - \widetilde{E_n}\right|. \end{aligned}$$

, , , . , Δ $\left|E - \widetilde{E}\right|$.
, E $\widetilde{E}$. , ξ_k $[x_{k-1}, x_k]$, $f(\xi_k)$, $(x_k - x_{k-1})$ $k = 1, \ldots, n$.
Ξ $\{\xi_1, \ldots, \xi_n\}$, . Δ Ξ, . $f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_n)(x_n - x_{n-1})$

$$\Sigma(f; a, b; \Delta; \Xi) = f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_n)(x_n - x_{n-1}).$$

: , $\xi_k = x_{k-1}$, , $\xi_k = x_k$.

: $\Delta = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ $[a, b]$ $\Xi = \{\xi_1, \ldots, \xi_n\}$ $\Sigma(f; a, b; \Delta; \Xi) = f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_n)(x_n - x_{n-1})$. $|E - \Sigma(f; a, b; \Delta; \Xi)|$. :

$$E \approx \Sigma(f; a, b; \Delta; \Xi) = f(\xi_1)(x_1 - x_0) + \cdots + f(\xi_n)(x_n - x_{n-1}).$$

, $\Sigma(f; a, b; \Delta; \Xi)$ E Δ , , $|E - \Sigma(f; a, b; \Delta; \Xi)|$ Δ .

. . $\left|E_k - \widetilde{E_k}\right|$ $\left|E_1 - \widetilde{E_1}\right| + \cdots + \left|E_n - \widetilde{E_n}\right|$, , . . $\left|E_k - \widetilde{E_k}\right|$ $[x_{k-1}, x_k]$, , n . , , : 10^{-5} n 10^{13}, $10^{13} \cdot 10^{-5} = 10^8$.
.
$y = f(x)$ A $\epsilon > 0$ $\delta > 0$ $|f(x') - f(x'')| < \epsilon$ x', x'' A $|x' - x''| < \delta$. ·
, $y = f(x)$ A, A. , , . , $y = f(x)$ A $y = f(x)$ A A. , , A , .
7.1, .

7.1 $y = f(x)$ $[a, b]$, $[a, b]$.

$\epsilon > 0$. $y = f(x)$ $[a, b]$, 7.1 $\delta > 0$ x', x'' $[a, b]$ $|x' - x''| < \delta$ $|f(x') - f(x'')| < \epsilon$.
$\Delta = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ $< \delta$, , $[x_{k-1}, x_k]$ $< \delta$. . , $\Xi = \{\xi_1, \ldots, \xi_n\}$, .
$\left|E_k - \widetilde{E_k}\right|$. - , $\xi_{k,1}, \xi_{k,2}$ $[x_{k-1}, x_k]$ $f(\xi_{k,1}) \le f(x) \le f(\xi_{k,2})$ x $[x_{k-1}, x_k]$.
. $\widetilde{A_{k,1}}$ $[x_{k-1}, x_k]$ $f(\xi_{k,1})$. $\widetilde{A_{k,2}}$ $[x_{k-1}, x_k]$ $f(\xi_{k,2})$, A_k $\widetilde{A_{k,1}}$ $\widetilde{A_{k,2}}$.
$\widetilde{E_{k,1}} = f(\xi_{k,1})(x_k - x_{k-1})$ $\widetilde{E_{k,2}} = f(\xi_{k,2})(x_k - x_{k-1})$,

$$\widetilde{E_{k,1}} \le E_k \le \widetilde{E_{k,2}}.$$

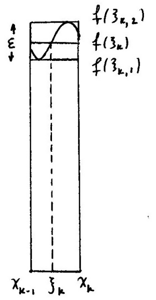
. $\widetilde{A_k}$ $[x_{k-1}, x_k]$ $f(\xi_k)$, $\widetilde{A_k}$ $\widetilde{A_{k,1}}$ $\widetilde{A_{k,2}}$. ,

$$\widetilde{E_{k,1}} \le \widetilde{E_k} \le \widetilde{E_{k,2}}.$$

$\xi_{k,1}, \xi_{k,2}$ $[x_{k-1}, x_k]$, $|\xi_{k,1} - \xi_{k,2}| \le x_k - x_{k-1} < \delta$. δ, $f(\xi_{k,2}) - f(\xi_{k,1}) < \epsilon$.

$$\begin{aligned} \left|E_k - \widetilde{E_k}\right| \le \widetilde{E_{k,2}} - \widetilde{E_{k,1}} &= f(\xi_{k,2})(x_k - x_{k-1}) - f(\xi_{k,1})(x_k - x_{k-1}) \\ &= (f(\xi_{k,2}) - f(\xi_{k,1}))(x_k - x_{k-1}) < (x_k - x_{k-1})\epsilon. \end{aligned}$$

E_k $\widetilde{E_k}$: $\left|E_k - \widetilde{E_k}\right|$ $(x_k - x_{k-1})\epsilon$.



Σχήμα 7.3: «», «» «» . .

$\left|E_1-\widetilde{E_1}\right|+\cdots+\left|E_n-\widetilde{E_n}\right|$, , $\left|E-\widetilde{E}\right|$. :

$$\begin{aligned}\left|E-\widetilde{E}\right| &\leq \left|E_1-\widetilde{E_1}\right|+\cdots+\left|E_n-\widetilde{E_n}\right| < (x_1-x_0)\epsilon+\cdots+(x_n-x_{n-1})\epsilon \\ &= (x_1-x_0+\cdots+x_n-x_{n-1})\epsilon=(b-a)\epsilon.\end{aligned}$$

$\left|E-\widetilde{E}\right|$ $(b-a)\epsilon$. , ϵ , $\left|E-\widetilde{E}\right|$. ¨ , , Δ · δ ϵ.

: (1) . , , A . $y=f(x)=c\geq 0$ $[a,b]$. $\Delta=\{x_0=a,x_1,\dots,x_{n-1},x_n=b\}$ $\Xi=\{\xi_1,\dots,\xi_n\}$:

$$\begin{aligned}\Sigma(f;a,b;\Delta;\Xi) &= f(\xi_1)(x_1-x_0)+\cdots+f(\xi_n)(x_n-x_{n-1}) \\ &= (x_1-x_0)c+\cdots+(x_n-x_{n-1})c \\ &= (b-a)c.\end{aligned}$$

, , $\Sigma(f;a,b;\Delta;\Xi)$ E Δ , $\Sigma(f;a,b;\Delta;\Xi)$ $(b-a)c$,

$$E=(b-a)c,$$

, , .

(2) . A $[a,b]$ $0\leq a<b$ $y=f(x)=x$. , $\Delta=\{a,a+\frac{b-a}{n},a+2\frac{b-a}{n},\dots,a+k\frac{b-a}{n},\dots,a+(n-1)\frac{b-a}{n},a+n\frac{b-a}{n}=b\}$, $\frac{b-a}{n}$. Δ $\frac{b-a}{n}$. , , $\Xi=\{a+\frac{b-a}{n},a+2\frac{b-a}{n},\dots,a+k\frac{b-a}{n},\dots,a+(n-1)\frac{b-a}{n},a+n\frac{b-a}{n}=b\}$, . $\Sigma(f;a,b;\Delta;\Xi)$

$$\begin{aligned}\Sigma(f;a,b;\Delta;\Xi) &= \Big(a+\frac{b-a}{n}\Big)\frac{b-a}{n}+\Big(a+2\frac{b-a}{n}\Big)\frac{b-a}{n}+\cdots \\ &\qquad\cdots+\Big(a+(n-1)\frac{b-a}{n}\Big)\frac{b-a}{n}+\Big(a+n\frac{b-a}{n}\Big)\frac{b-a}{n} \\ &= \Big(na+\big(1+2+\cdots+(n-1)+n\big)\frac{b-a}{n}\Big)\frac{b-a}{n} \\ &= \Big(na+\frac{n(n+1)}{2}\frac{b-a}{n}\Big)\frac{b-a}{n} \\ &= a(b-a)+\frac{n+1}{2n}(b-a)^2\,.\end{aligned}$$

(: $1+2+\cdots+(n-1)+n=\frac{n(n+1)}{2}$.) , Δ, $\frac{b-a}{n}$, , , n , $\Sigma(f;a,b;\Delta;\Xi)=a(b-a)+\frac{n+1}{2n}(b-a)^2$ E.

$$E=\lim_{n\to+\infty}\Big(a(b-a)+\frac{n+1}{2n}(b-a)^2\Big)=a(b-a)+\frac{1}{2}(b-a)^2=\frac{b^2-a^2}{2}.$$

(3) . A $[a,b]$ $y=f(x)=x^2$, , .
$\Delta=\{a,a+\frac{b-a}{n},a+2\frac{b-a}{n},\dots,a+k\frac{b-a}{n},\dots,a+(n-1)\frac{b-a}{n},a+n\frac{b-a}{n}=b\}$, $\frac{b-a}{n}$, , $\frac{b-a}{n}$. $\Xi=\{a+\frac{b-a}{n},a+2\frac{b-a}{n},\dots,a+k\frac{b-a}{n},\dots,a+(n-1)\frac{b-a}{n},a+n\frac{b-a}{n}=b\}$, . $\Sigma(f;a,b;\Delta;\Xi)$

$$\begin{aligned}\Sigma(f;a,b;\Delta;\Xi)&=\Big(a+\frac{b-a}{n}\Big)^2\frac{b-a}{n}+\Big(a+2\frac{b-a}{n}\Big)^2\frac{b-a}{n}+\cdots\\&\quad\cdots+\Big(a+(n-1)\frac{b-a}{n}\Big)^2\frac{b-a}{n}+\Big(a+n\frac{b-a}{n}\Big)^2\frac{b-a}{n}\\&=\Big(na^2+2a\big(1+2+\cdots+(n-1)+n\big)\frac{b-a}{n}\\&\quad+\big(1^2+2^2+\cdots+(n-1)^2+n^2\big)\frac{(b-a)^2}{n^2}\Big)\frac{b-a}{n}\\&=a^2(b-a)+\frac{n+1}{n}a(b-a)^2+\frac{(n+1)(2n+1)}{6n^2}(b-a)^3.\end{aligned}$$

(: $1^2+2^2+\cdots+(n-1)^2+n^2=\frac{n(n+1)(2n+1)}{6}$.) , Δ, $\frac{b-a}{n}$, , , n , $\Sigma(f;a,b;\Delta;\Xi)=a^2(b-a)+\frac{n+1}{n}a(b-a)^2+\frac{(n+1)(2n+1)}{6n^2}(b-a)^3$ E. ,

$$\begin{aligned}E&=\lim_{n\to+\infty}\Big(a^2(b-a)+\frac{n+1}{n}a(b-a)^2+\frac{(n+1)(2n+1)}{6n^2}(b-a)^3\Big)\\&=a^2(b-a)+a(b-a)^2+\frac{1}{3}(b-a)^3=\frac{b^3-a^3}{3}.\end{aligned}$$

.

1. $y=x^3$ $[a,b]$ $a\geq 0$. $1^3+2^3+\cdots+(n-1)^3+n^3=\frac{n^2(n+1)^2}{4}$.

7.2 Riemann.

, , . 7.4 .
$y=f(x)$ $[a,b]$. ≥ 0. $\Delta=\{x_0=a,x_1,\dots,x_{n-1},x_n=b\}$ $[a,b]$ $\Xi=\{\xi_1,\dots,\xi_n\}$

$$\Sigma(f;a,b;\Delta;\Xi)=f(\xi_1)(x_1-x_0)+\cdots+f(\xi_n)(x_n-x_{n-1}).$$

$\Sigma(f;a,b;\Delta;\Xi)$ **Riemann** $y=f(x)$ $[a,b]$, Δ Ξ . I $\Sigma(f;a,b;\Delta;\Xi)$ I , , $|\Sigma(f;a,b;\Delta;\Xi)-I|$ Δ . $y=f(x)$ **Riemann** $[a,b]$, I **Riemann** $y=f(x)$ $[a,b]$

$$\int_a^b f(x)\,dx.$$

: $y = f(x)$ *Riemann* $[a,b]$ *Riemann* $\int_a^b f(x)\,dx$ $\epsilon > 0$ $\delta > 0$

$$\left|\Sigma(f;a,b;\Delta;\Xi) - \int_a^b f(x)\,dx\right| < \epsilon$$

Δ $[a,b]$ $(\Delta) < \delta$ Ξ .

, , « Riemann» «Riemann » «» «», .

.

$$\int_a^b f(x)\,dx, \quad \int_a^b f(y)\,dy, \quad \int_a^b f(t)\,dt, \quad \int_a^b f(u)\,du$$

, $y = f(x)$ $[a,b]$.

Riemann 0.

$$: \quad \lim_{(\Delta)\to 0} \Sigma(f;a,b;\Delta;\Xi) = \int_a^b f(x)\,dx$$

. . Riemann $\Sigma(f;a,b;\Delta;\Xi) = f(\xi_1)(x_1-x_0)+\cdots+f(\xi_n)(x_n-x_{n-1})$, « », $\Sigma f(x)\Delta x$, (Σ) : $(f(x))$ (Δx). Sum (=), S $f(x)\Delta x$ Riemann , , Δx dx ( ) S «» $\int$.

, $y = f(x)$ $[a,b]$, (Δ) , $\Sigma(f;a,b;\Delta;\Xi) = f(\xi_1)(x_1-x_0)+\cdots+f(\xi_n)(x_n-x_{n-1})$ $I-$ $y = f(x)$ $[a,b]$. : ( ) , , $\Sigma(f;a,b;\Delta;\Xi)$. · : « » « ». , $y = f(x)$ $[a,b]$, $\int_a^b f(x)\,dx$ $\Sigma(f;a,b;\Delta;\Xi)$.

. .

7.2 $y = f(x)$ $[a,b]$, $[a,b]$.

.

7.3 $y = f(x)$ $[a,b]$, $[a,b]$.

7.2 7.3 ′ .

: (1) $y = f(x)$ $[a,b]$ $f(x) \geq 0$ x $[a,b]$. E A $[a,b]$, $y = f(x)$, . $(a,0)$ $(a,f(a))$. $(b,0)$ $(b,f(b))$.

$$E = \int_a^b f(x)\,dx$$

, , E , , $\Sigma(f;a,b;\Delta;\Xi) = f(\xi_1)(x_1-x_0)+\cdots+f(\xi_n)(x_n-x_{n-1})$.

(2) x- : $y = c \geq 0$ $y = x^2$ $[a,b]$ $y = x$ $[a,b]$ $0 \leq a < b$. :

$$\int_a^b 1\,dx = b-a, \qquad \int_a^b x\,dx = \frac{b^2-a^2}{2}, \qquad \int_a^b x^2\,dx = \frac{b^3-a^3}{3}.$$

Riemann . . Δ Ξ . $y = x$ $y = x^2$, 7.2.

$0 \le a < b$ $\int_a^b x\,dx = \frac{b^2-a^2}{2}$ $x \ge 0$ $[a,b]$ $\int_a^b x\,dx$. , , Riemann a, b $a < b$.

(3) – – , $\int_a^b \frac{1}{x}\,dx$ $0 < a < b$, . 7.2 , $y = \frac{1}{x}$ $[a,b]$. $[a,b]$, , Riemann .

n $\Delta = \{a, a\mu, a\mu^2, \dots, a\mu^{n-1}, a\mu^n = b\}$, μ $a\mu^n = b$: $\mu = \sqrt[n]{\frac{b}{a}} > 1$. : . , , $\Xi = \{a\mu, a\mu^2, \dots, a\mu^{n-1}, a\mu^n = b\}$, . k- $a\mu^k - a\mu^{k-1} = a\big(1-\frac{1}{\mu}\big)\mu^k$, $\mu > 1$, n-. : $(\Delta) = a(1-\frac{1}{\mu})\mu^n = b\big(1 - \sqrt[n]{\frac{a}{b}}\big)$. $\lim_{n\to+\infty} (\Delta) = \lim_{n\to+\infty} b\big(1-\sqrt[n]{\frac{a}{b}}\big) = b(1-1) = 0$. , n , Δ , Riemann . , , Riemann:

$$\begin{aligned}\Sigma(f;a,b;\Delta,\Xi) &= \frac{1}{a\mu}(a\mu - a) + \frac{1}{a\mu^2}(a\mu^2 - a\mu) + \cdots \\ &\quad \cdots + \frac{1}{a\mu^{n-1}}(a\mu^{n-1} - a\mu^{n-2}) + \frac{1}{a\mu^n}(a\mu^n - a\mu^{n-1}) \\ &= \Big(1-\frac{1}{\mu}\Big) + \Big(1-\frac{1}{\mu}\Big) + \cdots + \Big(1-\frac{1}{\mu}\Big) + \Big(1-\frac{1}{\mu}\Big) \\ &= n\Big(1-\frac{1}{\mu}\Big) = n\Big(1-\sqrt[n]{\frac{a}{b}}\Big).\end{aligned}$$

$$\int_a^b \frac{1}{x}\,dx = \lim_{n\to+\infty} n\Big(1-\sqrt[n]{\frac{a}{b}}\,\Big) = \log\frac{b}{a}\,.$$

$\lim_{x\to 0}\frac{a^x-1}{x} = \frac{d\,a^x}{dx}\big|_{x=0} = a^0\log a = \log a$. , $x_n = \frac{1}{n}$, $\lim_{n\to+\infty} n(\sqrt[n]{a}-1) = \log a$.

(4) . $[a,b]$, c $[a,b]$, $a \le c \le b$,

$$y = f(x) = \begin{cases} 1\,, & x = c, \\ 0\,, & a \le x \le b \ \ x \ne c. \end{cases}$$

$\Delta = \{x_0 = a, x_1, \dots, x_{n-1}, x_n = b\}$ $\Xi = \{\xi_1, \dots, \xi_n\}$. . c, $\Sigma(f;a,b;\Delta,\Xi) = f(\xi_1)(x_1-x_0) + \cdots + f(\xi_n)(x_n - x_{n-1}) = 0(x_1-x_0) + \cdots + 0(x_n-x_{n-1}) = 0$. c, $\xi_k = c$ k, $\Sigma(f;a,b;\Delta,\Xi) = f(\xi_1)(x_1-x_0) + \cdots + f(\xi_k)(x_k-x_{k-1}) + \cdots + f(\xi_n)(x_n-x_{n-1}) = 0(x_1-x_0) + \cdots + 1(x_k-x_{k-1}) + \cdots + 0(x_n-x_{n-1}) = x_k - x_{k-1}$. c, , . $\xi_k = \xi_{k+1} = x_k = c$ k, $\Sigma(f;a,b;\Delta,\Xi) = f(\xi_1)(x_1-x_0) + \cdots + f(\xi_n)(x_n-x_{n-1}) = 0(x_1-x_0) + \cdots + f(\xi_k)(x_k-x_{k-1}) + f(\xi_{k+1})(x_{k+1}-x_k) + \cdots + 0(x_n-x_{n-1}) = 1(x_k-x_{k-1}) + 1(x_{k+1}-x_k) = x_{k+1}-x_{k-1}$. c . , ,

$$0 \le \Sigma(f;a,b;\Delta,\Xi) \le 2\ (\Delta).$$

$\Sigma(f;a,b;\Delta,\Xi)$ 0 (Δ) . , $\epsilon > 0$ $\delta = \frac{\epsilon}{2} > 0$ Δ $(\Delta) < \delta$ Ξ $\big|\Sigma(f;a,b;\Delta,\Xi) - 0\big| = \Sigma(f;a,b;\Delta,\Xi) < 2\delta = \epsilon$. , ,

$$\int_a^b f(x)\,dx = 0.$$

$y = f(x)$ $[a,b]$ x- : $(a,0)$ $(b,0)$. $(c,0)$ $(c,1)$. . 0 . 0. . 0 , , $\int_a^b f(x)\,dx$.

.

1. – $\int_a^b x\,dx$ $\int_a^b x^2\,dx$ –

$$\int_a^b \alpha^x\,dx = \frac{\alpha^b - \alpha^a}{\log\alpha}$$

a, b $a < b$ $\alpha > 0$, $\alpha \neq 1$. ,

$$\int_a^b e^x\,dx = e^b - e^a\,.$$

2. $\int_a^b \frac{1}{x}\,dx$

$$\int_a^b x^\alpha\,dx = \frac{b^{\alpha+1} - a^{\alpha+1}}{\alpha+1}$$

a, b $0 < a < b$ α.

3.

$$\int_a^b \cos x\,dx = \sin b - \sin a, \qquad \int_a^b \sin x\,dx = \cos a - \cos b$$

a, b $a < b$.

$$\cos(p+q) + \cos(p+2q) + \cdots + \cos(p+nq) = \frac{\sin\frac{nq}{2}\cos(p + \frac{(n+1)q}{2})}{\sin\frac{q}{2}}\,,$$

$$\sin(p+q) + \sin(p+2q) + \cdots + \sin(p+nq) = \frac{\sin\frac{nq}{2}\sin(p + \frac{(n+1)q}{2})}{\sin\frac{q}{2}}\,.$$

12 1.4.

4. . .

(i) $\lim_{n\to+\infty}\big(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+(n-1)^2} + \frac{n}{n^2+n^2}\big)$.

(ii) $\lim_{n\to+\infty}\frac{\sqrt{n^2-0^2}+\sqrt{n^2-1^2}+\cdots+\sqrt{n^2-(n-1)^2}}{n^2}$.

(iii) $\lim_{n\to+\infty}\Big(\frac{1}{\sqrt{n^2+1^2}} + \frac{1}{\sqrt{n^2+2^2}} + \cdots + \frac{1}{\sqrt{n^2+(n-1)^2}} + \frac{1}{\sqrt{n^2+n^2}}\Big)$.

(iv) $\lim_{n\to+\infty}\frac{\sqrt{n+1}+\sqrt{n+2}+\cdots+\sqrt{n+(n-1)}+\sqrt{n+n}}{n\sqrt{n}}$.

(: (i) k- $\frac{n}{n^2+k^2} = \frac{1}{1+(\frac{k}{n})^2}\cdot\frac{1}{n} = f(\xi_k)(x_k - x_{k-1})$ $y = f(x)$ $[0,1]$, $\Delta = \{x_0, x_1, \ldots, x_n\}$ $[0,1]$ $\Xi = \{\xi_1, \ldots, \xi_n\}$. $(ii) - (iv)$.)

5. $\lim_{n\to+\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+(n-1)}+\frac{1}{n+n}\right)$ $[1,2]$.

(: .)

6. (*) $y=f(x)$ $[a,b]$ $A=f(a)$, $B=f(b)$. $x=f^{-1}(y)$ $[A,B]$.

$$\int_a^b f(x)\,dx+\int_A^B f^{-1}(y)\,dy=Bb-Aa.$$

(: $\{x_0=a,x_1,\dots,x_{n-1},x_n=b\}$ $[a,b]$ $\{y_0=A,y_1,\dots,y_{n-1},y_n=B\}$ $[A,B]$ $y_k=f(x_k)$ k. $y_1(x_1-x_0)+y_2(x_2-x_1)+\cdots+y_{n-1}(x_{n-1}-x_{n-2})+y_n(x_n-x_{n-1})$ $x_0(y_1-y_0)+x_1(y_2-y_1)+\cdots+x_{n-2}(y_{n-1}-y_{n-2})+x_{n-1}(y_n-y_{n-1})$; .)

;

(: $y=f(x)$ $x=f^{-1}(y)$.)

7.3 Riemann.

. .

7.1 $y=f(x)$ $[a,b]$ λ . $y=\lambda f(x)$ $[a,b]$

$$\boxed{\int_a^b \big(\lambda f(x)\big)\,dx=\lambda\int_a^b f(x)\,dx.}$$

: $\Delta=\{x_0=a,x_1,\dots,x_{n-1},x_n=b\}$ $\Xi=\{\xi_1,\dots,\xi_n\}$

$$\begin{aligned}\Sigma(\lambda f;a,b;\Delta;\Xi)&=\lambda f(\xi_1)(x_1-x_0)+\cdots+\lambda f(\xi_n)(x_n-x_{n-1})\\&=\lambda\Big(f(\xi_1)(x_1-x_0)+\cdots+f(\xi_n)(x_n-x_{n-1})\Big)\\&=\lambda\Sigma(f;a,b;\Delta;\Xi).\end{aligned}$$

$\epsilon>0$, $\delta>0$ Δ $(\Delta)<\delta$ Ξ $\left|\Sigma(f;a,b;\Delta;\Xi)-\int_a^b f(x)\,dx\right|<\frac{\epsilon}{|\lambda|+1}$. $\left|\Sigma(\lambda f;a,b;\Delta;\Xi)-\lambda\int_a^b f(x)\,dx\right|=\left|\lambda\Sigma(f;a,b;\Delta;\Xi)-\lambda\int_a^b f(x)\,dx\right|\le|\lambda|\frac{\epsilon}{|\lambda|+1}<\epsilon.$, $y=\lambda f(x)$ $\int_a^b\big(\lambda f(x)\big)\,dx=\lambda\int_a^b f(x)\,dx$.

7.1. $f(x)\ge 0$ x $[a,b]$ A $y=f(x)$ $[a,b]$ x-. $\lambda\ge 0$ B $y=\lambda f(x)$ $[a,b]$ x-, B $A-$ $[a,b]-$ $B-$ $\lambda f(x)-$ $A-$ $f(x)-$ λ. 7.1 B A λ.

. $f(x)\le 0$ x $[a,b]$ A $y=f(x)$ $[a,b]$ x-. A , , x-. B $y=-f(x)$ $[a,b]$ x-, B A x-. $-f(x)\ge 0$, $\int_a^b\big(-f(x)\big)\,dx$ B, , , A. 7.1 $-\int_a^b f(x)\,dx$ $\int_a^b\big(-f(x)\big)\,dx$, A. , $\int_a^b f(x)\,dx$ A.

7.2 $y=f(x)$ $y=g(x)$ $[a,b]$. $y=f(x)+g(x)$ $[a,b]$

$$\boxed{\int_a^b\big(f(x)+g(x)\big)\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx.}$$

: $\Delta = \{x_0 = a, x_1, \dots, x_{n-1}, x_n = b\}$ $\Xi = \{\xi_1, \dots, \xi_n\}$

$$\begin{aligned}\Sigma(f+g;a,b;\Delta;\Xi) &= \big(f(\xi_1)+g(\xi_1)\big)(x_1-x_0)+\cdots+\big(f(\xi_n)+g(\xi_n)\big)(x_n-x_{n-1})\\ &= f(\xi_1)(x_1-x_0)+\cdots+f(\xi_n)(x_n-x_{n-1})\\ &\quad +g(\xi_1)(x_1-x_0)+\cdots+g(\xi_n)(x_n-x_{n-1})\\ &= \Sigma(f;a,b;\Delta;\Xi)+\Sigma(g;a,b;\Delta;\Xi).\end{aligned}$$

$\epsilon > 0$, $\delta' > 0$ Δ $(\Delta) < \delta'$ Ξ $\left|\Sigma(f;a,b;\Delta;\Xi) - \int_a^b f(x)\,dx\right| < \frac{\epsilon}{2}$ $\delta'' > 0$ Δ $(\Delta) < \delta''$ Ξ $\left|\Sigma(g;a,b;\Delta;\Xi) - \int_a^b g(x)\,dx\right| < \frac{\epsilon}{2}$. $\delta = \min\{\delta', \delta''\}$, Δ $(\Delta) < \delta$ Ξ , , $\left|\Sigma(f+g;a,b;\Delta;\Xi) - \left(\int_a^b f(x)\,dx + \int_a^b g(x)\,dx\right)\right| = \left|\big(\Sigma(f;a,b;\Delta;\Xi)+\Sigma(g;a,b;\Delta;\Xi)\big) - \left(\int_a^b f(x)\,dx + \int_a^b g(x)\,dx\right)\right| \le \left|\Sigma(f;a,b;\Delta;\Xi) - \int_a^b f(x)\,dx\right| + \left|\Sigma(g;a,b;\Delta;\Xi) - \int_a^b g(x)\,dx\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. , $y = f(x) + g(x)$ $\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

$f(x), g(x) \ge 0$ x $[a,b]$. A $y = f(x)$ $[a,b]$ x-, B $y = g(x)$ $[a,b]$ C $y = f(x) + g(x)$ $[a,b]$, C – $f(x) + g(x)$ – A – $f(x)$ – B – $g(x)$. 7.2 C A B.

7.1 7.2, , $y = f(x)$ $y = g(x)$ $[a,b]$ λ μ , $y = \lambda f(x) + \mu g(x)$ $[a,b]$

$$\int_a^b \big(\lambda f(x) + \mu g(x)\big)\,dx = \lambda \int_a^b f(x)\,dx + \mu \int_a^b g(x)\,dx.$$

, , , $y = f_1(x)$, $\dots$, $y = f_m(x)$ $[a,b]$ $\lambda_1, \dots, \lambda_m$, $\lambda_1 f_1(x) + \cdots + \lambda_m f_m(x)$ $[a,b]$

$$\int_a^b \big(\lambda_1 f_1(x) + \cdots + \lambda_m f_m(x)\big)\,dx = \lambda_1 \int_a^b f_1(x)\,dx + \cdots + \lambda_m \int_a^b f_m(x)\,dx.$$

: (1) $\int_a^b (\lambda + \mu x + \nu x^2)\,dx = \lambda \int_a^b 1\,dx + \mu \int_a^b x\,dx + \nu \int_a^b x^2\,dx = \lambda(b-a) + \mu\frac{b^2-a^2}{2} + \nu\frac{b^3-a^3}{3} = \big(\lambda b + \mu\frac{b^2}{2} + \nu\frac{b^3}{3}\big) - \big(\lambda a + \mu\frac{a^2}{2} + \nu\frac{a^3}{3}\big)$. .

(2) $\int_a^b (\rho\frac{1}{x} + \lambda + \mu x + \nu x^2)\,dx = \rho\int_a^b \frac{1}{x}\,dx + \int_a^b (\lambda + \mu x + \nu x^2)\,dx = \rho\log\frac{b}{a} + \lambda(b-a) + \mu\frac{b^2-a^2}{2} + \nu\frac{b^3-a^3}{3} = \big(\rho\log b + \lambda b + \mu\frac{b^2}{2} + \nu\frac{b^3}{3}\big) - \big(\rho\log a + \lambda a + \mu\frac{a^2}{2} + \nu\frac{a^3}{3}\big)$ $0 < a < b$.

(3) $[a,b]$ m $c_1, \dots, c_m$ $[a,b]$. $y = f(x)$ $f(x) = 0$ x $[a,b]$ $c_1, \dots, c_m$. $y = f(x)$ $[a,b]$ $\int_a^b f(x)\,dx = 0$.

. $\lambda_1, \dots, \lambda_m$ $c_1, \dots, c_m$, . c_k $y = f_k(x) = \begin{cases} 1, & x = c_k, \\ 0, & a \le x \le b \;\; x \ne c_k. \end{cases}$ $f(x) = \lambda_1 f_1(x) + \cdots + \lambda_m f_m(x)$ x $[a,b]$, , $y = f_k(x)$ $[a,b]$ $\int_a^b f_k(x)\,dx = 0$, $y = f(x)$ $[a,b]$ $\int_a^b f(x)\,dx = \lambda_1 \int_a^b f_1(x)\,dx + \cdots + \lambda_m \int_a^b f_m(x)\,dx = \lambda_1 0 + \cdots + \lambda_m 0 = 0$.

$y = f(x)$ $[a,b]$ x- m . 0 .

.

7.3 $y = f(x)$ $y = g(x)$ $[a,b]$. $y = f(x)g(x)$ $[a,b]$.

, , . , $\int_a^b f(x)g(x)\,dx = \int_a^b f(x)\,dx \int_a^b g(x)\,dx$. , ,
.

: $\int_a^b 1{\cdot}1\,dx = \int_a^b 1\,dx = b{-}a$ $\int_a^b 1\,dx \int_a^b 1\,dx = (b{-}a)(b{-}a) = (b{-}a)^2$.
$b - a = (b-a)^2$!

7.4 $y = f(x)$ $[a,b]$. $m > 0$ $|f(x)| \geq m$ x $[a,b]$, $y = \frac{1}{f(x)}$ $[a,b]$.

, , . , $\int_a^b \frac{1}{f(x)}\,dx = \frac{1}{\int_a^b f(x)\,dx}$.

: , , $\int_a^b \frac{1}{1}\,dx = \int_a^b 1\,dx = b-a$ $\frac{1}{\int_a^b 1\,dx} = \frac{1}{b-a}$. $b - a = \frac{1}{b-a}$!

. .

7.5 $y = f(x)$ $y = g(x)$ $[a,b]$ $[a,b]$. $[a,b]$, $[a,b]$

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx.$$

: $y = f(x)$ $[a,b]$. $y = h(x) = g(x) - f(x)$, $h(x) = 0$ x $[a,b]$ $[a,b]$, , , $[a,b]$ $\int_a^b h(x)\,dx = 0$. $g(x) = f(x) + h(x)$ x $[a,b]$, $y = g(x)$ $[a,b]$ $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b h(x)\,dx = \int_a^b f(x)\,dx$.

7.5 .

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,

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$f(x), g(x) \geq 0$ x $[a,b]$. A $y = f(x)$ $[a,b]$ B $y = g(x)$ $[a,b]$, m . . 0, A B . 7.5.

. .

7.6 $y = f(x)$ $[a,b]$. $y = f(x)$ $[c,d]$ $[a,b]$.

7.6.

7.7 $y = f(x)$ $[a,b]$ $[b,c]$. $y = f(x)$ $[a,c]$

$$\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.}$$

: $y = f(x)$ $[a,b]$ $[b,c]$, , M' M'' $|f(x)| \leq M'$ x $[a,b]$ $|f(x)| \leq M''$ x $[b,c]$. $M = \max\{M', M''\}$ $|f(x)| \leq M' \leq M$ x $[a,b]$ $|f(x)| \leq M'' \leq M$ x $[b,c]$. $|f(x)| \leq M$ x $[a,c]$, $[a,c]$.

$\Delta = \{x_0 = a, x_1, \ldots, x_{n-1}, x_n = c\}$ $[a,c]$ b : $b = x_k$ k $1 \le k \le n-1$. $\Xi = \{\xi_1, \ldots, \xi_n\}$. $\Delta' = \{x_0 = a, x_1, \ldots, x_{k-1}, x_k = b\}$ $[a,b]$ $\Xi' = \{\xi_1, \ldots, \xi_k\}$. $\Delta'' = \{x_k = b, x_{k+1}, \ldots, x_{n-1}, x_n = c\}$ $[b,c]$ $\Xi'' = \{\xi_{k+1}, \ldots, \xi_n\}$. :

$$\begin{aligned}\Sigma(f;a,c;\Delta,\Xi) &= f(\xi_1)(x_1-x_0)+\cdots+f(\xi_k)(x_k-x_{k-1})\\ &\quad +f(\xi_{k+1})(x_{k+1}-x_k)+\cdots+f(\xi_n)(x_n-x_{n-1})\\ &= \Sigma(f;a,b;\Delta',\Xi')+\Sigma(f;b,c;\Delta'',\Xi'').\end{aligned}$$

$\epsilon > 0$, $\delta' > 0$ $\left|\Sigma(f;a,b;\Delta',\Xi') - \int_a^b f(x)\,dx\right| < \frac{\epsilon}{3}$ Δ' $(\Delta') < \delta'$ Ξ' $\delta'' > 0$ $\left|\Sigma(f;a,b;\Delta'',\Xi'') - \int_b^c f(x)\,dx\right| < \frac{\epsilon}{3}$ Δ'' $(\Delta'') < \delta''$ Ξ''. $\delta = \min\left\{\delta', \delta'', \frac{\epsilon}{6M+1}\right\}$ Δ $[a,c]$ $(\Delta) < \delta$ Ξ. .

1. Δ b . , , Δ Δ' $[a,b]$ Δ'' $[b,c]$ Ξ Ξ' Ξ''. , $(\Delta') \le (\Delta) < \delta \le \delta'$ $(\Delta'') \le (\Delta) < \delta \le \delta''$, , $\left|\Sigma(f;a,b;\Delta',\Xi') - \int_a^b f(x)\,dx\right| < \frac{\epsilon}{3}$ $\left|\Sigma(f;a,b;\Delta'',\Xi'') - \int_b^c f(x)\,dx\right| < \frac{\epsilon}{3}$. , , $\Sigma(f;a,c;\Delta,\Xi) = \Sigma(f;a,b;\Delta',\Xi') + \Sigma(f;b,c;\Delta'',\Xi'')$. ,

$$\begin{aligned}\left|\Sigma(f;a,c;\Delta,\Xi) - \left(\int_a^b f(x)\,dx + \int_b^c f(x)\,dx\right)\right| &= \left|\Big(\Sigma(f;a,b;\Delta',\Xi') + \Sigma(f;b,c;\Delta'',\Xi'')\Big)\right.\\ &\quad \left.-\left(\int_a^b f(x)\,dx + \int_b^c f(x)\,dx\right)\right|\\ &\le \left|\Sigma(f;a,b;\Delta',\Xi') - \int_a^b f(x)\,dx\right|\\ &\quad +\left|\Sigma(f;b,c;\Delta'',\Xi'') - \int_b^c f(x)\,dx\right|\\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3} < \epsilon.\end{aligned}$$

2. Δ b . $x_{k-1} < b < x_k$ k $1 \le k \le n$. $\Delta^* = \{x_0 = a, \ldots, x_{k-1}, b, x_k, \ldots, x_n = c\}$, b Δ. $(\Delta^*) \le (\Delta) < \delta$. , $\Xi^* = \{\xi_1, \ldots, \xi_{k-1}, \eta, \zeta, \xi_{k+1}, \ldots, \xi_n\}$, η $[x_{k-1}, b]$ ζ $[b, x_k]$ ξ_k $[x_{k-1}, x_k]$.

$$\begin{aligned}\left|\Sigma(f;a,c;\Delta,\Xi) - \Sigma(f;a,c;\Delta^*,\Xi^*)\right| &= \left|f(\xi_k)(x_k-x_{k-1}) - f(\eta)(b-x_{k-1}) - f(\zeta)(x_k-b)\right|\\ &= \Big|f(\xi_k)(b-x_{k-1}) + f(\xi_k)(x_k-b)\\ &\quad -f(\eta)(b-x_{k-1}) - f(\zeta)(x_k-b)\Big|\\ &\le \left|f(\xi_k)-f(\eta)\right|(b-x_{k-1}) + \left|f(\xi_k)-f(\zeta)\right|(x_k-b)\\ &\le 2M(b-x_{k-1}) + 2M(x_k-b) = 2M(x_k-x_{k-1})\\ &< 2M\delta < \frac{\epsilon}{3}.\end{aligned}$$

Δ^* b , 1,

$$\left|\Sigma(f;a,c;\Delta^*,\Xi^*) - \left(\int_a^b f(x)\,dx + \int_b^c f(x)\,dx\right)\right| < \frac{2\epsilon}{3}.$$

$\left|\Sigma(f;a,c;\Delta,\Xi) - \Sigma(f;a,c;\Delta^*,\Xi^*)\right| < \frac{\epsilon}{3}$,

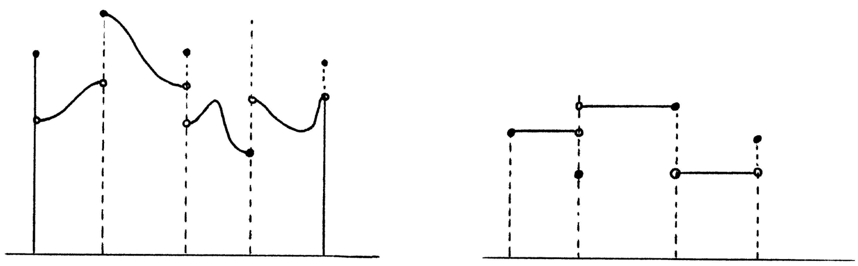
$$\begin{aligned}\left|\Sigma(f;a,c;\Delta,\Xi) - \left(\int_a^b f(x)\,dx + \int_b^c f(x)\,dx\right)\right| &\le \left|\Sigma(f;a,c;\Delta,\Xi) - \Sigma(f;a,c;\Delta^*,\Xi^*)\right|\\ &\quad +\left|\Sigma(f;a,c;\Delta^*,\Xi^*) - \left(\int_a^b f(x)\,dx + \int_b^c f(x)\,dx\right)\right|\\ &< \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.\end{aligned}$$

$\left|\Sigma(f;a,c;\Delta,\Xi) - \left(\int_a^b f(x)\,dx + \int_b^c f(x)\,dx\right)\right| < \epsilon.$
$y = f(x)$ $[a,c]$ $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$

$f(x) \geq 0$ x $[a,b]$. A, B C $y = f(x)$ $[a,b]$, $[b,c]$ $[a,c]$, , C A B A B . . . 0, C A B. 7.7.
7.7 . , $y = f(x)$ $[a_1,a_2]$, $[a_2,a_3]$, $\ldots$, $[a_{m-1},a_m]$, $y = f(x)$ $[a_1,a_m]$

$$\int_{a_1}^{a_m} f(x)\,dx = \int_{a_1}^{a_2} f(x)\,dx + \cdots + \int_{a_{m-1}}^{a_m} f(x)\,dx.$$

, . $[a,b]$ $y = f(x)$ ≥ 0 ≤ 0. ′ $y = f(x)$. $\int_a^b f(x)\,dx$ ≥ 0 ≤ 0.
7.7, .



Σχήμα 7.4: .

$y = f(x)$ $[a,b]$ $t_0 = a, t_1, \ldots, t_{m-1}, t_m = b$ $y = f(x)$ $(t_0,t_1), \ldots, (t_{m-1},t_m)$, $t_0,t_1,\ldots,t_{m-1},t_m$ − $t_0 = a$ $t_m = b$ t_k − .

7.8 $y = f(x)$ $[a,b]$, $[a,b]$.

: $y = f(x)$ $[a,b]$, . $y = f(x)$ $[t_{k-1},t_k]$. $y = g(x)$ $[t_{k-1},t_k]$, $g(x) = f(x)$ x (t_{k-1},t_k), $g(t_{k-1}) = \lim_{x\to t_{k-1}+} f(x) = \lim_{x\to t_{k-1}+} g(x)$ $g(t_k) = \lim_{x\to t_k-} f(x) = \lim_{x\to t_k-} g(x)$, $y = g(x)$ $[t_{k-1},t_k]$ $y = f(x)$: t_{k-1} t_k. $y = g(x)$ $[t_{k-1},t_k]$, $y = f(x)$ $[t_{k-1},t_k]$.

: $y = f(x)$ $[a,b]$. , $t_0 = a, t_1, \ldots, t_{m-1}, t_m = b$ $y = f(x)$ $(t_0,t_1), \ldots, (t_{m-1},t_m)$. λ_k $y = f(x)$ (t_{k-1},t_k). $y = f(x)$ $t_0,\ldots,t_m$. $y = f(x)$, , $[a,b]$, $\int_a^b f(x)\,dx$. $y = f(x)$ $[t_{k-1},t_k]$, $f(x) = \lambda_k$, , $\int_{t_{k-1}}^{t_k} f(x)\,dx = \int_{t_{k-1}}^{t_k} \lambda_k\,dx = \lambda_k(t_k - t_{k-1})$. $\int_a^b f(x)\,dx = \lambda_1(t_1 - t_0) + \cdots + \lambda_m(t_m - t_{m-1})$.

. .

7.9 $y=f(x)$ $y=g(x)$ $[a,b]$ $f(x)\le g(x)$ x $[a,b]$.

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

: $\Delta=\{x_0=a,x_1,\dots,x_{n-1},x_n=b\}$ $[a,b]$ $\Xi=\{\xi_1,\dots,\xi_n\}$.

$$\begin{aligned}\Sigma(f;a,b;\Delta,\Xi) &= f(\xi_1)(x_1-x_0)+\cdots+f(\xi_n)(x_n-x_{n-1})\\ &\le g(\xi_1)(x_1-x_0)+\cdots+g(\xi_n)(x_n-x_{n-1})\\ &= \Sigma(g;a,b;\Delta,\Xi).\end{aligned}$$

$\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$. ϵ $0<\epsilon\le\int_a^b f(x)\,dx-\int_a^b g(x)\,dx$. $\delta'>0$ $\left|\Sigma(f;a,b;\Delta,\Xi)-\int_a^b f(x)\,dx\right|<\frac{\epsilon}{2}$ Δ $(\Delta)<\delta'$ Ξ. , $\delta''>0$ $\left|\Sigma(g;a,b;\Delta,\Xi)-\int_a^b g(x)\,dx\right|<\frac{\epsilon}{2}$ Δ $(\Delta)<\delta''$ Ξ. $\delta=\min\{\delta',\delta''\}$, $\left|\Sigma(f;a,b;\Delta,\Xi)-\int_a^b f(x)\,dx\right|<\frac{\epsilon}{2}$ $\left|\Sigma(g;a,b;\Delta,\Xi)-\int_a^b g(x)\,dx\right|<\frac{\epsilon}{2}$ Δ $(\Delta)<\delta$ Ξ. , , $\Sigma(g;a,b;\Delta,\Xi)<\int_a^b g(x)\,dx+\frac{\epsilon}{2}\le\int_a^b f(x)-\frac{\epsilon}{2}<\Sigma(f;a,b;\Delta,\Xi)$, .

$0\le f(x)\le g(x)$ x $[a,b]$. A B $[a,b]$ $y=f(x)$ $y=g(x)$, , A B. 7.9 A B.

7.10 $y=f(x)$ $[a,b]$.

(1) u $y=f(x)$ $[a,b]$, $f(x)\le u$ x $[a,b]$, $\int_a^b f(x)\,dx\le(b-a)u$.

(2) l $y=f(x)$ $[a,b]$, $f(x)\ge l$ x $[a,b]$, $\int_a^b f(x)\,dx\ge(b-a)l$.

$y=h(x)=l$ $y=g(x)=u$ x $[a,b]$ 7.9, $\int_a^b h(x)\,dx=\int_a^b l\,dx=(b-a)l$ $\int_a^b g(x)\,dx=\int_a^b u\,dx=(b-a)u$.

: $y=\frac{x}{x^2+2}$ $[1,\sqrt{2}]$ $[\sqrt{2},4]$, $\frac{dy}{dx}=\frac{2-x^2}{(x^2+2)^2}$ >0 $(1,\sqrt{2})$ <0 $(\sqrt{2},4)$. $\frac{\sqrt{2}}{(\sqrt{2})^2+2}=\frac{\sqrt{2}}{4}$. , $\int_1^4\frac{x}{x^2+2}\,dx\le(4-1)\frac{\sqrt{2}}{4}=\frac{3\sqrt{2}}{4}$.

$0\le l\le f(x)\le u$ x $[a,b]$. A $y=f(x)$ $[a,b]$ B C $[a,b]$ l u, , A B C. 7.10 A B C.

7.11 $y=f(x)$ $[a,b]$. $y=|f(x)|$ $[a,b]$

$$\Big|\int_a^b f(x)\,dx\Big|\le\int_a^b|f(x)|\,dx.$$

$y=|f(x)|$ $[a,b]$. , , , $-|f(x)|\le f(x)\le|f(x)|$ x $[a,b]$ $-\int_a^b|f(x)|\,dx=\int_a^b f(x)\,dx\le\int_a^b|f(x)|\,dx$, , $\left|\int_a^b f(x)\,dx\right|\le\int_a^b|f(x)|\,dx$.

7.11 .

: $|\sin x|\le 1$ x $x>0$ $\left|\int_0^x\sin t\,dt\right|\le\int_0^x|\sin t|\,dt\le x$. , $|\sin x|\le|x|$ x $\left|\int_0^x\sin t\,dt\right|\le\int_0^x|\sin t|\,dt\le\int_0^x|t|\,dt=\int_0^x t\,dt=\frac{x^2}{2}$. $\left|\int_0^x\sin t\,dt\right|\le\min\left\{x,\frac{x^2}{2}\right\}=\begin{cases}\frac{x^2}{2}, & 0<x\le 2,\\ x, & x\ge 2.\end{cases}$

. .

$y_1, y_2, \ldots, y_n$, y_k ν_k , ,

$$\frac{\nu_1 y_1 + \cdots + \nu_n y_n}{\nu_1 + \cdots + \nu_n} = \frac{\nu_1}{\nu_1 + \cdots + \nu_n} y_1 + \cdots + \frac{\nu_n}{\nu_1 + \cdots + \nu_n} y_n = \mu_1 y_1 + \cdots + \mu_n y_n,$$

$\mu_k = \frac{\nu_k}{\nu_1 + \cdots + \nu_n}$ y_k $y_1, \ldots, y_n$. , , $y_1, \ldots, y_n$ ( ) $y_1, \ldots, y_n$.

. $y = f(x)$ $[a, b]$ $\Delta = \{x_0 = a, x_1, \ldots, x_{n-1}, x_n = b\}$ $[a, b]$ $\Xi = \{\xi_1, \ldots, \xi_n\}$. , , $f(\xi_1), \ldots, f(\xi_n)$. Δ , $[x_{k-1}, x_k]$, x ξ_k, $f(\xi_k)$ «» $f(x)$ x $[x_{k-1}, x_k]$, , $f(\xi_1), \ldots, f(\xi_n)$ «» . , , $[x_{k-1}, x_k]$ $[x_{l-1}, x_l]$, $f(\xi_k)$ «» «» $f(\xi_l)$. , , $f(x)$ «» $f(\xi_k)$ $\mu_k = \frac{x_k - x_{k-1}}{b-a}$ $[x_{k-1}, x_k]$ $b - a$. , $y = f(x)$,

$$\frac{x_1 - x_0}{b-a} f(\xi_1) + \cdots + \frac{x_n - x_{n-1}}{b-a} f(\xi_n)$$

«» $f(\xi_1), \ldots, f(\xi_n)$ Δ . , , ,

$$\frac{1}{b-a} \Sigma(f; a, b; \Delta; \Xi).$$

, , $\frac{1}{b-a} \int_a^b f(x)\, dx$ Δ .

, , $y = f(x)$ $[a, b]$, $[a, b]$

$$\boxed{y = f(x) \quad [a, b] = \frac{1}{b-a} \int_a^b f(x)\, dx.}$$

$y = f(x)$ $[a, b]$ ρ, $\int_a^b f(x)\, dx = (b-a)\rho = \int_a^b \rho\, dx$:

$y = f(x)$ $[a, b]$
$[a, b]$
$y = f(x)$.

. $f(x) \geq 0$ x $[a, b]$ A $y = f(x)$ $[a, b]$. $y = f(x)$ $[a, b]$ A $[a, b]$, , $[a, b]$ A.

: . , $y_1, \ldots, y_n$ $\mu_1, \ldots, \mu_n$. $x_0 = 0$, $x_1 = \mu_1$, $x_2 = \mu_1 + \mu_2$, ... , $x_{n-1} = \mu_1 + \cdots + \mu_{n-1}$ $x_n = \mu_1 + \cdots + \mu_{n-1} + \mu_n = 1$, , $y = f(x)$ $[0, 1]$ y_1 $[x_0 = 0, x_1)$, y_2 $[x_1, x_2)$, ... , y_n $[x_{n-1}, x_n = 1]$. $y = f(x)$ $[0, 1]$ $\frac{1}{1-0} \int_0^1 f(x)\, dx = y_1(x_1 - x_0) + y_2(x_2 - x_1) + \cdots + y_n(x_n - x_{n-1}) = \mu_1 y_1 + \mu_2 y_2 + \cdots + \mu_n y_n$.

7.10 . , , . , , , .

7.4 . $y = f(x)$ $[a, b]$. ξ $[a, b]$

$$\frac{1}{b-a} \int_a^b f(x)\, dx = f(\xi).$$

: - x_1 x_2 $[a,b]$ $f(x_1) \le f(x) \le f(x_2)$ x $[a,b]$. $(b-a)f(x_1) \le \int_a^b f(x)\,dx \le (b-a)f(x_2)$, $\frac{1}{b-a}\int_a^b f(x)\,dx$ $y = f(x)$. ξ $[a,b]$ $f(\xi)$.

: (1) $y = x^2$ $[0,1]$ $\int_0^1 x^2\,dx = \frac{1}{3}$. ξ $[0,1]$ $\xi^2 = \frac{1}{3}$ $\frac{1}{\sqrt{3}}$.

(2) $y = f(x)$ $[a,b]$, $[a,b]$. $y = \begin{cases} -1, & -1 \le x < 0, \\ 1, & 0 \le x \le 1, \end{cases}$ $[-1,1]$ $\frac{1}{1-(-1)}\left(\int_{-1}^0 (-1)\,dx + \int_0^1 1\,dx\right) = 0$, $[-1,1]$ 0.

.

. .

1. 1, 2 3 7.2, .

$$\int_{-1}^{2}(2-3x+4x^2)\,dx, \quad \int_{-2}^{4}(3x-2^x)\,dx, \quad \int_{\pi}^{2\pi}(3\cos x - 2\sin x)\,dx,$$

$$\int_0^{\pi}(3x-2\sin x)\,dx, \quad \int_1^3\left(\frac{2}{x} - x^2 + x^{\sqrt{2}} + 3e^x\right)dx.$$

. .

1. $\int_1^2 f(x)\,dx$ $y = f(x) = \begin{cases} 1+3x^2, & 1 < x < 2, \\ 0, & x = 1, \\ -2, & x = 2. \end{cases}$

2. $\int_{-1}^5 f(x)\,dx$ $y = f(x) = \begin{cases} x^2, & -1 \le x < 0, \\ 2x, & 0 \le x \le 2, \\ x+2, & 2 < x \le 5. \end{cases}$

3. $\int_{-2}^{\frac{7}{2}}[x]\,dx$.
   (: .)

4.
   (i) $\int_k^{k+1}\left(x - [x] - \frac{1}{2}\right)dx = 0$ k,
   (ii) $\int_k^{k+\frac{1}{2}}\left(x - [x] - \frac{1}{2}\right)dx = -\frac{1}{8}$ $\int_{k+\frac{1}{2}}^{k+1}\left(x - [x] - \frac{1}{2}\right)dx = \frac{1}{8}$ k,
   (*) (iii) $-\frac{1}{8} \le \int_a^b\left(x - [x] - \frac{1}{2}\right)dx \le \frac{1}{8}$ a, b $a < b$.
   (: a b.)

. .

1. $xe^{-2x} \le \int_x^{2x} e^{-t}\,dt \le xe^{-x}$ $x > 0$, .

2. $3e^{-2} \le \int_{\frac{1}{2}}^{2} xe^{-x}\,dx \le \frac{3}{2}e^{-1}$, .

3. $0 \leq \frac{x}{1-x+x^2} \leq \frac{4x}{3}$ x $[0,1]$ $0 \leq \frac{x}{1-x+x^2} \leq \frac{4}{3x}$ x $[1,+\infty)$.

,

(i) $0 \leq \int_0^x \frac{t}{1-t+t^2}\,dt \leq \frac{2x^2}{3}$ x $[0,1]$,

(ii) $0 \leq \int_0^x \frac{t}{1-t+t^2}\,dt \leq \frac{2}{3} + \frac{4}{3}\log x$ x $[1,+\infty)$.

4. , n

$$\int_0^{\pi} (\sin x)^{n+1}\,dx \leq \int_0^{\pi} (\sin x)^n\,dx, \quad \int_0^{\frac{\pi}{4}} (\tan x)^{n+1}\,dx \leq \int_0^{\frac{\pi}{4}} (\tan x)^n\,dx.$$

5. .

$$\lim_{x\to+\infty} \int_x^{x+\sqrt{x}} \frac{t}{1+t^2}\,dt, \quad \lim_{x\to 0+} \int_{1-x}^{1+x} \frac{t}{1+t^2}\,dt, \quad \lim_{x\to 0+} \frac{1}{2x} \int_{1-x}^{1+x} \frac{t}{1+t^2}\,dt.$$

(: $y = \frac{t}{1+t^2}$.)

6. (*) $|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$ x_1, x_2 $[0,1]$. , $y = f(x)$ Lipschitz- $[a,b]$ (8 6.9).

(i) $\left|\int_a^b f(x)\,dx - (b-a)f(b)\right| \leq M\frac{(b-a)^2}{2}$ $[a,b]$ $[0,1]$.

(: $(b-a)f(b) = \int_a^b f(b)\,dx$.)

(ii)

$$\left|\int_0^1 f(x)\,dx - \frac{1}{n}\sum_{k=1}^{n} f\Big(\frac{k}{n}\Big)\right| \leq \frac{M}{2n}$$

n.

(: (i) $[\frac{k-1}{n}, \frac{k}{n}]$ $k = 1, \ldots, n$.)

7. $y = f(x)$ $[a,b]$ $f(x) \geq 0$ x $[a,b]$. $0 \leq \int_c^d f(x)\,dx \leq \int_a^b f(x)\,dx$ $[c,d]$ $[a,b]$.

8. $\frac{1}{n+1} \leq \int_n^{n+1} \frac{1}{x}\,dx \leq \frac{1}{n}$, , $\frac{1}{n+1} \leq \log(n+1) - \log n \leq \frac{1}{n}$ n.

(x_n) $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n} - \log n$ n 0 , , .

$$\gamma = \lim_{n\to+\infty} \Big(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\Big)$$

Euler.

9. (*) $y = f(x)$ $y = g(x)$ $[a,b]$.

t, s $t^2 \int_a^b \big(f(x)\big)^2\,dx + 2ts\int_a^b f(x)g(x)\,dx + s^2\int_a^b \big(g(x)\big)^2\,dx = \int_a^b \big(tf(x) + sg(x)\big)^2\,dx \geq 0$.

Schwarz:

$$\Big(\int_a^b f(x)g(x)\,dx\Big)^2 \leq \int_a^b \big(f(x)\big)^2\,dx \int_a^b \big(g(x)\big)^2\,dx.$$

(: , $At^2+2Bts+Cs^2 \geq 0$ t,s, $B^2 \leq AC$.)

10. (*) $y=f(x)$ $y=g(x)$ $[a,b]$.

$$\frac{1}{2}\int_a^b \Big(\int_a^b \big(f(y)-f(x)\big)\big(g(y)-g(x)\big)\,dy\Big)\,dx$$
$$= (b-a)\int_a^b f(x)g(x)\,dx - \int_a^b f(x)\,dx \int_a^b g(x)\,dx.$$

$[a,b]$, $\int_a^b f(x)\,dx \int_a^b g(x)\,dx \leq (b-a)\int_a^b f(x)g(x)\,dx$.

$[a,b]$ $[a,b]$, $\int_a^b f(x)\,dx \int_a^b g(x)\,dx \geq (b-a)\int_a^b f(x)g(x)\,dx$.

(: $\big(f(y)-f(x)\big)\big(g(y)-g(x)\big)$.)

. .

1. .

(i) $y=x$ $[-1,1]$ $[0,1]$.

(ii) $y=x^2$ $[-1,1]$.

(iii) $y=\sin x$ $[0,\pi]$, $[0,\frac{\pi}{2}]$ $[0,2\pi]$. 3 .

2. (*) . $y=f(x)$ $y=w(x)$ $[a,b]$ $w(x)\geq 0$ x $[a,b]$, ξ $[a,b]$

$$\int_a^b f(x)w(x)\,dx = f(\xi)\int_a^b w(x)\,dx.$$

7.4 – $w(x)=1$ – .

′ , $y=w(x)$, $\int_a^b f(x)w(x)\,dx$ $y=f(x)$ - $y=w(x)$, $\int_a^b w(x)\,dx > 0$, $\frac{1}{\int_a^b w(x)\,dx}\int_a^b f(x)w(x)\,dx$ $y=f(x)$ $y=w(x)$.

3. (**) $z=f(y)$ I, $y=g(x)$ $[a,b]$ $g(x)$ I x $[a,b]$ – , $z=f\big(g(x)\big)$ $[a,b]$.

$$f\Big(\frac{1}{b-a}\int_a^b g(x)\,dx\Big) \leq \frac{1}{b-a}\int_a^b f\big(g(x)\big)\,dx.$$

(: $\Delta=\{a=x_0,x_1,\ldots,x_n=b\}$ $[a,b]$ $\Xi=\{\xi_1,\ldots,\xi_n\}$. 5 6.10 $f(\mu_1y_1+\cdots+\mu_ny_n) \leq \mu_1f(y_1)+\cdots+\mu_nf(y_n)$ $\mu_k=\frac{x_k-x_{k-1}}{b-a}$ $y_k=g(\xi_k)$. $\frac{1}{b-a}\int_a^b g(x)\,dx$ I. , Δ 0.)

:

(i) $e^{\frac{1}{b-a}\int_a^b g(x)\,dx} \leq \frac{1}{b-a}\int_a^b e^{g(x)}\,dx$,

(ii) $\log\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \geq \frac{1}{b-a}\int_a^b \log g(x)\,dx$ $g(x) > 0$ $[a,b]$,

(iii) $\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)^\alpha \leq \frac{1}{b-a}\int_a^b \big(g(x)\big)^\alpha\,dx$ $g(x) > 0$ $[a,b]$ $\alpha \geq 1$ $\alpha \leq 0$,

(iii) $\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)^\alpha \geq \frac{1}{b-a}\int_a^b \big(g(x)\big)^\alpha\,dx$ $g(x) > 0$ $[a,b]$ $0 \leq \alpha \leq 1$.

7.4 Riemann.

. .

– – $d = \frac{m}{l}$, m l .
, $d = \frac{m}{l}$. , . , , .
m .
x-, $[a,b]$. $d(x)$ x $[a,b]$, $y = d(x)$ $[a,b]$. $y = d(x)$ x , , $[a,b]$.
. $\Delta = \{x_0 = a, x_1, \dots, x_{n-1}, x_n = b\}$ $[a,b]$. m_k $[x_{k-1}, x_k]$,

$$m = m_1 + \cdots + m_n\,.$$

$y = d(x)$, x $[x_{k-1}, x_k]$, $y = d(x)$, , . , Δ $y = d(x)$. , ξ_k $[x_{k-1}, x_k]$, $d(x)$ $d(\xi_k)$, m_k $\widetilde{m_k} = d(\xi_k)(x_k - x_{k-1})$ $[x_{k-1}, x_k]$ $d(\xi_k)$. :

$$m_k \approx \widetilde{m_k} = d(\xi_k)(x_k - x_{k-1})\,.$$

, $m = m_1 + \cdots + m_n$ $\widetilde{m} = \widetilde{m_1} + \cdots + \widetilde{m_n} = d(\xi_1)(x_1 - x_0) + \cdots + d(\xi_n)(x_n - x_{n-1})$, ,

$$m \approx \widetilde{m} = d(\xi_1)(x_1 - x_0) + \cdots + d(\xi_n)(x_n - x_{n-1}).$$

Δ , $\left|m_k - \widetilde{m_k}\right|$, , $\left|m - \widetilde{m}\right|$. Riemann $\Sigma(d; a, b; \Delta, \Xi) = d(\xi_1)(x_1 - x_0) + \cdots + d(\xi_n)(x_n - x_{n-1})$ m Δ ,

$$\boxed{m = \int_a^b d(x)\,dx.}$$

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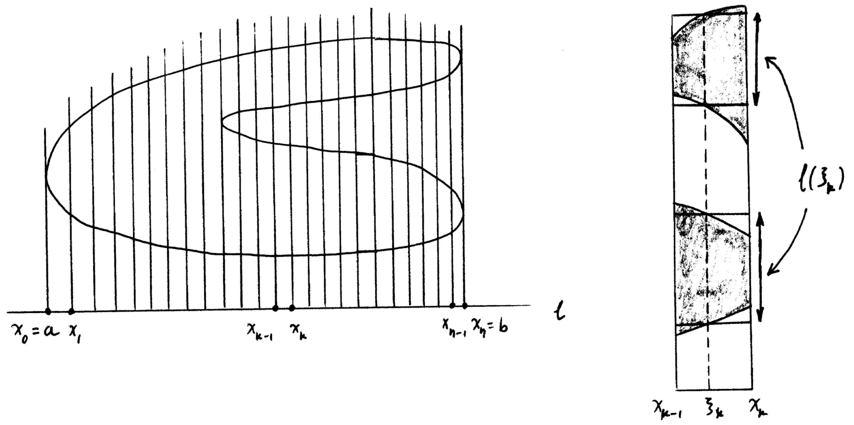
A , E. , .

(i). . x- l A. x l l_x ( ), $A^{(x)}$ A l_x A l. A ,



Σχήμα 7.5: .

$A^{(x)}$ $l(x)$, . A , $[a,b]$ l , x $[a,b]$, $A^{(x)}$, , $l(x)=0$. E $y=l(x)$ $[a,b]$.

$\Delta=\{x_0=a,x_1,\dots,x_{n-1},x_n=b\}$ $[a,b]$. A_k A $l_{x_{k-1}}$ l_{x_k} E_k A_k . , A $A_1,\dots,A_n$,

$$E=E_1+\cdots+E_n\,.$$

$[x_{k-1},x_k]$ ξ_k $A^{(\xi_k)}$. $l_{x_{k-1}}$ l_{x_k} . $[x_{k-1},x_k]$, A_k $\widetilde{A_k}$ $A^{(\xi_k)}$. x_k-x_{k-1} $l(\xi_k)$, $\widetilde{E_k}$ $\widetilde{A_k}$ $l(\xi_k)(x_k-x_{k-1})$. E_k A_k $\widetilde{E_k}$ $\widetilde{A_k}$, :

$$E_k\approx\widetilde{E_k}=l(\xi_k)(x_k-x_{k-1})\,.$$

, $E=E_1+\cdots+E_n$ $\widetilde{E}=\widetilde{E_1}+\cdots+\widetilde{E_n}=l(\xi_1)(x_1-x_0)+\cdots+l(\xi_n)(x_n-x_{n-1})$, ,

$$E\approx\widetilde{E}=l(\xi_1)(x_1-x_0)+\cdots+l(\xi_n)(x_n-x_{n-1}).$$

Δ , $\big|E_k-\widetilde{E_k}\big|$, , $\big|E-\widetilde{E}\big|$. Riemann $\Sigma(l;a,b;\Delta,\Xi)=l(\xi_1)(x_1-x_0)+\cdots+l(\xi_n)(x_n-x_{n-1})$ E Δ ,

$$\boxed{E=\int_a^b l(x)\,dx.}$$

:

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-
.

: (1) h, a b. x- l 0 l a h

l b . x $[0,h]$, x $[0,h]$ $l(x)$ $l(x) = \frac{b-a}{h}x + a$. $\int_0^h l(x)\,dx = \int_0^h (\frac{b-a}{h}x + a)\,dx = \frac{b-a}{h}\int_0^h x\,dx + a\int_0^h 1\,dx = \frac{b-a}{h}\frac{h^2}{2} + ah = \frac{b+a}{2}h$.
.

(2) $r > 0$ x- l . 0 l , $l(x)$ l x $2\sqrt{r^2 - x^2}$ x $[-r,r]$ 0 x $[-r,r]$. $2\int_{-r}^{r}\sqrt{r^2 - x^2}\,dx$. ! πr^2, .
, , .

. $y = f(x)$ $y = g(x)$ $[a,b]$ A , $(a, f(a))$ $(a, g(a))$ $(b, f(b))$ $(b, g(b))$. x $[a,b]$ $A^{(x)}$ $(x, f(x))$ $(x, g(x))$, $l(x) = |g(x) - f(x)|$, x $[a,b]$ $A^{(x)}$. A

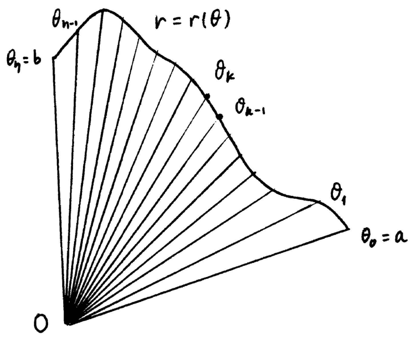
$$E = \int_a^b |g(x) - f(x)|\,dx.$$

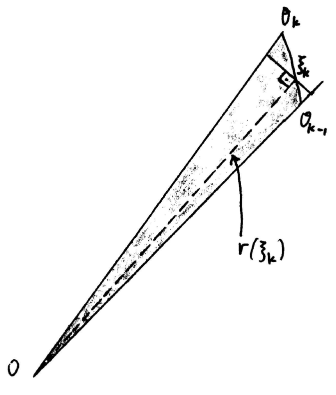
: $y = x$ $y = x^2$, $(-1,-1)$ $(-1,1)$ $(3,3)$ $(3,9)$ $\int_{-1}^3 |x^2 - x|\,dx$. $x^2 - x = x(x-1)$, : $\int_{-1}^3 |x^2 - x|\,dx = \int_{-1}^0 (x^2 - x)\,dx + \int_0^1 (x - x^2)\,dx + \int_1^3 (x^2 - x)\,dx = \frac{5}{6} + \frac{1}{6} + \frac{16}{3} = \frac{19}{3}$.

(ii). . , . ′ s_0 , θ $[0,2\pi]$, s_θ θ s_0. , $s_{2\pi}$ s_0, θ 0 2π, s_θ, , s_0 $s_{2\pi}$. θ $[a,b]$ $[0,2\pi]$ $A^{(\theta)}$ s_θ $r(\theta)$. $r = r(\theta)$ ( ) $[a,b]$, , $r(\theta) \geq 0$ θ. $A^{(\theta)}$ A s_a s_b. θ $[a,b]$ $A^{(\theta)}$, A s_θ, A . A $r = r(\theta)$ A.

$\Delta = \{\theta_0 = a, \theta_1, \ldots, \theta_{n-1}, \theta_n = b\}$ $[a,b]$. $[\theta_{k-1}, \theta_k]$ A_k A $s_{\theta_{k-1}}$ s_{θ_k}, A $A_1, \ldots, A_n$, E_k A_k,

$$E = E_1 + \cdots + E_n.$$





Σχήμα 7.6: .

$[\theta_{k-1},\theta_k]$ ξ_k $\widetilde{A_k}$, $s_{\theta_{k-1}}$ s_{θ_k} s_{ξ_k} $r(\xi_k)$. $[\theta_{k-1},\theta_k]$, A_k $\widetilde{A_k}$, E_k A_k $\widetilde{E_k}$ $\widetilde{A_k}$. , $\widetilde{E_k}$ $\frac{1}{2}(r(\xi_k))^2\big(\tan(\theta_k-\xi_k)+\tan(\xi_k-\theta_{k-1})\big)\approx \frac{1}{2}(r(\xi_k))^2\big((\theta_k-\xi_k)+(\xi_k-\theta_{k-1})\big)$. $\theta_k-\xi_k$ $\xi_k-\theta_{k-1}$ 0.

$$E_k\approx\frac{1}{2}(r(\xi_k))^2\big((\theta_k-\xi_k)+(\xi_k-\theta_{k-1})\big)=\frac{1}{2}(r(\xi_k))^2(\theta_k-\theta_{k-1}).$$

$k=1,\dots,n,$

$$E\approx\frac{1}{2}(r(\xi_1))^2(\theta_1-\theta_0)+\cdots+\frac{1}{2}(r(\xi_n))^2(\theta_n-\theta_{n-1}).$$

Riemann $\Sigma\big(\frac{1}{2}r^2;a,b;\Delta;\Xi\big)$ E A Δ ,

$$\boxed{E=\frac{1}{2}\int_a^b(r(\theta))^2\,d\theta.}$$

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: (1) $r>0$. : $r(\theta)=r$ θ $[0,2\pi]$. $\frac{1}{2}\int_0^{2\pi}r^2\,d\theta=\pi r^2$.

(2) , $r>0$ Θ ($[0,2\pi]$) . $r(\theta)=r$ $[0,\Theta]$ $r(\theta)=0$ $[0,\Theta]$. $\frac{1}{2}\int_0^{\Theta}r^2\,d\theta=\frac{1}{2}r^2\Theta$.

(3) $r_1\geq 0$ r_2 ($r_2>r_1$) . , , $\pi{r_2}^2-\pi{r_1}^2=\pi({r_2}^2-{r_1}^2)$. , r_1, r_2 Θ , $\frac{1}{2}{r_2}^2\Theta-\frac{1}{2}{r_1}^2\Theta=\frac{1}{2}({r_2}^2-{r_1}^2)\Theta$.

(4) $0\leq\lambda<\mu$. «» $x=x(\theta)=\lambda\theta\cos\theta,\ y=y(\theta)=\lambda\theta\sin\theta$ θ $[0,2\pi]$, $x=x(\theta)=\mu\theta\cos\theta,\ y=y(\theta)=\mu\theta\sin\theta$ θ $[0,2\pi]$. $(\lambda2\pi,0)$ $(\mu2\pi,0)$ $\frac{1}{2}\int_0^{2\pi}\mu^2\theta^2\,d\theta-\frac{1}{2}\int_0^{2\pi}\lambda^2\theta^2\,d\theta=\frac{4\pi^3}{3}(\mu^2-\lambda^2)$.

(iii). . A . $r\geq 0$ C_r r. $A^{(r)}$ C_r A A ( ). $A^{(r)}$ C_r, $l(r)$. A , $[a,b]$ $[0,+\infty)$, r $[a,b]$, $A^{(r)}$, , $l(r)=0$.
$\Delta=\{r_0=a,r_1,\dots,r_{n-1},r_n=b\}$ $[a,b]$ A_k A r_{k-1} r_k. E_k A_k , , :

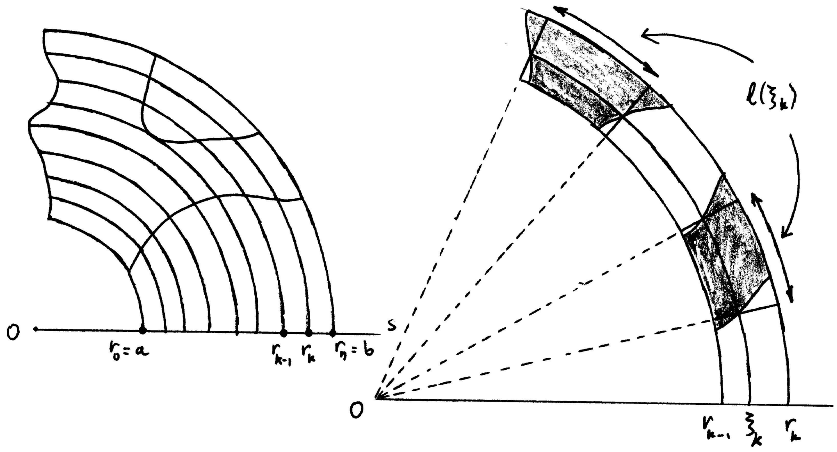
$$E=E_1+\cdots+E_n\,.$$

$[r_{k-1},r_k]$ ξ_k $A^{(\xi_k)}$. r_{k-1} r_k , . $\frac{1}{2}({r_k}^2-{r_{k-1}}^2)\theta$, θ . , $\frac{1}{2}({r_k}^2-{r_{k-1}}^2)\theta=\frac{r_k+r_{k-1}}{2}(r_k-r_{k-1})\theta\approx\xi_k(r_k-r_{k-1})\theta$, r_k r_{k-1} ξ_k. l $\xi_k\theta$, $l\cdot(r_k-r_{k-1})$. $\widetilde{A_k}$ $A^{(\xi_k)}$, $\widetilde{E_k}$ $\widetilde{A_k}$ $A^{(\xi_k)}$ r_k-r_{k-1} , , $l(\xi_k)(r_k-r_{k-1})$. $[r_{k-1},r_k]$, A_k $\widetilde{A_k}$, E_k A_k $\widetilde{E_k}$ $\widetilde{A_k}$. :

$$E_k\approx\widetilde{E_k}\approx l(\xi_k)(r_k-r_{k-1}).$$

:

$$E\approx l(\xi_1)(r_1-r_0)+\cdots+l(\xi_n)(r_n-r_{n-1}).$$



Σχήμα 7.7: .

Riemann $\Sigma(l;a,b;\Delta;\Xi)$ E Δ ,

$$\boxed{E=\int_a^b l(r)\,dr.}$$

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( ).

: $0<\lambda<\mu$. «» $x = x(r) = r\cos(\lambda r)$, $y = y(r) = r\sin(\lambda r)$ r $[0,\frac{2\pi}{\mu}]$, $x = x(r) = r\cos(\mu r)$, $y = y(r) = r\sin(\mu r)$ r $[0,\frac{2\pi}{\mu}]$ $(0,0)$ $\big(\frac{2\pi}{\mu}cos(\lambda\frac{2\pi}{\mu}), \frac{2\pi}{\mu}\sin(\lambda\frac{2\pi}{\mu}\big)$ $\big(\frac{2\pi}{\mu},0)$. r $(\mu-\lambda)r$, $\int_0^{\frac{2\pi}{\mu}}(\mu-\lambda)r\,dr = 2\pi^2\frac{\mu-\lambda}{\mu^2}$.

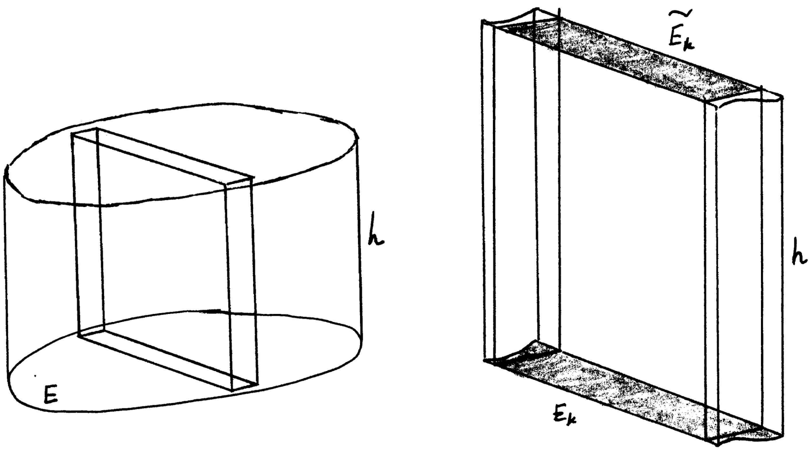
. .

B V. .

(i). . . A E L L, A, h L. B A h. V B

$$V = Eh.$$

l L x l l_x L l x. A , $[a,b]$ l x $[a,b]$ l_x A . $\Delta=\{x_0=a,x_1,\dots,x_{n-1},x_n=b\}$ $[a,b]$ A_k A $l_{x_{k-1}}$ l_{x_k}. E_k



Σχήμα 7.8: .

A_k , , :

$$E = E_1 + \cdots + E_n \,.$$

B_k A_k , B A, B $B_1, \ldots, B_n$, V_k B_k , :

$$V = V_1 + \cdots + V_n \,.$$

$[x_{k-1}, x_k]$, A_k $\widetilde{A_k}$, , B_k $\widetilde{B_k}$. , $\widetilde{B_k}$ L $\widetilde{A_k}$ h. $\widetilde{E_k}$ $\widetilde{A_k}$ $\widetilde{V_k}$ $\widetilde{B_k}$, :

$$V_k \approx \widetilde{V_k} \,, \quad E_k \approx \widetilde{E_k} \,, \quad \widetilde{V_k} = \widetilde{E_k} \cdot h.$$

. :

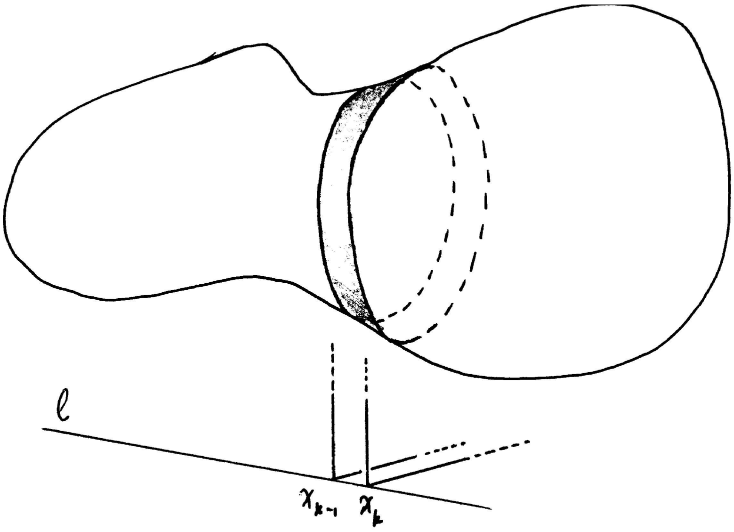
$$V \approx \widetilde{V_1} + \cdots + \widetilde{V_n} \,, \quad E \approx \widetilde{E_1} + \cdots + \widetilde{E_n}$$

$$\widetilde{V_1} + \cdots + \widetilde{V_n} = \big(\widetilde{E_1} + \cdots + \widetilde{E_n}\big)\, h.$$

, Δ , $\widetilde{V_1} + \cdots + \widetilde{V_n}$ V Eh. , , $V = Eh$.

(ii). . B.
l x l L_x l x $B^{(x)}$ L_x B. B B l. B , $[a,b]$ l x $[a,b]$ $B^{(x)}$ B . x $[a,b]$ $E(x)$ $B^{(x)}$. $\Delta = \{x_0 = a, x_1, \ldots, x_{n-1}, x_n = b\}$ $[a,b]$ B_k B $L_{x_{k-1}}$ L_{x_k} . B $B_1, \ldots, B_n$, , V_k B_k , :

$$V = V_1 + \cdots + V_n \,.$$



Σχήμα 7.9: .

$[x_{k-1}, x_k]$ ξ_k $\widetilde{B_k}$ $L_{x_{k-1}}$ L_{x_k} L_{ξ_k} , , $B^{(\xi_k)}$ B. , $[x_{k-1}, x_k]$, B_k $\widetilde{B_k}$, V_k B_k $\widetilde{V_k}$ $\widetilde{B_k}$. , $\widetilde{V_k} = E(\xi_k)(x_k - x_{k-1})$,

$$V_k \approx \widetilde{V_k} = E(\xi_k)(x_k - x_{k-1})$$

k.

$$V \approx E(\xi_1)(x_1 - x_0) + \cdots + E(\xi_n)(x_n - x_{n-1}).$$

, Δ , Riemann $\Sigma(E; a, b; \Delta; \Xi) = E(\xi_1)(x_1 - x_0) + \cdots + E(\xi_n)(x_n - x_{n-1})$ V , ,

$$\boxed{V = \int_a^b E(x)\, dx.}$$

:

-

.

: , – – h E. l L A . 0 l L h l . $E(x)$ E x $[0, h]$ 0 x $[0, h]$. $\int_0^h E\, dx = Eh$.

(iii). . l , x-, $[a, b]$ l. x $[a, b]$ l x x $r(x)$. $r = r(x)$ ( ) $[a, b]$ $r(x) \geq 0$ x $[a, b]$. B . B. L l l', y-, L l 0 l A L $y = r(x)$ $[a, b]$. B A 2π l. x $[a, b]$ B l x $r(x)$ x. $E(x) = \pi(r(x))^2$, B

$$\boxed{V = \pi \int_a^b (r(x))^2 \, dx.}$$

:

π –

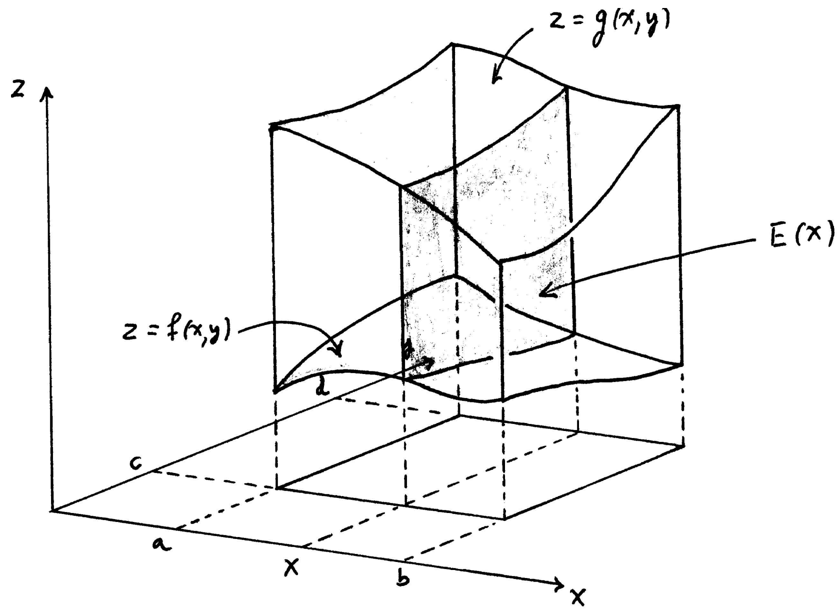
.

: (1) $R > 0$. l 0 l . l x $[-R, R]$ l x $r = \sqrt{R^2 - x^2}$. , $\pi \int_{-R}^{R} (R^2 - x^2)\, dx = \frac{4\pi}{3} R^3$.

(2) $B = \{(x, y, z) : x^2 + y^2 \leq z \leq 1\}$. , l z-, , $z < 0$ $z > 1$, $B^{(z)}$ B , $0 \leq z \leq 1$, $B^{(z)}$ $(0, 0, z)$, l $r(z) = \sqrt{z}$. B $\pi \int_0^1 z\, dz = \frac{\pi}{2}$. B $A = \{(x, z) : x^2 \leq z \leq 1\}$, xz-, z-.

(3) B $h > 0$ $R > 0$. l . 0 l h . x l $[0, h]$ $B^{(x)}$ x $[0, h]$ $B^{(x)}$ l x $r(x) = \frac{R}{h} x$. $\pi \int_0^h \frac{R^2}{h^2} x^2 \, dx = \frac{\pi}{3} R^2 h$.

(iv). . xy- (x, y) $a \leq x \leq b$ $c \leq y \leq d$. , $z = f(x, y)$ $z = g(x, y)$, (x, y) . , $(x, y, f(x, y))$, , , , $(x, y, g(x, y))$. B . $z = f(x, y)$ $z = g(x, y)$ x y. , , x y $f(x, y)$ $g(x, y)$.



Σχήμα 7.10: .

B.

l x- x L_x l x. L_x (x, y, z) x. x $[a, b]$ $B^{(x)}$ B l x L_x $z = f(x, y)$ $z = g(x, y)$ x y $[c, d]$. , $E(x) = \int_c^d |g(x, y) - f(x, y)|\, dy$. x $[a, b]$, $B^{(x)}$. V B $\int_a^b E(x)\, dx$, ,

$$\boxed{V = \int_a^b \Big(\int_c^d |g(x, y) - f(x, y)|\, dy \Big)\, dx.}$$

l y-, B

$$\boxed{V = \int_c^d \Big(\int_a^b |g(x, y) - f(x, y)|\, dx \Big)\, dy.}$$

: B $z = 2x + y$ $z = x + 2y$ $-1 \le x \le 1$ $-1 \le y \le 1$. $\int_{-1}^1 \big(\int_{-1}^1 |x + 2y - 2x - y|\, dy \big)\, dx = \int_{-1}^1 \big(\int_{-1}^1 |y - x|\, dy \big)\, dx$. $y - x$, x $[-1, 1]$ $y - x \ge 0$, y $[x, 1]$, $y - x \le 0$, y $[-1, x]$. , B $\int_{-1}^1 \big(\int_{-1}^x (x - y)\, dy + \int_x^1 (y - x)\, dy \big)\, dx = \int_{-1}^1 (x^2 + 1)\, dx = \frac{8}{3}$.

. .

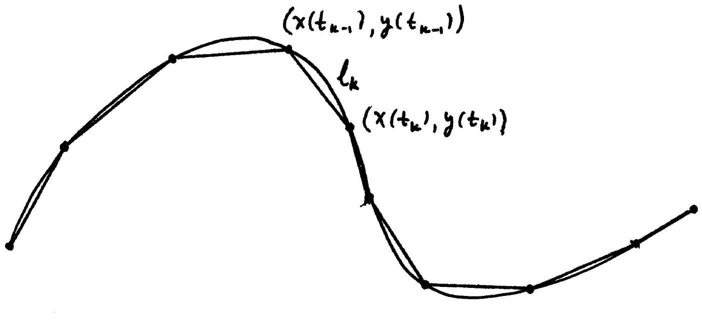
(x', y') (x'', y'') ,

$$\sqrt{(x'' - x')^2 + (y'' - y')^2}\,.$$

, (x', y', z') (x'', y'', z'') ,

$$\sqrt{(x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2}\,.$$

, – – , .
l . : .



Σχήμα 7.11: .

. $(x,y)=(x(t),y(t))$ t $[a,b]$, , $x=x(t)$ $y=y(t)$ $[a,b]$. . $\Delta=\{t_0=a,t_1,\dots,t_{n-1},t_n=b\}$ $[a,b]$ $k=0,1,\dots,n$ T_k $(x(t_k),y(t_k))$. T_0 T_n . l_k T_{k-1} T_k ,

$$l=l_1+\cdots+l_n\,.$$

$T_0,T_1,\dots,T_{n-1},T_n$, $[t_{k-1},t_k]$, T_{k-1} T_k . , l_k ,

$$l_k\approx\sqrt{(x(t_k)-x(t_{k-1}))^2+(y(t_k)-y(t_{k-1}))^2}\,.$$

ξ_k η_k $[t_{k-1},t_k]$ $x(t_k)-x(t_{k-1})=x'(\xi_k)(t_k-t_{k-1})$ $y(t_k)-y(t_{k-1})=y'(\eta_k)(t_k-t_{k-1})$.

$$\sqrt{(x(t_k)-x(t_{k-1}))^2+(y(t_k)-y(t_{k-1}))^2}=\sqrt{(x'(\xi_k))^2+(y'(\eta_k))^2}(t_k-t_{k-1}).$$

, $y'(t)$ $[t_{k-1},t_k]$ – ξ_k η_k – $y'(\eta_k)\approx y'(\xi_k)$,

$$\sqrt{(x(t_k)-x(t_{k-1}))^2+(y(t_k)-y(t_{k-1}))^2}\approx\sqrt{(x'(\xi_k))^2+(y'(\xi_k))^2}(t_k-t_{k-1}).$$

:

$$l\approx\sqrt{(x'(\xi_1))^2+(y'(\xi_1))^2}(t_1-t_0)+\cdots+\sqrt{(x'(\xi_n))^2+(y'(\xi_n))^2}(t_n-t_{n-1})$$

, , Riemann $\Sigma\big(\sqrt{x'^{\,2}+y'^{\,2}}\,;a,b;\Delta;\Xi\big)$ l Δ .

$$\boxed{l=\int_a^b\sqrt{(x'(t))^2+(y'(t))^2}\,dt.}$$

. $(x,y,z)=(x(t),y(t),z(t))$ t $[a,b]$

$$\boxed{l=\int_a^b\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt.}$$

:

-

.

: (1) , $x=x(t)=\kappa t+\lambda$ $y=y(t)=\mu t+\nu$ t $[a,b]$. $x'(t)=\kappa$ $y'(t)=\mu$ t $[a,b]$, $\int_a^b\sqrt{\kappa^2+\mu^2}\,dt=\sqrt{\kappa^2+\mu^2}(b-a)$. $=(\kappa a+\lambda,\mu a+\nu)$ $=(\kappa b+\lambda,\mu b+\nu)$ $\sqrt{(\kappa b+\lambda-\kappa a-\lambda)^2+(\mu b+\nu-\mu a-\nu)^2}=\sqrt{\kappa^2+\mu^2}(b-a)$.

(2) (x_0,y_0) r_0 $x=x(t)=r_0\cos t+x_0$ $y=y(t)=r_0\sin t+y_0$ t $[0,2\pi]$. $x'(t)=-r_0\sin t$ $y'(t)=r_0\cos t$ $\int_0^{2\pi}\sqrt{{r_0}^2(\sin t)^2+{r_0}^2(\cos t)^2}\,dt=2\pi r_0$.

(3) $x=x(t)=\kappa_0\cos t+x_0$ $y=y(t)=\mu_0\sin t+y_0$ t $[0,2\pi]$. $x'(t)=-\kappa_0\sin t$ $y'(t)=\mu_0\cos t$ $\int_0^{2\pi}\sqrt{{\kappa_0}^2(\sin t)^2+{\mu_0}^2(\cos t)^2}\,dt$.

(4) , , $y=f(x)$ $[a,b]$. $y=f(x)$ $[a,b]$,

$$\boxed{l = \int_a^b \sqrt{1+(f'(x))^2}\,dx.}$$

, $x = g(y)$ $[a,b]$,

$$\boxed{l = \int_a^b \sqrt{1+(g'(y))^2}\,dy.}$$

(5) $(x,y,z) = (r_0\cos t + x_0, r_0 \sin t + y_0, \frac{h_0}{2\pi}t + z_0)$, t $[0,2\pi]$, $x'(t) = -r_0\sin t$, $y'(t) = r_0\cos t$ $z'(t) = \frac{h_0}{2\pi}$, $\int_0^{2\pi}\sqrt{{r_0}^2(\sin t)^2 + {r_0}^2(\cos t)^2 + \frac{{h_0}^2}{4\pi^2}}\,dt = \sqrt{4\pi^2 {r_0}^2 + {h_0}^2}$.

. .

, $\overrightarrow{F}$ () , A B, W, $\overrightarrow{F}$ A B $+$ $-$, $\overrightarrow{F}$ $\overrightarrow{AB}$, , . ,

$$W = |\overrightarrow{F}||\overrightarrow{AB}|\cos\theta,$$

θ $[0,\pi]$ $\overrightarrow{F}$ $\overrightarrow{AB}$. , $0 \le \theta < \frac{\pi}{2}$, , $\frac{\pi}{2} < \theta \le \pi$, , , $\theta = \frac{\pi}{2}$, 0. $|\overrightarrow{F}||\overrightarrow{AB}|\cos\theta$ $\overrightarrow{F}$ $\overrightarrow{AB}$,

$$W = \overrightarrow{F}\cdot\overrightarrow{AB}.$$

W () A B .

, $(x,y) = (x(t),y(t))$ t $[a,b]$. $\overrightarrow{F} = (F_x,F_y)$ $(x,y) = (x(t),y(t))$ $\overrightarrow{F}(t) = (F_x(t),F_y(t))$ t $[a,b]$. , , $F_x = F_x(t)$ $F_y = F_y(t)$ $x = x(t)$ $y = y(t)$ $[a,b]$, , .

$\Delta = \{t_0 = a, t_1, \dots, t_{n-1}, t_n = b\}$ $[a,b]$. W_k $[t_{k-1},t_k]$. , :

$$W = W_1 + \cdots + W_n.$$

$[t_{k-1},t_k]$, $(F_x(t),F_y(t))$. ξ_k $[t_{k-1},t_k]$, W_k $\overrightarrow{F}(\xi_k) = (F_x(\xi_k),F_y(\xi_k))$ $A_{k-1} = (x(t_{k-1}),y(t_{k-1}))$ $A_k = (x(t_k),y(t_k))$, $\widetilde{W_k} = \overrightarrow{F}(\xi_k)\cdot\overrightarrow{A_{k-1}A_k} = F_x(\xi_k)\big(x(t_k)-x(t_{k-1})\big) + F_y(\xi_k)\big(y(t_k)-y(t_{k-1})\big)$. :

$$W_k \approx \widetilde{W_k} = F_x(\xi_k)\big(x(t_k)-x(t_{k-1})\big) + F_y(\xi_k)\big(y(t_k)-y(t_{k-1})\big).$$

η_k ζ_k $[t_{k-1},t_k]$ $x(t_k)-x(t_{k-1}) = x'(\eta_k)(t_k-t_{k-1})$ $y(t_k)-y(t_{k-1}) = y'(\zeta_k)(t_k-t_{k-1})$. $x'(t)$ $y'(t)$, $x'(\eta_k)$ $y'(\zeta_k)$ $x'(\xi_k)$ $y'(\xi_k)$,

$$W_k \approx \widetilde{W_k} \approx F_x(\xi_k)x'(\xi_k)(t_k-t_{k-1}) + F_y(\xi_k)y'(\xi_k)(t_k-t_{k-1}).$$

,

$$\begin{aligned} W \;&\approx\; \big(F_x(\xi_1)x'(\xi_1)+F_y(\xi_1)y'(\xi_1)\big)(t_1-t_0)+\cdots \\ &\quad \cdots + \big(F_x(\xi_n)x'(\xi_n)+F_y(\xi_n)y'(\xi_n)\big)(t_n-t_{n-1}). \end{aligned}$$

, , Riemann $\Sigma(F_xx'+F_yy';a,b;\Delta;\Xi)$ W Δ , ,

$$W = \int_a^b \big(F_x(t)x'(t) + F_y(t)y'(t)\big)\,dt.$$

, ,

$$W = \int_a^b \big(F_x(t)x'(t) + F_y(t)y'(t) + F_z(t)z'(t)\big)\,dt.$$

:

-

.

: , 0.
, $(x(t), y(t))$ $(x'(t), y'(t))$. $F_x(t)x'(t) + F_y(t)y'(t) = 0$ t , ,
$W = \int_a^b \big(F_x(t)x'(t) + F_y(t)y'(t)\big)\,dt = 0.$

.

. .

1. , $y = x^2$ $y = \sqrt{x}$ $[0, 1]$.

2. , $\frac{(x-x_0)^2}{{\kappa_0}^2} + \frac{(y-y_0)^2}{{\mu_0}^2} = 1.$

   .

3. $a > 0$, $x = x(\theta) = a\sqrt{\cos(2\theta)}\cos\theta$ $y = y(\theta) = a\sqrt{\cos(2\theta)}\sin\theta$ θ $[-\frac{\pi}{4}, \frac{\pi}{4}]$ θ $[\frac{3\pi}{4}, \frac{5\pi}{4}]$. , $(0, 0)$. .

   , «» .

4. $a > 0$, $x = x(\theta) = a\sqrt{\cos(3\theta)}\cos\theta$ $y = y(\theta) = a\sqrt{\cos(3\theta)}\sin\theta$ $[-\frac{\pi}{6}, \frac{\pi}{6}]$, $[\frac{\pi}{2}, \frac{5\pi}{6}]$ $[\frac{7\pi}{6}, \frac{3\pi}{2}]$ . .

   , «» .

5. $n \geq 2$ «».

. .

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

   , a, b, c , .

2. (*) $x^2 + y^2 = 4$ $z = 0$ $x + y + z = 0$.

   (: $y = x$ xy-.)

. .

1. $y = ax^2$.

2. $y = \frac{a}{x}$.

3. $\kappa > 0$. $y = \frac{\kappa}{2}(e^{\frac{x}{\kappa}} + e^{-\frac{x}{\kappa}}) = \kappa \cosh \frac{x}{\kappa}$ . . .

4. $x = x(\theta) = r(\theta)\cos\theta$ $y = y(\theta) = r(\theta)\sin\theta$, θ $[a, b]$, $r = r(\theta)$ $[a, b]$ $r(\theta) \geq 0$ t $[a, b]$. l

$$l = \int_a^b \sqrt{(r(\theta))^2 + (r'(\theta))^2}\, d\theta.$$

$c, \kappa > 0$, $x = x(\theta) = ce^{\kappa\theta}\cos\theta$ $y = y(\theta) = ce^{\kappa\theta}\sin\theta$, θ $(-\infty, +\infty)$, . .

$\kappa > 0$, $x = x(\theta) = \kappa\theta\cos\theta$ $y = y(\theta) = \kappa\theta\sin\theta$, θ $[0, +\infty)$, . .

$\kappa > 0$, $x = x(\theta) = \frac{\kappa}{\theta}\cos\theta$ $y = y(\theta) = \frac{\kappa}{\theta}\sin\theta$, θ $(0, +\infty)$, . .

5. $\frac{(x-x_0)^2}{{\kappa_0}^2} + \frac{(y-y_0)^2}{{\mu_0}^2} = 1$.

6.

$$\sqrt{(x(b)-x(a))^2 + (y(b)-y(a))^2} \leq \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt\, ?$$

, .

$$(x(b)-x(a))^2 + (y(b)-y(a))^2 = |x(b)-x(a)|\Big|\int_a^b x'(t)\, dt\Big| +$$

$$+|y(b)-y(a)|\Big|\int_a^b y'(t)\, dt\Big|$$

$$\leq \int_a^b \big(|x(b)-x(a)||x'(t)| + |y(b)-y(a)||y'(t)|\big)\, dt$$

$$\leq \int_a^b \sqrt{(x(b)-x(a))^2 + (y(b)-y(a))^2}\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

7. « » ;

. .

1. Newton $\overrightarrow{F} = (F_x, F_y, F_z)$ $(x, y, z) \neq (0, 0, 0)$ m

$$(F_x, F_y, F_z) = -\frac{cm}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,(x, y, z) = -\frac{cm}{r^3}\,(x, y, z)\,,$$

$r = \sqrt{x^2+y^2+z^2} > 0$ c $-$ $-$.

$x = x(t),\ y = y(t)\ \ z = z(t),\ \ t\ \ \ [a, b],$

$$-\frac{cm}{2}\int_a^b \frac{1}{(r(t))^3}\frac{d\,(r(t))^2}{dt}\,dt = -cm\int_a^b \frac{r'(t)}{(r(t))^2}\,dt,$$

$r(t) = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2} > 0$, , $(0, 0, 0)$.

2. Hooke $\overrightarrow{F} = F_x$ x x- 0 $F_x = -cx$, c – – . $x = x(t)$, t $[a, b]$, .

Κεφάλαιο 8

Riemann.

Riemann. . Riemann . Riemann , , . Riemann . , Riemann , Riemann , Riemann Riemann . Riemann. . : , , , .

8.1 Riemann.

′ , , «» «» « Riemann» «Riemann » .

. .

$y = f(x)$ I ( ). $y = F(x)$, I,

$$F'(x) = f(x) \qquad (x \quad I),$$

$y = F(x)$ $y = f(x)$ I.

: (1) n $y = \frac{x^{n+1}}{n+1}$ $y = x^n$ $(-\infty, +\infty)$.

(2) $y = x$ $y = 1$ $(-\infty, +\infty)$.

(3) $n \le -2$ $y = \frac{x^{n+1}}{n+1}$ $y = x^n$ $(-\infty, 0)$ $(0, +\infty)$.

(4) a $y = \frac{x^{a+1}}{a+1}$ $y = x^a$ $(0, +\infty)$.

(5) $y = \log|x|$ $y = \frac{1}{x}$ $(-\infty, 0)$ $(0, +\infty)$.

(6) $a > 0, a \ne 1$, $y = \frac{a^x}{\log a}$ $y = a^x$ $(-\infty, +\infty)$.

(7) $y = \sin x$ $y = \cos x$ $y = -\cos x$ $y = \sin x$ $(-\infty, +\infty)$.

(8) $y = \tan x$ $y = \frac{1}{(\cos x)^2}$ $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$ $(k \in \mathbf{Z})$. , $y = -\cot x$ $y = \frac{1}{(\sin x)^2}$ $(k\pi, \pi + k\pi)$ $(k \in \mathbf{Z})$.

(9) $y = \arcsin x$ $y = \frac{1}{\sqrt{1-x^2}}$ $(-1, 1)$. , $y = -\arccos x$ $y = \frac{1}{\sqrt{1-x^2}}$ $(-1, 1)$.

(10) $y=\arctan x$ $\frac{1}{1+x^2}$ $(-\infty,+\infty)$.

8.1 $y=F_1(x)$ $y=F_2(x)$ I $y=f(x)$ I, $y=f(x)$ I. , $y=F_1(x)$ $y=F_2(x)$ $y=f(x)$ I, I.

: . $F_2(x)-F_1(x)=c$ x I, c (x), $y=F_1(x)$ $y=f(x)$ I. $F_2{}'(x)=F_1{}'(x)+0=f(x)$ x I, $y=F_2(x)$ $y=f(x)$ I. , $y=F_1(x)$ $y=F_2(x)$ $y=f(x)$ I $y=h(x)=F_2(x)-F_1(x)$ I. $h'(x)=F_1{}'(x)-F_2{}'(x)=f(x)-f(x)=0$ x I, 6.7(i) $y=h(x)$ I.

.

$y=F(x)$ $y=f(x)$ I. $y=f(x)$ I -
$y=F(x)+c$, c
(x),
.

8.1 , , : .

: (1) $y=x^2$ $(-\infty,+\infty)$ $y=\frac{x^3}{3}+c$, c .
(2) c . , $y=\cos x$ $(-\infty,+\infty)$ $y=\sin x+c$, c .

. $y=g(x)$ 0 , .

: $y=g(x)=\begin{cases}1\,, & 0<x<1,\\ 2\,, & 1<x<3,\end{cases}$ 0 $(0,1)\cup(1,3)$, $(0,1)$ $(1,3)$. , $y=g(x)$ $(0,1)\cup(1,3)$.

8.1 «» « ».

: $y=\frac{1}{x}$ $(-\infty,0)$ $(0,+\infty)$ $y=\log|x|+c$, c . , $y=\frac{1}{x}$ $(-\infty,0)\cup(0,+\infty)$ $y=\begin{cases}\log|x|+c_1\,, & x<0,\\ \log|x|+c_2\,, & x>0,\end{cases}$ c_1 c_2 (x), .

. .

$y=f(x)$ $[a,b]$, $\int_a^b f(x)\,dx$. :

$$\int_b^a f(x)\,dx=-\int_a^b f(x)\,dx.$$

, , . , $y=f(x)$ a, , , $[a,a]=\{a\}$:

$$\int_a^a f(x)\,dx=0.$$

, $\int_a^b f(x)\,dx$ a b ¨ $y=f(x)$ $[a,b]$, $b>a$, $[b,a]$, $b<a$, a, $b=a$.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx,$$

$a < b < c$, a, b, c, $y = f(x)$. , . , $c < b < a$, $-\int_c^a f(x)\,dx = -\int_b^a f(x)\,dx - \int_c^b f(x)\,dx$, , $\int_c^a f(x)\,dx = \int_c^b f(x)\,dx + \int_b^a f(x)\,dx$. , $a = c < b$, $0 = \int_a^b f(x)\,dx + \int_b^a f(x)\,dx$ $\int_b^a f(x)\,dx$. .
. $y = f(x)$ $[a, b]$, $a < b$, $[b, a]$, $b < a$, , M $|f(x)| \le M$ x ,

$$\Big|\int_a^b f(x)\,dx\Big| \le M|b-a|.$$

, $a < b$, ($|b-a| = b-a$), . $b < a$, $\big|\int_a^b f(x)\,dx\big| = \big|-\int_b^a f(x)\,dx\big| = \big|\int_b^a f(x)\,dx\big| \le M(a-b) = M|b-a|$. , $a = b$, $\big|\int_a^b f(x)\,dx\big| \le M|b-a|$ $0 = 0$.
, $y = f(x)$ I ( ) . a I, x I x $\int_a^x f(t)\,dt$. x. , c (x) I

$$y = F(x) = \int_a^x f(t)\,dt + c.$$

$y = f(x)$ I. a $y = F(x) = \int_a^x f(t)\,dt + c$.
a a' I, :

$$F(x) = \int_a^x f(t)\,dt + c = \int_{a'}^x f(t)\,dt + \int_a^{a'} f(t)\,dt + c = \int_{a'}^x f(t)\,dt + c',$$

c' , $c' = \int_a^{a'} f(t)\,dt + c$. , a a' c c'. a $\int_a^x f(t)\,dt$.

: $y = x^2$ $(-\infty, +\infty)$, $a = 0$ $y = \int_0^x t^2\,dt = \frac{x^3}{3} - \frac{0^3}{3} = \frac{x^3}{3}$. $y = x^2$ $y = \frac{x^3}{3} + c$, c . , , a , $y = \int_a^x t^2\,dt$ $y = \int_a^x t^2\,dt = \frac{x^3}{3} - \frac{a^3}{3}$ $c = -\frac{a^3}{3}$.

8.2 $y = F_1(x)$ $y = F_2(x)$ I $y = f(x)$ I, $y = f(x)$ I. , $y = F_1(x)$ $y = F_2(x)$ $y = f(x)$ I, I.

, $y = F_2(x) - F_1(x) = c$ x I, c , x, $y = F_1(x)$ $y = f(x)$ I. , a_1 I c_1 $F_1(x) = \int_{a_1}^x f(t)\,dt + c_1$ x I. $F_2(x) = F_1(x) + c = \int_{a_1}^x f(t)\,dt + (c_1 + c) = \int_{a_2}^x f(t)\,dt + c_2$ x I, $a_2 = a_1$ $c_2 = c_1 + c$. $y = F_2(x)$ $y = f(x)$ I. , $y = F_1(x)$ $y = F_2(x)$ $y = f(x)$ I, a_1 a_2 I c_1 c_2 $F_1(x) = \int_{a_1}^x f(t)\,dt + c_1$ $F_2(x) = \int_{a_2}^x f(t)\,dt + c_2$ x I. $F_2(x) - F_1(x) = \int_{a_2}^x f(t)\,dt + c_2 - \int_{a_1}^x f(t)\,dt - c_1 = \int_{a_2}^{a_1} f(t) + c_2 - c_1$ x I, $y = F_2(x) - F_1(x)$ I.

8.2 , , : .

$$\int f(x)\,dx$$

$y = f(x)$ I , , $\int_a^x f(t)\,dt + c$ I, a I c . ,

$$\int f(x)\,dx = \int_a^x f(t)\,dt + c.$$

: , $y = x^2$ $(-\infty, +\infty)$ $y = \frac{x^3}{3} + c$, c . $\int x^2\,dx$, , $\int x^2\,dx = \frac{x^3}{3} + c$, c .

$\int f(x)\,dx$.
, , $y = x^2$: . .
$\int f(x)\,dx$. :

$$\boxed{\int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx.}$$

. a I $F(x) = \int_a^x f(t)\,dt$ $G(x) = \int_a^x g(t)\,dt$ I. $\int f(x)\,dx = F(x) + c_1$ $\int g(x)\,dx = G(x) + c_2$, c_1 c_2 . , $\int f(x)\,dx + \int g(x)\,dx = F(x) + c_1 + G(x) + c_2 = F(x) + G(x) + (c_1 + c_2) = \int_a^x (f(t) + g(t))\,dt + (c_1 + c_2)$. $\int_a^x (f(t) + g(t))\,dt$ $y = f(x) + g(x)$ I , $c_1 + c_2$, , , $\int_a^x (f(t) + g(t))\,dt + (c_1 + c_2)$ $\int (f(x) + g(x))\,dx$. , $\int f(x)\,dx + \int g(x)\,dx = \int (f(x) + g(x))\,dx$.

:

$$\boxed{\int \big(\lambda f(x)\big)\,dx = \lambda \int f(x)\,dx \qquad (\lambda \neq 0).}$$

, , $F(x) = \int_a^x f(t)\,dt$, $\int f(x)\,dx = F(x) + c$, c . $\lambda \int f(x)\,dx = \lambda F(x) + \lambda c = \int_a^x (\lambda f(t))\,dt + \lambda c$. $\int_a^x (\lambda f(t))\,dt$ $y = \lambda f(x)$ I λc (: $\lambda \neq 0$) . $\int_a^x (\lambda f(t))\,dt + \lambda c$ $\int (\lambda f(x))\,dx$, , $\lambda \int f(x)\,dx = \int (\lambda f(x))\,dx$.
, . , $\neq 0$.

: (1) $\int (x + x^2)\,dx = \int x\,dx + \int x^2\,dx = \frac{x^2}{2} + \frac{x^3}{3} + c$. $\int (x + x^2)\,dx = \int x\,dx + \int x^2\,dx = \frac{x^2}{2} + c_1 + \frac{x^3}{3} + c_2$.

(2) $\int (7x)\,dx = 7 \int x\,dx = 7\frac{x^2}{2} + c$. $\int (7x)\,dx = 7 \int x\,dx = 7\frac{x^2}{2} + 7c$.

(3) $\int (x + g(x))\,dx = \int x\,dx + \int g(x)\,dx = \frac{x^2}{2} + \int g(x)\,dx$. $\int (x + g(x))\,dx = \int x\,dx + \int g(x)\,dx = \frac{x^2}{2} + c + \int g(x)\,dx$ c «» $\int g(x)\,dx$ ′ .

(4) ! $\int x\,dx - \int x\,dx = c$ $= 0$. : $\int x\,dx - \int x\,dx = \int (x - x)\,dx = \int 0\,dx = c$, , $\int x\,dx - \int x\,dx = \frac{x^2}{2} + c_1 - \frac{x^2}{2} - c_2 = c_1 - c_2 = c$.

.

. .

1. $y = 2x + \sin x$ $(-\infty, +\infty)$. $y = 2x + \sin x$ $(-\infty, +\infty)$;
$y = 2x + \sin x$ $(-\infty, +\infty)$ $x = 1$ -2. ;

2. $y = F(x)$ $F'(x^2) = \frac{1}{x}$ x $(0, +\infty)$ $F(1) = 1$.

3. $y = F(x)$ $F'(\log x) = 1$ x $(0, 1]$ $F'(\log x) = x$ x $[1, +\infty)$ $F(1) = 1$.

4. $y = r(x)$ (a, b) $r'(x) = \frac{1}{x}$ x (a, b), $r(x) = \dots$ (a, b). ;

. .

1. $y = 1 - x^2$ $(-\infty, +\infty)$. $y = 1 - x^2$ $(-\infty, +\infty)$; , $\int (1 - t^2)\, dt$; $y = 1 - x^2$ $(-\infty, +\infty)$ $x = 2$ -1. ;

2. $y = f(x)$ I a I κ. $y = f(x)$ κ $x = a$;

3. $\int f(x)\, dx = \int g(x)\, dx + x^2 - 3$. $\int f(x)\, dx - \int g(x)\, dx$;

4. $y = f(x) = x - [x] - \frac{1}{2}$ $(-\infty, +\infty)$.
   (i) $y = f(x)$ 1.
   (ii) $F(x) = \int_0^x f(t)\, dt$ $[0, 1]$.
   (iii) $y = F(x) = \int_0^x f(t)\, dt$ $(-\infty, +\infty)$ 1. $y = F(x)$ $[x]$.
   (iv) $G(x) = \int_0^x (F(t) + \frac{1}{12})\, dt$ $[0, 1]$ $y = G(x)$ $(-\infty, +\infty)$ 1.

5. . , , , ;

8.2 .

. .

8.1 . $y = f(x)$ I ( ) . a I $y = F(x) = \int_a^x f(t)\, dt$ I. $y = f(x)$ ξ I, $y = F(x)$ ξ

$$F'(\xi) = f(\xi).$$

, $y = f(x)$ I, $y = F(x)$ I $F'(x) = f(x)$ x I.

: $\epsilon > 0$. $y = f(x)$ ξ, $\delta > 0$ $|f(x) - f(\xi)| < \epsilon$ x I $|x - \xi| < \delta$. , , x I $0 < |x - \xi| < \delta$. t x ξ , , $|t - \xi| < \delta$, , $|f(t) - f(\xi)| < \epsilon$. , ,

$$\begin{aligned}
\left|\frac{F(x) - F(\xi)}{x - \xi} - f(\xi)\right| &= \left|\frac{\int_a^x f(t)\, dt - \int_a^\xi f(t)\, dt}{x - \xi} - f(\xi)\right| = \left|\frac{\int_\xi^x f(t)\, dt - f(\xi)(x - \xi)}{x - \xi}\right| \\
&= \left|\frac{\int_\xi^x f(t)\, dt - \int_\xi^x f(\xi)\, dt}{x - \xi}\right| = \frac{\left|\int_\xi^x (f(t) - f(\xi))\, dt\right|}{|x - \xi|} \\
&\leq \frac{|x - \xi|\epsilon}{|x - \xi|} = \epsilon.
\end{aligned}$$

$$\lim_{x \to \xi} \frac{F(x) - F(\xi)}{x - \xi} = f(\xi),$$

$F'(\xi) = f(\xi)$.

, , .

$y = f(x)$ I, -
I I -
.

, 8.1 $y = F(x) = \int_a^x f(t)\,dt$ $y = f(x)$ I $y = f(x)$ I. , , $y = f(x)$ $y = F(x) + c,$ c , ( ) $y = f(x)$, , $y = F(x) + c,$ c ( ).

8.1. .

,
, . .

, $y = f(x)$ I , $y = \int_a^x f(t)\,dt + c,$ I , $y = f(x)$:

$$\frac{d\left(\int_a^x f(t)\,dt + c\right)}{dx} = f(x).$$

, , 8.1. , $y = F(x)$ I $y = \frac{d\,F(x)}{dx}$ I $I,$ $y = F(x)$ (x):

$$\int_a^x \frac{d\,F(t)}{dt}\,dt + c = F(x) + \big(c - F(a)\big).$$

. $y = f(x) = \frac{d\,F(x)}{dx}$ $I.$ $y = F(x)$ $y = f(x)$ $I.$ $\int_a^x \frac{d\,F(t)}{dt}\,dt + c = \int_a^x f(t)\,dt + c$, 8.1, $y = f(x)$ I , , c' $\int_a^x \frac{d\,F(t)}{dt}\,dt + c = F(x) + c'$ x $I.$ $x = a,$ $0 + c = F(a) + c'$, $c' = c - F(a)$, , $\int_a^x \frac{d\,F(t)}{dt}\,dt + c = F(x) + (c - F(a))$ x $I.$

, 8.1, .

8.3 $y = f(x)$ I $y = F(x)$ $y = f(x)$ $I.$
(1) $y = f(x)$ I $y = F(x) + c,$ c . ,

$$\boxed{\int f(x)\,dx = F(x) + c \qquad (x \quad I).}$$

(2) $y = f(x)$ $[a, b]$ I $y = F(x)$ $[a, b]$. ,

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) \qquad (a, b \quad I).}$$

, 8.3. (1) (2) , 8.1, $y = \int_a^x f(t)\,dt$ $y = f(x)$ $I.$ c $\int_a^x f(t)\,dt - F(x) = c$ x $I.$ $x = a,$ $0 - F(a) = c$, , $\int_a^x f(t)\,dt - F(x) = -F(a)$ x $I.$, $x = b$ $\int_a^b f(t)\,dt - F(b) = -F(a)$, , $\int_a^b f(t)\,dt = F(b) - F(a).$

. $y = f(x)$ I
$y = F(x)$ $y = f(x)$ I, -
I.

: .

(1)

$$\boxed{\int 1\, dx = x + c}$$

$(-\infty, +\infty)$.

(2) $\alpha \neq -1$ $\alpha \neq 0$,

$$\boxed{\int x^{\alpha}\, dx = \frac{x^{\alpha+1}}{\alpha+1} + c.}$$

(i) $(-\infty, +\infty)$ α > 0 , (ii) $(-\infty, 0) \cup (0, +\infty)$ α < 0 $\neq -1$, (iii) $[0, +\infty)$ α > 0 > 0 (iv) $(0, +\infty)$ α < 0 < 0.

(3) , $\alpha = -1$,

$$\boxed{\int \frac{1}{x}\, dx = \log|x| + c}$$

$(-\infty, 0)$ $(0, +\infty)$.

(4) $\alpha > 0$, $\alpha \neq 1$,

$$\boxed{\int \alpha^{x}\, dx = \frac{\alpha^{x}}{\log \alpha} + c}$$

$(-\infty, +\infty)$.

(5) $(-\infty, +\infty)$.

$$\boxed{\int \cos x\, dx = \sin x + c, \qquad \int \sin x\, dx = -\cos x + c}$$

(6)

$$\boxed{\int \frac{1}{(\cos x)^2}\, dx = \tan x + c, \qquad \int \frac{1}{(\sin x)^2}\, dx = -\cot x + c}$$

$(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$ $(k\pi, \pi + k\pi)$, k .

(7)

$$\boxed{\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + c, \qquad \int \frac{1}{\sqrt{1-x^2}}\, dx = -\arccos x + c}$$

$(-1, 1)$.

(8)

$$\boxed{\int \frac{1}{1+x^2}\,dx = \arctan x + c}$$

$(-\infty,+\infty)$.

. $y=f(x)$ I
$y=F(x)$ $y=f(x)$ I, -
-
I.

: .

(1) a,b $(-\infty,+\infty)$

$$\boxed{\int_a^b 1\,dx = b-a.}$$

(2) $\alpha\neq -1$ $\alpha\neq 0$,

$$\boxed{\int_a^b x^\alpha\,dx = \frac{b^{\alpha+1}-a^{\alpha+1}}{\alpha+1}\,.}$$

(i) a,b α >0 , (ii) $a,b<0$ $a,b>0$ α <0 $\neq -1$, (iii) $a,b\geq 0$ α >0 >0 (iv) $a,b>0$ α <0 <0.

(3) $\alpha=-1$, $a,b<0$ $a,b>0$

$$\boxed{\int_a^b \frac{1}{x}\,dx = \log|b| - \log|a| = \log\frac{b}{a}\,.}$$

(4) $\alpha>0$, $\alpha\neq 1$, a,b

$$\boxed{\int_a^b \alpha^x\,dx = \frac{\alpha^b-\alpha^a}{\log\alpha}\,.}$$

(5) a,b

$$\boxed{\int_a^b \cos x\,dx = \sin b - \sin a, \qquad \int_a^b \sin x\,dx = \cos a - \cos b.}$$

(6)

$$\boxed{\int_a^b \frac{1}{\cos^2 x}\,dx = \tan b - \tan a, \qquad \int_a^b \frac{1}{\sin^2 x}\,dx = \cot a - \cot b}$$

a,b $(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi)$ a,b $(k\pi, \pi+k\pi)$, k .

(7) a,b $(-1,1)$

$$\boxed{\int_a^b \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin b - \arcsin a, \qquad \int_a^b \frac{1}{\sqrt{1-x^2}}\,dx = \arccos a - \arccos b.}$$

(8) a, b

$$\boxed{\int_a^b \frac{1}{1+x^2}\,dx = \arctan b - \arctan a.}$$

8.1 , , .

, $\frac{f(b)-f(a)}{b-a} = f'(\xi)$ $\frac{1}{b-a}\int_a^b f'(x)\,dx = f'(\xi)$ $y = f(x)$ $[a, b]$.

, $\frac{1}{b-a}\int_a^b f(x)\,dx = f(\xi)$ $\frac{F(b)-F(a)}{b-a} = F'(\xi)$, $y = F(x)$ $y = f(x)$ $[a, b]$.

.

1.

(*i*) $\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$,

(: , $y = \frac{1}{a}\sin(ax)$.)

(*ii*) $\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$.

2.

(*i*) $\int \frac{1}{x\log x}\,dx = \log(\log x) + c$ $(1, +\infty)$,

(*ii*) $\int \frac{1}{x\log x\log(\log x)}\,dx = \log\big(\log(\log x)\big) + c$ $(e, +\infty)$.

3.

(*i*) $\int x^n e^{-x}\,dx = n!e^{-x}\big(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\big) + c$,

(*ii*) $\int x^n e^{x}\,dx = (-1)^{n-1}n!e^{x}\big(e^{-x} - 1 + x - \frac{x^2}{2!} + \cdots + (-1)^{n-1}\frac{x^n}{n!}\big) + c$.

4. :

(*i*) 1 7.4.

(*ii*) 3, 4 5 7.4 . ;

(*iii*) 3 7.4.

(*iv*) 4 7.4.

(*v*) 1 7.4.

;

«» ;

(*vi*) 2 7.4.

5. $y = f(x)$ $(-\infty, +\infty)$ a $\int_a^x f(t)\,dt = \sin x - \frac{\sqrt{3}}{2}$ x. ;

6. $y = f(x)$ $(-\infty, +\infty)$ $\int_0^x f(t)\,dt = e^x$ x;

$\int_0^x f(t)\,dt = e^x - 1$.

7. .

$$y=\int_0^x \frac{\sin t}{1+t^2}\,dt, \qquad y=\int_x^2 \frac{\sin t+e^t}{t^2+1}\,dt,$$

$$y=\int_1^{x^2-x} \frac{t^2-2t}{e^t+2t^2}\,dt, \qquad y=\int_{\sin x}^{x+\cos x} te^t\,dt.$$

, $y=\int_{g(x)}^{h(x)} f(t)\,dt$ ξ $y=f(x)$ $y=g(x)$ $y=h(x)$.

8. $y=f(x)$ $(-\infty,+\infty)$ $\int_0^{x^2} f(t)\,dt = 1-2^{x^2}$ x.

9. $\lim_{x\to+\infty} e^{-x^2}\int_0^x e^{t^2}\,dt = 0.$

(: .)

10. $\lim_{x\to+\infty} \frac{1}{x}\int_0^x e^{t-x}(2t+1)\,dt.$

11. (*) $a>0$ b $\lim_{x\to 0+} \frac{1}{bx-\sin x}\int_0^x \frac{t^2}{\sqrt{a+t}}\,dt = 1.$

12. 4 7.2.

$\lim_{n\to+\infty} \frac{1^\alpha+2^\alpha+\cdots+(n-1)^\alpha+n^\alpha}{n^{\alpha+1}}$.

13. , k :

(*i*) $\int_0^{2\pi}\sin(kx)\,dx = 0.$

(*ii*) $\int_0^{2\pi}\cos(kx)\,dx = 0,\ \ k\neq 0.$

, n, m :

(*iii*) $\int_0^{2\pi}\sin(nx)\cos(mx)\,dx = 0.$

(*iv*) $\int_0^{2\pi}\sin(nx)\sin(mx)\,dx = 0$ $n\neq m.$

(*v*) $\int_0^{2\pi}\cos(nx)\cos(mx)\,dx = 0$ $n\neq m.$

(*vi*) $\int_0^{2\pi}(\sin(nx))^2\,dx = \int_0^{2\pi}(\cos(nx))^2\,dx = \pi$ $n\neq 0.$

14. , $f(x) = a_0 + (a_1\cos x + b_1\sin x)+\cdots+(a_n\cos(nx)+b_n\sin(nx))$ $g(x) = c_0 + (c_1\cos x + d_1\sin x)+\cdots+(c_n\cos(nx)+d_n\sin(nx))$,

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)g(x)\,dx = a_0c_0 + \frac{a_1c_1+b_1d_1}{2}+\cdots+\frac{a_nc_n+b_nd_n}{2}.$$

15. 3 7.2 ,

(*i*) $\int_0^\pi \frac{\sin((n+\frac{1}{2})x)}{\sin\frac{x}{2}}\,dx = \pi$ $n.$

(*ii*) $\int_0^\pi \frac{\sin(nx)}{\sin x}\,dx = \pi$ 0, n , .

(*iii*) $\int_0^\pi \left(\frac{\sin(nx)}{\sin x}\right)^2 dx = n\pi$ $n.$

16. $a \neq \pm b$, $\lim_{T\to+\infty} \frac{1}{T}\int_0^T \sin(ax)\sin(bx)\,dx = 0$.

l' Hopitâl?

17. **Bernoulli** :

$$P_0(x) = 1, \quad {P_n}'(x) = nP_{n-1}(x), \quad \int_0^1 P_n(x)\,dx = 0 \ (n \geq 1).$$

$P_n(x)$ $n = 1, 2, 3, 4$.

$P_n(x)$ n x^n .

$P_n(0) = P_n(1)$ $n \geq 2$.

$P_n(x+1) - P_n(x) = nx^{n-1}$ $n \geq 1$.

$$1^n + 2^n + \cdots + k^n = \int_0^{k+1} P_n(x)\,dx = \frac{P_{n+1}(k+1) - P_{n+1}(0)}{n+1}$$

$n = 1$ $n = 2$.

$P_n(1-x) = (-1)^n P_n(x)$ $n \geq 1$.

$P_{2n+1}(0) = 0$ $P_{2n-1}(\frac{1}{2}) = 0$ $n \geq 1$.

18. $y = f(x)$ I a I.

(i) $y = \int_a^x f(t)\,dt$ I, $y = f(x)$ 0 I.

(ii) $\int_a^x f(t)\,dt = \int_x^b f(t)\,dt$ x I, $y = f(x)$ 0 I.

19. (**) $y = f(x)$ $[a,b]$ $f(x) \geq 0$ x $[a,b]$. $\int_a^b f(x)\,dx = 0$, $y = f(x)$ $[a,b]$.

(: : $\int_a^x f(t)\,dt + \int_x^b f(t)\,dt = \int_a^b f(t)\,dt = 0$ $\int_a^x f(t)\,dt = 0$ x $[a,b]$. : $f(\xi) > 0$ ξ $[a,b]$. $[c,d]$ $[a,b]$ $d-c > 0$ ξ $f(x) > \frac{f(\xi)}{2}$ x $[c,d]$. $\int_a^b f(x)\,dx \geq \int_c^d f(x)\,dx \geq \frac{f(\xi)}{2}(d-c) > 0$.)

: $y = f(x)$ $[a,b]$, $f(x) \geq 0$ x $[a,b]$ $\int_a^b f(x)\,dx = 0$, $y = f(x)$ 0 $[a,b]$.

20. (*) $y = f(x)$ $[a,b]$ M $f(x) \leq M$ x $[a,b]$. $\int_a^b f(x)\,dx \leq M(b-a)$.

, $\int_a^b f(x)\,dx = M(b-a)$, $f(x) = M$ x $[a,b]$.

(: $y = M - f(x)$.)

21. (*) $y = f(x)$ $[a,b]$. $\int_a^b (f(x))^2\,dx = 0$, $f(x) = 0$ x $[a,b]$.

(: .)

22. (*) $y = f(x)$ $[a,b]$. $\int_{x'}^{x''} f(t)\,dt \geq 0$ x', x'' $[a,b]$ $x' < x''$, $f(x) \geq 0$ x $[a,b]$.

(: : $y = \int_a^x f(t)\,dt$ $[a,b]$. : $f(\xi) < 0$ ξ $[a,b]$. $[c,d]$ $[a,b]$ $d-c > 0$ ξ $f(x) < \frac{f(\xi)}{2}$ x $[c,d]$. $\int_c^d f(x)\,dx \leq \frac{f(\xi)}{2}(d-c) < 0$.)

23. $^{(*)}$ $y = f(x)$ $[0, +\infty)$ $f(x) \neq 0$ $x > 0$ $(f(x))^2 = 2\int_0^x f(t)\,dt$ $x \geq 0$.

(i) $f(x) > 0$ $x > 0$.

(: $f(0) = 0$ $f(x) > 0$ $x > 0$ $f(x) < 0$ $x > 0$. $y = (f(x))^2 = 2\int_0^x f(t)\,dt$ $[0, +\infty)$.)

(ii) $y = f(x)$ $(0, +\infty)$.

(: $\sqrt{\cdots}$.)

(iii) $f(x) = x$ $x \geq 0$.

24. $^{(*)}$ $y = f(x)$ $[0, a]$ $f(0) = 0$.

(i) $y = g(x) = \begin{cases} \frac{(f(x))^2}{x}, & 0 < x \leq a, \\ 0, & x = 0, \end{cases}$ $[0, a]$.

(ii) $y = g(x)$ $\int_0^x (f'(t))^2\,dt$.

(iii) $(f(x))^2 \leq x\int_0^x (f'(t))^2\,dt$ x $[0, a]$.

(iv) $(f(a))^2 = a\int_0^a (f'(t))^2\,dt$, $y = \frac{f(x)}{x}$ $(0, a]$.

(v) $(f(a))^2 = a\int_0^a (f'(t))^2\,dt$ $f'(0) = 2$, $f(x) = 2x$ x $[0, a]$.

25. $^{(**)}$ $y = f(x)$ I. I.

(: $F(x) = \int_a^x f(t)\,dt + c$ ξ I. $[c, d]$ I a, ξ. M $|f(x)| \leq M$ x $[c, d]$. x $[c, d]$ $|F(x) - F(\xi)| = \big|\int_\xi^x f(t)\,dt\big| \leq M|x - \xi|$.)

26. $^{(*)}$ $y = f(x)$ $[a, b]$. ξ $[a, b]$ $\int_a^\xi f(x)\,dx = \int_\xi^b f(x)\,dx$.

(: $y = \int_a^x f(t)\,dt - \int_x^b f(t)\,dt$.)

8.3 Riemann.

. .

8.4 $z = f(y)$ J, $y = \phi(x)$ I $y = \phi(x)$ J ($z = f(\phi(x))$ I)

$$\boxed{\int f(\phi(x))\phi'(x)\,dx = \int f(y)\,dy\,\Big|_{y=\phi(x)} \qquad (x \;\; I).}$$

,

$$\boxed{\int_a^b f(\phi(x))\phi'(x)\,dx = \int_{\phi(a)}^{\phi(b)} f(y)\,dy \qquad (a, b \;\; I).}$$

$\int f(\phi(x))\phi'(x)\,dx = \int f(y)\,dy\,\big|_{y=\phi(x)}$. $z = f(\phi(x))\phi'(x)$ I $z = \int f(\phi(x))\phi'(x)\,dx$, , . , , x I. , $z = \int f(y)\,dy$, , $z = f(y)$, , y J. $y = \phi(x)$ x I, x I. , ′ .

: , $z = \int f(y)\,dy$ $z = f(y)$ J, $\int f(y)\,dy = G(y) + c$ J, $z = G(y)$ - $z = f(y)$

J c . , $\int f(y)\,dy\Big|_{y=\phi(x)} = G(\phi(x)) + c$ I. , $\frac{d\,(G(\phi(x)))}{dx} = G'(\phi(x))\phi'(x) = f(\phi(x))\phi'(x)$ I, $z = G(\phi(x))$ $z = f(\phi(x))\phi'(x)$ I. $z = \int f(\phi(x))\phi'(x)\,dx$ $z = f(\phi(x))\phi'(x)$ I, $\int f(\phi(x))\phi'(x)\,dx = G(\phi(x)) + c$ I, c . , , $\int f(\phi(x))\phi'(x)\,dx = \int f(y)\,dy\Big|_{y=\phi(x)}$ I.

$z = F(x) = \int_a^x f(\phi(t))\phi'(t)\,dt$ $z = H(x) = \int_{\phi(a)}^{\phi(x)} f(s)\,ds$ I $z = G(y) = \int_{\phi(a)}^{y} f(s)\,ds$ J. $F'(x) = f(\phi(x))\phi'(x)$ x I $G'(y) = f(y)$ y J. , $H(x) = G(\phi(x))$ I , , $H'(x) = G'(\phi(x))\phi'(x) = f(\phi(x))\phi'(x)$ I. $F'(x) = H'(x)$ I, c $F(x) = H(x) + c$ I. $x = a$ $0 = 0 + c$, $c = 0$, , $\int_a^x f(\phi(t))\phi'(t)\,dt = \int_{\phi(a)}^{\phi(x)} f(s)\,ds$ I. , $x = b$.

: (1) $\int (\sin x)^n \cos x\,dx$, n . $y = \sin x$, $\int (\sin x)^n \cos x\,dx = \int (\sin x)^n \frac{d\,\sin x}{dx}\,dx = \int y^n\,dy\big|_{y=\sin x} = \big(\frac{y^{n+1}}{n+1} + c\big)\big|_{y=\sin x} = \frac{(\sin x)^{n+1}}{n+1} + c$.

(2) $\int \frac{2x}{x^2+1}\,dx$ $y = x^2+1$, $\int \frac{2x}{x^2+1}\,dx = \int \frac{1}{x^2+1}\frac{d(x^2+1)}{dx}\,dx = \int \frac{1}{y}\,dy\big|_{y=x^2+1} = (\log|y| + c)\big|_{y=x^2+1} = \log(x^2+1) + c$.

(3) $\int_a^b \frac{1}{x\log x}\,dx$ $y = \log x$ $\int_a^b \frac{1}{x\log x}\,dx = \int_a^b \frac{1}{\log x}\frac{d\log x}{dx}\,dx = \int_{\log a}^{\log b} \frac{1}{y}\,dy = \log|\log b| - \log|\log a| = \log\big|\frac{\log b}{\log a}\big|$.
a, b $y = \log x$ a, b $z = \frac{1}{y}$, $(-\infty, 0)$ $(0, +\infty)$. $y = \log x$ $\log a, \log b$. $\log a, \log b > 0$, , $a, b > 1$ $\log a, \log b < 0$, , $0 < a, b < 1$. , $\log a, \log b$, $\int_a^b \frac{1}{x\log x}\,dx = \log\frac{\log b}{\log a}$.

. .

8.5 $y = f(x)$ $y = g(x)$ I,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (x\ \ I).$$

,

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx \qquad (a, b\ \ I).$$

: $y = F(x)$ $y = f'(x)g(x)$ I, $\int f'(x)g(x)\,dx = F(x) - c$ I, c . $\frac{d\,(f(x)g(x) - F(x))}{dx} = f'(x)g(x) + f(x)g'(x) - F'(x) = f(x)g'(x)$ I, $y = f(x)g(x) - F(x)$ $y = f(x)g'(x)$ I. $\int f(x)g'(x)\,dx = f(x)g(x) - F(x) + c = f(x)g(x) - \int f'(x)g(x)\,dx$ I.
$y = F(x) = \int_a^x f'(t)g(t)\,dt$ $y = G(x) = \int_a^x f(t)g'(t)\,dt$ I, $F'(x) + G'(x) = f'(x)g(x) + f(x)g'(x) = \frac{d\,(f(x)g(x))}{dx}$ I. c $F(x) + G(x) = f(x)g(x) + c$ I. $x = a$ $0 + 0 = f(a)g(a) + c$, $c = -f(a)g(a)$. $F(x) + G(x) = f(x)g(x) - f(a)g(a)$, , $\int_a^x f'(t)g(t)\,dt + \int_a^x f(t)g'(t)\,dt = f(x)g(x) - f(a)g(a)$ I. , $x = b$.

: (1) $\int \log x\,dx = \int \log x \frac{d\,x}{dx}\,dx = x\log x - \int \frac{d\,\log x}{dx}\,x\,dx = x\log x - \int \frac{1}{x}\,x\,dx = x\log x - \int 1\,dx = x\log x - x + c$ $(0, +\infty)$.

(2) $a \neq 0$, $\int e^{ax}\sin(bx)\,dx = \frac{1}{a}\int \frac{d\,e^{ax}}{dx}\sin(bx)\,dx = \frac{1}{a}e^{ax}\sin(bx) - \frac{1}{a}\int e^{ax}\frac{d\,\sin(bx)}{dx}\,dx = \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}\int e^{ax}\cos(bx)\,dx$ ( ) $= \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a^2}\int \frac{d\,e^{ax}}{dx}\cos(bx)\,dx = \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a^2}e^{ax}\cos(bx) + \frac{b}{a^2}\int e^{ax}\frac{d\,\cos(bx)}{dx}\,dx = \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a^2}e^{ax}\cos(bx) -$

$\frac{b^2}{a^2}\int e^{ax}\sin(bx)\,dx$. $\big(1+\frac{b^2}{a^2}\big)\int e^{ax}\sin(bx)\,dx = \frac{1}{a^2}e^{ax}\big(a\sin(bx)-b\cos(bx)\big)+c$, c , , $\int e^{ax}\sin(bx)\,dx = e^{ax}\big(\frac{a}{a^2+b^2}\sin(bx)-\frac{b}{a^2+b^2}\cos(bx)\big)+c$.
$a=0$ $b\neq 0$, $\int \sin(bx)\,dx = -\frac{1}{b}\cos(bx)+c$. :

$$\int e^{ax}\sin(bx)\,dx = \frac{a}{a^2+b^2}\,e^{ax}\sin(bx) - \frac{b}{a^2+b^2}\,e^{ax}\cos(bx)+c$$

a, b $a^2+b^2\neq 0$. , :

$$\int e^{ax}\cos(bx)\,dx = \frac{a}{a^2+b^2}\,e^{ax}\cos(bx) + \frac{b}{a^2+b^2}\,e^{ax}\sin(bx)+c.$$

(3) $\int_0^2 xe^x\,dx = \int_0^2 x\frac{d\,e^x}{dx}\,dx = 2e^2-0e^0-\int_0^2\frac{d\,x}{dx}e^x\,dx = 2e^2-\int_0^2 e^x\,dx = 2e^2-(e^2-e^0)=e^2+1$.

(4) $\int_0^\pi x\sin x\,dx = -\int_0^\pi x\frac{d\cos x}{dx}dx = -\pi\cos\pi+0\cos 0+\int_0^\pi\frac{dx}{dx}\cos x\,dx = \pi+\int_0^\pi\cos x\,dx = \pi+(\sin\pi-\sin 0)=\pi$.

. .

$$\int r(x)\,dx = \int \frac{a_mx^m+a_{m-1}x^{m-1}+\cdots+a_1x+a_0}{b_nx^n+b_{n-1}x^{n-1}+\cdots+b_1x+b_0}\,dx,$$

$r(x)=\frac{a_mx^m+a_{m-1}x^{m-1}+\cdots+a_1x+a_0}{b_nx^n+b_{n-1}x^{n-1}+\cdots+b_1x+b_0}$.

. . $m<n$, . , , $m\geq n$, $p(x)$ $q(x)$ $q(x)$ $<n$

$$a_mx^m+\cdots+a_0 = p(x)\big(b_nx^n+\cdots+b_0\big)+q(x)$$

x.

$$r(x) = p(x)+\frac{q(x)}{b_nx^n+b_{n-1}x^{n-1}+\cdots+b_1x+b_0}\,.$$

$\int p(x)\,dx$, $m<n$.

. . , , . .

$b_nx^n+b_{n-1}x^{n-1}+\cdots+b_1x+b_0$

$$b_nx^n+\cdots+b_0 = b_n(x-\alpha)^\kappa\cdots(x-\gamma)^\lambda((x-\mu)^2+\nu^2)^\rho\cdots((x-\epsilon)^2+\delta^2)^\tau$$

$$= b_n(x-\alpha)^\kappa\cdots(x-\gamma)^\lambda(x-\mu-i\nu)^\rho(x-\mu+i\nu)^\rho\cdots(x-\epsilon-i\delta)^\tau(x-\epsilon+i\delta)^\tau,$$

$\kappa,\ldots,\lambda,\rho,\ldots,\tau$ $\kappa+\cdots+\lambda+2\rho+\cdots+2\tau=n$ $\nu,\ldots,\delta$ >0.

$x-\alpha,\ldots,x-\gamma$ $\alpha,\ldots,\gamma$ $\kappa,\ldots,\lambda$. , $(x-\mu)^2+\nu^2,\ldots,(x-\epsilon)^2+\delta^2$ $\mu\pm i\nu,\ldots,\epsilon\pm i\delta$ $\rho,\ldots,\tau$.
, , $\alpha,\ldots,\gamma$: $-\infty,\alpha,\ldots,\gamma,+\infty$.

. :

$$\begin{aligned} r(x) = &\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_\kappa}{(x-\alpha)^\kappa} \\ &+ \cdots\cdots\cdots\cdots \\ &+ \frac{\Gamma_1}{x-\gamma} + \frac{\Gamma_2}{(x-\gamma)^2} + \cdots + \frac{\Gamma_\lambda}{(x-\gamma)^\lambda} \\ &+ \frac{M_1(x-\mu)+N_1}{(x-\mu)^2+\nu^2} + \cdots + \frac{M_\rho(x-\mu)+N_\rho}{((x-\mu)^2+\nu^2)^\rho} \\ &+ \cdots\cdots\cdots\cdots \\ &+ \frac{E_1(x-\epsilon)+\Delta_1}{(x-\epsilon)^2+\delta^2} + \cdots + \frac{E_\tau(x-\epsilon)+\Delta_\tau}{((x-\epsilon)^2+\delta^2)^\tau}\,. \end{aligned}$$

«» . $x-\alpha, \ldots, x-\gamma$ 1 $\kappa, \ldots, \lambda$, , . , $(x-\mu)^2+\nu^2, \ldots, (x-\epsilon)^2+\delta^2$ 1 $\rho, \ldots, \tau$. $A_1, A_2, \ldots, E_\tau, \Delta_\tau$. , , $b_n x^n + \cdots + b_1 x + b_0$. , n n $A_1, A_2, \ldots, E_\tau, \Delta_\tau$.

. , , :

$$\int \frac{1}{(x-\alpha)^k}\,dx, \qquad \int \frac{x-\mu}{((x-\mu)^2+\nu^2)^k}\,dx, \qquad \int \frac{1}{((x-\mu)^2+\nu^2)^k}\,dx,$$

k . .

(i) $\int \frac{1}{(x-\alpha)^k}\,dx$, $(-\infty, \alpha)$ $(\alpha, +\infty)$, $y = x-\alpha$

$$\int \frac{1}{(x-\alpha)^k}\,dx = \int \frac{1}{(x-\alpha)^k}\frac{d(x-\alpha)}{dx}\,dx = \int \frac{1}{y^k}\,dy\Big|_{y=x-\alpha}.$$

$k \geq 2$,

$$\int \frac{1}{(x-\alpha)^k}\,dx = \Big(-\frac{1}{k-1}\frac{1}{y^{k-1}} + c\Big)\Big|_{y=x-\alpha} = -\frac{1}{k-1}\frac{1}{(x-\alpha)^{k-1}} + c.$$

$k = 1$,

$$\int \frac{1}{x-\alpha}\,dx = \big(\log|y| + c\big)\Big|_{y=x-\alpha} = \log|x-\alpha| + c.$$

(ii) $\int \frac{x-\mu}{((x-\mu)^2+\nu^2)^k}\,dx$ $y = (x-\mu)^2+\nu^2$

$$\begin{aligned} \int \frac{x-\mu}{((x-\mu)^2+\nu^2)^k}\,dx &= \frac{1}{2}\int \frac{1}{((x-\mu)^2+\nu^2)^k}\frac{d\,((x-\mu)^2+\nu^2)}{dx}\,dx \\ &= \frac{1}{2}\int \frac{1}{y^k}\,dy\Big|_{y=(x-\mu)^2+\nu^2}. \end{aligned}$$

$k \geq 2$,

$$\begin{aligned} \int \frac{x-\mu}{((x-\mu)^2+\nu^2)^k}\,dx &= \frac{1}{2}\Big(-\frac{1}{k-1}\frac{1}{y^{k-1}} + c\Big)\Big|_{y=(x-\mu)^2+\nu^2} \\ &= -\frac{1}{2(k-1)}\frac{1}{((x-\mu)^2+\nu^2)^{k-1}} + c. \end{aligned}$$

$k=1$,

$$\int \frac{x-\mu}{(x-\mu)^2+\nu^2}\,dx = \frac{1}{2}\big(\log|y|+c\big)\Big|_{y=(x-\mu)^2+\nu^2} = \frac{1}{2}\log((x-\mu)^2+\nu^2)+c.$$

(iii) , $\int \frac{1}{((x-\mu)^2+\nu^2)^k}\,dx$ $y=\frac{x-\mu}{\nu}$

$$\begin{aligned}\int \frac{1}{((x-\mu)^2+\nu^2)^k}\,dx &= \frac{1}{\nu^{2k-1}}\int \frac{1}{\big((\frac{x-\mu}{\nu})^2+1\big)^k}\frac{d}{dx}\frac{x-\mu}{\nu}\,dx \\ &= \frac{1}{\nu^{2k-1}}\int \frac{1}{(y^2+1)^k}\,dy\Big|_{y=\frac{x-\mu}{\nu}}.\end{aligned}$$

$$I_k = \int \frac{1}{(y^2+1)^k}\,dy,$$

k . . ′ , $k=1$,

$$I_1 = \int \frac{1}{y^2+1}\,dy = \arctan y + c.$$

$k>1$,

$$\begin{aligned}I_k &= \int \frac{1}{(y^2+1)^k}\,dy = \int \frac{y^2+1}{(y^2+1)^k}\,dy - \int \frac{y^2}{(y^2+1)^k}\,dy \\ &= \int \frac{1}{(y^2+1)^{k-1}}\,dy - \int y\frac{y}{(y^2+1)^k}\,dy \\ &= I_{k-1} + \frac{1}{2(k-1)}\int y\frac{d}{dy}\frac{1}{(y^2+1)^{k-1}}\,dy \\ &= I_{k-1} + \frac{1}{2(k-1)}\frac{y}{(y^2+1)^{k-1}} - \frac{1}{2(k-1)}\int \frac{1}{(y^2+1)^{k-1}}\,dy \\ &= \frac{1}{2k-2}\frac{y}{(y^2+1)^{k-1}} + \frac{2k-3}{2k-2}\,I_{k-1}\,.\end{aligned}$$

I_k I_{k-1} , , I_1: $I_k = \frac{1}{2k-2}\frac{y}{(y^2+1)^{k-1}} + \frac{2k-3}{(2k-2)(2k-4)}\frac{y}{(y^2+1)^{k-2}} + \frac{(2k-3)(2k-5)}{(2k-2)(2k-4)}\,I_{k-2}$

$$\begin{aligned}I_k = \frac{1}{2k-2}\frac{y}{(y^2+1)^{k-1}} &+ \frac{2k-3}{(2k-2)(2k-4)}\frac{y}{(y^2+1)^{k-2}} + \cdots \\ \cdots + \frac{(2k-3)(2k-5)\cdots 3}{(2k-2)(2k-4)\cdots 2}\frac{y}{y^2+1} &+ \frac{(2k-3)(2k-5)\cdots 1}{(2k-2)(2k-4)\cdots 2}\arctan y + c.\end{aligned}$$

$\int r(x)\,dx$, $\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_\kappa}{(x-\alpha)^\kappa}$

$$A_1\log|x-\alpha| - \frac{A_2}{x-\alpha} - \cdots - \frac{A_\kappa}{(k-1)(x-\alpha)^{\kappa-1}}$$

, $\frac{M_1(x-\mu)}{(x-\mu)^2+\nu^2} + \frac{M_2(x-\mu)}{((x-\mu)^2+\nu^2)^2} + \cdots + \frac{M_\rho(x-\mu)}{((x-\mu)^2+\nu^2)^\rho}$

$\frac{M_1}{2}\log((x-\mu)^2+\nu^2) - \frac{M_2}{2((x-\mu)^2+\nu^2)} - \cdots - \frac{M_\rho}{2(\rho-1)((x-\mu)^2+\nu^2)^{\rho-1}}$

, , $\frac{N_1}{(x-\mu)^2+\nu^2} + \frac{N_2}{((x-\mu)^2+\nu^2)^2} + \cdots + \frac{N_\rho}{((x-\mu)^2+\nu^2)^\rho}$

$$N_1{}'\arctan\frac{x-\mu}{\nu} + \frac{N_2{}'(x-\mu)}{(x-\mu)^2+\nu^2} + \cdots + \frac{N_\rho{}'(x-\mu)}{((x-\mu)^2+\nu^2)^{\rho-1}},$$

$N_1{}', \ldots, N_\rho{}'$ $N_1, \ldots, N_\rho$.

: (1) $\int \frac{2}{x-3}\,dx = 2\log|x-3| + c$ $(-\infty, 0)$ $(0, +\infty)$.

(2) $\int \frac{-5}{(x+2)^3}\,dx = \frac{5}{2}\frac{1}{(x+2)^2} + c$ $(-\infty, -2)$ $(-2, +\infty)$.

(3) $\int \frac{1}{(x+1)^2+9}\,dx$ $y = \frac{x+1}{3}$,

$$\begin{aligned}\int \frac{1}{(x+1)^2+9}\,dx &= \frac{1}{3}\int \frac{1}{y^2+1}\,dy\Big|_{y=\frac{x+1}{3}} = \Big(\frac{1}{3}\arctan y + c\Big)\Big|_{y=\frac{x+1}{3}} \\ &= \frac{1}{3}\arctan\frac{x+1}{3} + c\end{aligned}$$

$(-\infty, +\infty)$.

(4) $\int \frac{x-2}{(x-2)^2+4}\,dx$ $y = (x-2)^2+4$,

$$\begin{aligned}\int \frac{x-2}{(x-2)^2+4}\,dx &= \frac{1}{2}\int \frac{1}{y}\,dy\Big|_{y=(x-2)^2+4} = \frac{1}{2}(\log|y| + c)\big|_{y=(x-2)^2+4} \\ &= \frac{1}{2}\log((x-2)^2+4) + c\end{aligned}$$

$(-\infty, +\infty)$.

(5) $\int \frac{x-2}{((x-2)^2+4)^4}\,dx$ $y = (x-2)^2+4$:

$$\begin{aligned}\int \frac{x-2}{((x-2)^2+4)^4}\,dx &= \frac{1}{2}\int \frac{1}{y^4}\,dy\Big|_{y=(x-2)^2+4} = \frac{1}{2}\Big(-\frac{1}{3}\frac{1}{y^3} + c\Big)\Big|_{y=(x-2)^2+4} \\ &= -\frac{1}{6}\frac{1}{((x-2)^2+4)^3} + c.\end{aligned}$$

$(-\infty, +\infty)$.

(6) $\int \frac{x^3-2x^2+2}{x^2-3x+2}\,dx$, ′ $x^3 - 2x^2 + 2$ $x^2 - 3x + 2$ $x^3 - 2x^2 + 2 = (x^2-3x+2)(x+1) + x$, , $\frac{x^3-2x^2+2}{x^2-3x+2} = x + 1 + \frac{x}{x^2-3x+2}$.

$$\begin{aligned}\int \frac{x^3-2x^2+2}{x^2-3x+2}\,dx &= \int (x+1)\,dx + \int \frac{x}{x^2-3x+2}\,dx \\ &= \frac{1}{2}x^2 + x + \int \frac{x}{x^2-3x+2}\,dx.\end{aligned}$$

$\int \frac{x}{x^2-3x+2}\,dx$ $\frac{x}{x^2-3x+2}$. x^2-3x+2 1 2, $x^2-3x+2=(x-1)(x-2)$. $\frac{x}{x^2-3x+2}$:

$$\frac{x}{x^2-3x+2}=\frac{A}{x-1}+\frac{B}{x-2}\,,$$

A, B . $(x-1)(x-2)$ $x=A(x-2)+B(x-1)$, , $x=(A+B)x+(-2A-B)$. $A+B=1$ $-2A-B=0$. $A=-1$, $B=2$.

$$\frac{x}{x^2-3x+2}=-\frac{1}{x-1}+\frac{2}{x-2}\,.$$

,

$$\int \frac{x}{x^2-3x+2}\,dx=-\int\frac{1}{x-1}\,dx+2\int\frac{1}{x-2}\,dx=-\log|x-1|+2\log|x-2|+c$$

, ,

$$\int \frac{x^3-2x^2+2}{x^2-3x+2}\,dx=\frac{1}{2}x^2+x-\log|x-1|+2\log|x-2|+c$$

$(-\infty,1)$, $(1,2)$ $(2,+\infty)$.

(7) $\int \frac{2x^2+1}{x^3+x^2-x-1}\,dx$ $\frac{2x^2+1}{x^3+x^2-x-1}$. x^3+x^2-x-1 : $x^3+x^2-x-1=x^2(x+1)-(x+1)=(x^2-1)(x+1)=(x-1)(x+1)^2$. , x^3+x^2-x-1 1 -1. $\frac{2x^2+1}{x^3+x^2-x-1}$:

$$\frac{2x^2+1}{x^3+x^2-x-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}\,.$$

$(x-1)(x+1)^2$ $2x^2+1=A(x+1)^2+B(x-1)(x+1)+C(x-1)$, , $2x^2+1=(A+B)x^2+(2A+C)x+(A-B-C)$. $A+B=2$, $2A+C=0$ $A-B-C=1$. $A=\frac{3}{4}$, $B=\frac{5}{4}$, $C=-\frac{3}{2}$.

$$\frac{2x^2+1}{x^3+x^2-x-1}=\frac{3}{4}\frac{1}{x-1}+\frac{5}{4}\frac{1}{x+1}-\frac{3}{2}\frac{1}{(x+1)^2}\,,$$

$$\begin{aligned}\int \frac{2x^2+1}{x^3+x^2-x-1}\,dx&=\frac{3}{4}\int\frac{1}{x-1}\,dx+\frac{5}{4}\int\frac{1}{x+1}\,dx-\frac{3}{2}\int\frac{1}{(x+1)^2}\,dx\\&=\frac{3}{4}\log|x-1|+\frac{5}{4}\log|x+1|+\frac{3}{2}\frac{1}{x+1}+c\end{aligned}$$

$(-\infty,-1)$, $(-1,1)$ $(1,+\infty)$.

(8) $\int \frac{x}{x^4-x^2-2x+2}\,dx$ $\frac{x}{x^4-x^2-2x+2}$. x^4-x^2-2x+2 : $x^4-x^2-2x+2=x^2(x^2-1)-2(x-1)=x^2(x-1)(x+1)-2(x-1)=(x-1)(x^3+x^2-2)=(x-1)(x^3-x^2+2x^2-2)=(x-1)(x^2(x-1)+2(x-1)(x+1))=(x-1)^2(x^2+2x+2)$. , x^4-x^2-2x+2 1 , $x^2+2x+2=(x+1)^2+1$ () . $\frac{x}{x^4-x^2-2x+2}$:

$$\frac{x}{x^4-x^2-2x+2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C(x+1)+D}{(x+1)^2+1}\,.$$

$(x-1)^2((x+1)^2+1)$ $x=(A+C)x^3+(A+B-C+D)x^2+(2B-C-2D)x+(-2A+2B+C+D)$. $A+C=0$, $A+B-C+D=0$, $2B-C-2D=1$ $-2A+2B+C+D=0$. $A=\frac{1}{25}$, $B=\frac{1}{5}$, $C=-\frac{1}{25}$, $D=-\frac{7}{25}$.

$$\frac{x}{x^4-x^2-2x+2}=\frac{1}{25}\frac{1}{x-1}+\frac{1}{5}\frac{1}{(x-1)^2}-\frac{1}{25}\frac{x+1}{(x+1)^2+1}-\frac{7}{25}\frac{1}{(x+1)^2+1},$$

$$\begin{aligned}\int\frac{x}{x^4-x^2-2x+2}\,dx &= \frac{1}{25}\int\frac{1}{x-1}\,dx+\frac{1}{5}\int\frac{1}{(x-1)^2}\,dx\\ &\quad -\frac{1}{25}\int\frac{x+1}{(x+1)^2+1}\,dx-\frac{7}{25}\int\frac{1}{(x+1)^2+1}\,dx\\ &= \frac{1}{25}\log|x-1|-\frac{1}{5}\frac{1}{x-1}-\frac{1}{50}\log((x+1)^2+1)\\ &\quad -\frac{7}{25}\arctan(x+1)+c.\end{aligned}$$

$(-\infty,1)$ $(1,+\infty)$.

(9) $\int\frac{x^7+6x^6-x}{x^5-x^4+2x^3-2x^2+x-1}\,dx$.

: $\frac{x^7+6x^6-x}{x^5-x^4+2x^3-2x^2+x-1}=x^2+7x+5+\frac{-7x^4+3x^3+4x^2+x+5}{x^5-x^4+2x^3-2x^2+x-1}$.

$$\begin{aligned}\int\frac{x^7+6x^6-x}{x^5-x^4+2x^3-2x^2+x-1}\,dx &= \frac{1}{3}x^3+\frac{7}{2}x^2+5x\\ &\quad +\int\frac{-7x^4+3x^3+4x^2+x+5}{x^5-x^4+2x^3-2x^2+x-1}\,dx.\end{aligned}$$

: $x^5-x^4+2x^3-2x^2+x-1=x^4(x-1)+2x^2(x-1)+(x-1)=(x^4+2x^2+1)(x-1)=(x^2+1)^2(x-1)$. $\frac{-7x^4+3x^3+4x^2+x+5}{x^5-x^4+2x^3-2x^2+x-1}=\frac{-7x^4+3x^3+4x^2+x+5}{(x^2+1)^2(x-1)}$
:

$$\frac{-7x^4+3x^3+4x^2+x+5}{(x^2+1)^2(x-1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}.$$

$(x^2+1)^2(x-1)$ $-7x^4+3x^3+4x^2+x+5=(A+B)x^4+(-B+C)x^3+(2A+B-C+D)x^2+(-B+C-D+E)x+(A-C-E)$ $A+B=-7$, $-B+C=3$, $2A+B-C+D=4$, $-B+C-D+E=1$ $A-C-E=5$. $A=\frac{3}{2}$, $B=-\frac{17}{2}$, $C=-\frac{11}{2}$, $D=4$ $E=2$.

$$\begin{aligned}\int\frac{-7x^4+3x^3+4x^2+x+5}{x^5-x^4+2x^3-2x^2+x-1}\,dx &= \frac{3}{2}\int\frac{1}{x-1}\,dx-\frac{1}{2}\int\frac{17x+11}{x^2+1}\,dx\\ &\quad +2\int\frac{2x+1}{(x^2+1)^2}\,dx\\ &= \frac{3}{2}\log|x-1|-\frac{17}{4}\log(x^2+1)\\ &\quad -\frac{11}{2}\arctan x-\frac{2}{x^2+1}\\ &\quad +2\int\frac{1}{(x^2+1)^2}\,dx.\end{aligned}$$

$(-\infty, 1)$ $(1, +\infty)$. .

$$\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{x^2+1}{(x^2+1)^2}\,dx - \int \frac{x^2}{(x^2+1)^2}\,dx$$
$$= \int \frac{1}{x^2+1}\,dx - \int x\frac{x}{(x^2+1)^2}\,dx$$
$$= \arctan x + \frac{1}{2}\int x\frac{d}{dx}\frac{1}{x^2+1}\,dx$$
$$= \arctan x + \frac{x}{2(x^2+1)} - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$
$$= \frac{1}{2}\arctan x + \frac{x}{2(x^2+1)} + c.$$

$(-\infty, +\infty)$. :

$$\int \frac{x^7+6x^6-x}{x^5-x^4+2x^3-2x^2+x-1}\,dx = \frac{1}{3}x^3 + \frac{7}{2}x^2 + 5x + \frac{x-2}{x^2+1} + \frac{3}{2}\log|x-1| - \frac{17}{4}\log(x^2+1) - \frac{9}{2}\arctan x + c.$$

$(-\infty, 1)$ $(1, +\infty)$.

. .

$$\int r(\cos x, \sin x)\,dx,$$

$r(s,t)$ $s,\ t.$, $y = f(x) = r(\cos x, \sin x)$ $\frac{f_1(x)}{f_2(x)}$, $y = f_1(x)$ $y = f_2(x)$ $a\,(\cos x)^k(\sin x)^l$, a k, l .

: $\int \big(2\sin x\cos x - \frac{\sin x + (\cos x)^3 - (\sin x)^2\cos x}{\sin x + (\cos x)^2}\big)\,dx$ $f(x) = 2\sin x\cos x - \frac{\sin x+(\cos x)^3-(\sin x)^2\cos x}{\sin x+(\cos x)^2} = \frac{2(\cos x)^3\sin x + 3\cos x(\sin x)^2 - (\cos x)^3 - \sin x}{(\cos x)^2+\sin x}$, $f_1(x) = 2(\cos x)^3\sin x + 3\cos x(\sin x)^2 - (\cos x)^3 - \sin x$, $f_2(x) = (\cos x)^2 + \sin x$ $r(s,t) = \frac{2s^3t+3st^2-s^3-t}{s^2+t}$.

$y = f_1(x)$ $y = f_2(x)$ $(-\infty, +\infty)$ $y = f(x)$ $y = f_2(x)$ 0 , , $y = f(x)$, $y = f(x)$. , , $y = f(x)$ 2π .

1. $y = f(x)$ $-\pi$, 0 $-\pi$. 2π, π. , $(-\pi, \pi)$, $(-\pi, \pi)$, $(-\pi, \pi)$, $(-\pi, \pi)$. $y = f(x)$ $(-\pi, \pi)$ $u = \tan\frac{x}{2}$. : $u = \tan\frac{x}{2}$ $\pm\pi$. $u = \tan\frac{x}{2}$, $\frac{du}{dx} = \frac{1}{2(\cos\frac{x}{2})^2} > 0$ x $(-\pi, \pi)$. $\lim_{x\to-\pi+}\tan\frac{x}{2} = -\infty$ $\lim_{x\to\pi-}\tan\frac{x}{2} = +\infty$, $u = \tan\frac{x}{2}$ $(-\infty, +\infty)$. $x = 2\arctan u$, $(-\infty, +\infty)$ $(-\pi, \pi)$. , x $(-\pi, \pi)$, u $(-\infty, +\infty)$. , $\cos x = \frac{1-(\tan\frac{x}{2})^2}{1+(\tan\frac{x}{2})^2} = \frac{1-u^2}{1+u^2}$ $\sin x = \frac{2\tan\frac{x}{2}}{1+(\tan\frac{x}{2})^2} = \frac{2u}{1+u^2}$, $y = f(x) = r(\cos x, \sin x)$

$(-\pi,\pi)$ $y=g(u)=r\big(\frac{1-u^2}{1+u^2},\frac{2u}{1+u^2}\big)$ $(-\infty,\infty)$. u.

: $y=f(x)=\frac{2(\cos x)^3\sin x+3\cos x(\sin x)^2-(\cos x)^3-\sin x}{(\cos x)^2+\sin x}$ $(-\pi,\pi)$ $y=g(u)=\frac{2\big(\frac{1-u^2}{1+u^2}\big)^3\frac{2u}{1+u^2}+3\frac{1-u^2}{1+u^2}\big(\frac{2u}{1+u^2}\big)^2-\big(\frac{1-u^2}{1+u^2}\big)^3-\frac{2u}{1+u^2}}{\big(\frac{1-u^2}{1+u^2}\big)^2+\frac{2u}{1+u^2}}=\frac{-1+2u+14u^2-18u^3+6u^5-14u^6+2u^7+u^8}{1+2u+6u^3-2u^4+6u^5+2u^7+u^8}$ $(-\infty,+\infty)$.

$f(x)=g(u)=g(\tan\frac{x}{2})$ $g(u)=f(x)=f(2\arctan u)$, , $y=f(x)$ $y=g(u)$. , $y=f(x)$ x $(-\pi,\pi)$, $y=g(u)$ u $(-\infty,+\infty)$. , $y=f(x)$ x $(-\pi,\pi)$, $y=g(u)$ u $(-\infty,+\infty)$.
, $y=f(x)$ $(-\pi,\pi)$ $y=g(u)$ $(-\infty,\infty)$, . , $y=g(u)$, , $(-\infty,\infty)$, $y=f(x)$ $(-\pi,\pi)$. $y=g(u)$ $u_1,\dots,u_n$ $-\infty<u_1<\dots<u_n<+\infty$, $(-\infty,u_1),(u_1,u_2),\dots,(u_{n-1},u_n),(u_n,+\infty)$, $y=f(x)$ $x_1,\dots,x_n$ $-\pi<x_1<\dots<x_n<\pi$ $(-\pi,x_1),(x_1,x_2),\dots,(x_{n-1},x_n),(x_n,\pi)$. x_i u_i $u_i=\tan\frac{x_i}{2}$ $x_i=2\arctan u_i$.
, $\frac{du}{dx}=\frac{1}{2(\cos\frac{x}{2})^2}=\frac{1}{2}\big(1+(\tan\frac{x}{2})^2\big)=\frac{1+u^2}{2}$. , $\int r(\sin x,\cos x)\,dx$ ′ $(-\pi,\pi)$,

$$\int r(\sin x,\cos x)\,dx=\int f(x)\,dx=\int g(u)\frac{2}{1+u^2}\,du\Big|_{u=\tan\frac{x}{2}}$$
$$=\int r\Big(\frac{1-u^2}{1+u^2},\frac{2u}{1+u^2}\Big)\frac{2}{1+u^2}\,du\Big|_{u=\tan\frac{x}{2}}.$$

$y=f(x)=r(\cos x,\sin x)$ $(-\pi,\pi)$ u $(-\infty,+\infty)$.
, , $\int g(u)\frac{2}{1+u^2}\,du=G(u)+c$, $y=G(u)$ u $(-\infty,+\infty)$. , ,

$$\int r(\cos x,\sin x)\,dx=\int f(x)\,dx=G\Big(\tan\frac{x}{2}\Big)+c=F(x)+c$$

$(-\pi,\pi)$.
, , $(-\pi,\pi)$ $(-\pi+k2\pi,\pi+k2\pi)$ $(k\in\mathbf{Z})$. , $y=f(x)$ 2π, $y=F(x)=G\big(\tan\frac{x}{2}\big)$ 2π. , $\int f(x)\,dx=F(x)+c$ (a,b) $(-\pi,\pi)$ $y=f(x)$. , , $F'(x)=f(x)$. $(a+k2\pi,b+k2\pi)$ $(-\pi+k2\pi,\pi+k2\pi)$, x $(a+k2\pi,b+k2\pi)$ $x-k2\pi$ (a,b) , , $F'(x)=F'(x-k2\pi)=f(x-k2\pi)=f(x)$. $\int f(x)\,dx=F(x)+c$ $(a+k2\pi,b+k2\pi)$.
, , $\int r(\cos x,\sin x)\,dx$ $(-\infty,+\infty)$ $y=f(x)=r(\cos x,\sin x)$.

: $\int\frac{1}{\sin x}\,dx$.
1 $y=\frac{1}{\sin x}$ $-\pi$. ′ $(-\pi,\pi)$.
$u=\tan\frac{x}{2}$ $y=\frac{1}{\sin x}$ $(-\pi,\pi)$ $y=\frac{1+u^2}{2u}$ $(-\infty,+\infty)$. $y=\frac{1+u^2}{2u}$ 0 $(-\infty,+\infty)$ $y=\frac{1}{\sin x}$ 0 $(-\pi,\pi)$. $\int\frac{1}{\sin x}\,dx=\int\frac{1+u^2}{2u}\frac{2}{1+u^2}\,du\big|_{u=\tan\frac{x}{2}}=\int\frac{1}{u}\,du\big|_{u=\tan\frac{x}{2}}=\log\big|\tan\frac{x}{2}\big|+c$ $(-\pi,0)$ $(0,\pi)$ $(-\pi,\pi)$.
$y=\frac{1}{\sin x}$ $y=\log\big|\tan\frac{x}{2}\big|$ 2π, $\int\frac{1}{\sin x}\,dx=\log\big|\tan\frac{x}{2}\big|+c$ $(-\pi+k2\pi,k2\pi)$ $(k2\pi,\pi+k2\pi)$ $(-\pi+k2\pi,\pi+k2\pi)$ $(k\in\mathbf{Z})$.

, : $\int \frac{1}{\sin x}\,dx = \log\left|\tan\frac{x}{2}\right| + c$ $(k\pi, \pi + k\pi)$ $(k \in \mathbf{Z})$.

, , , $y = r(\cos x, \sin x)$ x_0 $\neq -\pi$.

, $z = x - x_0 - \pi$ $\cos x = \cos(z + x_0 + \pi) = -\cos(z + x_0) = -\cos x_0 \cos z + \sin x_0 \sin z = p\cos z + q\sin z$, $p = -\cos x_0$ $q = \sin x_0$, , , $\sin x = \sin(z + x_0 + \pi) = -\sin(z + x_0) = -\sin x_0 \cos z - \cos x_0 \sin z = -q\cos z + p\sin z$. $r(\cos x, \sin x) = r(p\cos z + q\sin z, -q\cos z + p\sin z)$ $\frac{dz}{dx} = 1$,

$$\int r(\cos x, \sin x)\,dx = \int r(p\cos z + q\sin z, -q\cos z + p\sin z)\,dz\Big|_{z=x-x_0-\pi}.$$

x $(x_0, x_0 + 2\pi)$ z $(-\pi, \pi)$, $y = r(\cos x, \sin x)$ x_0, $y = r(p\cos z + q\sin z, -q\cos z + p\sin z)$ $-\pi$. , , .

: $\int \frac{1}{(\cos x)^2}\,dx$. .

$y = \frac{1}{(\cos x)^2}$ $-\frac{\pi}{2}$ ($-\pi$). $z = x - (-\frac{\pi}{2}) - \pi = x - \frac{\pi}{2}$ $y = \frac{1}{(\cos x)^2}$ $y = \frac{1}{(\cos(z+\frac{\pi}{2}))^2} = \frac{1}{(\sin z)^2}$, , $\int \frac{1}{(\cos x)^2}\,dx = \int \frac{1}{(\sin z)^2}\,dz\big|_{z=x-\frac{\pi}{2}}$.

, $y = \frac{1}{(\sin z)^2}$ $-\pi$ $'$ $(-\pi, \pi)$.

$u = \tan\frac{z}{2}$ $y = \frac{1}{(\sin z)^2}$ $(-\pi, \pi)$ $y = \frac{(1+u^2)^2}{4u^2}$ $(-\infty, +\infty)$. $y = \frac{(1+u^2)^2}{4u^2}$ 0 $(-\infty, +\infty)$ $y = \frac{1}{(\sin z)^2}$ 0 $(-\pi, \pi)$. , $\int \frac{1}{(\sin z)^2}\,dz = \int \frac{(1+u^2)^2}{4u^2}\frac{2}{1+u^2}\,du\big|_{u=\tan\frac{z}{2}} = \int \frac{1+u^2}{2u^2}\,du\big|_{u=\tan\frac{z}{2}} = \left(-\frac{1}{2u} + \frac{u}{2}\right)\big|_{u=\tan\frac{z}{2}} + c = -\frac{1}{2}\cot\frac{z}{2} + \frac{1}{2}\tan\frac{z}{2} + c = -\cot z + c$ $(-\pi, 0)$ $(0, \pi)$.

z $(-\pi, 0)$ $(0, \pi)$, $x = z + \frac{\pi}{2}$ $(-\frac{\pi}{2}, \frac{\pi}{2})$ $(\frac{\pi}{2}, \frac{3\pi}{2})$. $'$ $\int \frac{1}{(\cos x)^2}\,dx = \int \frac{1}{(\sin z)^2}\,dz\big|_{z=x-\frac{\pi}{2}} = -\cot(x - \frac{\pi}{2}) + c = \tan x + c$.

$y = \frac{1}{(\cos x)^2}$ $y = \tan x$ 2π, $\int \frac{1}{(\cos x)^2}\,dx = \tan x + c$ $(-\frac{\pi}{2} + k2\pi, \frac{\pi}{2} + k2\pi)$ $(\frac{\pi}{2} + k2\pi, \frac{3\pi}{2} + k2\pi)$ $(-\frac{\pi}{2} + k2\pi, \frac{3\pi}{2} + k2\pi)$ $(k \in \mathbf{Z})$.

$\int \frac{1}{(\cos x)^2}\,dx = \tan x + c$ $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$ $(k \in \mathbf{Z})$.

2. $y = f(x) = r(\cos x, \sin x)$ $(-\infty, +\infty)$, . , , $y = f(x)$ $(-\infty, +\infty)$. $\int f(x)\,dx = \int r(\cos x, \sin x)\,dx$ $(-\infty, +\infty)$, $\int f(x)\,dx = F(x) + c$, $y = F(x)$ $F'(x) = f(x)$ $(-\infty, +\infty)$. , , $y = F(x)$.

1, , , $(-\pi, \pi)$ $u = \tan\frac{x}{2}$ $x = 2\arctan u$. , $y = f(x) = r(\cos x, \sin x)$ $(-\pi, \pi)$ $y = g(u) = r\left(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}\right)$ $(-\infty, \infty)$. , , $y = g(u)$. $y = f(x)$ π, $\lim_{u\to+\infty} g(u) = \lim_{x\to\pi-} f(x) = f(\pi)$, , $y = g(u)$. $y = g_1(u) = g(u)\frac{2}{1+u^2} = \frac{a_0 + a_1u + \cdots + a_nu^n}{b_0 + b_1u + \cdots + b_mu^m}$ ($a_n, b_m \neq 0$) $m \geq n + 2$. $\lim_{u\to+\infty} ug_1(u) = 0$. $y = g(u)$ $(-\infty, +\infty)$, $b_0 + b_1u + \cdots + b_mu^m$. $y = g_1(u)$

$$\begin{aligned} g_1(u) = {} & \frac{M_1(u-\mu) + N_1}{(u-\mu)^2 + \nu^2} + \cdots + \frac{M_\rho(u-\mu) + N_\rho}{((u-\mu)^2 + \nu^2)^\rho} + \\ & + \cdots\cdots\cdots\cdots + \\ & + \frac{E_1(u-\epsilon) + \Delta_1}{(u-\epsilon)^2 + \delta^2} + \cdots + \frac{E_\tau(u-\epsilon) + \Delta_\tau}{((u-\epsilon)^2 + \delta^2)^\tau}, \end{aligned}$$

, , $\nu, \ldots, \delta > 0$.

u $u \to +\infty$, $M_1 + \cdots + E_1 = 0.$ $\int g_1(u)\,du$

$$\begin{aligned}\int g_1(u)\,du &= G(u) + c\\ &= \frac{M_1}{2}\log((u-\mu)^2+\nu^2) + \cdots + \frac{E_1}{2}\log((u-\epsilon)^2+\delta^2) +\\ &\quad + {N_1}'\arctan\frac{u-\mu}{\nu} + \cdots + {\Delta_1}'\arctan\frac{u-\epsilon}{\delta} +\\ &\quad + h(u) + c,\end{aligned}$$

$y = h(u)$ u $(-\infty, +\infty)$. , $M_1 + \cdots + E_1 = 0$

$$\kappa = ({N_1}' + \cdots + {\Delta_1}')\pi,$$

$$\lim_{x\to\pi-} G\Big(\tan\frac{x}{2}\Big) = \lim_{u\to+\infty} G(u) = \frac{\kappa}{2}, \quad \lim_{x\to-\pi+} G\Big(\tan\frac{x}{2}\Big) = \lim_{u\to-\infty} G(u) = -\frac{\kappa}{2}.$$

$\int f(x)\,dx$ $(-\pi, \pi)$:

$$\int f(x)\,dx = \int g_1(u)\,du\big|_{u=\tan\frac{x}{2}} = G\Big(\tan\frac{x}{2}\Big) + c.$$

$y = f(x)$ $y = G(\tan\frac{x}{2})$ 2π, $\int f(x)\,dx = G(\tan\frac{x}{2}) + c$ $(-\pi + k2\pi, \pi + k2\pi)$ $(k \in \mathbf{Z})$.

$$y = \Phi(x) = \begin{cases} G(\tan\frac{x}{2}) + \kappa[\frac{x+\pi}{2\pi}], & x \neq \pi + k2\pi \ (k \in \mathbf{Z}),\\ \frac{\kappa}{2\pi}x, & x = \pi + k2\pi \ (k \in \mathbf{Z}).\end{cases}$$

$I_k = (-\pi + k2\pi, \pi + k2\pi)$ $\Phi(x) = G(\tan\frac{x}{2}) + k\kappa$, $y = \Phi(x)$, , $\Phi'(x) = f(x)$. $y = \Phi(x)$ $\pi + k2\pi$ I_k I_{k+1}. $y = G(\tan\frac{x}{2})$ $\lim_{x\to(\pi+k2\pi)+}\Phi(x) = \lim_{x\to(\pi+k2\pi)+}\big(G(\tan\frac{x}{2}) + (k+1)\kappa\big) = \lim_{x\to-\pi+} G(\tan\frac{x}{2}) + (k+1)\kappa = -\frac{1}{2}\kappa + (k+1)\kappa = (k+\frac{1}{2})\kappa = \Phi(\pi + k2\pi)$ $\lim_{x\to(\pi+k2\pi)-}\Phi(x) = \lim_{x\to(\pi+k2\pi)-}\big(G(\tan\frac{x}{2}) + k\kappa\big) = \lim_{x\to\pi-} G(\tan\frac{x}{2}) + k\kappa = \frac{1}{2}\kappa + k\kappa = (k+\frac{1}{2})\kappa = \Phi(\pi + k2\pi)$. $y = \Phi(x)$ $\pi + k2\pi$ $(k \in \mathbf{Z})$.

, , $y = \Phi(x)$ $(-\infty, +\infty)$, $(-\pi + k2\pi, \pi + k2\pi)$ $(k \in \mathbf{Z})$ $\Phi'(x) = f(x)$. $y = F(x)$ $y = f(x)$ $(-\infty, +\infty)$ – – $y = F(x)$ $F'(x) = f(x)$ $(-\infty, +\infty)$. $y = \Phi(x) - F(x)$ $(-\infty, +\infty)$, $(-\pi + k2\pi, \pi + k2\pi)$ $(k \in \mathbf{Z})$ $(\Phi - F)'(x) = 0$. $y = \Phi(x) - F(x)$ $[-\pi + k2\pi, \pi + k2\pi]$ $(k \in \mathbf{Z})$. , $y = \Phi(x) - F(x)$ $(-\infty, +\infty)$. c_0 $\Phi(x) = F(x) + c_0$ x $(-\infty, +\infty)$. , $y = \Phi(x)$ $y = f(x)$ $(-\infty, +\infty)$ $y = F(x)$ $y = \Phi(x)$. , ,

$$\int r(\cos x, \sin x)\,dx = \Phi(x) + c$$

$(-\infty, +\infty)$.

: $\int \frac{1}{2+\sin x}\,dx$.

$y = \frac{1}{2+\sin x}$ $(-\infty, +\infty)$. ′ $(-\pi, \pi)$ $u = \tan\frac{x}{2}$ $\int \frac{1}{2+\sin x}\, dx = \int \frac{1}{2+\frac{2u}{1+u^2}} \frac{2}{1+u^2}\, du\big|_{u=\tan\frac{x}{2}} = \int \frac{1}{u^2+u+1}\, du\big|_{u=\tan\frac{x}{2}}$. , $\int \frac{1}{u^2+u+1}\, du = \int \frac{1}{(u+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\, du = \frac{2}{\sqrt{3}} \arctan\frac{2u+1}{\sqrt{3}} + c$ $(-\infty, +\infty)$. $\int \frac{1}{2+\sin x}\, dx = \frac{2}{\sqrt{3}} \arctan\big(\frac{2}{\sqrt{3}} \tan\frac{x}{2} + \frac{1}{\sqrt{3}}\big) + c$ $(-\pi, \pi)$.

$y = \frac{1}{2+\sin x}$ $\frac{2}{\sqrt{3}} \arctan\big(\frac{2}{\sqrt{3}} \tan\frac{x}{2} + \frac{1}{\sqrt{3}}\big)$ 2π , , $\int \frac{1}{2+\sin x}\, dx = \frac{2}{\sqrt{3}} \arctan\big(\frac{2}{\sqrt{3}} \tan\frac{x}{2} + \frac{1}{\sqrt{3}}\big) + c$ $(-\pi + k2\pi, \pi + k2\pi)$. , $\lim_{x\to\pi-} \frac{2}{\sqrt{3}} \arctan\big(\frac{2}{\sqrt{3}} \tan\frac{x}{2} + \frac{1}{\sqrt{3}}\big) = \frac{\pi}{\sqrt{3}}$ $\lim_{x\to-\pi+} \frac{2}{\sqrt{3}} \arctan\big(\frac{2}{\sqrt{3}} \tan\frac{x}{2} + \frac{1}{\sqrt{3}}\big) = -\frac{\pi}{\sqrt{3}}$. , $y = \Phi(x) = \begin{cases} \frac{2}{\sqrt{3}} \arctan\big(\frac{2}{\sqrt{3}} \tan\frac{x}{2} + \frac{1}{\sqrt{3}}\big) + \frac{2\pi}{\sqrt{3}}[\frac{x+\pi}{2\pi}] \\ \frac{1}{\sqrt{3}} x, \end{cases}$

$y = \Phi(x)$ $(-\infty, +\infty)$ $\int \frac{1}{2+\sin x}\, dx = \Phi(x) + c$ $(-\infty, +\infty)$.

. .

$$\int r\big(x, \sqrt{1-x^2}\big)\, dx, \qquad \int r\big(x, \sqrt{x^2-1}\big)\, dx, \qquad \int r\big(x, \sqrt{x^2+1}\big)\, dx.$$

$r(s,t)$ s, t.

(*i*) ′ $[-1,1]$ $x = \sin t$ t $[-\frac{\pi}{2}, \frac{\pi}{2}]$. $\sqrt{1-x^2} = \cos t$ $\int r(\sin t, \cos t) \cos t\, dt$, , , $u = \tan\frac{t}{2}$. x u $u = \frac{x}{1+\sqrt{1-x^2}}$. , , ′ $u = \frac{x}{1+\sqrt{1-x^2}}$. $[-1,1]$, $\frac{du}{dx} = \frac{1}{1-x^2+\sqrt{1-x^2}} > 0$ x $(-1,1)$, , $[-1,1]$. , $x = \frac{2u}{1+u^2}$. , $\sqrt{1-x^2} = \frac{1-u^2}{1+u^2}$ $\frac{du}{dx} = \frac{(1+u^2)^2}{2(1-u^2)}$. ,

$$\int r\big(x, \sqrt{1-x^2}\big)\, dx = \int r\Big(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\Big) \frac{2(1-u^2)}{(1+u^2)^2}\, dy\Big|_{u=\frac{x}{1+\sqrt{1-x^2}}}.$$

u $[-1,1]$.

, $r(x, \sqrt{1-x^2})$ x $[-1,1]$. , , .

: $\int \frac{1}{x+\sqrt{1-x^2}}\, dx$ $[-1, -\frac{1}{\sqrt{2}})$ $(-\frac{1}{\sqrt{2}}, 1]$ $[-1,1]$.

$[-1,1]$, $x + \sqrt{1-x^2} \neq 0$ $x \neq -\frac{1}{\sqrt{2}}$. , $u = \frac{x}{1+\sqrt{1-x^2}}$ $[-1, 1-\sqrt{2})$ $(1-\sqrt{2}, 1]$, , : $\int \frac{1}{x+\sqrt{1-x^2}}\, dx = 2\int \frac{1-u^2}{(1+2u-u^2)(1+u^2)}\, du\big|_{u=\frac{x}{1+\sqrt{1-x^2}}}$. , $2\int \frac{1-u^2}{(1+2u-u^2)(1+u^2)}\, du$ $\frac{1}{2}\log\frac{|1+2u-u^2|}{1+u^2} + \arctan u + c$. , , $\int \frac{1}{x+\sqrt{1-x^2}}\, dx = \frac{1}{2}\log|x+\sqrt{1-x^2}| + \arctan\frac{x}{1+\sqrt{1-x^2}} + c$.

(*ii*) $[1,+\infty)$ $(-\infty,-1]$. $[1,+\infty)$ $x = \frac{1}{\sin t}$ t $(0, \frac{\pi}{2}]$. $\sqrt{x^2-1} = \frac{\cos t}{\sin t}$ $-\int r(\frac{1}{\sin t}, \frac{\cos t}{\sin t}) \frac{\cos t}{(\sin t)^2}\, dt$, , , $u = \tan\frac{t}{2}$. x u $u = x + \sqrt{x^2-1}$, ′ t. , , $u = x + \sqrt{x^2-1}$. $[1,+\infty)$, $\frac{du}{dx} = 1 + \frac{x}{\sqrt{x^2-1}} > 0$ x $(1,+\infty)$. $\lim_{x\to+\infty}(x+\sqrt{x^2-1}) = +\infty$, $[1,+\infty)$. $x = \frac{u^2+1}{2u}$ $\sqrt{x^2-1} = \frac{u^2-1}{2u}$ $\frac{du}{dx} = \frac{2u^2}{u^2-1}$.

$$\int r\big(x, \sqrt{x^2-1}\big)\, dx = \int r\Big(\frac{u^2+1}{2u}, \frac{u^2-1}{2u}\Big) \frac{u^2-1}{2u^2}\, du\Big|_{u=x+\sqrt{x^2-1}}$$

u $[1,+\infty)$.
$(-\infty,-1]$, $u = x-\sqrt{x^2-1}$ u $(-\infty,-1]$. .
, $(-\infty,-1]$ $[1,+\infty)$, $r(x,\sqrt{x^2-1})$.

: $\int \frac{1}{x+\sqrt{x^2-1}}\,dx$ $[1,+\infty)$.
, $x+\sqrt{x^2-1}\neq 0$ $[1,+\infty)$. $\int \frac{1}{x+\sqrt{x^2-1}}\,dx = \int(\frac{1}{2u}-\frac{1}{2u^3})\,du\big|_{u=x+\sqrt{x^2-1}}$. , $\int(\frac{1}{2u}-\frac{1}{2u^3})\,du = \frac{1}{2}\log|u|+\frac{1}{4u^2}+c$, $\int \frac{1}{x+\sqrt{x^2-1}}\,dx = \frac{1}{2}\log|x+\sqrt{x^2-1}| + \frac{1}{4(x+\sqrt{x^2-1})^2}+c$.

(iii) $(-\infty,+\infty)$ $x = -\cot t$ t $(0,\pi)$. $\sqrt{x^2+1} = \frac{1}{\sin t}$ $\int r(-\frac{\cos t}{\sin t},\frac{1}{\sin t})\frac{1}{(\sin t)^2}\,dt$, , , $u=\tan\frac{t}{2}$. , $u = x+\sqrt{x^2+1}$, ′ . $u = x+\sqrt{x^2+1}$ $\frac{du}{dx} = 1+\frac{x}{\sqrt{x^2+1}} > 0$ x $(-\infty,+\infty)$. $\lim_{x\to-\infty}(x+\sqrt{x^2+1}) = 0$ $\lim_{x\to+\infty}(x+\sqrt{x^2+1}) = +\infty$, $(0,+\infty)$. $x = \frac{u^2-1}{2u}$. , $\sqrt{x^2+1} = \frac{u^2+1}{2u}$ $\frac{du}{dx} = \frac{2u^2}{u^2+1}$.

$$\int r\big(x,\sqrt{x^2+1}\big)\,dx = \int r\Big(\frac{u^2-1}{2u},\frac{u^2+1}{2u}\Big)\frac{u^2+1}{2u^2}\,du\Big|_{u=x+\sqrt{x^2+1}}$$

u $(0,+\infty)$.

: $\int \frac{1}{x\sqrt{x^2+1}}\,dx$ $(-\infty,0)$ $(0,+\infty)$.
x , $u = x+\sqrt{x^2+1}$, , $(0,1)$ $(1,+\infty)$. $\int \frac{1}{x\sqrt{x^2+1}}\,dx = 2\int \frac{1}{u^2-1}\,du\big|_{u=x+\sqrt{x^2+1}}$ $2\int\frac{1}{u^2-1}\,du$ $(0,1)$ $(1,+\infty)$ $\log|u-1|-\log(u+1)+c$. , $\int \frac{1}{x\sqrt{x^2+1}}\,dx = \log\frac{|x|}{1+\sqrt{x^2+1}}+c$.

,

$$\int r\big(x,\sqrt{\kappa x^2+\lambda x+\mu}\big)\,dx,$$

κ,λ,μ , $\kappa\neq 0$ $r(s,t)$ s,t.
, $\kappa x^2+\lambda x+\mu = \kappa\big((x+\frac{\lambda}{2\kappa})^2+\frac{4\kappa\mu-\lambda^2}{4\kappa^2}\big)$, .

1: $\kappa>0$ $4\kappa\mu-\lambda^2>0$. $u = \frac{2\kappa}{\sqrt{4\kappa\mu-\lambda^2}}(x+\frac{\lambda}{2\kappa})$, $\frac{\sqrt{4\kappa\mu-\lambda^2}}{2\kappa}\int r\big(-\frac{\lambda}{2\kappa}+\frac{\sqrt{4\kappa\mu-\lambda^2}}{2\kappa}u,\frac{\sqrt{4\kappa\mu-\lambda^2}}{2\sqrt{\kappa}}\sqrt{u^2+1}\big)\,du = \int R(u,\sqrt{u^2+1})\,du$, $R(s,t)$ s,t.

2: $\kappa>0$ $4\kappa\mu-\lambda^2<0$. $u = \frac{2\kappa}{\sqrt{\lambda^2-4\kappa\mu}}(x+\frac{\lambda}{2\kappa})$ $\frac{\sqrt{\lambda^2-4\kappa\mu}}{2\kappa}\int r\big(-\frac{\lambda}{2\kappa}+\frac{\sqrt{\lambda^2-4\kappa\mu}}{2\kappa}u,\frac{\sqrt{\lambda^2-4\kappa\mu}}{2\sqrt{\kappa}}\sqrt{u^2-1}\big)\,du = \int R(u,\sqrt{u^2-1})\,du$, $R(s,t)$ s,t.

3: $\kappa<0$ $4\kappa\mu-\lambda^2<0$. $u = \frac{-2\kappa}{\sqrt{\lambda^2-4\kappa\mu}}(x+\frac{\lambda}{2\kappa})$ $-\frac{\sqrt{\lambda^2-4\kappa\mu}}{2\kappa}\int r\big(-\frac{\lambda}{2\kappa}-\frac{\sqrt{\lambda^2-4\kappa\mu}}{2\kappa}u,\frac{\sqrt{\lambda^2-4\kappa\mu}}{2\sqrt{-\kappa}}\sqrt{1-u^2}\big)\,du = \int R(u,\sqrt{1-u^2})\,du$, $R(s,t)$ s,t.

$\kappa < 0$ $4\kappa\mu - \lambda^2 > 0$ $\sqrt{\kappa x^2 + \lambda x + \mu}$. , κ $4\kappa\mu - \lambda^2$ 0, .

$\int r(x, \sqrt{\kappa x^2 + \lambda x + \mu})\, dx$ $\kappa \neq 0$ $\int r(x, a(x))\, dx$, $y = a(x)$ x.

Abel . $y = a(x) = \sqrt{\rho x^4 + \sigma x^3 + \kappa x^2 + \lambda x + \mu}$, ρ σ $\neq 0$, , . 8 .

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. .

1. .

   (i) $\int_{a+c}^{b+c} f(x-c)\, dx = \int_a^b f(x)\, dx$.

   (ii) $\int_{\lambda a}^{\lambda b} f\left(\frac{x}{\lambda}\right) dx = \lambda \int_a^b f(x)\, dx$ $\lambda > 0$.

   , $f(x) \geq 0$ x $[a, b]$.

2. $y = f(x)$, $\int_{-b}^{-a} f(x)\, dx = \int_a^b f(x)\, dx$.

   $y = f(x)$, $\int_{-b}^{-a} f(x)\, dx = -\int_a^b f(x)\, dx$.

   $y = f(x)$, $\int_{-b}^{b} f(x)\, dx = 2\int_0^b f(x)\, dx$.

   $y = f(x)$, $\int_{-b}^{b} f(x)\, dx = 0$.

   ;

3. $y = f(x)$ $T > 0$,

   (i) $\int_{a+T}^{b+T} f(x)\, dx = \int_a^b f(x)\, dx$.

   (ii) $\int_a^{a+T} f(x)\, dx = \int_b^{b+T} f(x)\, dx$.

   ;

. .

1. .

$$\int x^3 \cos(x^4)\, dx, \quad \int (\cos x)^2 \sin x\, dx, \quad \int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx, \quad \int \frac{x}{\sqrt{x^2+1}}\, dx,$$

$$\int \sqrt{2x+1}\, dx, \quad \int x\sqrt{x+1}\, dx, \quad \int x^2\sqrt{2x+1}\, dx, \quad \int \frac{x}{\sqrt{1-x}}\, dx,$$

$$\int \frac{x+1}{(x^2+2x+5)^2}\, dx, \quad \int \frac{2^{\sqrt{x}}}{\sqrt{x}}\, dx, \quad \int x\sqrt[3]{x-1}\, dx, \quad \int \frac{x^5}{\sqrt{1-x^6}}\, dx,$$

$$\int \cos(2x)\sqrt{4-\sin(2x)}\, dx, \quad \int \frac{\sin x}{(2+\cos x)^3}\, dx, \quad \int \frac{\sin x + \cos x}{(\sin x - \cos x)^3}\, dx,$$

$$\int \frac{x}{\sqrt{x}+1}\, dx, \quad \int \frac{e^x}{1+e^{2x}}\, dx, \quad \int x^2 e^{x^3}\, dx, \quad \int e^{3\sin x}\cos x\, dx,$$

$$\int \tan x\,dx, \quad \int \frac{1}{x^2}\sin\frac{1}{x}\,dx, \quad \int \sqrt{1+3(\cos x)^2}\,\sin(2x)\,dx,$$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx, \quad \int \frac{1}{4+x^2}\,dx, \quad \int \frac{1}{\sqrt{1-x-x^2}}\,dx, \quad \int \frac{1}{x^2-x+2}\,dx,$$

$$\int \frac{1}{x(x^4+1)}\,dx, \quad \int x\sin(x^2)\cos(x^2)\,dx, \quad \int (\sin x)^3\,dx, \quad \int \frac{(\cos x)^3}{\sin x}\,dx,$$

$$\int \frac{\log x}{x\sqrt{1+\log x}}\,dx, \quad \int \frac{\arctan\sqrt{x}}{(1+x)\sqrt{x}}\,dx, \quad \int \frac{1}{1+e^x}\,dx.$$

2. .

$$\int e^{-2x}\sin(3x)\,dx, \quad \int e^x\cos(5x)\,dx, \quad \int x^3e^{2x}\,dx, \quad \int x^3e^{-x^2}\,dx,$$

$$\int e^{\sqrt{x}}\,dx, \quad \int x^2\sin x\,dx, \quad \int x\log x\,dx, \quad \int x^2(\log x)^4\,dx,$$

$$\int \arcsin x\,dx, \quad \int x^2\arccos x\,dx, \quad \int \arctan x\,dx, \quad \int x^2\arcsin x\,dx,$$

$$\int x(\arctan x)^2\,dx, \quad \int \arctan\sqrt{x}\,dx, \quad \int (\cos x)^2\,dx, \quad \int (\sin x)^4\,dx,$$

$$\int (\sin x)^3\sin(5x)\,dx, \quad \int \frac{x}{(\cos x)^2}\,dx, \quad \int (\tan x)^2\,dx,$$

$$\int \frac{x^2}{(x^2+1)^2}\,dx, \quad \int \frac{\arctan(e^x)}{e^x}\,dx, \quad \int \frac{xe^{\arctan x}}{(x^2+1)^{\frac{3}{2}}}\,dx.$$

3. $I_k = \int \frac{1}{(y^2+1)^k}\,dy.$ $I_1, \ldots, I_5$.

4. .

$$\int \frac{5x+3}{x^2+2x-3}\,dx, \quad \int \frac{x+2}{x^2-4x+4}\,dx, \quad \int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx,$$

$$\int \frac{x^2+2x+3}{x^3+x^2-x-1}\,dx, \quad \int \frac{3x^2+2x-2}{x^3-1}\,dx, \quad \int \frac{x^2+1}{(2x-1)^3}\,dx,$$

$$\int \frac{1}{x^4-1}\,dx, \quad \int \frac{1}{(x^2-4x+4)(x^2-4x+5)}\,dx, \quad \int \frac{x^4}{x^4+5x^2+4}\,dx,$$

$$\int \frac{8x^3+7}{x^4+2x^3-2x-1}\,dx, \quad \int \frac{1}{x^4-2x^2+1}\,dx, \quad \int \frac{x^2}{(x^2+2x+2)^2}\,dx,$$

$$\int \frac{1}{x^4+1}\,dx, \quad \int \frac{x^4-x^3+2x^2-x+2}{(x-1)(x^4+4x^2+4)}\,dx, \quad \int \frac{x^2+x+1}{(x-1)^4}\,dx,$$

$$\int \frac{1}{(x^2-2x+1)(x^4+2x^2+1)}\,dx, \quad \int \frac{1}{(x+1)(x+2)^2(x+3)^3}\,dx.$$

5. $\sin x$ $\cos x$.

$$\int \frac{1}{(1+\cos x)^2}\,dx, \quad \int \frac{1}{1+2\sin x}\,dx, \quad \int \frac{1}{5+3\cos x}\,dx,$$

$$\int \frac{(\sin x)^2}{1+(\sin x)^2}\,dx, \quad \int \frac{\sin x}{1+\sin x+\cos x}\,dx, \quad \int \frac{1}{2\sin x-\cos x+5}\,dx.$$

6. .

$$\int \sqrt{1-x^2}\,dx, \quad \int \sqrt{x^2-1}\,dx, \quad \int \frac{1}{\sqrt{x^2-1}}\,dx, \quad \int \sqrt{x^2+1}\,dx,$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx, \quad \int \frac{1}{\sqrt{(x-1)(x-2)}}\,dx, \quad \int \frac{x}{\sqrt{x^2+x+1}}\,dx,$$

$$\int \frac{1}{\sqrt{x-1}+\sqrt{x+1}}\,dx, \quad \int \frac{1}{(x+1)\sqrt{1+2x-x^2}}\,dx.$$

7. :

(i) 2 7.4 .

(ii) 3, 4 5 7.4 .

(iii) 1 7.4.

(iv) 1 7.4.

(v) 4 7.4.

8. 5 2 7.4 .

. .

1. $y = f(x)$ $(-\infty, +\infty)$. $y = f_n(x)$ n, $f_1(x) = \int_0^x f(t)\,dt$ $f_n(x) = \int_0^x f_{n-1}(t)\,dt$.

$f_n(x) = \frac{1}{(n-1)!}\int_0^x f(t)(x-t)^{n-1}\,dt$ n.

2. $J_n(x) = \int (\cos x)^n\,dx$ $I_n(x) = \int (\sin x)^n\,dx$, $n \geq 2$:

(i) $J_n(x) = \frac{\sin x(\cos x)^{n-1}}{n} + \frac{n-1}{n}J_{n-2}(x)$,

(ii) $I_n(x) = -\frac{\cos x(\sin x)^{n-1}}{n} + \frac{n-1}{n}I_{n-2}(x)$.

$y = J_1(x), \ldots, J_6(x)$ $y = I_1(x), \ldots, I_6(x)$.

.

3. $I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n\,dx$ n. $I_n = \frac{n-1}{n}I_{n-2}$ $n \geq 2$.

$I_0 = \frac{\pi}{2}$ $I_1 = 1$. n :

(i) $I_{2n} = \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2n(2n-2)\cdots 4\cdot 2}\cdot\frac{\pi}{2}$,

(ii) $I_{2n+1} = \frac{2n(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 5\cdot 3}$.

n :

(iii) $\frac{\pi}{2} = \frac{(2\cdot4\cdot6\cdots(2n))^2}{(3\cdot5\cdots(2n-1))^2(2n+1)} \frac{I_{2n}}{I_{2n+1}}$,

(iv) $(n+1)I_n I_{n+1} = \frac{\pi}{2}$.

$I_{2n+1} \le I_{2n} \le I_{2n-1} = \frac{2n+1}{2n} I_{2n+1}$, $1 \le \frac{I_{2n}}{I_{2n+1}} \le 1 + \frac{1}{2n}$, , $\lim_{n\to+\infty} \frac{I_{2n}}{I_{2n+1}} = 1$.

Wallis:

$$\frac{\pi}{2} = \lim_{n\to+\infty} \frac{\big(2\cdot4\cdot6\cdots(2n-2)\cdot(2n)\big)^2}{\big(3\cdot5\cdots(2n-3)\cdot(2n-1)\big)^2(2n+1)}$$

$$\sqrt{\pi} = \lim_{n\to+\infty} \frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n}} .$$

4. $I_n(x) = \int (\tan x)^n \, dx$, $I_n(x) = \frac{(\tan x)^{n-1}}{n-1} - I_{n-2}(x)$ $n \ge 2$.
   $y = I_1(x), \ldots, I_6(x)$.

5. $I_n(x) = \int x^n e^{-x} \, dx$ $J_n(x) = \int x^n e^x \, dx$, $n \ge 1$
   (i) $I_n(x) = -x^n e^{-x} + nI_{n-1}(x)$,
   (ii) $J_n(x) = x^n e^x - nJ_{n-1}(x)$.
   $y = I_n(x)$ $y = J_n(x)$ 3 8.2.

6.
$$\int x^n e^{-x^2} \, dx = p_{n-1}(x) e^{-x^2} + c \qquad (n \quad)$$
$$\int x^n e^{-x^2} \, dx = p_{n-1}(x) e^{-x^2} + \int e^{-x^2} \, dx \qquad (n \quad)$$
$(-\infty, +\infty)$, $p_{n-1}(x)$ $n-1$ c .
, $\int e^{-x^2} \, dx$.
$$y = \int_0^x e^{-t^2} \, dt$$
, .

7. $I_{m,n}(x) = \int x^m (1-x)^n \, dx$,
   (i) $I_{m,n}(x) = \frac{x^{m+1}(1-x)^n}{m+1} + \frac{n}{m+1} I_{m+1,n-1}(x)$ $m \ne -1, \, n \ne 0$,
   (ii) $I_{m,n}(x) = -\frac{x^m(1-x)^{n+1}}{n+1} + \frac{m}{n+1} I_{m-1,n+1}(x)$ $m \ne 0, \, n \ne -1$.
   $\int_0^1 x^m (1-x)^n \, dx = \frac{m!n!}{(m+n+1)!}$ $m, n \ge 0$.

8. $I_n(x) = \int x^n \sin x \, dx$ $J_n(x) = \int x^n \cos x \, dx$, n
   (i) $I_n(x) = -x^n \cos x + nx^{n-1} \sin x - n(n-1)I_{n-2}(x)$,
   (ii) $J_n(x) = x^n \sin x + nx^{n-1} \cos x - n(n-1)J_{n-2}(x)$.
   $y = I_n(x)$ $y = J_n(x)$.

9. $I_{m,n}(x) = \int (\cos x)^m (\sin x)^n \, dx$, m, n

(i) $I_{m,n}(x) = \frac{(\cos x)^{m-1}(\sin x)^{n+1}}{m+n} + \frac{m-1}{m+n} I_{m-2,n}(x)$,

(ii) $I_{m,n}(x) = -\frac{(\cos x)^{m+1}(\sin x)^{n-1}}{m+n} + \frac{n-1}{m+n} I_{m,n-2}(x)$.

$I_{m,n} = \int_0^{\frac{\pi}{2}} (\cos x)^m (\sin x)^n \, dx$, m, n

(i) $I_{m,n} = \frac{(m-1)(m-3)\cdots 1 \cdot (n-1)(n-3)\cdots 1}{(m+n)(m+n-2)\cdots 2} \, \frac{\pi}{2}$ m, n ,

(ii) $I_{m,n} = \frac{(m-1)(m-3)\cdots(1\ 2)\cdot(n-1)(n-3)\cdots(1\ 2)}{(m+n)(m+n-2)\cdots(1\ 2)}$ m, n .

10. $I_{m,n}(x) = \int (\sin x)^m \sin(nx) \, dx$, , $n \neq \pm m$, $I_{m,n}(x) = -\frac{n(\sin x)^m \cos(nx)}{n^2-m^2} + \frac{m(\sin x)^{m-1}\cos x \sin(nx)}{n^2-m^2} - \frac{m(m-1)}{n^2-m^2} I_{m-2,n}(x)$.

. .

1. $y = f(x)$ $[0, \pi]$ $\int_0^\pi \big(f(x) + \frac{1}{n^2} f''(x)\big) \sin(nx) \, dx = \frac{f(0)+(-1)^{n-1}f(\pi)}{n}$ n.

2. (**) . $y = f(x)$ $[a, b]$ $y = g(x)$ $[a, b]$. ξ $[a, b]$

$$\int_a^b f(x)g(x) \, dx = f(a) \int_a^\xi g(x) \, dx + f(b) \int_\xi^b g(x) \, dx.$$

$G(x) = \int_a^x g(t) \, dt$, $\int_a^b f(x)g(x) \, dx = \int_a^b f(x)G'(x) \, dx$, , 7.4 2 7.3.

3. (**) $y = \phi(x)$ $[a, b]$ $y = \phi'(x)$ $[a, b]$ $m > 0$ $\phi'(x) \geq m$ x $[a, b]$, $\big|\int_a^b \sin(\phi(x)) \, dx\big| \leq \frac{4}{m}$.

(: $\big|\int_a^b \sin(\phi(x)) \, dx\big| = \big|\int_a^b \frac{1}{\phi'(x)} \phi'(x) \sin(\phi(x)) \, dx\big|$. .)

$\big|\int_a^b \sin(x^2) \, dx\big| \leq \frac{2}{a}$ a, b $0 < a < b$.

4. (*) $x = x(t)$ $[a, b]$, $y = y(t)$ $[a, b]$ $y(t) \geq 0$ t $[a, b]$.
$x = x(t)$ $y = y(t)$, t $[a, b]$, x- $x = x(a)$ $x = x(b)$

$$E = \int_a^b y(t)x'(t) \, dt.$$

(: $\int_a^b y(t)x'(t) \, dt = \int_a^b y\big(x^{-1}(x(t))\big)x'(t) \, dt$.)

5. (**) C $x = x(t)$ $y = y(t)$, t $[a, b]$, , : $(x(b), y(b)) = (x(a), y(a))$. , t , $(x, y) = (x(t), y(t))$ C C . $x = x(t)$ $y = y(t)$ $[a, b]$, , , E C

$$E = -\int_a^b y(t)x'(t) \, dt = \int_a^b x(t)y'(t) \, dt = \frac{1}{2} \int_a^b \big(x(t)y'(t) - y(t)x'(t)\big) \, dt.$$

(: t_1, t_2, t_3, t_4 $[a, b]$ $x(t_1) \leq x(t) \leq x(t_3)$ $y(t_2) \leq y(t) \leq y(t_4)$ t $[a, b]$. t_1, t_2, t_3, t_4 $x = x(t)$ $y = y(t)$ $[a, b]$.)

6. $x = x(t) = r_0 \cos t + x_0$ $y = y(t) = r_0 \sin t + y_0$ (x_0, y_0) $r_0 > 0$.
$x = x(t) = \kappa_0 \cos t + x_0$ $y = y(t) = \mu_0 \sin t + y_0$

7. (*) $P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}\big((x^2-1)^n\big)$.
$P_n(x)$ n.
$\int_{-1}^{1} p(x)P_n(x)\,dx = 0$ $p(x)$ n.
$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0, & n \neq m, \\ \frac{2}{2n+1}, & n = m. \end{cases}$

8.4 Riemann.

«»

$$\int_0^1 \frac{1}{x}\,dx, \qquad \int_1^{+\infty} x\,dx.$$

. «» $y = \frac{1}{x}$ $[0,1]$, , $\lim_{x\to 0+} \frac{1}{x} = +\infty$. «» $[1,+\infty)$. «», ¨; .
$y = f(x)$ $[a,b)$ $[a,c]$ $[a,b)$.

$$\int_a^{b-} f(x)\,dx$$

$y = f(x)$ $[a,b)$. $\lim_{c\to b-} \int_a^c f(x)\,dx$, $\int_a^{b-} f(x)\,dx$

$$\int_a^{b-} f(x)\,dx = \lim_{c\to b-} \int_a^c f(x)\,dx.$$

, $\lim_{c\to b-} \int_a^c f(x)\,dx$. $\lim_{c\to b-} \int_a^c f(x)\,dx$, $\int_a^{b-} f(x)\,dx$. $\lim_{c\to b-} \int_a^c f(x)\,dx$ $\pm\infty$, $\int_a^{b-} f(x)\,dx$ $\pm\infty$, . $\lim_{c\to b-} \int_a^c f(x)\,dx$, $\int_a^{b-} f(x)\,dx$.
: $[a,+\infty)$, $(a,b]$ $(-\infty,b]$. , , ( ) :

$$\int_a^{+\infty} f(x)\,dx = \lim_{c\to +\infty} \int_a^c f(x)\,dx,$$

$$\int_{a+}^{b} f(x)\,dx = \lim_{c\to a+} \int_c^b f(x)\,dx, \qquad \int_{-\infty}^{b} f(x)\,dx = \lim_{c\to -\infty} \int_c^b f(x)\,dx.$$

$\int_{a+}^{b-} f(x)\,dx$, $y = f(x)$ (a,b). d ($a < d < b$), ( ) $\int_{a+}^{d} f(x)\,dx$ $\int_{d}^{b-} f(x)\,dx$ ( ) . $\int_{a+}^{b-} f(x)\,dx = \int_{a+}^{d} f(x)\,dx + \int_{d}^{b-} f(x)\,dx$.
$\int_{a+}^{+\infty} f(x)\,dx$ $\int_{-\infty}^{b-} f(x)\,dx$ $\int_{-\infty}^{+\infty} f(x)\,dx$.
$\int_a^{b-} f(x)\,dx$; $f(x) \geq 0$ x $[a,b)$. c (a,b) $\int_a^c f(x)\,dx$ E_c A_c $y = f(x)$, $[a,c]$ x-, . $(a,0)$ $(a,f(a))$. $(c,0)$ $(c,f(c))$. c b, . $x = b$, , A_c « » A $y = f(x)$, $[a,b)$ x-, . $(a,0)$ $(a,f(a))$

$x = b$. A_c A E_c A_c E A. $\int_a^{b-} f(x)\,dx = \lim_{c\to b-}\int_a^c f(x)\,dx$
E_c c b , , E. , $y = f(x)$ $[a,b)$, A .
. , $f(x) \ge 0$ x $[a,+\infty)$, $\int_a^{+\infty} f(x)\,dx$ A $y = f(x)$,
$[a,+\infty)$ x- . $(a,0)$ $(a, f(a))$. ′ A .
. .
, .

: (1) p . $c > 1$ $\int_1^c \frac{1}{x^p}\,dx = \frac{c^{1-p}-1}{1-p}$, $p \neq 1$, $= \log c$, $p = 1$. , $\lim_{c\to+\infty}\int_1^c \frac{1}{x^p}\,dx = +\infty$, $p \le 1$, $= \frac{1}{p-1}$, $p > 1$.

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & p > 1, \\ +\infty, & p \le 1. \end{cases}$$

(2) p . c $0 < c < 1$ $\int_c^1 \frac{1}{x^p}\,dx = \frac{1-c^{1-p}}{1-p}$, $p \neq 1$, $= \log\frac{1}{c}$, $p = 1$. , $\lim_{c\to 0+}\int_c^1 \frac{1}{x^p}\,dx = +\infty$, $p \ge 1$, $= \frac{1}{1-p}$, $p < 1$.

$$\int_{0+}^{1} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p}, & p < 1, \\ +\infty, & p \ge 1. \end{cases}$$

, $\int_{0+}^{+\infty} \frac{1}{x^p}\,dx = +\infty$ p.

(3) $c > 0$ $\int_0^c \frac{1}{x^2+1}\,dx = \arctan c - \arctan 0 = \arctan c$, , $\lim_{c\to+\infty}\int_0^c \frac{1}{x^2+1}\,dx = \frac{\pi}{2}$.

$$\int_0^{+\infty} \frac{1}{x^2+1}\,dx = \frac{\pi}{2}.$$

$\int_{-\infty}^0 \frac{1}{x^2+1}\,dx = \frac{\pi}{2}$, ,

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx = \pi.$$

(4) t $c > 0$ $\int_0^c e^{-tx}\,dx = \frac{1-e^{-tc}}{t}$, $t \neq 0$, $= c$, $t = 0$. $\lim_{c\to+\infty}\int_0^c e^{-tx}\,dx = \frac{1}{t}$, $t > 0$, $= +\infty$, $t \le 0$. ,

$$\int_0^{+\infty} e^{-tx}\,dx = \begin{cases} \frac{1}{t}, & t > 0, \\ +\infty, & t \le 0. \end{cases}$$

(5) c $0 < c < 1$ $\int_0^c \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin c - \arcsin 0 = \arcsin c$. , $\lim_{c\to 1-}\int_0^c \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}$.

$$\int_0^{1-} \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}.$$

$\int_{-1+}^{0} \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}$, ,

$$\boxed{\int_{-1+}^{1-} \frac{1}{\sqrt{1-x^2}}\,dx = \pi.}$$

, , 7.1, 7.2 7.9 «» .

8.6 $\int_a^{b-} f(x)\,dx$, λ $\lambda \int_a^{b-} f(x)\,dx$. $\int_a^{b-} \lambda f(x)\,dx$

$$\int_a^{b-} \lambda f(x)\,dx = \lambda \int_a^{b-} f(x)\,dx.$$

, , .

: c $[a,b)$ $\int_a^c \lambda f(x)\,dx = \lambda \int_a^c f(x)\,dx$ $c \to b-$.

8.7 $\int_a^{b-} f(x)\,dx$ $\int_a^{b-} g(x)\,dx$ $\int_a^{b-} f(x)\,dx + \int_a^{b-} g(x)\,dx$. $\int_a^{b-} (f(x) + g(x))\,dx$

$$\int_a^{b-} (f(x) + g(x))\,dx = \int_a^{b-} f(x)\,dx + \int_a^{b-} g(x)\,dx.$$

, , .

: c $[a,b)$ $\int_a^c (f(x)+g(x))\,dx = \int_a^c f(x)\,dx + \int_a^c g(x)\,dx$ $c \to b-$.

8.8 $\int_a^{b-} f(x)\,dx$ $\int_a^{b-} g(x)\,dx$ $f(x) \leq g(x)$ x $[a,b)$.

$$\int_a^{b-} f(x)\,dx \leq \int_a^{b-} g(x)\,dx.$$

, , .

: c $[a,b)$ $\int_a^c f(x)\,dx \leq \int_a^c g(x)\,dx$ $c \to b-$.

8.2 .

8.2 $y = f(x)$ $y = g(x)$ $[a,c]$ $[a,b)$. $|f(x)| \leq g(x)$ x $[a,b)$ $\int_a^{b-} g(x)\,dx$, $\int_a^{b-} f(x)\,dx$

$$\Big| \int_a^{b-} f(x)\,dx \Big| \leq \int_a^{b-} g(x)\,dx.$$

, , .

: $f(x) + g(x) \geq 0$ x $[a,b)$, $a < c_1 < c_2 < b$ $\int_a^{c_2} (f(x)+g(x))\,dx = \int_a^{c_1} (f(x)+g(x))\,dx + \int_{c_1}^{c_2} (f(x)+g(x))\,dx \geq \int_a^{c_1} (f(x)+g(x))\,dx$. $\int_a^c (f(x)+g(x))\,dx$ c $[a,b)$. $\lim_{c\to b-} \int_a^c (f(x)+g(x))\,dx = \int_a^{b-} (f(x)+g(x))\,dx$. , $0 \leq f(x)+g(x) \leq 2g(x)$ x $[a,b)$, $0 \leq \int_a^{b-} (f(x)+g(x))\,dx \leq \int_a^{b-} 2g(x)\,dx = 2\int_a^{b-} g(x)\,dx$. , $\int_a^{b-} g(x)\,dx < +\infty$, $0 \leq \int_a^{b-} (f(x)+g(x))\,dx < +\infty$. $\int_a^{b-} f(x)\,dx = \int_a^{b-} (f(x)+g(x))\,dx - \int_a^{b-} g(x)\,dx$.

, $-g(x) \leq f(x) \leq g(x)$ $-\int_a^{b-} g(x)\,dx = \int_a^{b-} (-g(x))\,dx \leq \int_a^{b-} f(x)\,dx \leq \int_a^{b-} g(x)\,dx$, , $\big|\int_a^{b-} f(x)\,dx\big| \leq \int_a^{b-} g(x)\,dx$.

: (1) $\int_1^{+\infty} \frac{1}{x^2}\,dx$ 1. $|\frac{\sin x}{x^2}| \le \frac{1}{x^2}$ $x \ge 1$, $\int_1^{+\infty} \frac{\sin x}{x^2}\,dx$, , $|\int_1^{+\infty} \frac{\sin x}{x^2}\,dx| \le \int_1^{+\infty} \frac{1}{x^2}\,dx = 1$.

(2) $\int_0^{+\infty} e^{-x^2}\,dx$.

, $0 \le e^{-x^2} \le e^{1-x}$ $x \ge 0$. $\int_0^{+\infty} e^{1-x}\,dx$, , $\int_0^{+\infty} e^{1-x}\,dx = e\int_0^{+\infty} e^{-x}\,dx = e$. , $\int_1^{+\infty} e^{-x^2}\,dx$.

.

1. , , .

$$\int_{0+}^{1} \frac{1}{x}\,dx,\quad \int_{1}^{+\infty} \frac{1}{x}\,dx,\quad \int_{0+}^{1} \frac{1}{x^2}\,dx,\quad \int_{1}^{+\infty} \frac{1}{x^2}\,dx,\quad \int_{0+}^{1} \frac{1}{\sqrt{x}}\,dx,$$

$$\int_{1}^{+\infty} \frac{1}{\sqrt{x}}\,dx,\quad \int_{0}^{+\infty} \frac{x}{x^2+1}\,dx,\quad \int_{0}^{+\infty} e^{-tx}x^2\,dx,\quad \int_{0}^{+\infty} \sin x\,dx,$$

$$\int_{0}^{+\infty} e^{-tx}\sin x\,dx,\quad \int_{0+}^{1} \log x\,dx,\quad \int_{e}^{+\infty} \frac{1}{x\log x}\,dx,\quad \int_{e}^{+\infty} \frac{1}{x(\log x)^2}\,dx,$$

$$\int_{0}^{+\infty} \frac{1}{(x^2+1)^2}\,dx,\quad \int_{0}^{+\infty} \frac{1}{x^4+1}\,dx \quad \int_{1}^{+\infty} \frac{1}{x\sqrt{x^2+1}}\,dx.$$

2. , , .

$$\int_{0+}^{+\infty} \frac{1}{x^p}\,dx,\ \int_{-\infty}^{+\infty} \frac{x}{x^2+1}\,dx,\ \int_{-\infty}^{+\infty} |x|\,dx,\ \int_{-\infty}^{+\infty} x\,dx,\ \int_{-\infty}^{+\infty} e^{-x}x\,dx,$$

$$\int_{-\infty}^{+\infty} e^{-|x|}\,dx,\ \int_{-\infty}^{+\infty} e^{-|x|}x\,dx,\ \int_{0+}^{1-} \frac{1}{x(1-x)}\,dx,\ \int_{0+}^{1-} \frac{1}{\sqrt{x(1-x)}}\,dx.$$

3. $y = \int_0^x (-1)^{[t]}\,dt$ x.
$\int_0^{+\infty} (-1)^{[x]}\,dx$.

4. (*) $\int_1^{+\infty} \frac{\sin x}{x}\,dx$.
(: $\int_1^c \frac{\sin x}{x}\,dx = \cos 1 - \frac{\cos c}{c} - \int_1^c \frac{\cos x}{x^2}\,dx$, , $\int_1^{+\infty} \frac{\cos x}{x^2}\,dx$.)

5. (**) $\int_0^{+\infty} \sin(x^2)\,dx$ $\int_0^{+\infty} \cos(x^2)\,dx$, **Fresnel**, .
(: $\int_1^c \sin(x^2)\,dx = -\int_1^c \frac{1}{2x}\big(\cos(x^2)\big)'\,dx = -\frac{\cos(c^2)}{2c} + \frac{\cos 1}{2} - \frac{1}{2}\int_1^c \frac{\cos(x^2)}{x^2}\,dx$, , $\int_1^{+\infty} \frac{\cos(x^2)}{x^2}\,dx$. , $\int_0^c \sin(x^2)\,dx = \int_0^1 \sin(x^2)\,dx + \int_1^c \sin(x^2)\,dx$.)

6. (*) $\Gamma(n)$ $\int_0^{+\infty} e^{-x}x^{n-1}\,dx$ n:

$$\Gamma(n) = \int_0^{+\infty} e^{-x}x^{n-1}\,dx \qquad (n\ ).$$

$\Gamma(1) = 1$.

$\Gamma(n)$, $\Gamma(n+1)$ $\Gamma(n+1) = n\Gamma(n)$.

(: $\int_0^c e^{-x}x^n\,dx = -e^{-c}c^n + n\int_0^c e^{-x}x^{n-1}\,dx$.)

$\Gamma(n) = (n-1)!$ n.

7. $^{(**)}$ $\Gamma(t)$ $\int_0^{+\infty} e^{-x}x^{t-1}\,dx$ $t \geq 1$ $\int_{0+}^{+\infty} e^{-x}x^{t-1}\,dx$ t $0 < t < 1$:

$$\Gamma(t) = \begin{cases} \int_0^{+\infty} e^{-x}x^{t-1}\,dx, & t \geq 1, \\ \int_{0+}^{+\infty} e^{-x}x^{t-1}\,dx, & 0 < t < 1. \end{cases}$$

$\Gamma(t)$ $\Gamma(n)$.

$\Gamma(t)$, , $\Gamma(t)$ t $(0, +\infty)$. $y = \Gamma(t)$ $(0, +\infty)$ **(Euler)**.

(: $t \geq 1$, $n = [t] + 1$ $|e^{-x}x^{t-1}| \leq e^{-x}x^{n-1}$ x $[1, +\infty)$. $0 < t < 1$, $|e^{-x}x^{t-1}| \leq x^{t-1}$ x $(0, 1]$ $|e^{-x}x^{t-1}| \leq e^{-x}$ x $[1, +\infty)$.)

$\Gamma(t+1) = t\Gamma(t)$ $t > 0$.

8.5 .

$$F(x, y, y', y'', \dots) = 0,$$

$F(x, y, y', y'', \dots)$ $x, y', y'', \dots$ x I, y $x-$ $y = f(x)-$ $y', y'', \dots$ $y' = f'(x), y'' = f''(x), \dots$. $y = f(x)$ ( ) $F(x, y, y', y'', \dots) = 0$

$$F\big(x, f(x), f'(x), f''(x), \dots\big) = 0 \qquad x \quad I.$$

: (1) $y' + 2xy = 0$ $y = e^{-x^2}$ $(-\infty, +\infty)$, $\frac{d\,e^{-x^2}}{dx} + 2xe^{-x^2} = -2xe^{-x^2} + 2xe^{-x^2} = 0$ x $(-\infty, +\infty)$.

(2) $y'' + y = 0$ $y = \sin x$ $(-\infty, +\infty)$, $\frac{d^2 \sin x}{dx^2} + \sin x = -\sin x + \sin x = 0$ x $(-\infty, +\infty)$.

′. .

, $y' - q(x) = 0$

$$y' = q(x),$$

$y = q(x)$ I. I $y = f(x)$ I

$$f'(x) = q(x) \qquad (x \quad I),$$

$y = q(x)$ I. $y = q(x)$ I, , , , $y = q(x)$ I, $y = q(x)$ I. , $y' = q(x)$ I

$$y = f(x) = \int q(t)\,dt = \int_{x_0}^{x} q(t)\,dt + c \qquad (x \quad I),$$

x_0 I c . y_0 y_0 x_0 ,

$$f'(x) = q(x) \quad (x \ \ I), \qquad f(x_0) = y_0\,.$$

, $x = x_0$ $y_0 = f(x_0) = \int_{x_0}^{x_0} q(t)\,dt + c = c$. ,

$$y = f(x) = \int_{x_0}^{x} q(t)\,dt + y_0 \qquad (x \ \ I).$$

. . .

. .

$$g(y)y' = h(x).$$

.

$y = f(x)$ I. ,

$$g\big(f(x)\big)f'(x) = h(x) \qquad (x \ \ I).$$

$y = f(x)$ I J $g(y)$ $g\big(f(x)\big)$.

$G(y)$ $g(y)$ J $H(x)$ $h(x)$ I. , , $g(y)$ $h(x)$, $G(y)$ $g(y)$ J $H(x)$ $h(x)$ I. $G'(y) = g(y)$ J $H'(x) = h(x)$ I, $y = f(x)$

$$\Big(G\big(f(x)\big)\Big)' = G'\big(f(x)\big)f'(x) = H'(x) \qquad (x \ \ I).$$

$$G\big(f(x)\big) = H(x) + c \qquad (x \ \ I)$$

c. , , $y = f(x)$, , $y = f(x)$. , $G(y)$ $H(x)$, , − $g(y)$ $h(x)$ − $G\big(f(x)\big) = H(x) + c$ $y = f(x)$.

, $y = f(x)$ $G\big(f(x)\big) = H(x) + c$ I I. , , $G'\big(f(x)\big)f'(x) = H'(x)$ I , $G'(y) = g(y)$ J $H'(x) = h(x)$ I, $g\big(f(x)\big)f'(x) = h(x)$ I.

, , $g(y)y' = h(x)$ I ( ) $G(y) = H(x) + c$ (c) I.

: (1) $yy' = x$.

$\frac{1}{2}y^2$ y $(-\infty, +\infty)$ $\frac{1}{2}x^2$ x $(-\infty, +\infty)$. $y = f(x)$ $yy' = x$ I I $\frac{1}{2}(f(x))^2 = \frac{1}{2}x^2 + c$ (c) . $\frac{1}{2}(f(x))^2 = \frac{1}{2}x^2 + c$ ( ) $(f(x))^2 = x^2 + c$ ( ).

, . $c > 0$, $y = f(x) = \sqrt{x^2 + c}$ $y = f(x) = -\sqrt{x^2 + c}$ $(-\infty, +\infty)$. $c = 0$, $y = f(x) = x$ $y = f(x) = -x$ $(-\infty, +\infty)$. $c < 0$, $y = f(x) = \sqrt{x^2 + c}$ $y = f(x) = -\sqrt{x^2 + c}$ $(-\infty, -\sqrt{|c|})$ $(\sqrt{|c|}, +\infty)$.

(2) $y^2y' = x$.

$\frac{1}{3}y^3$ y^2 $(-\infty, +\infty)$ $\frac{1}{2}x^2$ x $(-\infty, +\infty)$. $y = f(x)$ $y^2y' = x$ I I $\frac{1}{3}(f(x))^3 = \frac{1}{2}x^2 + c$ (c) . $\frac{1}{3}(f(x))^3 = \frac{1}{2}x^2 + c$ ( ) $f(x) = \sqrt[3]{\frac{3}{2}x^2 + c}$ ( ).

$c > 0$, $y = f(x) = \sqrt[3]{\frac{3}{2}x^2 + c}$ $(-\infty, +\infty)$. $c = 0$, $y = f(x) = \sqrt[3]{\frac{3}{2}}x^{\frac{2}{3}}$ $(-\infty, 0)$ $(0, +\infty)$. $c < 0$, $y = f(x) = \sqrt[3]{\frac{3}{2}x^2 + c}$ $\big(-\infty, -\sqrt{\frac{2}{3}|c|}\big)$

$\big(-\sqrt{\frac{2}{3}|c|}, \sqrt{\frac{2}{3}|c|}\big)$ $\big(\sqrt{\frac{2}{3}|c|}, +\infty\big)$. : .

(3) $y' + p(x)y = 0$, $y = p(x)$ I. $\frac{1}{y}\,y' = -p(x)$ $y \neq 0$, .
$\log|y|$ $\frac{1}{y}$ $(-\infty, 0)$ $(0, +\infty)$ $-\int_{x_0}^{x} p(t)\,dt$ $-p(x)$ I, x_0 I. $y = f(x)$ $\frac{1}{y}\,y' = -p(x)$ I $f(x) \neq 0$ I I $\log|f(x)| = -\int_{x_0}^{x} p(t)\,dt + c$ (c) . $|f(x)| = e^{-\int_{x_0}^{x} p(t)\,dt + c}$, $y = f(x)$ I. I , , $y = f(x) = e^c e^{-\int_{x_0}^{x} p(t)\,dt}$ $y = f(x) = -e^c e^{-\int_{x_0}^{x} p(t)\,dt}$ I. I. c $\pm e^c$, $y' + p(x)y = 0$ I I

$$y = f(x) = ce^{-\int_{x_0}^{x} p(t)\,dt} \qquad (x \quad I),$$

$c \neq 0$. $c = 0$, 0 I.

. .

$$y' + p(x)y = q(x),$$

$y' = q(x)$. $y = p(x)$ $y = q(x)$ I.
$y = \mu(x) = e^{\int_{x_0}^{x} p(t)\,dt}$ x I, x_0 I $\mu'(x) = p(x)e^{\int_{x_0}^{x} p(t)\,dt} = p(x)\mu(x)$ x I. () $y = f(x)$ $y' + p(x)y = q(x)$ I,

$$f'(x) + p(x)f(x) = q(x) \qquad (x \quad I),$$

$$\mu(x)f'(x) + p(x)\mu(x)f(x) = q(x)\mu(x) \qquad (x \quad I)$$

$$\mu(x)f'(x) + \mu'(x)f(x) = q(x)\mu(x) \qquad (x \quad I)$$

$$\big(\mu(x)f(x)\big)' = q(x)\mu(x) \qquad (x \quad I),$$

$y = \mu(x)f(x)$ $y' = q(x)\mu(x)$ I.

$$\mu(x)f(x) = \int_{x_0}^{x} q(t)\mu(t)\,dt + c \qquad (x \quad I),$$

, ,

$$y = f(x) = \frac{1}{\mu(x)}\int_{x_0}^{x} q(t)\mu(t)\,dt + \frac{c}{\mu(x)} \qquad (x \quad I),$$

x_0 I c .

$$f'(x) + p(x)f(x) = q(x) \quad (x \quad I), \qquad f(x_0) = y_0\,,$$

$x = x_0$ $y_0 = f(x_0) = \frac{1}{\mu(x_0)}\int_{x_0}^{x_0} q(t)\mu(t)\,dt + \frac{c}{\mu(x_0)} = c$, $\mu(x_0) = e^{\int_{x_0}^{x_0} p(t)\,dt} = 1$.

$$y = f(x) = \frac{1}{\mu(x)}\int_{x_0}^{x} q(t)\mu(t)\,dt + \frac{y_0}{\mu(x)} \qquad (x \quad I).$$

$y'+p(x)y=q(x)$. , , .

: (1) $y'-2xy=0$.

$y=p(x)=-2x$ $y=q(x)=0$ $(-\infty,+\infty)$ $y=\mu(x)=e^{\int_0^x(-2t)\,dt}=e^{-x^2}$ $(-\infty,+\infty)$. $y=f(x)$ $y'-2xy=0$ $(-\infty,+\infty)$, $f'(x)-2xf(x)=0$ x, $e^{-x^2}f'(x)-2xe^{-x^2}f(x)=0$, , $\left(e^{-x^2}f(x)\right)'=0$, , $e^{-x^2}f(x)=c$ x, c . $y'-2xy=0$ $(-\infty,+\infty)$ $y=f(x)=ce^{x^2}$ c.

$f(0)=-2$, $x=0$ $f(x)=ce^{x^2}$. $-2=f(0)=c$ $y=f(x)=-2e^{x^2}$.

(2) $x^2y'+y=\begin{cases}e^{-\frac{1}{|x|}}, & x\neq 0,\\ 0, & x=0.\end{cases}$

$y'+\frac{1}{x^2}y=\frac{1}{x^2}e^{-\frac{1}{|x|}}$ $x\neq 0$. $y=p(x)=\frac{1}{x^2}$ $y=q(x)=\frac{1}{x^2}e^{-\frac{1}{|x|}}$ $(-\infty,0)$ $(0,+\infty)$.

$'$ $(0,+\infty)$ $y=\mu(x)=e^{\int_1^x\frac{1}{t^2}\,dt}=e^{1-\frac{1}{x}}$ $(0,+\infty)$. $y=f(x)$ $y'+\frac{1}{x^2}y=\frac{1}{x^2}e^{-\frac{1}{|x|}}$ $(0,+\infty)$, $f'(x)+\frac{1}{x^2}f(x)=\frac{1}{x^2}e^{-\frac{1}{x}}$ $x>0$, $e^{1-\frac{1}{x}}f'(x)+\frac{1}{x^2}e^{1-\frac{1}{x}}f(x)=\frac{1}{x^2}e^{1-\frac{2}{x}}$, , $\left(e^{1-\frac{1}{x}}f(x)\right)'=\frac{1}{x^2}e^{1-\frac{2}{x}}$ $x>0$. $e^{1-\frac{1}{x}}f(x)=\int_1^x\frac{1}{t^2}e^{1-\frac{2}{t}}\,dt+c=\frac{e}{2}e^{-\frac{2}{x}}+c$ $x>0$, c . $x^2y'+y=e^{-\frac{1}{|x|}}$ $(0,+\infty)$ $y=f(x)=\frac{1}{2}e^{-\frac{1}{x}}+ce^{\frac{1}{x}}$ c.

$(-\infty,0)$ $x_0=-1$, $x^2y'+y=e^{-\frac{1}{|x|}}$ $(-\infty,0)$ $y=f(x)=-\frac{1}{x}e^{\frac{1}{x}}+c'e^{\frac{1}{x}}$ c'.

, $x^2y'+y=\begin{cases}e^{-\frac{1}{|x|}}, & x\neq 0,\\ 0, & x=0,\end{cases}$ $(-\infty,+\infty)$. $y=f(x)$, , , $(-\infty,0)$ $(0,+\infty)$. $y=f(x)=\frac{1}{2}e^{-\frac{1}{x}}+ce^{\frac{1}{x}}$ $(0,+\infty)$ c $y=f(x)=-\frac{1}{x}e^{\frac{1}{x}}+c'e^{\frac{1}{x}}$ $(-\infty,0)$ c'. $y=f(x)$ 0, $\lim_{x\to 0+}f(x)=f(0)=\lim_{x\to 0-}f(x)$. $\lim_{x\to 0+}f(x)=\lim_{x\to 0+}\left(\frac{1}{2}e^{-\frac{1}{x}}+ce^{\frac{1}{x}}\right)=c(+\infty)$ $\lim_{x\to 0-}f(x)=\lim_{x\to 0-}\left(-\frac{1}{x}e^{\frac{1}{x}}+c'e^{\frac{1}{x}}\right)=0$, $c=0$ $f(0)=0$. , $(-\infty,+\infty)$ $y=f(x)=\begin{cases}\frac{1}{2}e^{-\frac{1}{x}}, & x>0,\\ 0, & x=0,\\ -\frac{1}{x}e^{\frac{1}{x}}+c'e^{\frac{1}{x}}, & x<0.\end{cases}$ $\lim_{x\to 0+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0+}\frac{1}{2x}e^{-\frac{1}{x}}=0$ $y=f(x)$ 0 $f'_+(0)=0$. , $\lim_{x\to 0-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0-}\left(-\frac{1}{x^2}e^{\frac{1}{x}}+c'\frac{1}{x}e^{\frac{1}{x}}\right)=0$. $y=f(x)$ 0 $f'_-(0)=0$. $y=f(x)$ 0 $f'(0)=0$, , 0. . $(-\infty,0)$ $(0,+\infty)$. , $(-\infty,+\infty)$, , , $[0,+\infty)$.

(3) $y'+p(x)y=q(x)$.

$$y'+ky=q(x),$$

k $y=q(x)$ I. $y=\mu(x)=e^{\int_{x_0}^x k\,dt}=e^{k(x-x_0)}$ I. $y=f(x)$ I, $f'(x)+kf(x)=q(x)$ x I. $\left(e^{k(x-x_0)}f(x)\right)'=q(x)e^{k(x-x_0)}$, $e^{k(x-x_0)}f(x)=\int_{x_0}^x q(t)e^{k(t-x_0)}\,dt+c$, ,

$$f(x)=e^{-kx}\int_{x_0}^x q(t)e^{kt}\,dt+ce^{-k(x-x_0)}$$

x I, c . y_0 x_0, $f(x_0)=y_0$, $x=x_0$ $c=y_0$.

(4) : « t (k) q ».
$y = q(t)$, t, t, $q'(t) = -kq(t)$ t. $q'(t)$, , ≤ 0 k > 0. , $x_0 = 0$, $q(t) = ce^{-kt}$ c. q_0 0 $t = 0$, $q_0 = q(0) = c$, $q(t) = q_0 e^{-kt}$.

. .

,

$$y'' + ky' + ly = q(x),$$

k l . $y = q(x)$ I.
() $t^2 + kt + l = 0$.

1. $t^2 + kt + l = 0$ () , κ_1 κ_2, $k = -\kappa_1 - \kappa_2$ $l = \kappa_1\kappa_2$. $y = f(x)$ $y'' + ky' + ly = q(x)$ I,

$$f''(x) + kf'(x) + lf(x) = q(x) \qquad (x \quad I),$$

, x_0 I ,

$$f''(x) - (\kappa_1 + \kappa_2)f'(x) + \kappa_1\kappa_2 f(x) = q(x) \qquad (x \quad I),$$

$$\big(f'(x) - \kappa_1 f(x)\big)' - \kappa_2\big(f'(x) - \kappa_1 f(x)\big) = q(x) \qquad (x \quad I),$$

$$f'(x) - \kappa_1 f(x) = e^{\kappa_2 x}\int_{x_0}^{x} q(t)e^{-\kappa_2 t}\,dt + c_1 e^{\kappa_2(x-x_0)} \qquad (x \quad I),$$

$$\begin{aligned} f(x) &= e^{\kappa_1 x}\int_{x_0}^{x}\Big(e^{\kappa_2 t}\int_{x_0}^{t} q(s)e^{-\kappa_2 s}\,ds + c_1 e^{\kappa_2(t-x_0)}\Big)e^{-\kappa_1 t}\,dt + c_2 e^{\kappa_1(x-x_0)} \\ &= e^{\kappa_1 x}\int_{x_0}^{x} e^{(\kappa_2-\kappa_1)t}\int_{x_0}^{t} q(s)e^{-\kappa_2 s}\,ds\,dt + c_1 e^{-\kappa_2 x_0 + \kappa_1 x}\int_{x_0}^{x} e^{(\kappa_2-\kappa_1)t}\,dt \\ &\quad + c_2 e^{\kappa_1(x-x_0)} \qquad (x \quad I) \end{aligned}$$

, ,

$$\begin{aligned} f(x) &= e^{\kappa_1 x}\frac{e^{(\kappa_2-\kappa_1)x}}{\kappa_2-\kappa_1}\int_{x_0}^{x} q(s)e^{-\kappa_2 s}\,ds - e^{\kappa_1 x}\int_{x_0}^{x}\frac{e^{(\kappa_2-\kappa_1)t}}{\kappa_2-\kappa_1}q(t)e^{-\kappa_2 t}\,dt \\ &\quad + c_1 e^{-\kappa_2 x_0 + \kappa_1 x}\frac{e^{(\kappa_2-\kappa_1)x} - e^{(\kappa_2-\kappa_1)x_0}}{\kappa_2-\kappa_1} + c_2 e^{\kappa_1(x-x_0)} \\ &= \int_{x_0}^{x}\frac{e^{\kappa_2(x-t)} - e^{\kappa_1(x-t)}}{\kappa_2-\kappa_1}q(t)\,dt + c_1\frac{e^{\kappa_2(x-x_0)} - e^{\kappa_1(x-x_0)}}{\kappa_2-\kappa_1} + c_2 e^{\kappa_1(x-x_0)} \\ &\quad (x \quad I), \end{aligned}$$

c_1 c_2 . x_0:

$$f(x_0) = y_0\,, \quad f'(x_0) = {y_0}'.$$

$x = x_0$ $c_2 = y_0$. $x = x_0$ $c_1 + \kappa_1 c_2 = {y_0}'$, , $c_1 = -\kappa_1 y_0 + {y_0}'$.

2. $t^2+kt+l=0$, κ, $k=-2\kappa$ $l=\kappa^2$. $y=f(x)$ $y''+ky'+ly=q(x)$ I, , ,

$$f''(x)-2\kappa f'(x)+\kappa^2 f(x)=q(x) \qquad (x \quad I),$$

$$\big(f'(x)-\kappa f(x)\big)'-\kappa\big(f'(x)-\kappa f(x)\big)=q(x) \qquad (x \quad I),$$

$$f'(x)-\kappa f(x)=e^{\kappa x}\int_{x_0}^{x} q(t)e^{-\kappa t}\,dt+c_1e^{\kappa(x-x_0)} \qquad (x \quad I),$$

$$\begin{aligned} f(x) &= e^{\kappa x}\int_{x_0}^{x}\Big(e^{\kappa t}\int_{x_0}^{t} q(s)e^{-\kappa s}\,ds+c_1e^{\kappa(t-x_0)}\Big)e^{-\kappa t}\,dt+c_2e^{\kappa(x-x_0)} \\ &= e^{\kappa x}\int_{x_0}^{x}(t-x_0)'\int_{x_0}^{t} q(s)e^{-\kappa s}\,ds\,dt+c_1e^{\kappa(x-x_0)}\int_{x_0}^{x}dt+c_2e^{\kappa(x-x_0)} \\ &= e^{\kappa x}(x-x_0)\int_{x_0}^{x} q(s)e^{-\kappa s}\,ds-e^{\kappa x}\int_{x_0}^{x}(t-x_0)q(t)e^{-\kappa t}\,dt \\ &\quad +c_1(x-x_0)e^{\kappa(x-x_0)}+c_2e^{\kappa(x-x_0)} \\ &= \int_{x_0}^{x}(x-t)e^{\kappa(x-t)}q(t)\,dt+c_1(x-x_0)e^{\kappa(x-x_0)}+c_2e^{\kappa(x-x_0)} \quad (x \quad I), \end{aligned}$$

c_1 c_2 . $y=f(x)$ x_0, $f(x_0)=y_0$ $f'(x_0)={y_0}'$, $c_2=y_0$ $c_1=-\kappa y_0+{y_0}'$.

3. $t^2+kt+l=0$, $\kappa+i\lambda$ $\kappa-i\lambda$ $\lambda>0$, $k=-2\kappa$ $l=\kappa^2+\lambda^2$. $y=f(x)$ $y''+ky'+ly=q(x)$ I, $f''(x)-2\kappa f'(x)+(\kappa^2+\lambda^2)f(x)=q(x)$ x I. $e^{-\kappa x}$ $g(x)=e^{-\kappa x}f(x)$,

$$g''(x)+\lambda^2 g(x)=e^{-\kappa x}q(x) \qquad (x \quad I).$$

$\sin(\lambda x)$ $\cos(\lambda x)$

$$\big(g'(x)\sin(\lambda x)-\lambda g(x)\cos(\lambda x)\big)'=e^{-\kappa x}q(x)\sin(\lambda x) \qquad (x \quad I),$$

$$\big(g'(x)\cos(\lambda x)+\lambda g(x)\sin(\lambda x)\big)'=e^{-\kappa x}q(x)\cos(\lambda x) \qquad (x \quad I)$$

, ,

$$g'(x)\sin(\lambda x)-\lambda g(x)\cos(\lambda x)=\int_{x_0}^{x} e^{-\kappa t}q(t)\sin(\lambda t)\,dt+c_1 \qquad (x \quad I),$$

$$g'(x)\cos(\lambda x)+\lambda g(x)\sin(\lambda x)=\int_{x_0}^{x} e^{-\kappa t}q(t)\cos(\lambda t)\,dt+c_2 \qquad (x \quad I)$$

c_1 c_2. $-\frac{1}{\lambda}\cos(\lambda x)$ $\frac{1}{\lambda}\sin(\lambda x)$,

$$g(x)=\int_{x_0}^{x} e^{-\kappa t}\frac{\sin(\lambda(x-t))}{\lambda}q(t)\,dt+\frac{-c_1\cos(\lambda x)+c_2\sin(\lambda x)}{\lambda} \qquad (x \quad I)$$

, ,

$$f(x)=\int_{x_0}^{x} e^{\kappa(x-t)}\frac{\sin(\lambda(x-t))}{\lambda}q(t)\,dt+\frac{-c_1e^{\kappa x}\cos(\lambda x)+c_2e^{\kappa x}\sin(\lambda x)}{\lambda} \qquad (x \quad I).$$

$f(x_0) = y_0$ $f'(x_0) = {y_0}'$, $c_1 = (-\kappa\sin(\lambda x_0) - \lambda\cos(\lambda x_0))e^{-\kappa x_0}y_0 + \sin(\lambda x_0)e^{-\kappa x_0}{y_0}'$ $c_2 = (\lambda\sin(\lambda x_0) - \kappa\cos(\lambda x_0))e^{-\kappa x_0}y_0 + \cos(\lambda x_0)e^{-\kappa x_0}{y_0}'$.

: .

: (1) $y'' - 5y' + 6y = x$ $(-\infty, +\infty)$ $y = x$.
$t^2 - 5t + 6 = 0$ 2 3, , $y = f(x)$ $y'' - 5y' + 6y = x$ $(-\infty, +\infty)$,

$$f''(x) - (2+3)f'(x) + 2\cdot 3f(x) = x$$

$$\big(f'(x) - 2f(x)\big)' - 3\big(f'(x) - 2f(x)\big) = x$$

$$f'(x) - 2f(x) = e^{3x}\int_0^x te^{-3t}\,dt + c_1e^{3x} = -\frac{x}{3} - \frac{1}{9} + \Big(\frac{1}{9} + c_1\Big)e^{3x}$$

$$f'(x) - 2f(x) = -\frac{x}{3} - \frac{1}{9} + c_1e^{3x}$$

$$\begin{aligned} f(x) &= e^{2x}\int_0^x e^{-2t}\Big(-\frac{t}{3} - \frac{1}{9} + c_1e^{3t}\Big)\,dt + c_2e^{2x} \\ &= \frac{x}{6} + \frac{5}{36} - \Big(\frac{5}{36} + c_1 - c_2\Big)e^{2x} + c_1e^{3x} \end{aligned}$$

$$f(x) = \frac{x}{6} + \frac{5}{36} + c_2e^{2x} + c_1e^{3x}$$

c_1 c_2.
$f(0) = -1$ $f'(0) = 2$, $x = 0$ $-1 = f(0) = \frac{5}{36} + c_2 + c_1$. $x = 0$, $2 = f'(0) = \frac{1}{6} + 2c_2 + 3c_1$. $c_1 = \frac{37}{9}$ $c_2 = -\frac{21}{4}$. $f(x) = \frac{x}{6} + \frac{5}{36} - \frac{21}{4}e^{2x} + \frac{37}{9}e^{3x}$.

(2) $y'' - 2y' + y = 4$ $(-\infty, +\infty)$ $y = 4$.
$t^2 - 2t + 1 = 0$ 1, , $y = f(x)$ $y'' - 2y' + y = 4$ $(-\infty, +\infty)$,

$$f''(x) - 2f'(x) + f(x) = 4$$

$$\big(f'(x) - f(x)\big)' - \big(f'(x) - f(x)\big) = 4$$

$$f'(x) - f(x) = e^x\int_0^x 4e^{-t}\,dt + c_1e^x = -4 + 4e^x + c_1e^x$$

$$f'(x) - f(x) = -4 + c_1e^x$$

$$f(x) = e^x\int_0^x e^{-t}\big(-4 + c_1e^t\big)\,dt + c_2e^x = 4 + c_1xe^x + (c_2 - 4)e^x$$

$$f(x) = 4 + c_1xe^x + c_2e^x$$

c_1 c_2.
$f(0) = 3$ $f'(0) = -2$, $x = 0$ $3 = f(0) = 4 + c_2$. , $x = 0$ $-2 = f'(0) = c_2 + c_1$. $c_1 = c_2 = -1$. $f(x) = 4 - xe^x - e^x$.

(3) $y'' + 2y' + 2 = \sin x$ $(-\infty, +\infty)$ $y = \sin x$.

$t^2+2t+2=0$, $-1+i$ $-1-i$. $y=f(x)$ $y''+2y'+2y=\sin x$ $(-\infty,+\infty)$, $f''(x)+2f'(x)+2f(x)=\sin x$ x $(-\infty,+\infty)$. e^x $g(x)=e^x f(x)$,

$$g''(x)+g(x)=e^x\sin x.$$

$\sin x$ $\cos x$

$$\big(g'(x)\sin x-g(x)\cos x\big)'=e^x(\sin x)^2\,,$$

$$\big(g'(x)\cos x+g(x)\sin x\big)'=e^x\sin x\cos x.$$

,

$$g'(x)\sin x-g(x)\cos x=\int_0^x e^t(\sin t)^2\,dt+c_1=\frac{e^x}{2}-\frac{e^x}{5}\sin(2x)-\frac{e^x}{10}\cos(2x)+c_1\,,$$

$$g'(x)\cos x+g(x)\sin x=\int_0^x e^t\sin t\cos t\,dt+c_2=\frac{e^x}{10}\sin(2x)-\frac{e^x}{5}\cos(2x)+c_2\,.$$

$-\cos x$ $\sin x$,

$$g(x)=\frac{e^x}{5}\sin x-\frac{2e^x}{5}\cos x-c_1\cos x+c_2\sin x$$

, ,

$$f(x)=\frac{1}{5}\sin x-\frac{2}{5}\cos x-c_1e^{-x}\cos x+c_2e^{-x}\sin x$$

c_1 c_2.

$f(0)=1$ $f'(0)=1$, $x=0$ $1=f(0)=-\frac{2}{5}-c_1$. , $x=0$ $1=f'(0)=\frac{1}{5}+c_1+c_2$. $c_1=-\frac{7}{5}$ $c_2=\frac{11}{5}$. $f(x)=\frac{1}{5}\sin x-\frac{2}{5}\cos x+\frac{7}{5}e^{-x}\cos x+\frac{11}{5}e^{-x}\sin x$.

.

1. c $y=x\tan(x+c)$ – – $xy'-x^2-y^2-y=0$.

2. $f(x)=1+\frac{1}{x}\int_1^x f(t)\,dt$ $(0,+\infty)$.
   (: , .)

3. $y=f(x)$ $(x^2+1)y'=1$ $(-\infty,+\infty)$ $\lim_{x\to\pm\infty}f(x)$. $\lim_{x\to+\infty}f(x)-\lim_{x\to-\infty}f(x)$.

4. $y=f(x)$ $x+y^3+cxy^3=0$ x I, c , $y=f(x)$ $y^4+3x^2y'=0$ I.
   (: $x+y^3+cxy^3=0$ c.)

5. $y=f(x)$ $x(x+y)=ce^y$ x I, c , $y=f(x)$ $x(x+y-1)y'=2x+y$ I.

. .

1. : . : .

$$y' = e^{-y}\,,\quad y^2y' = 1,\quad (1+x^2)yy' = 1+y^2\,,\quad yy'\sqrt{1-x^2} = x,$$
$$xyy' = (1+x^2)(1+y^2),\quad x + yy' = x(xy'-y)y,\quad (1+x^2)y' = 1+y^2\,,$$
$$y' = y^2\,,\quad y' = (y-1)(y-2),\quad (x-1)y' = xy,\quad (x^2-4)y' = y,$$
$$(x+1)y' + y^2 = 0,\quad y^2 + (y')^2 = 1.$$

. .

1. $y' + ky = 0$ $(-\infty, +\infty)$;

2. $k > 0$ $y = f(x)$ $y' + ky = 0$ $(-\infty, +\infty)$ $\lim_{x\to+\infty} f(x) = 0$.

3. m $y = f(x)$ $y' + 3y = e^{mx}$ $(-\infty, +\infty)$ $\lim_{x\to+\infty} f(x) = 0$;

4. : .

$$y' + y = xe^{2x}\,,\quad xy' - y = 1,\quad xy' - y = x,\quad y' + xy = x^3\,,$$
$$x(x-1)y' + 3y = x(x-2),\quad y' + y\tan x = \cos x,\quad y' + y\cot x = \cos x.$$

5. $f(x) = 1 - x\int_1^x f(t)\,dt$ $(0, +\infty)$.

6. .

$$y' - x\cot y = \frac{x+1}{\sin y}\,,\quad yy' + (1+x^2)y^2 = e^x\,,\quad 1 + e^y + 2xe^yy' = 0.$$

7. **Bernoulli**. $\alpha \neq 1$,

$$y' + p(x)y = q(x)y^\alpha$$

$z = y^{1-\alpha}$.

.

$$y' + y = e^x\sqrt{y}\,,\quad y - y' = xy^2\,,\quad xy' - y = x\sqrt{y}\,,\quad xy' + y = xy^2\log x.$$

8. **Riccati**. $y = f(x)$

$$y' + p(x)y + r(x)y^2 = q(x)$$

$y = g(x)$ – ; – , $y = f(x) + \frac{1}{g(x)}$ $y' + p(x)y + r(x)y^2 = q(x)$. $y' + 3y - y^2 = 2$. – – $y' + 3y - y^2 = 2$. , . $y' + 3y - y^2 = 2$. ;

. .

1. $y'' + ky' + ly = 0$ $(-\infty, +\infty)$ (i) $k^2 - 4l > 0$, (ii) $k^2 - 4l = 0$ (iii) $k^2 - 4l < 0$;

2. k l $y'' + ky' + ly = 0$

(i) $y = e^{-x} + 3e^{2x}$.

(ii) $y = (1 + 2x)e^{x}$.

(iii) $y = e^{-2x}\sin(3x)$.

;

3. $k > 0$ $l > 0$ $y = f(x)$ $y''+ky'+ly = 0$ $(-\infty, +\infty)$ $\lim_{x\to+\infty} f(x) = 0$.

4. $y = f(x)$ $y'' + ky' + ly = 0$ $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$;

5. .

$$y'' - 4y' + 3y = e^{x}\,, \quad y'' + 6y' + 9y = xe^{3x}\,, \quad y'' - 2y' + 5y = \sin(2x).$$

6. m $y = f(x)$ $y'' + 4y' + 3y = e^{mx}$ $\lim_{x\to+\infty} f(x) = 0$;

$y'' + 6y' + 9y = e^{mx}$ $y'' + 2y' + 5y = e^{mx}$.

m;

8.6 , .

. .

«» :

$$\log x = \int_1^x \frac{1}{t}\,dt \qquad (x > 0).$$

$y = \frac{1}{t}$ $[x, 1]$, $0 < x < 1$, $[1, x]$, $x > 1$, . , . , $y = \frac{1}{x}$ $(0, +\infty)$, $y = \log x$ $(0, +\infty)$ $\frac{d\,\log x}{dx} = \frac{1}{x}$ $x > 0$. $y = \log x$ $(0, +\infty)$. $\log(ab) = \log a + \log b$ $a, b > 0$. $y = h(x) = \log(xb) - \log x$ $h'(x) = b\frac{1}{xb} - \frac{1}{x} = 0$, $y = h(x)$ $(0, +\infty)$. , , $\log 1 = 0$, $h(1) = \log b$, , $\log(xb) - \log x = h(x) = h(1) = \log b$ $x > 0$. $x = a$ $\log(ab) = \log a + \log b$. , . $\log(ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt = \log a + \int_1^{b} \frac{1}{s}\,ds = \log a + \log b$, $t = as$. $\log(ab) = \log a + \log b$, $b = \frac{1}{a}$ $\log\frac{1}{a} = -\log a$. , n $\log(2^n) = n\log 2$, $\log 2 > \log 1 = 0$, $\lim_{n\to+\infty}\log(2^n) = +\infty$. $y = \log x$, $\lim_{x\to+\infty}\log x$, , $\lim_{n\to+\infty} 2^n = +\infty$, $\lim_{x\to+\infty}\log x = \lim_{n\to+\infty}\log(2^n) = +\infty$. , $\lim_{x\to0+}\log x = \lim_{t\to+\infty}\log\frac{1}{t} = -\lim_{t\to+\infty}\log t = -\infty$. , $x \to 0+$ $x \to +\infty$ $y = \log x$ $(0, +\infty)$ $(-\infty, +\infty)$.

, . , , $y = e^{x}$ $(-\infty, +\infty)$ $(0, +\infty)$

$$y = e^{x} \qquad\qquad x = \log y.$$

, $y = e^{x}$, $(-\infty, +\infty)$ $\frac{d\,e^x}{dx} = \frac{1}{\frac{d\,\log y}{dy}\Big|_{y=e^x}} = e^{x}$ x. $e^{a+b} = e^{a}e^{b}$ a, b. $c = e^{a}$ $d = e^{b}$, $a + b = \log c + \log d = \log(cd)$, , $e^{a+b} = cd = e^{a}e^{b}$.

, $y = h(x) = e^{a+b-x}e^x$ $h'(x) = -e^{a+b-x}e^x + e^{a+b-x}e^x = 0$ x, $y = h(x)$ $(-\infty, +\infty)$. $e^{a+b-x}e^x = h(x) = h(0) = e^{a+b}e^0 = e^{a+b}$ x , $x = b$, $e^{a+b} = e^a e^b$. $e^0 = 1$ $e^{-a} = \frac{1}{e^a}$.
′ e^x e. , ,

$$e = e^1 ,$$

, , $\log e = 1$.
, a x

$$a^x = e^{x \log a} .$$

, $a = e$, a^x $e^{x \log e} = e^x$, . , $a^1 = e^{1 \log a} = e^{\log a} = a$.
: $a^{x+y} = e^{(x+y)\log a} = e^{x \log a + y \log a} = e^{x \log a}e^{y \log a} = a^x a^y$ $(ab)^x = e^{x \log(ab)} = e^{x \log a + x \log b} = e^{x \log a}e^{x \log b} = a^x b^x$ $a^{xy} = e^{xy \log a} = e^{x \log(e^{y \log a})} = e^{x \log(a^y)} = (a^y)^x$. , , , $a > 1$ $x > 0$, $\log a > \log 1 = 0$, , $a^x = e^{x \log a} > e^{0 \cdot 0} = e^0 = 1$. − − .
, , : , , ; . , $\log x = \int_1^x \frac{1}{t}\, dt$ − − . . , $y = \log x$ ( ) $\frac{1}{x}$. , , $\log 1 = 0$ 1, . , ′ , . , , − , , , , . . $y = f_1(x)$ $y = f_2(x)$ $y = h(x) = f_1(x)f_2(-x)$, $h'(x) = f_1{}'(x)f_2(-x) - f_1(x)f_2{}'(-x) = f_1(x)f_2(-x) - f_1(x)f_2(-x) = 0$ x, $y = h(x)$. $f_1(x)f_2(-x) = h(x) = h(0) = f_1(0)f_2(0) = 1 \cdot 1 = 1$, , $f_1(x) = f_2(x)$ x. , . , , . a^x ′ , , , n $a^n = a^{1+\cdots+1} = a^1 \cdots a^1 = a \cdots a$, , a^n a . . , $x^n = a$, n $a > 0$, $x = a^{\frac{1}{n}}$ − − $\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n} \cdot n} = a^1 = a$. $a^{\frac{1}{n}}$ $\sqrt[n]{a}$. , $r = \frac{m}{n}$ − − $a^r = a^{\frac{m}{n}} = a^{\frac{1}{n}m} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m$, , a^r a^r , . , $y = a^x$ − − . $y = f_1(x)$ $y = a^x$ $y = f_2(x)$ $y = a^x$, $a > 1$ x, $f_1(s) < f_1(x) < f_1(t)$ $f_2(s) < f_2(x) < f_2(t)$ s, t $s < x < t$. , $f_1(x)$ $f_2(x)$ $f_1(s) < \xi < f_1(t)$ s, t $s < x < t$. 1.3 $f_2(x) = f_1(x)$. , $a > 1$, a^x x. $0 < a \leq 1$, , .
a^x a x .

- : , , «» , , , , . «» , , ! 1 . ′ , , − , , , . 1 , , , , − , , . , , , 1, . , , . − − « » . , , !

. .

. - x

$$\arctan x = \int_0^x \frac{1}{t^2+1}\, dt.$$

$y = \frac{1}{t^2+1}$ $[0, x]$, $x > 0$, $[x, 0]$, $x < 0$, . $y = \frac{1}{x^2+1}$ $(-\infty, +\infty)$, $y = \arctan x$ $(-\infty, +\infty)$ $\frac{d \arctan x}{dx} = \frac{1}{x^2+1}$ x. $y = \arctan x$ $(-\infty, +\infty)$. , $\arctan(-x) = \int_0^{-x} \frac{1}{t^2+1}\, dt = -\int_0^x \frac{1}{(-s)^2+1}\, ds = -\arctan x$. , $\frac{1}{x^2+1} \leq 1$ $[0, 1]$ $\frac{1}{x^2+1} \leq \frac{1}{x^2}$ $[1, +\infty)$. $x \geq 1$ $\arctan x = \int_0^1 \frac{1}{t^2+1}\, dt + \int_1^x \frac{1}{t^2+1}\, dt \leq \int_0^1 1\, dt + \int_1^x \frac{1}{t^2}\, dt = 1 + (1 - \frac{1}{x}) < 2$. $y = \arctan x$ $[1, +\infty)$, ,

$\lim_{x\to+\infty}\arctan x$. , , π

$$\pi = 2\lim_{x\to+\infty}\arctan x.$$

$\lim_{x\to+\infty}\arctan x = \frac{\pi}{2}$ $\lim_{x\to-\infty}\arctan x = \lim_{t\to+\infty}\arctan(-t) = -\lim_{t\to+\infty}\arctan t = -\frac{\pi}{2}$, , $y = \arctan x$ $(-\infty,+\infty)$ $(-\frac{\pi}{2},\frac{\pi}{2})$. $y = \arctan x$ $\arctan 0 = \int_0^0 \frac{1}{t^2+1}\,dt = 0$.

$y = \arctan x$. $s = \frac{1}{t}$, $\int_1^x \frac{1}{t^2+1}\,dt = -\int_1^{\frac{1}{x}} \frac{1}{s^2+1}\,ds = \int_{\frac{1}{x}}^1 \frac{1}{s^2+1}\,ds$, $\frac{\pi}{2} = \lim_{x\to+\infty}\int_0^x \frac{1}{t^2+1}\,dt = \int_0^1 \frac{1}{t^2+1}\,dt + \lim_{x\to+\infty}\int_1^x \frac{1}{t^2+1}\,dt = \int_0^1 \frac{1}{t^2+1}\,dt + \lim_{x\to+\infty}\int_{\frac{1}{x}}^1 \frac{1}{t^2+1}\,dt = 2\int_0^1 \frac{1}{t^2+1}\,dt$. $\arctan 1 = \int_0^1 \frac{1}{t^2+1}\,dt = \frac{\pi}{4}$. , $\arctan(-1) = -\frac{\pi}{4}$.

, $y = \tan x$ $x = \arctan y$. ,

$$y = \tan x \qquad\qquad x = \arctan y.$$

$y = \tan x$ $(-\frac{\pi}{2},\frac{\pi}{2})$ $(-\infty,+\infty)$, $(-\frac{\pi}{2},\frac{\pi}{2})$ $\frac{d\tan x}{dx} = \frac{1}{\frac{d\arctan y}{dy}\big|_{y=\tan x}} = (\tan x)^2 + 1$. $x = \arctan y$, $y = \tan x$. , $\tan 0 = 0$, $\tan\frac{\pi}{4} = 1$ $\tan(-\frac{\pi}{4}) = -1$.

k $\mathbf{Z}$, $(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi)$ π, , k $k+1$, $(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi)$ $\big(-\frac{\pi}{2}+(k+1)\pi,\frac{\pi}{2}+(k+1)\pi\big)$ $\frac{\pi}{2}+k\pi = -\frac{\pi}{2}+(k+1)\pi$, . – $k=0$ – $(-\frac{\pi}{2},\frac{\pi}{2})$ $y = \tan x$. $(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi)$ $(-\frac{\pi}{2},\frac{\pi}{2})$ $k\pi$. , , $y = \tan x$

$$\tan x = \tan(x-k\pi) \qquad -\frac{\pi}{2}+k\pi < x < \frac{\pi}{2}+k\pi.$$

′ $y = \tan x$ $(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi)$, k , π. , $y = \tan x$ $\frac{\pi}{2}+k\pi$ $\frac{\pi}{2}$, $+\infty$. , $y = \tan x$ $\frac{\pi}{2}+k\pi$ $-\frac{\pi}{2}$, $-\infty$. , , $\frac{d\tan x}{dx} = (\tan x)^2 + 1$ $(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi)$.

, , $\frac{x}{2}$ $(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi)$ x $(-\pi+k2\pi,\pi+k2\pi)$. , $y = \cos x$ $y = \sin x$ $(-\pi+k2\pi,\pi+k2\pi)$, k ,

$$\cos x = \frac{1-(\tan\frac{x}{2})^2}{1+(\tan\frac{x}{2})^2}, \qquad \sin x = \frac{2\tan\frac{x}{2}}{1+(\tan\frac{x}{2})^2}.$$

$(-\pi+k2\pi,\pi+k2\pi)$, $\frac{d\cos x}{dx} = -\frac{2\tan\frac{x}{2}}{1+(\tan\frac{x}{2})^2} = -\sin x$ $\frac{d\sin x}{dx} = \frac{1-(\tan\frac{x}{2})^2}{1+(\tan\frac{x}{2})^2} = \cos x$.

$(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi)$, k $\mathbf{Z}$, $(-\pi+k2\pi,\pi+k2\pi)$ 2π, , k $k+1$, $(-\pi+k2\pi,\pi+k2\pi)$ $\big(-\pi+(k+1)2\pi,\pi+(k+1)2\pi\big)$ $\pi+k2\pi = -\pi+(k+1)2\pi$, . $y = \cos x$ $y = \sin x$ $(-\infty,+\infty)$ $\pi+k2\pi$, k.

$y = \tan x$ $\frac{\pi}{2}+k\pi$, $y = \cos x$ $y = \sin x$ $\pi+k2\pi$ $y = \cos x$ ( ) $\pi+k2\pi$ -1. , $y = \sin x$ ( ) $\pi+k2\pi$ 0. , , $\cos(\pi+k2\pi) = -1$ $\sin(\pi+k2\pi) = 0$ k , , $y = \cos x$ $y = \sin x$ $(-\infty,+\infty)$.

, l' Hopitâl, $\xi = \pi+k2\pi$ $\lim_{x\to\xi}\frac{\cos x-\cos\xi}{x-\xi} = \lim_{x\to\xi}\frac{-\sin x}{1} = 0 = -\sin\xi$ $\lim_{x\to\xi}\frac{\sin x-\sin\xi}{x-\xi} = \lim_{x\to\xi}\frac{\cos x}{1} = -1 = \cos\xi$. $y = \cos x$ $y = \sin x$

$\xi = \pi + k2\pi$ $\frac{d \cos x}{dx}\Big|_{x=\xi} = -\sin\xi$ $\frac{d \sin x}{dx}\Big|_{x=\xi} = \cos\xi$. , $(-\infty, +\infty)$ $\frac{d \cos x}{dx} = -\sin x$ $\frac{d \sin x}{dx} = \cos x$.

$y = \cos x$ $y = \sin x$ $y = \tan x$ π $y = \cos x$ $y = \sin x$ 2π. , $y = \tan x$, $y = \cos x$ $y = \sin x$.

$y = \tan x$, : $\cos 0 = 1$, $\cos(\pm\frac{\pi}{2}) = 0$ $\sin 0 = 0$, $\sin\frac{\pi}{2} = 1$, $\sin(-\frac{\pi}{2}) = -1$.

$y = \tan x$ $(-\frac{\pi}{2}, \frac{\pi}{2})$ $-\frac{\pi}{4}$, 0 $\frac{\pi}{4}$, $y = \cos x$ $y = \sin x$, , .

· , $\cos(a+b) = \cos a\cos b - \sin a \sin b$. $y = h(x) = \cos(a+b-x)\cos x - \sin(a+b-x)\sin x$ $h'(x) = \sin(a+b-x)\cos x - \cos(a+b-x)\sin x + \cos(a+b-x)\sin x - \sin(a+b-x)\cos x = 0$, $y = h(x)$ $(-\infty, +\infty)$. $\cos(a+b-x)\cos x - \sin(a+b-x)\sin x = h(x) = h(0) = \cos(a+b)$ x. $x = b$ $\cos(a+b) = \cos a\cos b - \sin a\sin b$. $\sin(a+b) = \sin a\cos b + \cos a \sin b$. $\cos(a-b) = \cos a\cos b + \sin a \sin b$ $\sin(a-b) = \sin a\cos b - \cos a\sin b$. $(\cos a)^2 + (\sin a)^2 = 1$.

, , 1, , $'$, . . $y = c_1(x)$ $y = s_1(x)$ $y = \cos x$ $y = \sin x$ $y = c_2(x)$ $y = s_2(x)$, $y = h(x) = (c_1(x) - c_2(x))^2 + (s_1(x) - s_2(x))^2$ $h'(x) = 2(c_1(x) - c_2(x))(-s_1(x) + s_2(x)) + 2(s_1(x) - s_2(x))(c_1(x) - c_2(x) = 0$ x. $y = h(x)$, , $(c_1(x) - c_2(x))^2 + (s_1(x) - s_2(x))^2 = h(x) = h(0) = (1-1)^2 + (0-0)^2 = 0$ x. $c_1(x) = c_2(x)$ $s_1(x) = s_2(x)$ x.

- : 1 , . , 1 .

$y = \arctan x = \int_0^x \frac{1}{t^2+1}\,dt$ $y = \arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$. .

, 10.

Κεφάλαιο 9

•

Taylor Lagrange . Newton. Riemann: , , Simpson.

9.1 Taylor.

$y = f(x)$ ξ, $\lim_{x\to\xi} f(x) = f(\xi)$, , $f(x)$ $f(\xi)$ x ξ. :

$$f(x) \approx f(\xi) \quad (x \quad \xi).$$

$f(\xi)$ – – $f(x)$.

: $4,00001$ 4 $y = \sqrt{x}$ 4, $\sqrt{4,00001} \approx \sqrt{4} = 2$.

, , , $f(\xi)$, $f(x) - f(\xi)$ $f(x)$ $f(\xi)$. x ξ. , x ξ , η x ξ $\frac{f(x)-f(\xi)}{x-\xi} = f'(\eta)$, ,

$$f(x) = f(\xi) + f'(\eta)(x - \xi).$$

, , l u , , x ξ, $l(x-\xi) \le f(x) - f(\xi) \le u(x-\xi)$, $x > \xi$, $u(x-\xi) \le f(x) - f(\xi) \le l(x-\xi)$, $x < \xi$. , $M \ge 0$, $|f(x) - f(\xi)| = |f'(\eta)||x - \xi|$ $M|x - \xi|$.

: $\sqrt{4,00001}$ $\sqrt{4} = 2$. $y = \sqrt{x}$ $y = \frac{1}{2\sqrt{x}}$ $0 < \frac{1}{2\sqrt{x}} \le \frac{1}{2\sqrt{4}} = \frac{1}{4}$ $x \ge 4$, x 4 $4,00001$. , $0 \le \sqrt{4,00001} - 2 \le \frac{1}{4}(4,00001 - 4) = 0,0000025$. $\sqrt{4,00001}$ $\sqrt{4} = 2$ $0,0000025$. , $2 \le \sqrt{4,00001} \le 2 + 0,0000025$ $2,00000$ $\sqrt{4,00001}$.

$y = f(x)$ ξ (x ξ) ξ, $f'(\eta)$, $f(x) = f(\xi) + f'(\eta)(x - \xi)$, $f'(\xi)$. ,

$$f(x) \approx f(\xi) + f'(\xi)(x - \xi)$$

$f(x)$.

: $y = \sqrt{x}$, $\sqrt{4,00001} \approx 2 + \frac{1}{4}(4,00001 - 4) = 2,0000025.$
$\sqrt{4,00001}$, 2 2,0000025, $\sqrt{4,00001}$ ′ . $\sqrt{4,00001}$
;

$f(x) = f(\xi) + f'(\eta)(x-\xi)$ $f'(\eta)$ $f'(\frac{\xi+x}{2})$ $f'(\xi)$, $f(x) \approx f(\xi) + f'(\frac{\xi+x}{2})(x-\xi)$. , ζ ξ $\frac{\xi+x}{2}$ $f'(\frac{\xi+x}{2}) = f'(\xi) + f''(\zeta)(\frac{\xi+x}{2} - \xi) = f'(\xi) + f''(\zeta)\frac{x-\xi}{2}$. , , ξ $\frac{\xi+x}{2}$. , , $f(x) \approx f(\xi)+f'(\xi)(x-\xi)+\frac{f''(\zeta)}{2}(x-\xi)^2$. , , ξ, $f''(\zeta) \approx f''(\xi)$,

$$f(x) \approx f(\xi) + f'(\xi)(x-\xi) + \frac{f''(\xi)}{2}(x-\xi)^2 .$$

, , $f(x)$ x ξ:

$$f(x) \approx \begin{cases} f(\xi), \\ f(\xi) + f'(\xi)(x-\xi), \\ f(\xi) + f'(\xi)(x-\xi) + \frac{f''(\xi)}{2}(x-\xi)^2 . \end{cases}$$

. .

9.1 ***Taylor Lagrange.*** *n I ξ. $y = f(x)$ n I ( ) $n+1$ I. x I η x ξ*

$$\boxed{f(x) = f(\xi) + \frac{f'(\xi)}{1!}(x-\xi) + \cdots + \frac{f^{(n)}(\xi)}{n!}(x-\xi)^n + \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-\xi)^{n+1} .}$$

, , $|f^{(n+1)}(x)| \leq M$ x I,

$$\boxed{\Big| f(x) - \Big(f(\xi) + \frac{f'(\xi)}{1!}(x-\xi) + \cdots + \frac{f^{(n)}(\xi)}{n!}(x-\xi)^n \Big) \Big| \leq \frac{M}{(n+1)!}|x-\xi|^{n+1}}$$

x I.

$f(\xi) + \frac{f'(\xi)}{1!}(x-\xi) + \cdots + \frac{f^{(n)}(\xi)}{n!}(x-\xi)^n$ **Taylor** n $y = f(x)$ I $\frac{f^{(n+1)}(\eta)}{(n+1)!}(x-\xi)^{n+1}$ n **Lagrange**. Taylor n x $\leq n$.

$n = 0$, 0 , Taylor $f(x) = f(\xi) + \frac{f'(\eta)}{1!}(x-\xi)$.
9.1

$$f(x) - \Big(f(\xi) + \frac{f'(\xi)}{1!}(x-\xi) + \cdots + \frac{f^{(n)}(\xi)}{n!}(x-\xi)^n \Big) = \big((x-\xi)^{n+1}\big)$$

ξ. 11 6.12 7 6.13.

: 9.1 x ξ A $f(x) = f(\xi) + \frac{f'(\xi)}{1!}(x-\xi) + \cdots + \frac{f^{(n)}(\xi)}{n!}(x-\xi)^n + \frac{A}{(n+1)!}(x-\xi)^{n+1}$.

$$y = g(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - \frac{A}{(n+1)!}(x-t)^{n+1}$$

t x ξ. $y = g(t)$

$$g'(t) = \frac{A - f^{(n+1)}(t)}{n!}(x-t)^n .$$

, $g(x) = 0$, A, $g(\xi) = 0$. η x ξ $g'(\eta) = 0$, $A = f^{(n+1)}(\eta)$. A , Taylor. $|f^{(n+1)}(\eta)| \leq M$.

: $0,0000025$ $\sqrt{4,00001}$ $\sqrt{4} = 2$.

9.1 $y = \sqrt{x}$ $[4, 4,00001]$ $\xi = 4$, $x = 4,00001$ $n = 1$ $\sqrt{4,00001} = \sqrt{4} + \frac{1}{1!}\frac{1}{2\sqrt{4}}(4,00001-4) - \frac{1}{2!}\frac{1}{4\sqrt{\eta^3}}(4,00001-4)^2 = 2,0000025 - \frac{10^{-10}}{8\sqrt{\eta^3}}$ η $4 < \eta < 4,00001$. $0 < \frac{10^{-10}}{8\sqrt{\eta^3}} < \frac{10^{-10}}{8\sqrt{4^3}} = 0,0000000000015625$, $2,0000024999984375 < \sqrt{4,00001} < 2,0000025$. $2,00000249999$ $\sqrt{4,00001}$.

, 9.1 $n = 2$. $\sqrt{4,00001} = \sqrt{4} + \frac{1}{1!}\frac{1}{2\sqrt{4}}(4,00001-4) - \frac{1}{2!}\frac{1}{4\sqrt{4^3}}(4,00001-4)^2 + \frac{1}{3!}\frac{3}{8\sqrt{\eta^5}}(4,00001-4)^3 = 2,0000024999984375 + \frac{10^{-15}}{16\sqrt{\eta^5}}$ η $4 < \eta < 4,00001$. $0 < \frac{10^{-15}}{16\sqrt{\eta^5}} < \frac{10^{-15}}{16\sqrt{4^5}} = 0,00000000000000001953125$, $2,0000024999984375 < \sqrt{4,00001} < 2,0000024999998437501953125$. , $2,00000249999843750$ $\sqrt{4,00001}$.

9.2 ***Taylor*** **.** n I ξ. $y = f(x)$ $n+1$ I ( ). x I

$$\boxed{f(x) = f(\xi) + \frac{f'(\xi)}{1!}(x-\xi) + \cdots + \frac{f^{(n)}(\xi)}{n!}(x-\xi)^n + \frac{1}{n!}\int_\xi^x f^{(n+1)}(t)(x-t)^n\, dt.}$$

$\frac{1}{n!}\int_\xi^x f^{(n+1)}(t)(x-t)^n\, dt$.

$n = 0$, Taylor $f(x) = f(\xi) + \frac{1}{0!}\int_\xi^x f'(t)\, dt$ (ii) 8.3.

: $\int_\xi^x f^{(n+1)}(t)(x-t)^n\, dt$.

.

1. Taylor Lagrange $y = \sqrt{x}$ $[4, 4,00001]$ $\xi = 4$ $x = 4,00001$.
   n $\sqrt{4,00001}$;

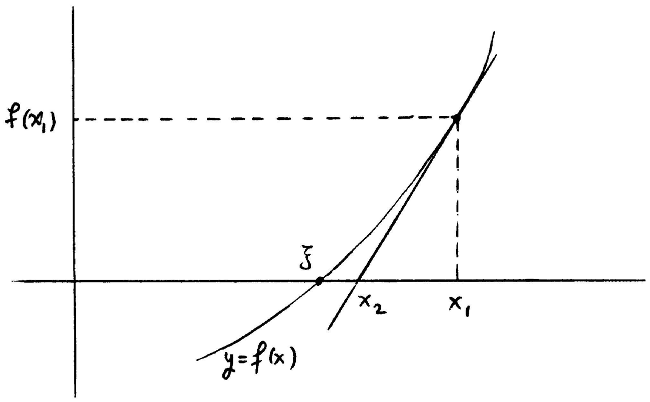
2. $\sin(1^o)$ $\sin(31^o)$. ($1^o = 0 + \frac{\pi}{180}$, $31^o = \frac{\pi}{6} + \frac{\pi}{180}$.)

3. (*) Taylor Lagrange $\frac{f^{(n+1)}(\eta)}{(n+1)!}(x-\xi)^{n+1}$ η , , x. , $n+2$ $y = f(x)$ I ξ, $\lim_{x\to\xi}\frac{\eta-\xi}{x-\xi} = \frac{1}{n+2}$, , $\eta - \xi \approx \frac{x-\xi}{n+2}$.

9.2 .

$$f(x) = 0,$$

$y = f(x)$ (a,b) (a,b) ξ . , $y = f(x)$ (a,b) (a,b) $y = f(x)$, Bolzano $f(x) = 0$ (a,b). ξ.



Σχήμα 9.1: .

x_1 (a,b) ξ, $\xi \approx x_1$, $(x_1, f(x_1))$ $y = f(x)$. x_1 , $(x_1, f(x_1))$ $y = f(x_1) + f'(x_1)(x - x_1)$. $0 = f(\xi) \approx f(x_1) + f'(x_1)(\xi - x_1)$,

$$\xi \approx x_1 - \frac{f(x_1)}{f'(x_1)}\,.$$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ x- $y = f(x)$ $(x_1, f(x_1))$.
x_1, () ξ, x_2, ξ. , $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$ x_2, $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$ x_3 ′ . , (x_n). **Newton.** ¨; (x_n) ξ , , $x_n - \xi$.

9.1 $y = f(x)$ (a,b) $0 < m \leq |f'(x)|$ $|f''(x)| \leq M$ x (a,b). $\mu < \frac{m}{M}$, $[c-\mu, c+\mu]$ (a,b), $\nu = \frac{m}{M}\sqrt{\frac{4M\mu}{m} + 1} - \mu - \frac{m}{M}$. ν $0 < \nu < \mu$ $[c-\nu, c+\nu]$ ξ $f(\xi) = 0$.
Newton x_1 $[c-\mu, c+\mu]$, x_n $[c-\mu, c+\mu]$

$$\lim_{n \to +\infty} x_n = \xi.$$

,

$$|x_n - \xi| \leq \frac{2m}{M}\Big(\frac{M\mu}{m}\Big)^{2^{n-1}}$$

n.

: x_1 $[c-\mu, c+\mu]$ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. 9.1, η x_1 ξ $0 = f(\xi) = f(x_1) + f'(x_1)(\xi - x_1) + \frac{f''(\eta)}{2}(\xi - x_1)^2$, , $x_2 = x_1 + (\xi - x_1) + \frac{f''(\eta)}{2f'(x_1)}(\xi - x_1)^2$, , $x_2 - \xi = \frac{f''(\eta)}{2f'(x_1)}(x_1 - \xi)^2$. $|x_2 - \xi| = \frac{|f''(\eta)|}{2|f'(x_1)|}|x_1 - \xi|^2 \leq \frac{M}{2m}|x_1 - \xi|^2$. x_1 $[c-\mu, c+\mu]$ ξ $[c-\nu, c+\nu]$, $|x_1 - \xi| \leq \mu + \nu$, $|x_2 - \xi| \leq \frac{M}{2m}(\mu+\nu)^2$. $|x_2 - c| \leq |x_2 - \xi| + |\xi - c| \leq \frac{M}{2m}(\mu+\nu)^2 + \nu = \mu$, , x_2 $[c-\mu, c+\mu]$. , x_2 x_3, x_3 $[c-\mu, c+\mu]$, , x_n $[c-\mu, c+\mu]$. , $|x_n - \xi| \leq \frac{M}{2m}|x_{n-1} - \xi|^2$ $n \geq 2$.

$|x_1-\xi|\leq \mu+\nu\leq 2\mu$ $|x_n-\xi|\leq \frac{2m}{M}\big(\frac{M\mu}{m}\big)^{2^{n-1}}$ $n\geq 1$. $\lim_{n\to+\infty}2^{n-1}=+\infty$ $0\leq\frac{M\mu}{m}<1$, $\lim_{n\to+\infty}\frac{2m}{M}\big(\frac{M\mu}{m}\big)^{2^{n-1}}=0$, , $\lim_{n\to+\infty}x_n=\xi$.

(x_n) ξ. , $\big(\frac{M\mu}{m}\big)^{2^{n-1}}$, , $|x_n-\xi|$ -, 2^{n-1} . Newton, , n, ξ.

ξ (a,b) $f(\xi)=0$. , ξ_1 ξ_2 (a,b) $f(\xi_1)=f(\xi_2)=0$, η $f'(\eta)=0$ $0<m\leq|f'(\eta)|$.

, 9.1 . (a,b) m M. a_1 b_1 $a<a_1<b_1<b$ $y=f(x)$ a_1,b_1 – ξ a_1,b_1 . $\mu<\min\{\frac{m}{M},a_1-a,b-b_1\}$. , c $[a_1,b_1]$ ( ) $[c-\mu,c+\mu]$ (a,b). $\nu=\frac{m}{M}\sqrt{\frac{4M\mu}{m}+1}-\mu-\frac{m}{M}$ $[a_1,b_1]$ $\leq 2\nu$. , I, $y=f(x)$, ξ I. c I, I $[c-\nu,c+\nu]$, ξ $[c-\nu,c+\nu]$, $[c-\mu,c+\mu]$ (a,b). , Newton x_1 $[c-\mu,c+\mu]$.

.

1. Newton $x^2-2=0$ $\sqrt{2}$ $[1,2]$.
$x_1=2$ x_2,x_3,x_4 . $\sqrt{2}$.
n x_n $\sqrt{2}$;

9.3 Riemann.

.
$y=f(x)$ $[a,b]$. $[a,b]$ n

$$x_k=a+k\frac{b-a}{n}\quad(0\leq k\leq n).$$

, $[x_{k-1},x_k]$ $x_k-x_{k-1}=\frac{b-a}{n}$. $[x_{k-1},x_k]$, $\frac{x_{k-1}+x_k}{2}$,

$$x_{\frac{2k-1}{2}}=\frac{x_{k-1}+x_k}{2}=a+\Big(k-\frac{1}{2}\Big)\frac{b-a}{n}\,.$$

, ,

$$x_0\,,x_{\frac{1}{2}}\,,x_1\,,\ldots\,,x_{k-1}\,,x_{\frac{2k-1}{2}}\,,x_k\,,\ldots\,,x_{n-1}\,,x_{\frac{2n-1}{2}}\,,x_n$$

$y_i=f(x_i)$ $y=f(x)$:

$$y_0\,,y_{\frac{1}{2}}\,,y_1\,,\ldots\,,y_{k-1}\,,y_{\frac{2k-1}{2}}\,,y_k\,,\ldots\,,y_{n-1}\,,y_{\frac{2n-1}{2}}\,,y_n\,.$$

$\int_a^b f(x)\,dx$ $y=f(x)$, , . $y=f(x)$ $[x_{k-1},x_k]$ $\int_{x_{k-1}}^{x_k}f(x)\,dx$. , : , , , *Simpson* .
n $[a,b]$ $\int_a^b f(x)\,dx$, .

$$||\leq c\frac{(b-a)^{m+2}M_{m+1}}{n^{m+1}}=c(b-a)M_{m+1}{h_n}^{m+1}\,,$$

$h_n = \frac{b-a}{n}$, c , m M_{m+1} $m+1$ $y=f(x)$ $[a,b]$. 0 n , , $\lim_{n\to+\infty} h_n = \lim_{n\to+\infty} \frac{b-a}{n} = 0$. , , , Simpson . , «» ( ) $y = f(x)$.
. , − , , − $[x_{k-1}, x_k]$, $[a,b]$ $[x_{k-1}, x_k]$. , .

. .

$y=f(x)$ $[a,b]$ $|f'(x)| \le M_1$ x $[a,b]$.

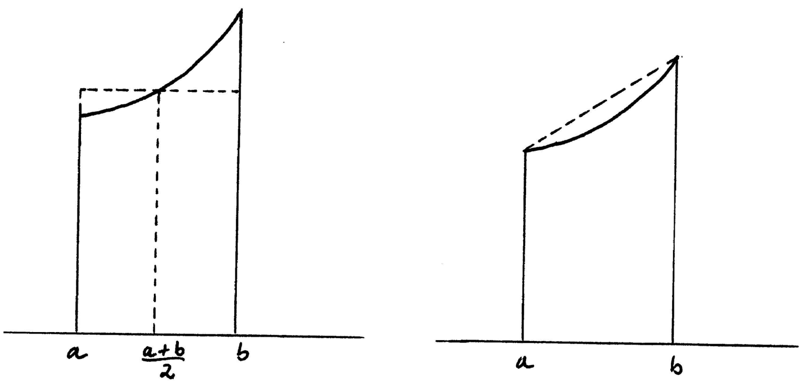
$$\Big|\int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big)(b-a)\Big| \le \frac{(b-a)^2 M_1}{4}\,.$$

: $y=f(x)$ (0) $y=p(x)=f(\frac{a+b}{2})$. . x $[a,b]$ ξ x $\frac{a+b}{2}$ $f(x)-p(x)=f(x)-f(\frac{a+b}{2}) = f'(\xi)(x-\frac{a+b}{2})$. $|f(x)-p(x)| \le M_1|x-\frac{a+b}{2}|$ x $[a,b]$. $\int_a^b p(x)\,dx = f(\frac{a+b}{2})(b-a)$, $\big|\int_a^b f(x)\,dx - f(\frac{a+b}{2})(b-a)\big| = \big|\int_a^b (f(x)-p(x))\,dx\big| \le M_1 \int_a^b |x-\frac{a+b}{2}|\,dx = \frac{(b-a)^2M_1}{4}$.

9.2 $y=f(x)$ $[a,b]$ $|f'(x)|\le M_1$ x $[a,b]$,

$$\boxed{\int_a^b f(x)\,dx \approx \Big(y_{\frac{1}{2}} + \cdots + y_{\frac{2n-1}{2}}\Big)\frac{b-a}{n}\,.}$$

: $|| \le \frac{1}{4}\frac{(b-a)^2M_1}{n}$.



Σχήμα 9.2: .

: $[x_{k-1}, x_k]$, $\big|\int_{x_{k-1}}^{x_k} f(x)\,dx - y_{\frac{2k-1}{2}}\frac{b-a}{n}\big| \le \frac{(b-a)^2M_1}{4n^2}$. $\int_a^b f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$ $\big|\int_a^b f(x)\,dx - \big(y_{\frac{1}{2}} + \cdots + y_{\frac{2n-1}{2}}\big)\frac{b-a}{n}\big| \le n\frac{(b-a)^2M_1}{4n^2} = \frac{(b-a)^2M_1}{4n}$.

. .

, $y = f(x)$ $[a,b]$ $|f''(x)| \le M_2$ x $[a,b]$.

$$\Big| \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a) \Big| \le \frac{(b-a)^3 M_2}{12}\,.$$

: $y = f(x)$ 1, $y = f(x)$ a b . $y = p(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$. x (a,b) c $f(x) - p(x) = c(x-a)(x-b)$. $y = g(t) = f(t) - p(t) - c(t-a)(t-b)$ t $[a,b]$. $g(a) = g(x) = g(b) = 0$, ξ (a,x) η (x,b) $g'(\xi) = g'(\eta) = 0$. ζ (ξ,η) , , (a,b) $g''(\zeta) = 0$. , $g''(t) = f''(t) - 2c$, $c = \frac{f''(\zeta)}{2}$. , , x (a,b) ζ (a,b) $f(x) - p(x) = \frac{f''(\zeta)}{2}(x-a)(x-b)$. x (a,b) $|f(x) - p(x)| \le \frac{M_2}{2}(x-a)(b-x)$. , , $x = a$ $x = b$. $\int_a^b p(x)\,dx = \frac{f(a)+f(b)}{2}(b-a)$, $\big| \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a) \big| = \big| \int_a^b (f(x)-p(x))\,dx \big| \le \frac{M_2}{2} \int_a^b (x-a)(b-x)\,dx = \frac{(b-a)^3 M_2}{12}$.

9.3 $y = f(x)$ $[a,b]$ $|f''(x)| \le M_2$ x $[a,b]$,

$$\boxed{\int_a^b f(x)\,dx \approx \Big(\frac{y_0}{2} + y_1 + \cdots + y_{n-1} + \frac{y_n}{2}\Big)\frac{b-a}{n}\,.}$$

: $|| \le \frac{1}{12}\frac{(b-a)^3 M_2}{n^2}$.

: $[x_{k-1}, x_k]$, $\big| \int_{x_{k-1}}^{x_k} f(x)\,dx - \frac{y_{k-1}+y_k}{2}\frac{b-a}{n} \big| \le \frac{(b-a)^3 M_2}{12n^3}$. , $\big| \int_a^b f(x)\,dx - \big(\frac{y_0+y_1}{2} + \cdots + \frac{y_{n-1}+y_n}{2}\big)\frac{b-a}{n} \big| \le n\frac{(b-a)^3 M_2}{12n^3} = \frac{(b-a)^3 M_2}{12n^2}$.

. .

$y = f(x)$ $[a,b]$ $|f''(x)| \le M_2$ x $[a,b]$.

$$\Big| \int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big)(b-a) \Big| \le \frac{(b-a)^3 M_2}{24}\,.$$

: $y = f(x)$ 1, $y = f(x)$ $\frac{a+b}{2}$ $y = f(x)$. $y = p(x) = f(\frac{a+b}{2}) + f'(\frac{a+b}{2})(x - \frac{a+b}{2})$ Taylor. , x $[a,b]$ ξ x $\frac{a+b}{2}$ $f(x) = f(\frac{a+b}{2}) + f'(\frac{a+b}{2})(x - \frac{a+b}{2}) + \frac{f''(\xi)}{2}(x - \frac{a+b}{2})^2$, , $|f(x) - p(x)| \le \frac{M_2}{2}(x - \frac{a+b}{2})^2$. $\int_a^b p(x)\,dx = f(\frac{a+b}{2})(b-a)$, $\big| \int_a^b f(x)\,dx - f(\frac{a+b}{2})(b-a) \big| = \big| \int_a^b (f(x) - p(x))\,dx \big| \le \frac{M_2}{2}\int_a^b (x - \frac{a+b}{2})^2\,dx = \frac{(b-a)^3 M_2}{24}$.

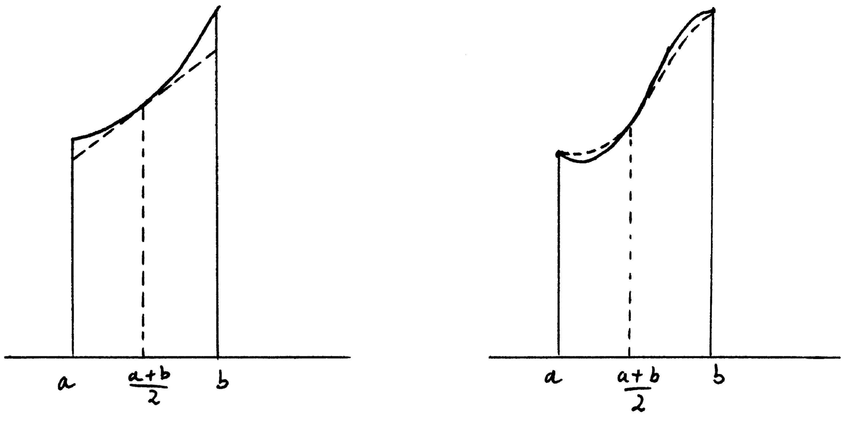
9.4 $y = f(x)$ $[a,b]$ $|f''(x)| \le M_2$ x $[a,b]$,

$$\boxed{\int_a^b f(x)\,dx \approx \Big(y_{\frac{1}{2}} + \cdots + y_{\frac{2n-1}{2}}\Big)\frac{b-a}{n}\,.}$$

: $|| \le \frac{1}{24}\frac{(b-a)^3 M_2}{n^2}$.

: $[x_{k-1}, x_k]$ $\big| \int_{x_{k-1}}^{x_k} f(x)\,dx - y_{\frac{2k-1}{2}}\frac{b-a}{n} \big| \le \frac{(b-a)^3 M_2}{24n^3}$. , $\big| \int_a^b f(x)\,dx - \big(y_{\frac{1}{2}} + \cdots + y_{\frac{2n-1}{2}}\big)\frac{b-a}{n} \big| \le n\frac{(b-a)^3 M_2}{24n^3} = \frac{(b-a)^3 M_2}{24n^2}$.

. `Simpson`.



Σχήμα 9.3: Simpson.

, $y = f(x)$ $[a,b]$ $|f^{(4)}(x)| \leq M_4$ x $[a,b]$.

$$\Big|\int_a^b f(x)\,dx - \Big(f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b)\Big)\frac{b-a}{6}\Big| \leq \frac{(b-a)^5 M_4}{2880}\,.$$

: , $y = f(x)$ a, $\frac{a+b}{2}$ b $\frac{a+b}{2}$ $y = f(x)$. $y = p(x) = c_0 + c_1(x-a) + c_2(x-a)(x-\frac{a+b}{2}) + c_3(x-a)(x-\frac{a+b}{2})(x-b)$, c_0, c_1, c_2, c_3 $c_0 = f(a)$, $c_0 + c_1(\frac{a+b}{2} - a) = f(\frac{a+b}{2})$, $c_0 + c_1(b-a) + c_2(b-a)(b-\frac{a+b}{2}) = f(b)$ $c_1 + c_2(\frac{a+b}{2} - a) + c_3(\frac{a+b}{2} - a)(\frac{a+b}{2} - b) = f'(\frac{a+b}{2})$. $y = f(x)$ $y = p(x)$. x (a,b) $x \neq \frac{a+b}{2}$ c $f(x) - p(x) = c(x-a)(x-\frac{a+b}{2})^2(x-b)$. $y = g(t) = f(t) - p(t) - c(t-a)(t-\frac{a+b}{2})^2(t-b)$ $[a,b]$ $g(a) = g(\frac{a+b}{2}) = g(x) = g(b) = 0$ $g'(\frac{a+b}{2}) = 0$. , $a < x < \frac{a+b}{2}$, ξ (a,x), η $(x, \frac{a+b}{2})$ ζ $(\frac{a+b}{2}, b)$ $g'(\xi) = g'(\eta) = g'(\zeta) = 0$. $g'(\frac{a+b}{2}) = 0$ κ (ξ,η), λ $(\eta, \frac{a+b}{2})$ μ $(\frac{a+b}{2}, \zeta)$ $g''(\kappa) = g''(\lambda) = g''(\mu) = 0$. ν (κ,λ) ρ (λ,μ) $g^{(3)}(\nu) = g^{(3)}(\rho) = 0$, (!), σ (ν,ρ) $g^{(4)}(\sigma) = 0$. , $g^{(4)}(\sigma) = f^{(4)}(\sigma) - 24c$, , $c = \frac{f^{(4)}(\sigma)}{24}$. $\frac{a+b}{2} < x < b$ x (a,b) $x \neq \frac{a+b}{2}$ σ (a,b) $f(x) - p(x) = \frac{f^{(4)}(\sigma)}{24}(x-a)(x-\frac{a+b}{2})^2(x-b)$, $|f(x) - p(x)| \leq \frac{M_4}{24}(x-a)(x-\frac{a+b}{2})^2(b-x)$. $x = a$, $x = \frac{a+b}{2}$ $x = b$. $\int_a^b p(x)\,dx = \frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}(b-a)$. , $\big|\int_a^b f(x)\,dx - \frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}(b-a)\big| = \big|\int_a^b (f(x)-p(x))\,dx\big| \leq \frac{M_4}{24}\int_a^b (x-a)(x-\frac{a+b}{2})^2(b-x)\,dx = \frac{(b-a)^5 M_4}{2880}$.

9.5 $y = f(x)$ $[a,b]$ $|f^{(4)}(x)| \leq M_4$ x $[a,b]$,

$$\boxed{\int_a^b f(x)\,dx \approx \frac{y_0 + y_n + 2(y_1 + \cdots + y_{n-1}) + 4\big(y_{\frac{1}{2}} + \cdots + y_{\frac{2n-1}{2}}\big)}{6}\,\frac{b-a}{n}\,.}$$

: $|| \leq \frac{1}{2880}\frac{(b-a)^5 M_4}{n^4}$.

: $[x_{k-1}, x_k]$ $\Big|\int_{x_{k-1}}^{x_k} f(x)\,dx - \frac{y_{k-1}+4y_{\frac{2k-1}{2}}+y_k}{6}\frac{b-a}{n}\Big| \leq \frac{(b-a)^5 M_4}{2880n^5}$, , $\Big|\int_a^b f(x)\,dx - \Big(\frac{y_0+4y_{\frac{1}{2}}+y_1}{6} + \cdots + \frac{y_{n-1}+4y_{\frac{2n-1}{2}}+y_n}{6}\Big)\frac{b-a}{n}\Big| \leq n\frac{(b-a)^5 M_4}{2880n^5} = \frac{(b-a)^5 M_4}{2880n^4}$.

.

1. $y = p(x)$ ≤ 3 $\int_a^b p(x)\,dx = \big(p(a) + 4p(\frac{a+b}{2}) + p(b)\big)\,\frac{b-a}{6}$.

2. $\log 2$ $\log 2 = \int_1^2 \frac{1}{x}\,dx$ Simpson.

3. π $\pi = 4\int_0^1 \frac{1}{x^2+1}\,dx$ Simpson.

Κεφάλαιο 10

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(). , ... Cauchy. p- . , , , . , . . Taylor . : , , , - (Newton). : , , , .

10.1 .

(x_n),

$$s_1 = x_1\,,\; s_2 = x_1 + x_2\,,\; s_3 = x_1 + x_2 + x_3\,,\; \dots\; , s_n = x_1 + \cdots + x_n\,,\; \dots$$

(s_n).

$$\sum_{n=1}^{+\infty} x_n \qquad\qquad x_1 + x_2 + \cdots + x_n + \cdots$$

(x_n) , , x_n ( ). x_n n- n- s_n n- x_n . , $\sum_{n=1}^{+\infty} x_n$ $\sum_{k=1}^{+\infty} x_k$ $\sum_{j=1}^{+\infty} x_j$.

: (1) $\sum_{n=1}^{+\infty} 1$ $1 + 1 + 1 + \cdots + 1 + \cdots$. (1) $s_1 = 1$, $s_2 = 1 + 1 = 2$, $s_3 = 1 + 1 + 1 = 3$, , $s_n = \underbrace{1 + \cdots + 1}_{n} = n$ $n \geq 1$.

(2) a $1 + \sum_{n=2}^{+\infty} a^{n-1}$ $1 + a + a^2 + \cdots + a^{n-1} + \cdots$. (a^n) $s_1 = 1$, $s_2 = 1 + a$, $s_3 = 1 + a + a^2$, , $s_n = 1 + a + \cdots + a^{n-1}$ $n \geq 2$.

(3) $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ $1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$, p . $(\frac{1}{n^p})$ $s_1 = 1$, $s_2 = 1 + \frac{1}{2^p}$, $s_3 = 1 + \frac{1}{2^p} + \frac{1}{3^p}$, , $s_n = 1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p}$ $n \geq 1$. $\sum_{n=1}^{+\infty} \frac{1}{n}$.

(s_n) s, , s

$$\sum_{n=1}^{+\infty} x_n = s\,.$$

(s_n) , . , (s_n) $+\infty$ $-\infty$, $+\infty$ $-\infty$, , $+\infty$ $-\infty$

$$\sum_{n=1}^{+\infty} x_n = +\infty \qquad \sum_{n=1}^{+\infty} x_n = -\infty\,.$$

, $\sum_{n=1}^{+\infty} x_n$ $\pm\infty$, $\pm\infty$, . , $+\infty$ $-\infty$, .
, , $\sum_{n=1}^{+\infty} x_n$. ′ , . ′, (s_n) $\pm\infty$, , (s_n).

: (1) $\sum_{n=1}^{+\infty} 1$ $+\infty$, $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} n = +\infty$. ,

$$\boxed{\sum_{n=1}^{+\infty} 1 = +\infty.}$$

(2) a .

$$\boxed{1 + \sum_{n=2}^{+\infty} a^{n-1} \begin{cases} = +\infty, & a \geq 1, \\ = \frac{1}{1-a}\,, & -1 < a < 1, \\ \,, & a \leq -1. \end{cases}}$$

10.1 $\sum_{n=1}^{+\infty} x_n$, $\lim_{n\to+\infty} x_n = 0$.

: $s_n = x_1 + \cdots + x_n$. $\sum_{n=1}^{+\infty} x_n$ s, $\lim_{n\to+\infty} s_n = s$. , , $x_n = s_n - s_{n-1}$ $n \geq 2$. $\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} s_n - \lim_{n\to+\infty} s_{n-1} = s - s = 0$.

: $\sum_{n=1}^{+\infty} \frac{n}{n+1}$ $\lim_{n\to+\infty} \frac{n}{n+1} = 1 \neq 0$.

$\sum_{n=1}^{+\infty} x_n$ (:) $\lim_{n\to+\infty} x_n = 0$. , 10.1.

10.2 . $\sum_{n=1}^{+\infty} x_n$ $\sum_{n=1}^{+\infty} y_n$ $\sum_{n=1}^{+\infty} x_n + \sum_{n=1}^{+\infty} y_n$, $\sum_{n=1}^{+\infty}(x_n + y_n)$

$$\boxed{\sum_{n=1}^{+\infty}(x_n + y_n) = \sum_{n=1}^{+\infty} x_n + \sum_{n=1}^{+\infty} y_n\,.}$$

: n- $s_n = x_1 + \cdots + x_n$ $t_n = y_1 + \cdots + y_n$, $\lim_{n\to+\infty} s_n = \sum_{n=1}^{+\infty} x_n$ $\lim_{n\to+\infty} t_n = \sum_{n=1}^{+\infty} y_n$. , n- $\sum_{n=1}^{+\infty}(x_n + y_n)$

$$u_n = (x_1 + y_1) + \cdots + (x_n + y_n) = (x_1 + \cdots + x_n) + (y_1 + \cdots + y_n) = s_n + t_n\,.$$

$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty}(s_n + t_n) = \lim_{n\to+\infty} s_n + \lim_{n\to+\infty} t_n = \sum_{n=1}^{+\infty} x_n + \sum_{n=1}^{+\infty} y_n$. $\sum_{n=1}^{+\infty}(x_n + y_n)$ $\sum_{n=1}^{+\infty} x_n + \sum_{n=1}^{+\infty} y_n$.

10.3 . $\sum_{n=1}^{+\infty} x_n$, λ $\lambda\sum_{n=1}^{+\infty} x_n$, $\sum_{n=1}^{+\infty}(\lambda x_n)$

$$\boxed{\sum_{n=1}^{+\infty}(\lambda x_n) = \lambda\sum_{n=1}^{+\infty} x_n \qquad \Big(\lambda \neq 0 \quad \sum_{n=1}^{+\infty} x_n = \pm\infty\Big)\,.}$$

: $s_n = x_1 + \cdots + x_n$, $\lim_{n\to+\infty} s_n = \sum_{n=1}^{+\infty} x_n$. n- $\sum_{n=1}^{+\infty}(\lambda x_n)$

$$w_n = \lambda x_1 + \cdots + \lambda x_n = \lambda(x_1 + \cdots + x_n) = \lambda s_n\,.$$

$\lim_{n\to+\infty} w_n = \lim_{n\to+\infty}(\lambda s_n) = \lambda \lim_{n\to+\infty} s_n = \lambda \sum_{n=1}^{+\infty} x_n$, , $\sum_{n=1}^{+\infty}(\lambda x_n)$ $\lambda \sum_{n=1}^{+\infty} x_n$.

:

$$\sum_{n=1}^{+\infty}(\lambda x_n + \mu y_n) = \lambda \sum_{n=1}^{+\infty} x_n + \mu \sum_{n=1}^{+\infty} y_n\,.$$

, .

10.4 , . $\sum_{n=1}^{+\infty} x_n$ $\sum_{n=1}^{+\infty} y_n$ $x_n \le y_n$ $n \ge 1$,

$$\boxed{\sum_{n=1}^{+\infty} x_n \le \sum_{n=1}^{+\infty} y_n\,.}$$

: n- $s_n = x_1 + \cdots + x_n$ $t_n = y_1 + \cdots + y_n$, $\lim_{n\to+\infty} s_n = \sum_{n=1}^{+\infty} x_n$ $\lim_{n\to+\infty} t_n = \sum_{n=1}^{+\infty} y_n$. ,

$$s_n = x_1 + \cdots + x_n \le y_1 + \cdots + y_n = t_n$$

n, $\sum_{n=1}^{+\infty} x_n = \lim_{n\to+\infty} s_n \le \lim_{n\to+\infty} t_n = \sum_{n=1}^{+\infty} y_n$.

.

1. ( ).

$$\sum_{n=1}^{+\infty}\Big(-\frac{1}{2}\Big),\quad \sum_{n=1}^{+\infty}(-1)^n\,,\quad \sum_{n=1}^{+\infty} n,\quad \sum_{n=1}^{+\infty} n^2\quad \sum_{n=1}^{+\infty}\Big(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\Big),$$

$$\sum_{n=1}^{+\infty}(-1)^{n-1} n,\quad \sum_{n=1}^{+\infty}(-1)^{n-1} n^2\,.$$

2. .

$$\sum_{n=1}^{+\infty}\frac{n}{2n+1}\,,\ \sum_{n=1}^{+\infty}\Big(\frac{n}{n+1}\Big)^n\,,\ \sum_{n=1}^{+\infty}\sqrt[n]{n}\,,\ \sum_{n=1}^{+\infty} n\sin\frac{1}{n}\,,\ \sum_{n=1}^{+\infty} n\log\Big(1+\frac{1}{n}\Big).$$

(: n- .)

3. , ( ).

$$\sum_{n=1}^{+\infty}\Big(\frac{2}{3}\Big)^{n+2},\quad \sum_{n=1}^{+\infty}\Big(\frac{4}{3}\Big)^{n-3},\quad \sum_{n=1}^{+\infty}(-1)^{n-4}\,,\quad \sum_{n=3}^{+\infty}\Big(-\frac{2}{3}\Big)^n,$$

$$\sum_{n=4}^{+\infty}(-3)^n\,,\quad \sum_{n=1}^{+\infty}\frac{2}{3^{n-1}}\,,\quad \sum_{n=1}^{+\infty}\frac{2^{n-1}+3^{n+1}}{6^n}\,,\quad \sum_{n=1}^{+\infty}\frac{1+2^{\frac{n}{2}}}{2^n}\,.$$

4. $\sum_{n=1}^{+\infty}(b_n - b_{n+1})$.

s_n , , $\lim_{n\to+\infty} b_n$ $\lim_{n\to+\infty} b_n$. $\lim_{n\to+\infty} b_n$;

( ).

$$\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}, \quad \sum_{n=1}^{+\infty} \frac{1}{(2n-1)(2n+1)}, \quad \sum_{n=1}^{+\infty} \frac{1}{n(n+1)(n+2)},$$

$$\sum_{n=1}^{+\infty} \log \frac{n}{n+1}, \quad \sum_{n=1}^{+\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}, \quad \sum_{n=1}^{+\infty} \left(\sqrt[n]{n} - \sqrt[n+1]{n+1}\right),$$

$$\sum_{n=1}^{+\infty} \left((-1)^{n-1}\frac{n}{n+1} - (-1)^n \frac{n+1}{n+2}\right), \quad \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{2n+1}{n(n+1)}.$$

5. 11 12 2.4 , , .

10.2 .

10.1 $x_n \geq 0$ $n \geq 1$, $\sum_{n=1}^{+\infty} x_n$ $+\infty$. $0 \leq \sum_{n=1}^{+\infty} x_n \leq +\infty$.
: (s_n) , , (s_n) , $+\infty$.

: $x_n \geq 0$ $n \geq 1$, $s_{n+1} = x_1 + \cdots + x_n + x_{n+1} = s_n + x_{n+1} \geq s_n$ $n \geq 1$. (s_n) , , $+\infty$. , $s_n = x_1 + \cdots + x_n \geq 0$ $n \geq 1$, $\lim_{n\to+\infty} s_n \geq 0$. , (s_n) , $\sum_{n=1}^{+\infty} x_n = \lim_{n\to+\infty} s_n$, (s_n) , $\sum_{n=1}^{+\infty} x_n = \lim_{n\to+\infty} s_n = +\infty$.

$\sum_{n=1}^{+\infty} x_n$ $+\infty$. , $\sum_{n=1}^{+\infty} x_n < +\infty$.

10.5 , . (1) $0 \leq x_n \leq y_n$ $n \geq 1$.

$$\boxed{0 \leq \sum_{n=1}^{+\infty} x_n \leq \sum_{n=1}^{+\infty} y_n\,.}$$

, , $\sum_{n=1}^{+\infty} y_n$, $\sum_{n=1}^{+\infty} x_n$.
(2) $x_n \geq 0$ $y_n > 0$ $n \geq 1$ $\left(\frac{x_n}{y_n}\right)$, , . $\sum_{n=1}^{+\infty} y_n$, $\sum_{n=1}^{+\infty} x_n$.

: (1) $\sum_{n=1}^{+\infty} x_n$ $\sum_{n=1}^{+\infty} y_n$, 10.4 $0 \leq \sum_{n=1}^{+\infty} x_n \leq \sum_{n=1}^{+\infty} y_n$. $\sum_{n=1}^{+\infty} y_n$, $\sum_{n=1}^{+\infty} y_n < +\infty$, $\sum_{n=1}^{+\infty} x_n < +\infty$, , $\sum_{n=1}^{+\infty} x_n$.
, (2) (1). , u $\frac{x_n}{y_n} \leq u$, , $0 \leq x_n \leq uy_n$ $n \geq 1$. $\sum_{n=1}^{+\infty} y_n$, $\sum_{n=1}^{+\infty}(uy_n)$, $\sum_{n=1}^{+\infty} x_n$.

: (1) $\sum_{n=1}^{+\infty} \frac{2^n+3}{3^{n-1}+n}$ $\sum_{n=1}^{+\infty} \frac{2^n}{3^{n-1}}$. «» $\frac{2^n+3}{3^{n-1}+n}$ 2^n 3^{n-1} , .
$\frac{2^n+3}{3^{n-1}+n} = \frac{2^n}{3^{n-1}} \frac{1+3\cdot 2^{-n}}{1+3n3^{-n}}$, $\lim_{n\to+\infty} \frac{\frac{2^n+3}{3^{n-1}+n}}{\frac{2^n}{3^{n-1}}} = 1$. $\sum_{n=1}^{+\infty} \frac{2^n}{3^{n-1}} = 2\sum_{n=1}^{+\infty}(\frac{2}{3})^{n-1}$, $\sum_{n=1}^{+\infty} \frac{2^n+3}{3^{n-1}+n}$.

(2) $\sum_{n=1}^{+\infty} \frac{1}{n!}$.
$n \geq 2$

$$n! = 1 \cdot 2 \cdots n \geq 1 \cdot \underbrace{2 \cdots 2}_{n-1} = 2^{n-1}$$

$n! \geq 2^{n-1}$ $n = 1$. , $0 \leq \frac{1}{n!} \leq \frac{1}{2^{n-1}}$ $n \geq 1$. $\sum_{n=1}^{+\infty} \frac{1}{n!} \leq \sum_{n=1}^{+\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots = 2 < +\infty$, , $\sum_{n=1}^{+\infty} \frac{1}{n!} < +\infty$.

. :

$$\boxed{1 + \sum_{n=1}^{+\infty} \frac{1}{n!} = e.}$$

: $s_n = \frac{1}{1!} + \cdots + \frac{1}{n!}$ $t_n = \left(1 + \frac{1}{n}\right)^n$. , Newton,

$$\begin{aligned} t_n &= 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{k}\frac{1}{n^k} + \cdots + \binom{n}{n}\frac{1}{n^n} \\ &= 1 + \frac{1}{1!} + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{k-1}{n}\right) + \cdots \\ &\quad \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right). \end{aligned}$$

1, $t_n \leq 1 + \frac{1}{1!} + \cdots + \frac{1}{n!} = 1 + s_n$ $n \geq 1$. , $1 \leq k \leq n$, () k-,

$$t_n \geq 1 + \frac{1}{1!} + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{k-1}{n}\right).$$

$n \to +\infty$, $e \geq 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!} = 1 + s_k$ $k \geq 1$, , $e \geq 1 + s_n$ $n \geq 1$.
$t_n \leq 1 + s_n \leq e$ $n \geq 1$, , $\lim_{n\to+\infty} s_n = e - 1$. s_n n- $\sum_{n=1}^{+\infty} \frac{1}{n!}$, $\sum_{n=1}^{+\infty} \frac{1}{n!} = e - 1$.

′ 2 e:

e .

e $e = \frac{m}{n}$ m, n.

$$(n-1)!m = n!e = n!1 + \frac{n!}{1!} + \cdots + \frac{n!}{n!} + \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \frac{n!}{(n+3)!} + \cdots.$$

$(n-1)!m$, $n!1$, $\frac{n!}{1!}$, . . . , $\frac{n!}{n!}$, $s = \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \frac{n!}{(n+3)!} + \cdots$. ,

$$\begin{aligned} 0 < s &= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \\ &\leq \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots = \frac{1}{n+1}\frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n} \\ &< 1 \end{aligned}$$

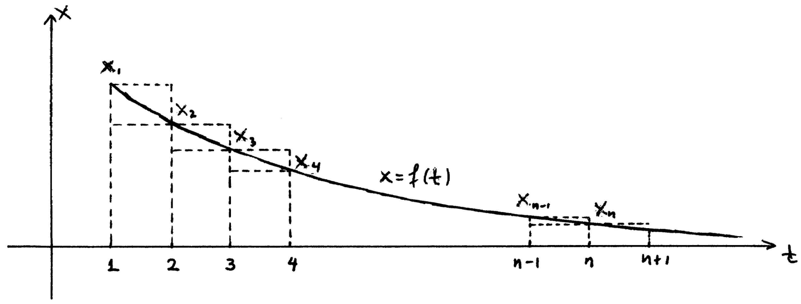
0 1.
, , 10.6 10.7 .

10.6 . (x_n) $x_n \geq 0$ $n \geq 1$. $x = f(t)$ $[1, +\infty)$: $f(n) = x_n$ $n \geq 1$. $\int_1^{+\infty} f(t)\, dt$ $+\infty$
(i) $\sum_{n=1}^{+\infty} x_n < +\infty$ $\int_1^{+\infty} f(t)\, dt < +\infty$,
(ii) $\sum_{n=1}^{+\infty} x_n = +\infty$ $\int_1^{+\infty} f(t)\, dt = +\infty$.
,

$$\boxed{\int_1^{n+1} f(t)\, dt \leq x_1 + \cdots + x_n \leq x_1 + \int_1^{n} f(t)\, dt}$$

n

$$\boxed{\int_1^{+\infty} f(t)\, dt \leq \sum_{n=1}^{+\infty} x_n \leq x_1 + \int_1^{+\infty} f(t)\, dt.}$$



Σχήμα 10.1: .

: $t \geq 1$ $n \geq t$ (, $[t]+1$). $x = f(t)$, $f(t) \geq f(n) = x_n \geq 0$. , , $f(t) \geq 0$ $t \geq 1$, $\int_1^{u_2} f(t)\, dt - \int_1^{u_1} f(t)\, dt = \int_{u_1}^{u_2} f(t)\, dt \geq 0$ u_1 u_2 $1 \leq u_1 < u_2$. $F(u) = \int_1^{u} f(t)\, dt$ u $[1, +\infty)$, , $\lim_{u \to +\infty} F(u) = \lim_{u \to +\infty} \int_1^{u} f(t)\, dt$, $\int_1^{+\infty} f(t)\, dt$, $+\infty$.

k t $[k, k+1]$ $f(k+1) \leq f(t) \leq f(k)$, $f(k+1) \leq \int_k^{k+1} f(t)\, dt \leq f(k)$, , $x_{k+1} \leq \int_k^{k+1} f(t)\, dt \leq x_k$. $k = 1, \ldots, n-1$ $k = 1, \ldots, n$ $x_2 + \cdots + x_n \leq \int_1^{n} f(t)\, dt$ $\int_1^{n+1} f(t)\, dt \leq x_1 + \cdots + x_n$, . , $\int_1^{n+1} f(t)\, dt \leq x_1 + \cdots + x_n \leq x_1 + \int_1^{n} f(t)\, dt$.

$n \to +\infty$, $\int_1^{+\infty} f(t)\, dt \leq \sum_{n=1}^{+\infty} x_n \leq x_1 + \int_1^{+\infty} f(t)\, dt$. (i) (ii) .

: (1) $\sum_{n=1}^{+\infty} \frac{1}{n^p}$, p .
, $+\infty$.

$p \leq 0$, $\frac{1}{n^p} \geq 1$ $n \geq 1$, , $\sum_{n=1}^{+\infty} \frac{1}{n^p} \geq \sum_{n=1}^{+\infty} 1 = +\infty$. $+\infty$.

$p > 0$. $(\frac{1}{n^p})$. $x = f(t) = \frac{1}{t^p}$, $[1, +\infty)$, , $f(n) = \frac{1}{n^p}$ n. $\int_1^{+\infty} \frac{1}{t^p}\, dt = +\infty$, $0 < p \leq 1$, $\int_1^{+\infty} \frac{1}{t^p}\, dt = \frac{1}{p-1} < +\infty$, $p > 1$. ,

$$\boxed{\sum_{n=1}^{+\infty} \frac{1}{n^p} \quad \begin{cases} < +\infty, & p > 1, \\ = +\infty, & p \leq 1. \end{cases}}$$

, $\sum_{n=1}^{+\infty}\frac{1}{n}$ $+\infty$ $\sum_{n=1}^{+\infty}\frac{1}{n^2}$. ,

$$\frac{1}{p-1}\leq\sum_{n=1}^{+\infty}\frac{1}{n^p}\leq 1+\frac{1}{p-1}\qquad (p>1),$$

$$\log(n+1)\leq 1+\frac{1}{2}+\cdots+\frac{1}{n}\leq 1+\log n$$

$$\frac{(n+1)^{1-p}-1}{1-p}\leq 1+\frac{1}{2^p}+\cdots+\frac{1}{n^p}\leq 1+\frac{n^{1-p}-1}{1-p}\qquad (0\leq p<1).$$

, $\sum_{n=1}^{+\infty}\frac{1}{n^p}$ $0<p\leq 1$, $\sum_{n=1}^{+\infty}x_n$ $\lim_{n\to+\infty}x_n=0$.
$\sum_{n=1}^{+\infty}\frac{1}{n^p}$ «» .

(2) $\sum_{n=1}^{+\infty}\frac{2n-1}{n^2+3n+1}$ $\sum_{n=1}^{+\infty}n^{-1}$. «» $\frac{2n-1}{n^2+3n+1}$ $2n$ n^2 , .
$\lim_{n\to+\infty}\frac{n^{-1}}{\frac{2n-1}{n^2+3n+1}}=\frac{1}{2}$, $\sum_{n=1}^{+\infty}\frac{2n-1}{n^2+3n+1}$.

(3) $\sum_{n=1}^{+\infty}\frac{\sqrt{n}+1}{2n^2+3}$ $\sum_{n=1}^{+\infty}n^{-\frac{3}{2}}$. $\lim_{n\to+\infty}\frac{\frac{\sqrt{n}+1}{2n^2+3}}{n^{-\frac{3}{2}}}=\frac{1}{2}$, $\sum_{n=1}^{+\infty}\frac{\sqrt{n}+1}{2n^2+3}$.

10.7 ***Cauchy***. (x_n) $x_n\geq 0$ $n\geq 1$.
(i) $\sum_{n=1}^{+\infty}x_n<+\infty$ $\sum_{k=1}^{+\infty}2^k x_{2^k}<+\infty$,
(ii) $\sum_{n=1}^{+\infty}x_n=+\infty$ $\sum_{k=1}^{+\infty}2^k x_{2^k}=+\infty$.

: $\sum_{n=1}^{+\infty}x_n$ $\sum_{k=1}^{+\infty}2^k x_{2^k}$, .
$n\geq 2$ 2, $2^k\leq n<2^{k+1}$. (x_n) ,

$$\begin{aligned} x_1+\cdots+x_n &= x_1+(x_2+x_3)+(x_4+x_5+x_6+x_7)+\cdots\\ &\quad\cdots+(x_{2^{k-1}}+\cdots+x_{2^k-1})+(x_{2^k}+\cdots+x_n)\\ &\leq x_1+2x_2+4x_4+\cdots+2^{k-1}x_{2^{k-1}}+2^k x_{2^k}\leq x_1+\sum_{k=1}^{+\infty}2^k x_{2^k}\,. \end{aligned}$$

$n\to+\infty$, $\sum_{n=1}^{+\infty}x_n\leq x_1+\sum_{k=1}^{+\infty}2^k x_{2^k}$.
,

$$\begin{aligned} 2x_1+2x_2+4x_4+\cdots+2^k x_{2^k} &\leq 2x_1+2x_2+2(x_3+x_4)+\cdots+2(x_{2^{k-1}+1}+\cdots+x_{2^k})\\ &= 2(x_1+x_2+\cdots+x_{2^k})\leq 2\sum_{n=1}^{+\infty}x_n\,. \end{aligned}$$

$k\to+\infty$, $2x_1+\sum_{k=1}^{+\infty}2^k x_{2^k}\leq 2\sum_{n=1}^{+\infty}x_n$.

$$x_1+\frac{1}{2}\sum_{k=1}^{+\infty}2^k x_{2^k}\leq\sum_{n=1}^{+\infty}x_n\leq x_1+\sum_{k=1}^{+\infty}2^k x_{2^k}\,.$$

$\sum_{k=1}^{+\infty}2^k x_{2^k}$ $\sum_{n=1}^{+\infty}x_n$ $+\infty$.

: $\sum_{n=1}^{+\infty}\frac{1}{n^p}$.

′ $p \le 0$: $\sum_{n=1}^{+\infty} \frac{1}{n^p} \ge \sum_{n=1}^{+\infty} 1 = +\infty$, $+\infty$.

$p > 0$, $(\frac{1}{n^p})$. $\sum_{k=1}^{+\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=1}^{+\infty} \big(\frac{1}{2^{p-1}}\big)^k$. $\frac{1}{2^{p-1}}$, $\frac{1}{2^{p-1}} < 1$, , $p > 1$ $+\infty$, $\frac{1}{2^{p-1}} \ge 1$, , $0 < p \le 1$.

$\sum_{n=1}^{+\infty} \frac{1}{n^p}$, $p > 1$, $+\infty$, $p \le 1$.

.

1. 10.5.

$$\sum_{n=1}^{+\infty} \frac{n\sqrt{n}+2n+1}{2n^2+1}, \quad \sum_{n=1}^{+\infty} \frac{2n^2+3n+1}{n^4-n^2+4}, \quad \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n(n+1)}},$$

$$\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}, \quad \sum_{n=1}^{+\infty} (\sqrt{1+n^2}-n), \quad \sum_{n=1}^{+\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n},$$

$$\sum_{n=1}^{+\infty} \frac{1}{n^{1+\frac{1}{n}}}, \quad \sum_{n=1}^{+\infty} \log\Big(1+\frac{1}{n^2}\Big), \quad \sum_{n=1}^{+\infty} \frac{1}{n}\sin\frac{1}{n},$$

$$\sum_{n=1}^{+\infty} \sin\frac{1}{n}, \quad \sum_{n=1}^{+\infty} \Big(1-\cos\frac{1}{n}\Big), \quad \sum_{n=1}^{+\infty} n\Big(1-\cos\frac{1}{n}\Big).$$

(: $\sum_{n=1}^{+\infty} \frac{1}{n^p}$. $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$, $\lim_{x\to 0} \frac{\sin x}{x} = 1$, $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.)

2.

$$\sum_{n=1}^{+\infty} n^a\Big(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\Big), \qquad \sum_{n=1}^{+\infty} n^a\big(\sqrt{n+1}-2\sqrt{n}+\sqrt{n-1}\big),$$

$$\sum_{n=2}^{+\infty} \frac{1}{n^a-n^b} \quad (0<b<a), \qquad \sum_{n=1}^{+\infty} \frac{1}{a^n-b^n} \quad (0<b<a).$$

, $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ $\sum_{n=1}^{+\infty} p^n$ p. a, b .

3. $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = 1$ (− 4).

$\frac{1}{n^2} \le \frac{2}{n(n+1)}$ n $\lim_{n\to+\infty} \frac{\frac{1}{n^2}}{\frac{1}{n(n+1)}} = 1$, 10.5, $\sum_{n=1}^{+\infty} \frac{1}{n^2}$.

4. . . $+\infty$.

, Cauchy.

$$\sum_{n=1}^{+\infty} \frac{1}{n^2+1}, \quad \sum_{n=1}^{+\infty} \frac{n}{n^2+1}, \quad \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n(n+1)}}, \quad \sum_{n=1}^{+\infty} \frac{1}{(n+1)\sqrt{n}},$$

$$\sum_{n=1}^{+\infty} ne^{-n}, \quad \sum_{n=1}^{+\infty} \frac{e^n}{1+e^{2n}}, \quad \sum_{n=2}^{+\infty} \frac{1}{n\log n}, \quad \sum_{n=2}^{+\infty} \frac{1}{n(\log n)^2},$$

$$\sum_{n=3}^{+\infty} \frac{1}{n\log n \log(\log n)}\,, \quad \sum_{n=3}^{+\infty} \frac{1}{n\log n(\log(\log n))^2}\,.$$

5. Cauchy

$$\sum_{n=2}^{+\infty} \frac{1}{n(\log n)^p}\,, \qquad \sum_{n=3}^{+\infty} \frac{1}{n\log n\big(\log(\log n)\big)^p}$$

p.

6. $\lim_{p\to 1+}(p-1)\sum_{n=1}^{+\infty}\frac{1}{n^p} = 1.$

$\lim_{n\to+\infty}\frac{1}{\log n}\big(1+\frac{1}{2}+\cdots+\frac{1}{n}\big) = 1.$

$0 \le p < 1, \quad \lim_{n\to+\infty} n^{p-1}\big(1+\frac{1}{2^p}+\cdots+\frac{1}{n^p}\big) = \frac{1}{1-p}\,.$

7. $\log\frac{n+1}{m} \le \frac{1}{m}+\frac{1}{m+1}+\cdots+\frac{1}{n-1}+\frac{1}{n} \le \frac{1}{m}+\log\frac{n}{m}\,.$

$\lim_{n\to+\infty}\big(\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}\big) = \log 2$, , p, $\lim_{n\to+\infty}\big(\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{pn-1}+\frac{1}{pn}\big) = \log p.$

8. $p > 1.$ $\frac{1}{(p-1)n^{p-1}} \le \frac{1}{n^p}+\frac{1}{(n+1)^p}+\cdots \le \frac{1}{n^p}+\frac{1}{(p-1)n^{p-1}}\,.$

, $\lim_{n\to+\infty} n^{p-1}\big(\frac{1}{n^p}+\frac{1}{(n+1)^p}+\cdots\big) = \frac{1}{p-1}\,.$

9. $x_n \ge 0$ $n.$ $\sum_{n=1}^{+\infty} x_n$, $\sum_{n=1}^{+\infty}\sqrt{x_n x_{n+1}}$.

(: $ab \le \frac{1}{2}a^2+\frac{1}{2}b^2$.)

, , (x_n) , .

10. $x_n > 0$ $n.$ $\sum_{n=1}^{+\infty} x_n$, $\sum_{n=1}^{+\infty}\frac{\sqrt{x_n}}{n}$.

(: $ab \le \frac{1}{2}a^2+\frac{1}{2}b^2$.)

11. $x_n > 0$ $n.$ $\sum_{n=1}^{+\infty} x_n$, $\sum_{n=1}^{+\infty} {x_n}^2$, $\sum_{n=1}^{+\infty}\frac{x_n}{1+x_n}$ $\sum_{n=1}^{+\infty}\frac{{x_n}^2}{1+{x_n}^2}$.

12. (*) (x_n) . $\sum_{n=1}^{+\infty} x_n < +\infty,$ $\lim_{n\to+\infty} nx_n = 0.$

(: $\frac{n}{2}x_n \le x_{[\frac{n}{2}]+1}+\cdots+x_n$.)

13. (*) (x_n) , $\lim_{n\to+\infty} x_n = 0$ $x_n - 2x_{n+1} + x_{n+2} \ge 0$ $n \ge 1.$

(1) $\sum_{n=1}^{+\infty}(x_n - x_{n+1}) = x_1$.

(2) $\lim_{n\to+\infty} n(x_n - x_{n+1}) = 0.$

(: .)

(3) $\sum_{n=1}^{+\infty} n(x_n - 2x_{n+1} + x_{n+2}) = x_1$.

(: $\sum_{k=1}^{n} k(x_k - 2x_{k+1} + x_{k+2}) = x_1 - (m+1)(x_{m+1} - x_{m+2}) - x_{m+2}$.)

10.3 p- .

. p- .

$0,1,\ldots,9$. , p ≥ 2, $0,1,\ldots,p-1$ p- .

: $0,1$, $0,1,2$ $0,1,\ldots,15$.

10.8 $p\geq 2$. x

$$\boxed{x = X_N p^N + X_{N-1}p^{N-1} + \cdots + X_1 p + X_0}$$

$X_N,\ldots,X_0$ $\{0,1,\ldots,p-1\}$, p- , $X_N \neq 0$.

: $N=[\log_p x]$. $x\geq 1$ $p>1$, $\log_p x \geq 0$, , $N\geq 0$. N $p^N \leq x < p^{N+1}$, $1\leq \frac{x}{p^N} < p$. , $X_N = \left[\frac{x}{p^N}\right]$, , , $1\leq X_N \leq p-1$. X_N

$$X_N p^N \leq x < X_N p^N + p^N ,$$

$0\leq \frac{x-X_Np^N}{p^{N-1}} < p$.

, $X_{N-1} = \left[\frac{x-X_Np^N}{p^{N-1}}\right]$, , , $0\leq X_{N-1}\leq p-1$. X_{N-1}

$$X_Np^N + X_{N-1}p^{N-1} \leq x < X_Np^N + X_{N-1}p^{N-1} + p^{N-1},$$

$0\leq \frac{x-X_Np^N-X_{N-1}p^{N-1}}{p^{N-2}} < p$.

, $X_{N-2} = \left[\frac{x-X_Np^N-X_{N-1}p^{N-1}}{p^{N-2}}\right]$, , , $0\leq X_{N-2}\leq p-1$. X_{N-2}

$$X_Np^N + X_{N-1}p^{N-1} + X_{N-2}p^{N-2} \leq x < X_Np^N + X_{N-1}p^{N-1} + X_{N-2}p^{N-2} + p^{N-2},$$

$0\leq \frac{x-X_Np^N-X_{N-1}p^{N-1}-X_{N-2}p^{N-2}}{p^{N-3}} < p$.

$X_0 = \left[\frac{x-X_Np^N-\cdots-X_1p}{p^0}\right]$, , , $0\leq X_0\leq p-1$. X_0

$$X_Np^N+\cdots+X_1p+X_0 \leq x < X_Np^N+\cdots+X_1p+X_0+1$$

, $X_Np^N+\cdots+X_1p+X_0$ x ,

$$X_Np^N+\cdots+X_1p+X_0 = x.$$

$Y_M,\ldots,Y_0$ $\{0,1,\ldots,p-1\}$ $Y_M\neq 0$ $x=Y_Mp^M+Y_{M-1}p^{M-1}+\cdots+Y_1p+Y_0$. $N<M$,

$$x = X_Np^N+\cdots+X_1p+X_0 \leq (p-1)p^N+\cdots+(p-1)p+(p-1) = p^{N+1}-1$$

$$x = Y_Mp^M+Y_{M-1}p^{M-1}+\cdots+Y_1p+Y_0 \geq 1p^M+0p^{M-1}+\cdots+0p+0 = p^M \geq p^{N+1}$$

. $N>M$, $N=M$. $x=Y_Np^N+Y_{N-1}p^{N-1}+\cdots+Y_1p+Y_0$ $Y_N\neq 0$.

, , n $X_n\neq Y_n$ n_0 n , $X_{n_0}\neq Y_{n_0}$ $X_n=Y_n$ $n=N,\ldots,n_0+1$. , , $X_{n_0}<Y_{n_0}$, $X_{n_0}+1\leq Y_{n_0}$.

$$X_Np^N+\cdots+X_1p+X_0 = x = Y_Np^N+\cdots+Y_1p+Y_0$$

$$X_{n_0}p^{n_0}+\cdots+X_1p+X_0 = Y_{n_0}p^{n_0}+\cdots+Y_1p+Y_0 .$$

,

$$\begin{aligned} X_{n_0}p^{n_0}+\cdots+X_1p+X_0 &\leq X_{n_0}p^{n_0}+(p-1)p^{n_0-1}+\cdots+(p-1)p+(p-1) \\ &= X_{n_0}p^{n_0}+p^{n_0}-1 \end{aligned}$$

$$\begin{aligned} Y_{n_0}p^{n_0} + \cdots + Y_1 p + Y_0 &\geq Y_{n_0}p^{n_0} + 0p^{n_0-1} + \cdots + 0p + 0 \\ &= Y_{n_0}p^{n_0} \\ &\geq X_{n_0}p^{n_0} + p^{n_0} \end{aligned}$$

. $X_{n_0} > Y_{n_0}$, $Y_N, \ldots, Y_0$ $X_N, \ldots, X_0$, .

$X_N, \ldots, X_0$ $x = X_N p^N + X_{N-1}p^{N-1} + \cdots + X_1 p + X_0$ p- x. $X_N p^N + X_{N-1}p^{N-1} + \cdots + X_1 p + X_0$ p- x $\overline{X_N X_{N-1} \ldots X_1 X_0}^p$,

$$x = \overline{X_N X_{N-1} \ldots X_1 X_0}^p .$$

: (1) , $p = 1p + 0$, p- p $1p + 0 = \overline{10}^p$. , p- p^2 $1p^2 + 0p + 0 = \overline{100}^p$.

(2) p , $9+1$, $X_N X_{N-1} \ldots X_1 X_0$ $\overline{X_N X_{N-1} \ldots X_1 X_0}^p$ $x = X_N p^N + X_{N-1}p^{N-1} + \cdots + X_1 p + X_0$.
′ – – 10. , , 10 , $9+1$.
, $10+1$ $1 \cdot 10 + 1$, 11. $10+2$ $1 \cdot 10 + 2$, 12. , , .

(3) $25 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2 + 1$, 25 $\overline{11001}^2$.

(4) $735 = 2 \cdot 16^2 + 13 \cdot 16 + 15$, 735 $\overline{2\,13\,15}^{16}$ – .

p- x 10.8. , , . $x = a_1 p + X_0$, a_1 X_0 $0 \leq X_0 \leq p-1$. $0 \leq a_1 < x$. $a_1 = 0$, , $a_1 > 0$, $a_1 = a_2 p + X_1$, a_2 X_1 $0 \leq X_1 \leq p-1$. $0 \leq a_2 < a_1$. $a_2 = 0$, , $a_2 > 0$, $a_2 = a_3 p + X_2$, a_3 X_2 $0 \leq X_2 \leq p-1$. $0 \leq a_3 < a_2$. , $a_1, a_2, a_3, \ldots$, 0 . , , $N \geq 0$ $a_{N+1} = 0$,

$$x = a_1 p + X_0$$

$$a_1 = a_2 p + X_1$$

$$\cdots\cdots\cdots$$

$$a_{N-1} = a_N p + X_{N-1}$$

$$a_N = 0p + X_N = X_N .$$

$$\begin{aligned} x &= a_1 p + X_0 \\ &= a_2 p^2 + X_1 p + X_0 \\ &= \cdots\cdots\cdots\cdots \\ &= a_{N-1}p^{N-1} + X_{N-2}p^{N-2} + \cdots + X_1 p + X_0 \\ &= a_N p^N + X_{N-1}p^{N-1} + X_{N-2}p^{N-2} + \cdots + X_1 p + X_0 \\ &= X_N p^N + X_{N-1}p^{N-1} + X_{N-2}p^{N-2} + \cdots + X_1 p + X_0 . \end{aligned}$$

: (1) 28 : $28 = 14 \cdot 2 + 0$, $14 = 7 \cdot 2 + 0$, $7 = 3 \cdot 2 + 1$, $3 = 1 \cdot 2 + 1$ $1 = 0 \cdot 2 + 1$. $28 = \overline{11100}^2$.

(2) 32137 : $32137 = 2008 \cdot 16 + 9$, $2008 = 125 \cdot 16 + 8$, $125 = 7 \cdot 16 + 13$

$7 = 0 \cdot 16 + 7$. $32137 = \overline{7\,13\,8\,9}^{16}$ – .

(3) 12 : $12 = 0 \cdot 15 + 12$. $12 = \overline{12}^{15}$ – .

. p- $[0,1)$.

, $0,25$

$$2 \cdot 10^{-1} + 5 \cdot 10^{-2} = \frac{2}{10} + \frac{5}{100} = \frac{20}{100} + \frac{5}{100} = \frac{25}{100} = \frac{1}{4}\,.$$

, $0,5403$

$$5 \cdot 10^{-1} + 4 \cdot 10^{-2} + 0 \cdot 10^{-3} + 3 \cdot 10^{-4} = \frac{5}{10} + \frac{4}{100} + \frac{0}{1000} + \frac{3}{10000} = \frac{5403}{10000}\,.$$

$0,25$ $0,5403$ $0,25000\ldots$ $0,5403000\ldots$, 0 . $\frac{1}{4}$ $\frac{5403}{10000}$.
, , $\frac{3}{7}$ $0,42857\ldots$ ′ 0 .
: $\frac{3}{7}$ $0,42857\ldots$; , ′ ,

$$4 \cdot 10^{-1} + 2 \cdot 10^{-2} + 8 \cdot 10^{-3} + 5 \cdot 10^{-4} + 7 \cdot 10^{-5} + \cdots$$

$\frac{3}{7}$. . , , ,

$$4 \cdot 10^{-1} + 2 \cdot 10^{-2} + 8 \cdot 10^{-3} + 5 \cdot 10^{-4} + 7 \cdot 10^{-5} + \cdots = \frac{3}{7}\,,$$

$4 \cdot 10^{-1} + 2 \cdot 10^{-2} + 8 \cdot 10^{-3} + 5 \cdot 10^{-4} + 7 \cdot 10^{-5} + \cdots$ $\frac{3}{7}$, , $\frac{3}{7}$.
10.9 $[0,1)$ $0, \ldots$. , 10.9 p-, p ≥ 2 $0, 1, \ldots, p-1$. ,
, :

$$\sum_{n=1}^{+\infty} \frac{p-1}{p^n} = \frac{p-1}{p} \sum_{n=1}^{+\infty} \left(\frac{1}{p}\right)^{n-1} = \frac{p-1}{p} \frac{1}{1-\frac{1}{p}} = 1$$

, , m

$$\sum_{n=m}^{+\infty} \frac{p-1}{p^n} = \frac{p-1}{p^m} \sum_{n=m}^{+\infty} \left(\frac{1}{p}\right)^{n-m} = \frac{p-1}{p^m} \sum_{n=1}^{+\infty} \left(\frac{1}{p}\right)^{n-1} = \frac{p-1}{p^m} \frac{1}{1-\frac{1}{p}} = \frac{1}{p^{m-1}}\,.$$

10.9 $p \geq 2$.
(1) p- (x_n) x_n $p-1$. $\sum_{n=1}^{+\infty} \frac{x_n}{p^n}$ $[0,1)$.
(2) x $[0,1)$ p- (x_n) x_n $p-1$

$$\boxed{x = \sum_{n=1}^{+\infty} \frac{x_n}{p^n} = x_1 p^{-1} + x_2 p^{-2} + x_3 p^{-3} + \cdots\,.}$$

: (1) $\sum_{n=1}^{+\infty} \frac{x_n}{p^n}$, , , $0 \leq x_n \leq p-1$ n,

$$0 \leq \sum_{n=1}^{+\infty} \frac{x_n}{p^n} \leq \sum_{n=1}^{+\infty} \frac{p-1}{p^n} = 1.$$

$\sum_{n=1}^{+\infty} \frac{x_n}{p^n}$, x, : $0 \le x \le 1$. $0 \le x < 1$, , , m $x_m \ne p-1$, , $x_m \le p-2$.

$$\begin{aligned} x &= \sum_{n=1}^{+\infty} \frac{x_n}{p^n} = \sum_{n=1}^{m-1} \frac{x_n}{p^n} + \frac{x_m}{p^m} + \sum_{n=m+1}^{+\infty} \frac{x_n}{p^n} \\ &\le \sum_{n=1}^{m-1} \frac{p-1}{p^n} + \frac{p-2}{p^m} + \sum_{n=m+1}^{+\infty} \frac{p-1}{p^n} = \sum_{n=1}^{+\infty} \frac{p-1}{p^n} - \frac{1}{p^m} = 1 - \frac{1}{p^m} \\ &< 1. \end{aligned}$$

(2) $0 \le x < 1$. $x_1 = [px]$, $x_1 \le px < x_1 + 1$, ,

$$\frac{x_1}{p} \le x < \frac{x_1}{p} + \frac{1}{p}.$$

$0 \le px < p$, x_1 $\{0, 1, \ldots, p-1\}$. $s_1 = \frac{x_1}{p}$ $x_2 = [p^2(x - s_1)]$. $x_2 \le p^2(x - s_1) < x_2 + 1$,

$$\frac{x_1}{p} + \frac{x_2}{p^2} \le x < \frac{x_1}{p} + \frac{x_2}{p^2} + \frac{1}{p^2}.$$

$0 \le p^2(x - s_1) < p$, x_2 $\{0, 1, \ldots, p-1\}$. $s_2 = \frac{x_1}{p} + \frac{x_2}{p^2}$ $x_3 = [p^3(x - s_2)]$. $x_3 \le p^3(x - s_2) < x_3 + 1$,

$$\frac{x_1}{p} + \frac{x_2}{p^2} + \frac{x_3}{p^3} \le x < \frac{x_1}{p} + \frac{x_2}{p^2} + \frac{x_3}{p^3} + \frac{1}{p^3}.$$

$0 \le p^3(x - s_2) < p$, x_3 $\{0, 1, \ldots, p-1\}$.

x_n , , n. , $x_1, \ldots, x_n$ $\{0, 1, \ldots, p-1\}$

$$\frac{x_1}{p} + \cdots + \frac{x_n}{p^n} \le x < \frac{x_1}{p} + \cdots + \frac{x_n}{p^n} + \frac{1}{p^n}.$$

$s_n = \frac{x_1}{p} + \cdots + \frac{x_n}{p^n}$ $x_{n+1} = [p^{n+1}(x - s_n)]$. $x_{n+1} \le p^{n+1}(x - s_n) < x_{n+1} + 1$,

$$\frac{x_1}{p} + \cdots + \frac{x_n}{p^n} + \frac{x_{n+1}}{p^{n+1}} \le x < \frac{x_1}{p} + \cdots + \frac{x_n}{p^n} + \frac{x_{n+1}}{p^{n+1}} + \frac{1}{p^{n+1}}.$$

$0 \le p^{n+1}(x - s_n) < p$, x_{n+1} $\{0, 1, \ldots, p-1\}$.

, , p- (x_n) $s_n = \frac{x_1}{p} + \cdots + \frac{x_n}{p^n}$ $\sum_{n=1}^{+\infty} \frac{x_n}{p^n}$ $s_n \le x < s_n + \frac{1}{p^n}$. $x - \frac{1}{p^n} < s_n \le x$, , $\lim_{n \to +\infty} s_n = x$. ,

$$\sum_{n=1}^{+\infty} \frac{x_n}{p^n} = x.$$

, , x_n $p-1$, m $x_n = p-1$ $n \ge m$. $m = 1$,

$$x = \sum_{n=1}^{+\infty} \frac{x_n}{p^n} = \sum_{n=1}^{+\infty} \frac{p-1}{p^n} = 1,$$

, , $m \ge 2$,

$$x = \sum_{n=1}^{+\infty} \frac{x_n}{p^n} = \frac{x_1}{p} + \cdots + \frac{x_{m-1}}{p^{m-1}} + \sum_{n=m}^{+\infty} \frac{x_n}{p^n} = s_{m-1} + \sum_{n=m}^{+\infty} \frac{p-1}{p^n} = s_{m-1} + \frac{1}{p^{m-1}},$$

, , .

, p- (y_n) , (x_n), y_n $p-1$

$$x = \sum_{n=1}^{+\infty} \frac{y_n}{p^n}.$$

(x_n) (y_n) , n $x_n \neq y_n$. n_0 n , $x_{n_0} \neq y_{n_0}$ $x_n = y_n$ $n = 1, \ldots, n_0 - 1$. ,
, $x_{n_0} < y_{n_0}$, , $x_{n_0} + 1 \leq y_{n_0}$. ,

$$\sum_{n=1}^{+\infty} \frac{x_n}{p^n} = x = \sum_{n=1}^{+\infty} \frac{y_n}{p^n}$$

$$\sum_{n=n_0}^{+\infty} \frac{x_n}{p^n} = \sum_{n=n_0}^{+\infty} \frac{y_n}{p^n} .$$

, $m \geq n_0 + 1$ $x_m \neq p - 1$, , $x_m \leq p - 2$.

$$\begin{aligned} \sum_{n=n_0}^{+\infty} \frac{x_n}{p^n} &= \frac{x_{n_0}}{p^{n_0}} + \sum_{n=n_0+1}^{m-1} \frac{x_n}{p^n} + \frac{x_m}{p^m} + \sum_{n=m+1}^{+\infty} \frac{x_n}{p^n} \\ &\leq \frac{x_{n_0}}{p^{n_0}} + \sum_{n=n_0+1}^{m-1} \frac{p-1}{p^n} + \frac{p-2}{p^m} + \sum_{n=m+1}^{+\infty} \frac{p-1}{p^n} \\ &= \frac{x_{n_0}}{p^{n_0}} + \sum_{n=n_0+1}^{+\infty} \frac{p-1}{p^n} - \frac{1}{p^m} \\ &= \frac{x_{n_0}}{p^{n_0}} + \frac{1}{p^{n_0}} - \frac{1}{p^m} \end{aligned}$$

$$\sum_{n=n_0}^{+\infty} \frac{y_n}{p^n} = \frac{y_{n_0}}{p^{n_0}} + \sum_{n=n_0+1}^{+\infty} \frac{y_n}{p^n} \geq \frac{y_{n_0}}{p^{n_0}} \geq \frac{x_{n_0}}{p^{n_0}} + \frac{1}{p^{n_0}}$$

. $x_{n_0} > y_{n_0}$, (x_n) (y_n) .

(x_n) p- x_n $p - 1$

$$x = \sum_{n=1}^{+\infty} \frac{x_n}{p^n} ,$$

(x_n) p- x , , x_n p- x. , $\sum_{n=1}^{+\infty} \frac{x_n}{p^n}$ p- x $\overline{0, x_1 x_2 x_3 \, \ldots}^{\,p}$,

$$x = \overline{0, x_1 x_2 x_3 \, \ldots}^{\,p} .$$

$p = 10$ $x = 0, x_1 x_2 x_3 \, \ldots$ $x = \overline{0, x_1 x_2 x_3 \, \ldots}^{\,10}$.
, , 10.9

$[0, 1)$ p- $\overline{0, x_1 x_2 x_3 \, \ldots}^{\,p}$ **,**
p-
$p - 1$**.**

$x = \sum_{n=1}^{+\infty} \frac{x_n}{p^n}$ p- x – , , $[0, 1)$ –

$$s_n = \frac{x_1}{p} + \frac{x_2}{p^2} + \cdots + \frac{x_n}{p^n}$$

$\sum_{n=1}^{+\infty} \frac{x_n}{p^n}$

$$\boxed{s_n \leq x < s_n + \frac{1}{p^n}\ .}$$

10.9, . ′ ,

$$x = \sum_{n=1}^{+\infty} \frac{x_n}{p^n} = \sum_{n=1}^{m} \frac{x_n}{p^n} + \sum_{n=m+1}^{+\infty} \frac{x_n}{p^n} \geq \sum_{n=1}^{m} \frac{x_n}{p^n} = s_m\,.$$

, x_n $p-1$, $k \geq m+1$ $x_k \leq p-2$,

$$\begin{aligned} x = \sum_{n=1}^{+\infty} \frac{x_n}{p^n} &= s_m + \sum_{n=m+1}^{+\infty} \frac{x_n}{p^n} \\ &= s_m + \sum_{n=m+1}^{k-1} \frac{x_n}{p^n} + \frac{x_k}{p^k} + \sum_{n=k+1}^{+\infty} \frac{x_n}{p^n} \\ &\leq s_m + \sum_{n=m+1}^{k-1} \frac{p-1}{p^n} + \frac{p-2}{p^k} + \sum_{n=k+1}^{+\infty} \frac{p-1}{p^n} \\ &= s_m + \sum_{n=m+1}^{+\infty} \frac{p-1}{p^n} - \frac{1}{p^k} = s_m + \frac{1}{p^m} - \frac{1}{p^k} \\ &< s_m + \frac{1}{p^m}\,. \end{aligned}$$

$s_n \leq x < s_n + \frac{1}{p^n}$ $x - \frac{1}{p^n} < s_n \leq x$ s_n n- p- x.

: $\frac{3}{7}$ $0,42857\ldots$ $\frac{3}{7}$.

$$0,4 \leq \frac{3}{7} < 0,5\,, \qquad 0,42 \leq \frac{3}{7} < 0,43\,, \qquad 0,428 \leq \frac{3}{7} < 0,429\,,$$

$$0,4285 \leq \frac{3}{7} < 0,4286\,, \qquad 0,42857 \leq \frac{3}{7} < 0,42858\,,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$0,4,\ 0,42,\ 0,428,\ 0,4285,\ 0,42857,\ \ldots$ $\frac{3}{7}$ $4,2,8,5,7,\ \ldots$ $\frac{3}{7}$.

10.9 p- $\overline{0,x_1x_2x_3\ldots}^{\,p} = \sum_{n=1}^{+\infty} \frac{x_n}{p^n}$ x $[0,1)$. p- . $x_1 = [px]$, n, x_1 , … , x_n , $x_{n+1} = [p^{n+1}(x - s_n)]$, $s_n = \frac{x_1}{p} + \cdots + \frac{x_n}{p^n}$. «»

$$x_1 = [px], \qquad x_{n+1} = [p^{n+1}(x - s_n)] \quad (n \geq 1),$$

$(n+1)$- .

: (1) $\frac{13}{16}$.

$$x_1 = \left[10 \cdot \frac{13}{16}\right] = \left[\frac{65}{8}\right] = 8, \quad s_1 = \frac{8}{10} = \frac{4}{5}\,,$$

$$x_2 = \left[10^2\left(\frac{13}{16} - \frac{4}{5}\right)\right] = \left[\frac{5}{4}\right] = 1, \quad s_2 = s_1 + \frac{1}{10^2} = \frac{81}{100}\,,$$
$$x_3 = \left[10^3\left(\frac{13}{16} - \frac{81}{100}\right)\right] = \left[\frac{5}{2}\right] = 2, \quad s_3 = s_2 + \frac{2}{10^3} = \frac{203}{250}\,,$$
$$x_4 = \left[10^4\left(\frac{13}{16} - \frac{203}{250}\right)\right] = [5] = 5, \quad s_4 = s_3 + \frac{5}{10^4} = \frac{13}{16}\,,$$
$$x_5 = \left[10^5\left(\frac{13}{16} - \frac{13}{16}\right)\right] = [0] = 0, \quad s_5 = s_4 + \frac{0}{10^5} = \frac{13}{16}\,,$$
$$x_6 = \left[10^6\left(\frac{13}{16} - \frac{13}{16}\right)\right] = [0] = 0, \quad s_6 = s_5 + \frac{0}{10^6} = \frac{13}{16}$$

′. $\frac{13}{16}$ $0,8125000\ldots$. s_4 $\frac{13}{16}$ $x_5\,, x_6\,, \ldots$ 0.

(2) $\frac{1}{\sqrt{2}}$.

$$x_1 = \left[10 \cdot \frac{1}{\sqrt{2}}\right] = 7, \quad s_1 = \frac{7}{10}\,,$$
$$x_2 = \left[10^2\left(\frac{1}{\sqrt{2}} - \frac{7}{10}\right)\right] = 0, \quad s_2 = s_1 + \frac{0}{10^2} = \frac{7}{10}\,,$$
$$x_3 = \left[10^3\left(\frac{1}{\sqrt{2}} - \frac{7}{10}\right)\right] = 7, \quad s_3 = s_2 + \frac{7}{10^3} = \frac{707}{1000}\,,$$
$$x_4 = \left[10^4\left(\frac{1}{\sqrt{2}} - \frac{707}{1000}\right)\right] = 1, \quad s_4 = s_3 + \frac{1}{10^4} = \frac{7071}{10000}\,,$$
$$x_5 = \left[10^5\left(\frac{1}{\sqrt{2}} - \frac{7071}{10000}\right)\right] = 0, \quad s_5 = s_4 + \frac{0}{10^5} = \frac{7071}{10000}\,,$$
$$x_6 = \left[10^6\left(\frac{1}{\sqrt{2}} - \frac{7071}{10000}\right)\right] = 6, \quad s_6 = s_5 + \frac{6}{10^6} = \frac{707106}{1000000}$$

. $\frac{1}{\sqrt{2}}$ $0,707106\ldots$.

(3) $\frac{3}{5}$.

$$x_1 = \left[2 \cdot \frac{3}{5}\right] = 1, \quad s_1 = \frac{1}{2}\,,$$
$$x_2 = \left[2^2\left(\frac{3}{5} - \frac{1}{2}\right)\right] = 0, \quad s_2 = s_1 + \frac{0}{2^2} = \frac{1}{2}\,,$$
$$x_3 = \left[2^3\left(\frac{3}{5} - \frac{1}{2}\right)\right] = 0, \quad s_3 = s_2 + \frac{0}{2^3} = \frac{1}{2}\,,$$
$$x_4 = \left[2^4\left(\frac{3}{5} - \frac{1}{2}\right)\right] = 1, \quad s_4 = s_3 + \frac{1}{2^4} = \frac{9}{16}\,,$$
$$x_5 = \left[2^5\left(\frac{3}{5} - \frac{9}{16}\right)\right] = 1, \quad s_5 = s_4 + \frac{1}{2^5} = \frac{19}{32}\,,$$
$$x_6 = \left[2^6\left(\frac{3}{5} - \frac{19}{32}\right)\right] = 0, \quad s_6 = s_5 + \frac{0}{2^6} = \frac{19}{32}$$

′. $\frac{3}{5}$ $0,100110\ldots$.

. p- .

p- $[0,1)$. $x \geq 1$, $x = [x] + (x - [x])$, $[x]$ x, , $x - [x]$ $[0,1)$. p- : $\overline{X_N \ldots X_0}^p$ $[x]$ $\overline{0, x_1 x_2 \ldots}^p$ $x - [x]$. $[x] = X_N p^N + \cdots + X_1 p + X_0$ $x - [x] = x_1 p^{-1} + x_2 p^{-2} + \cdots$,

$$x = X_N p^N + \cdots + X_1 p + X_0 + x_1 p^{-1} + x_2 p^{-2} + \cdots .$$

p- x $\overline{X_N \ldots X_0, x_1 x_2 \ldots}^p$

$$x = \overline{X_N \ldots X_0, x_1 x_2 \ldots}^p .$$

n- p- x

$$s_n = [x] + \frac{x_1}{p} + \cdots + \frac{x_n}{p^n} = X_N p^N + \cdots + X_1 p + X_0 + \frac{x_1}{p} + \cdots + \frac{x_n}{p^n}$$

, ,

$$s_n \leq x < s_n + \frac{1}{p^n}$$

n.

p- $[0,1)$, , , ≥ 0, p- p- $p-1$. p- (x_n) x_n $p-1$, m $x_n = p-1$ $n \geq m$ n_0 m. $n_0 = 1$, $x_n = p-1$ n, $\sum_{n=1}^{+\infty} \frac{x_n}{p^n}$

$$\sum_{n=1}^{+\infty} \frac{x_n}{p^n} = \sum_{n=1}^{+\infty} \frac{p-1}{p^n} = 1.$$

$\overline{0, p-1\, p-1\, p-1 \ldots}^p$, p- 1, p- $\overline{1,000\ldots}^p$.
$n_0 \geq 2$ $x_n = p-1$ $n \geq n_0$ $x_{n_0-1} \leq p-2$. $\sum_{n=1}^{+\infty} \frac{x_n}{p^n}$

$$\begin{aligned}
x = \sum_{n=1}^{+\infty} \frac{x_n}{p^n} &= \sum_{n=1}^{n_0-1} \frac{x_n}{p^n} + \sum_{n=n_0}^{+\infty} \frac{x_n}{p^n} = \sum_{n=1}^{n_0-1} \frac{x_n}{p^n} + \sum_{n=n_0}^{+\infty} \frac{p-1}{p^n} \\
&= \sum_{n=1}^{n_0-1} \frac{x_n}{p^n} + \frac{1}{p^{n_0-1}} = \sum_{n=1}^{n_0-2} \frac{x_n}{p^n} + \frac{x_{n_0-1}+1}{p^{n_0-1}} \\
&= \sum_{n=1}^{+\infty} \frac{y_n}{p^n} ,
\end{aligned}$$

(y_n) p- $y_n = x_n$ $n = 1, \ldots, n_0 - 2$, $y_{n_0-1} = x_{n_0-1} + 1$ $y_n = 0$ $n \geq n_0$. , $\overline{0, x_1 \ldots x_{n_0-1}\, p-1\, p-1 \ldots}^p$, p- x p- $\overline{0, x_1 \ldots x_{n_0-1}+1\, 00 \ldots}^p$ – p- $p-1$.

: (1) $0,35699999\ldots$ $0,35700000\ldots$, $\frac{357}{1000}$.
(2) $\overline{0,1010111111\ldots}^2$ $\overline{0,1011000000\ldots}^2$, $\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^4} = \frac{11}{16}$.

. p- **.**

p- $\overline{X_N \ldots X_0, x_1 x_2 \ldots}^p$ m k $x_{n+k} = x_n$ $n \geq m$. $x_m x_{m+1} \ldots, x_{m+k-1}$ p- $x_m x_{m+1} \ldots, x_{m+k-1}$ ′. , p-

$$\overline{X_N \ldots X_0, \ldots \underbrace{x_m \ldots x_{m+k-1}}\underbrace{x_m \ldots x_{m+k-1}}\underbrace{x_m \ldots x_{m+k-1}} \ldots}^p .$$

$\overline{X_N \dots X_0, \dots \overline{x_m \, \dots \, x_{m+k-1}}}^{\,p}$.

10.10 $p = 10$, , (,).

10.10 $p \geq 2$ $x \geq 0$. x p- .

: x p- :

$$x = \overline{X_N \dots X_0, \dots \underbrace{x_m \, \dots \, x_{m+k-1}}\underbrace{x_m \, \dots \, x_{m+k-1}}\underbrace{x_m \, \dots \, x_{m+k-1}} \dots}^{\,p} .$$

$$\begin{aligned} x &= X_N p^N + \cdots + X_0 + \frac{x_1}{p} + \cdots + \frac{x_{m-1}}{p^{m-1}} + \Big(\frac{x_m}{p^m} + \cdots + \frac{x_{m+k-1}}{p^{m+k-1}}\Big)\Big(1 + \frac{1}{p^k} + \frac{1}{p^{2k}} + \cdots\Big) \\ &= X_N p^N + \cdots + X_0 + \frac{x_1}{p} + \cdots + \frac{x_{m-1}}{p^{m-1}} + \Big(\frac{x_m}{p^m} + \cdots + \frac{x_{m+k-1}}{p^{m+k-1}}\Big)\frac{1}{1 - \frac{1}{p^k}} \\ &= \frac{X_N p^{N+m-1} + \cdots + X_0 p^{m-1} + x_1 p^{m-2} + \cdots + x_{m-1}}{p^{m-1}} + \frac{x_m p^{k-1} + \cdots + x_{m+k-1}}{p^{m-1}(p^k - 1)} , \end{aligned}$$

x .

, $x \geq 0$: $x = \frac{a}{b}$ $a \geq 0$ $b \geq 1$. $p = {p_1}^{n_1} \cdots {p_r}^{n_r}$, $p_1, \dots, p_r$ p $n_1, \dots, n_r$. , $b = {p_1}^{l_1} \cdots {p_r}^{l_r} b'$, $l_1, \dots, l_r$ ≥ 0 (p_j b, l_j 0) b' p – , p b' > 1. m $(m-1)n_j \geq l_j$ $j = 1, \dots, r$. $m_j = (m-1)n_j - l_j$, m_j ≥ 0. $x = \frac{a}{b} = \frac{a}{{p_1}^{l_1} \cdots {p_r}^{l_r} b'} = \frac{a {p_1}^{m_1} \cdots {p_r}^{m_r}}{({p_1}^{n_1} \cdots {p_r}^{n_r})^{m-1} b'} = \frac{a'}{p^{m-1} b'}$, a' ≥ 0. , $p, p^2, p^3, \dots$ b' . – $0, \dots, b'-1$ – , b'. , t s $t < s$ $p^t = q_t b' + z$ $p^s = q_s b' + z$, q_t q_s z $0, \dots, b'-1$. $p^t(p^{s-t} - 1) = p^s - p^t = (q_s - q_t) b'$, b' $p^t(p^{s-t} - 1)$. b' p , b' $p^{s-t} - 1$, b'' $b'b'' = p^{s-t} - 1$. $k = s - t$ $b'b'' = p^k - 1$, , $x = \frac{a'b''}{p^{m-1} b' b''} = \frac{a''}{p^{m-1}(p^k - 1)}$, a'' ≥ 0. , a'' $p^k - 1$, $a'' = w(p^k - 1) + u$, w ≥ 0 u $0, \dots, p^k - 2$. , p- w u $w = X_N p^{N+m-1} + \cdots + X_0 p^{m-1} + x_1 p^{m-2} + \cdots + x_{m-1}$ $u = x_m p^{k-1} + \cdots + x_{m+k-1}$ $x_m, \dots, x_{m+k-1}$ $p - 1$, $u = (p-1)p^{k-1} + \cdots + (p-1)p + (p-1) = p^k - 1$.

$$\begin{aligned} x &= \frac{w(p^k - 1) + u}{p^{m-1}(p^k - 1)} = \frac{w}{p^{m-1}} + \frac{u}{p^{m-1}(p^k - 1)} \\ &= \frac{X_N p^{N+m-1} + \cdots + X_0 p^{m-1} + x_1 p^{m-2} + \cdots + x_{m-1}}{p^{m-1}} + \frac{x_m p^{k-1} + \cdots + x_{m+k-1}}{p^{m-1}(p^k - 1)} \\ &= X_N p^N + \cdots + X_0 + \frac{x_1}{p} + \cdots + \frac{x_{m-1}}{p^{m-1}} + \Big(\frac{x_m}{p^m} + \cdots + \frac{x_{m+k-1}}{p^{m+k-1}}\Big)\frac{1}{1 - \frac{1}{p^k}} \\ &= X_N p^N + \cdots + X_0 + \frac{x_1}{p} + \cdots + \frac{x_{m-1}}{p^{m-1}} + \Big(\frac{x_m}{p^m} + \cdots + \frac{x_{m+k-1}}{p^{m+k-1}}\Big)\Big(1 + \frac{1}{p^k} + \frac{1}{p^{2k}} + \cdots\Big) \\ &= \overline{X_N \dots X_0, \dots \underbrace{x_m \, \dots \, x_{m+k-1}}\underbrace{x_m \, \dots \, x_{m+k-1}}\underbrace{x_m \, \dots \, x_{m+k-1}} \dots}^{\,p} , \end{aligned}$$

x p- .

: $\frac{3}{14}$.

$\frac{3}{14} = \frac{3}{2 \cdot 7} = \frac{15}{10 \cdot 7}$ 7 10. $10-1, 10^2-1, 10^3-1, \dots$ 7 10^6-1 $10^6-1 = 7 \cdot 142857$. $\frac{3}{14} = \frac{15 \cdot 142857}{10(10^6-1)} = \frac{2142855}{10(10^6-1)}$. $2142855 = 2(10^6 - 1) + 142857$, ,

$$\begin{aligned} \frac{3}{14} &= \frac{2}{10} + \frac{142857}{10(10^6 - 1)} = \frac{2}{10} + \frac{142857}{10^7}\frac{10^6}{10^6 - 1} \\ &= \frac{2}{10} + \Big(\frac{1}{10^2} + \frac{4}{10^3} + \frac{2}{10^4} + \frac{8}{10^5} + \frac{5}{10^6} + \frac{7}{10^7}\Big)\frac{1}{1 - \frac{1}{10^6}} \\ &= \frac{2}{10} + \Big(\frac{1}{10^2} + \frac{4}{10^3} + \frac{2}{10^4} + \frac{8}{10^5} + \frac{5}{10^6} + \frac{7}{10^7}\Big)\Big(1 + \frac{1}{10^6} + \frac{1}{10^{12}} + \cdots\Big) \\ &= 0,2\,\underbrace{142857}\underbrace{142857}\underbrace{142857} \dots = 0,2\overline{142857} . \end{aligned}$$

.

. p- .

1. , , 11 87.

2. $p \geq 2$ $N \geq 0$. p- $p^{N+1} - 1$.

   (: $10^{23} - 1$.)

3. $p \geq 2$. $X_N, \ldots, X_0$ $\{0, 1, \ldots, p-1\}$ $X_N \neq 0$, $p^N \leq X_N p^N + \cdots + X_1 p + X_0 \leq p^{N+1} - 1 < p^{N+1}$.

4. $p \geq 2$. x N . p- x $N+1$ $p^N \leq x < p^{N+1}$.

   (: .)

5. (*) $k_1 < k_2 < k_3 < \ldots$ 0 . $\sum_{n=1}^{+\infty} \frac{1}{k_n}$ 90.

   (: (k_n) 10.)

   0 ;

. p- $[0, 1)$.

1. , $\frac{7}{16}$ $\frac{31}{32}$.

2. $p \geq 2$ x, y $[0, 1)$. n n- p- x, y , $|x - y| < \frac{1}{p^n}$.

3. $p \geq 2$ x $[0, 1)$. s_n n- p- x, p- $x - s_n$ $p^n(x - s_n)$;

4. $p \geq 2$ x, y $[0, 1)$. $x + y$ x, y n- p- $\frac{2}{p^n}$.

   xy $\frac{2}{p^n} - \frac{1}{p^{2n}}$.

5. $p \geq 2$ p- k. $[0, 1)$ n- p- k p^{n-1} $[a, b)$. ; ;

   (: $n = 1, 2, 3, \ldots$.)

6. $p \geq 2$ x $[0, 1)$. x p- 0 $p - 1$. ′ ;

   (: $x = \sum_{n=1}^{+\infty} \frac{x_n}{p^n}$ $x = \sum_{n=1}^{+\infty} \frac{y_n}{p^n}$ n_0 n $x_n \neq y_n$. $x_n = y_n$ $n < n_0$ $x_{n_0} < y_{n_0}$, $x_{n_0} + 1 \leq y_{n_0}$. $x \leq \sum_{n=1}^{n_0 - 1} \frac{x_n}{p^n} + \frac{x_{n_0}}{p^{n_0}} + \sum_{n=n_0+1}^{+\infty} \frac{p-1}{p^n} = \sum_{n=1}^{n_0} \frac{x_n}{p^n} + \frac{1}{p^{n_0}}$ $x \geq \sum_{n=1}^{n_0 - 1} \frac{x_n}{p^n} + \frac{x_{n_0} + 1}{p^{n_0}} + \sum_{n=n_0+1}^{+\infty} \frac{0}{p^n} = \sum_{n=1}^{n_0} \frac{x_n}{p^n} + \frac{1}{p^{n_0}}$. $y_{n_0} = x_{n_0} + 1$ $x_n = p - 1$ $y_n = 0$ $n > n_0$.)

   x $[0, 1)$ p- $x = \frac{m}{p^n}$, n $m < p^n$.

. p- .

1. $\sqrt{2}$.

2. $p \geq 2$. s_n n- p- $x \geq 0$ $t_n = s_n + \frac{1}{p^n}$, (s_n) , (t_n) x.

. p- .

1. $32, 34\underbrace{239}\underbrace{239}\underbrace{239}\ldots$, $\overline{2, 0\underbrace{1201}\underbrace{1201}\underbrace{1201}\ldots}^3$ $\overline{1001, \underbrace{101}\underbrace{101}\underbrace{101}\ldots}^2$.

2. , $\frac{313}{150}$.

10.4 .

10.11 . (b_n) $\lim_{n\to+\infty} b_n = 0$, $\sum_{n=1}^{+\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots$.

: $s_n = b_1 - b_2 + \cdots + (-1)^{n-1}b_n$ $s_2, s_4, s_6, \ldots$. $s_{2k+2} = s_{2k} + b_{2k+1} - b_{2k+2} \geq s_{2k}$, . , $s_{2k} = b_1 - b_2 + b_3 - b_4 + \cdots + b_{2k-1} - b_{2k} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2k-2} - b_{2k-1}) - b_{2k} \leq b_1$. $s_2, s_4, s_6, \ldots$, , s' : $\lim_{k\to+\infty} s_{2k} = s'$. $s_1, s_3, s_5, \ldots$. $s_{2k+1} = s_{2k-1} - b_{2k} + b_{2k+1} \leq s_{2k-1}$, . , $s_{2k-1} = b_1 - b_2 + b_3 - b_4 + \cdots - b_{2k-2} + b_{2k-1} = (b_1 - b_2) + (b_3 - b_4) + \cdots + (b_{2k-3} - b_{2k-2}) + b_{2k-1} \geq 0$. $s_1, s_3, s_5, \ldots$, , s'' : $\lim_{k\to+\infty} s_{2k-1} = s''$. , $s_{2k-1} = s_{2k} + b_{2k} \geq s_{2k}$, $k \to +\infty$, $s'' \geq s'$.
, , :

$$s_2 \leq s_4 \leq \ldots \leq s_{2k} \leq \ldots \leq s' \leq s'' \leq \ldots \leq s_{2k-1} \leq \ldots \leq s_3 \leq s_1 .$$

$0 \leq s'' - s' \leq s_{2k-1} - s_{2k} = b_{2k} \leq b_{2k-1}$ $k \geq 1$, , $0 \leq s'' - s' \leq b_n$ $n \geq 1$. $n \to +\infty$, $s'' - s' = 0$, $s' = s''$.
$s = s' = s''$,

$$s_2 \leq s_4 \leq \ldots \leq s_{2k} \leq \ldots \leq s \leq \ldots \leq s_{2k-1} \leq \ldots \leq s_3 \leq s_1 .$$

$0 \leq s - s_{2k} \leq s_{2k-1} - s_{2k} = b_{2k}$ $0 \leq s_{2k-1} - s \leq s_{2k-1} - s_{2k-2} = b_{2k-1}$ $k \geq 1$. , $|s_n - s| \leq b_n$ $n \geq 1$ $\lim_{n\to+\infty} |s_n - s| = 0$, , $\lim_{n\to+\infty} s_n = s$. $\sum_{n=1}^{+\infty}(-1)^{n-1}b_n$ s.

: $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$.

$\sum_{n=1}^{+\infty} x_n$ ( ) $\sum_{n=1}^{+\infty} |x_n|$, , $\sum_{n=1}^{+\infty} |x_n| < +\infty$.

10.2 . $\sum_{n=1}^{+\infty} x_n$,

$$\boxed{\Big|\sum_{n=1}^{+\infty} x_n\Big| \leq \sum_{n=1}^{+\infty} |x_n|.}$$

: $0 \leq x_n + |x_n| \leq 2|x_n|$ n, , $\sum_{n=1}^{+\infty} 2|x_n|$, $\sum_{n=1}^{+\infty}(x_n + |x_n|)$. , $\sum_{n=1}^{+\infty} x_n$, , $\sum_{n=1}^{+\infty} x_n = \sum_{n=1}^{+\infty}(x_n + |x_n|) - \sum_{n=1}^{+\infty} |x_n|$.
, $-|x_n| \leq x_n \leq |x_n|$ $n \geq 1$, $-\sum_{n=1}^{+\infty} |x_n| = \sum_{n=1}^{+\infty}(-|x_n|) \leq \sum_{n=1}^{+\infty} x_n \leq \sum_{n=1}^{+\infty} |x_n|$, , $\big|\sum_{n=1}^{+\infty} x_n\big| \leq \sum_{n=1}^{+\infty} |x_n|$.

$\big|\sum_{n=1}^{+\infty} x_n\big| \leq \sum_{n=1}^{+\infty} |x_n|$ $|x_1 + x_2| \leq |x_1| + |x_2|$, $|x_1 + x_2 + x_3| \leq |x_1| + |x_2| + |x_3|$, .

: (1) $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2}$: $\sum_{n=1}^{+\infty} \Big|\frac{(-1)^{n-1}}{n^2}\Big| = \sum_{n=1}^{+\infty} \frac{1}{n^2}$.

(2) $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}$. , $\sum_{n=1}^{+\infty} \Big|\frac{(-1)^{n-1}}{n}\Big| = \sum_{n=1}^{+\infty} \frac{1}{n}$ $+\infty$.

, $\sum_{n=1}^{+\infty} x_n$.

10.12 **,** . (1) $|x_n| \le y_n$ $n \ge 1$ $\sum_{n=1}^{+\infty} y_n$, $\sum_{n=1}^{+\infty} x_n$, , . ,

$$\boxed{\Big|\sum_{n=1}^{+\infty} x_n\Big| \le \sum_{n=1}^{+\infty} y_n\,.}$$

(2) $y_n > 0$ n $\left(\frac{|x_n|}{y_n}\right)$, , . $\sum_{n=1}^{+\infty} y_n$, $\sum_{n=1}^{+\infty} x_n$, , .

: (1) $\sum_{n=1}^{+\infty} y_n$, $\sum_{n=1}^{+\infty} |x_n|$, $\sum_{n=1}^{+\infty} x_n$, , $\left|\sum_{n=1}^{+\infty} x_n\right| \le \sum_{n=1}^{+\infty} |x_n| \le \sum_{n=1}^{+\infty} y_n$.
(2) 10.5 10.2.

: $\sum_{n=1}^{+\infty} \frac{(-2)^n}{3^n+2^n}$ $\sum_{n=1}^{+\infty} (\frac{2}{3})^n$. $\lim_{n\to+\infty} \frac{|\frac{(-2)^n}{3^n+2^n}|}{(\frac{2}{3})^n} = 1,$ $\sum_{n=1}^{+\infty} \frac{(-2)^n}{3^n+2^n}$, , .

10.13 ***d' Alembert.*** $\lim_{n\to+\infty} \left|\frac{x_{n+1}}{x_n}\right|$.
(i) $\lim_{n\to+\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1,$ $\sum_{n=1}^{+\infty} x_n$, , .
(ii) $\lim_{n\to+\infty} \left|\frac{x_{n+1}}{x_n}\right| > 1,$ $\sum_{n=1}^{+\infty} x_n$.

: (i) $\lim_{n\to+\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1.$ a $\lim_{n\to+\infty} \left|\frac{x_{n+1}}{x_n}\right| < a < 1.$ n_0 $\left|\frac{x_{n+1}}{x_n}\right| \le a$ $n \ge n_0$. $|x_{n_0+1}| \le a|x_{n_0}|$, $|x_{n_0+2}| \le a|x_{n_0+1}| \le a^2|x_{n_0}|$, $|x_{n_0+3}| \le a|x_{n_0+2}| \le a^3|x_{n_0}|$, , $|x_n| \le a^{n-n_0}|x_{n_0}|$ $n \ge n_0$. $c = a^{-n_0+1}|x_{n_0}|$, $|x_n| \le ca^{n-1}$ $n \ge n_0$. $C = \max\left\{c, |x_1|, \frac{|x_2|}{a}, \dots, \frac{|x_{n_0-1}|}{a^{n_0-2}}\right\}$ $|x_n| \le Ca^{n-1}$ $n \ge 1$. , $\sum_{n=1}^{+\infty} |x_n| \le C\sum_{n=1}^{+\infty} a^{n-1} < +\infty$ $0 < a < 1$. $\sum_{n=1}^{+\infty} |x_n| < +\infty$.
(ii) $\lim_{n\to+\infty} \left|\frac{x_{n+1}}{x_n}\right| > 1.$ n_0 $\left|\frac{x_{n+1}}{x_n}\right| \ge 1$ $n \ge n_0$. , $|x_{n_0+1}| \ge |x_{n_0}|$, $|x_{n_0+2}| \ge |x_{n_0+1}| \ge |x_{n_0}|$, , $|x_n| \ge |x_{n_0}| > 0$ $n \ge n_0$. , , $\lim_{n\to+\infty} x_n = 0,$ $\sum_{n=1}^{+\infty} x_n$.

: (1) $\sum_{n=1}^{+\infty} \frac{2^n}{n!}$, $\lim_{n\to+\infty} \left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right| = \lim_{n\to+\infty} \frac{2}{n+1} = 0 < 1.$

(2) $\sum_{n=1}^{+\infty} \frac{n^n}{n!}$: $\lim_{n\to+\infty} \left|\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\right| = \lim_{n\to+\infty} (1+\frac{1}{n})^n = e > 1.$

10.13 $\lim_{n\to+\infty} \left|\frac{x_{n+1}}{x_n}\right| = 1.$ ′ .

: (1) $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ $\lim_{n\to+\infty} \left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| = 1.$

(2) $\sum_{n=1}^{+\infty} \frac{1}{n}$ $\lim_{n\to+\infty} \left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = 1.$

10.14 ***Cauchy.*** $\lim_{n\to+\infty} \sqrt[n]{|x_n|}$.
(i) $\lim_{n\to+\infty} \sqrt[n]{|x_n|} < 1,$ $\sum_{n=1}^{+\infty} x_n$, , .
(ii) $\lim_{n\to+\infty} \sqrt[n]{|x_n|} > 1,$ $\sum_{n=1}^{+\infty} x_n$.

: (i) $\lim_{n\to+\infty} \sqrt[n]{|x_n|} < 1,$ a $\lim_{n\to+\infty} \sqrt[n]{|x_n|} < a < 1.$ n_0 $\sqrt[n]{|x_n|} \le a$ $n \ge n_0$, $|x_n| \le a^n$ $n \ge n_0$. $C = \max\{a, |x_1|, \frac{|x_2|}{a}, \dots, \frac{|x_{n_0-1}|}{a^{n_0-2}}\}$, $|x_n| \le Ca^{n-1}$ $n \ge 1$. $\sum_{n=1}^{+\infty} |x_n| \le C\sum_{n=1}^{+\infty} a^{n-1} < +\infty$ $0 < a < 1$. $\sum_{n=1}^{+\infty} |x_n| < +\infty$.
(ii) $\lim_{n\to+\infty} \sqrt[n]{|x_n|} > 1,$ n_0 $\sqrt[n]{|x_n|} \ge 1$ $n \ge n_0$. $|x_n| \ge 1$ $n \ge n_0$, $\lim_{n\to+\infty} x_n = 0$, , $\sum_{n=1}^{+\infty} x_n$.

: (1) $\sum_{n=1}^{+\infty}\frac{n^3}{2^n}$ $\lim_{n\to+\infty}\sqrt[n]{|\frac{n^3}{2^n}|}=\lim_{n\to+\infty}\frac{(\sqrt[n]{n})^3}{2}=\frac{1}{2}<1.$

(2) $\sum_{n=1}^{+\infty}\frac{(-2)^n}{n}$: $\lim_{n\to+\infty}\sqrt[n]{|\frac{(-2)^n}{n}|}=\lim_{n\to+\infty}\frac{2}{\sqrt[n]{n}}=2>1.$

, , $\lim_{n\to+\infty}\sqrt[n]{|x_n|}=1.$ ’ .

: (1) $\sum_{n=1}^{+\infty}\frac{1}{n^2}$ $\lim_{n\to+\infty}\sqrt[n]{|\frac{1}{n^2}|}=1.$

(2) $\sum_{n=1}^{+\infty}\frac{1}{n}$ $\lim_{n\to+\infty}\sqrt[n]{|\frac{1}{n}|}=1.$

.

1. .

$$\sum_{n=1}^{+\infty}\frac{n+2}{(\sqrt{2})^n},\quad \sum_{n=1}^{+\infty}n^3e^n,\quad \sum_{n=1}^{+\infty}\frac{n!}{3^n},\quad \sum_{n=1}^{+\infty}\frac{3^n}{n!},\quad \sum_{n=1}^{+\infty}\frac{2^n}{n^n},$$

$$\sum_{n=1}^{+\infty}\frac{2^nn!}{n^n},\quad \sum_{n=1}^{+\infty}\frac{3^nn!}{n^n},\quad \sum_{n=1}^{+\infty}\frac{(n!)^2}{(2n)!},\quad \sum_{n=1}^{+\infty}\frac{(n!)^2}{2^{n^2}},$$

$$\sum_{n=1}^{+\infty}\frac{2\cdot5\cdot8\cdots(3n-1)}{1\cdot5\cdot9\cdots(4n-3)},\quad \sum_{n=1}^{+\infty}\frac{e^nn!}{n^n},\quad \sum_{n=1}^{+\infty}\frac{4^n(n!)^2}{(2n)!}.$$

2. .

$$\sum_{n=1}^{+\infty}\Big(\frac{n+1}{2n-1}\Big)^n,\quad \sum_{n=1}^{+\infty}\Big(\frac{3n-1}{2n+1}\Big)^{2n-1},\quad \sum_{n=1}^{+\infty}\frac{n^3}{e^n},\quad \sum_{n=1}^{+\infty}n^32^n,\quad \sum_{n=1}^{+\infty}\frac{2^n}{n^n},$$

$$\sum_{n=2}^{+\infty}\frac{1}{(\log n)^n},\quad \sum_{n=1}^{+\infty}\big(\sqrt[n]{n}-1\big)^n,\quad \sum_{n=1}^{+\infty}\frac{n}{\big(\sqrt[n]{n}+1\big)^n},\quad \sum_{n=1}^{+\infty}\frac{n3^n}{\big(\sqrt[n]{n}+1\big)^n},$$

$$\sum_{n=1}^{+\infty}\Big(\frac{n}{n+1}\Big)^{n^2},\quad \sum_{n=1}^{+\infty}\frac{n2^n}{\big(\sqrt[n]{n}+1\big)^n},\quad \sum_{n=1}^{+\infty}e^n\Big(\frac{n}{n+1}\Big)^{n^2}.$$

3. p $\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1}{n^p}$; ;

4. p,q $\sum_{n=1}^{+\infty}p^nn^q$; ;

5. . .

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^{\frac{12}{11}}},\quad \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^{\frac{11}{12}}},\quad \sum_{n=2}^{+\infty}\frac{(-1)^n}{n\log n},\quad \sum_{n=2}^{+\infty}\frac{(-1)^n}{n(\log n)^2},$$

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}\sqrt{n}}{n+1},\quad \sum_{n=1}^{+\infty}\frac{(-1)^{\frac{n(n-1)}{2}}}{2^n},\quad \sum_{n=1}^{+\infty}\frac{(-1)^{\frac{n(n-1)}{2}}}{n},\quad \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{\sqrt[n]{n}},$$

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{\sqrt{n}+1}\,,\quad \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{\sqrt{n}+(-1)^{n-1}}\,,\quad \sum_{n=1}^{+\infty}(-1)^{n-1}\sin\frac{1}{n}\,,$$

$$\sum_{n=1}^{+\infty}(-1)^{n-1}\Big(\sin\frac{1}{n}\Big)^{\frac{3}{2}}\,,\quad \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}\tan\frac{1}{n}\,,\quad \sum_{n=1}^{+\infty}(-1)^{n-1}\Big(1-\cos\frac{1}{n}\Big)\,.$$

6. x $\sum_{n=1}^{+\infty}|x_n|<+\infty$. $\sum_{n=1}^{+\infty}x_n\cos(nx)$ $\sum_{n=1}^{+\infty}x_n\sin(nx)$.

7. $^{(*)}$ $x\neq -1$ $\sum_{n=1}^{+\infty}\frac{1}{1+x^n}$.
(: $x\neq -1$ $\lim_{n\to+\infty}\frac{1}{1+x^n}=0$.)

8. $\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot5\cdot8\cdots(3n-1)}\leq(\frac{2}{3})^n$ n , , $\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot5\cdot8\cdots(3n-1)}$.

9. $^{(*)}$ (i) $\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\geq\int_1^{n+1}\frac{1}{2x}\,dx=\log\sqrt{n+1}$. , $1+x\leq e^x$, $\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\leq e^{-\frac{1}{2}-\frac{1}{4}-\cdots-\frac{1}{2n}}\leq e^{-\log\sqrt{n+1}}=\frac{1}{\sqrt{n+1}}$.
$\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}$.
(ii) $1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\leq 1+\int_1^{n}\frac{1}{2x-1}\,dx=1+\log\sqrt{2n-1}$. , $1+x\leq e^x$, $2\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\leq e^{1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}}\leq e^{1+\log\sqrt{2n-1}}=e\sqrt{2n-1}$.
$\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}$.

10. (b_n) , $\lim_{n\to+\infty}b_n=0$, $s=\sum_{n=1}^{+\infty}(-1)^{n-1}b_n$ $s_n=b_1-b_2+\cdots+(-1)^{n-1}b_n$. $0\leq(-1)^n(s-s_n)\leq b_{n+1}$ n.

10.5 .

$$a_0+\sum_{n=1}^{+\infty}a_n(x-\xi)^n=a_0+a_1(x-\xi)+a_2(x-\xi)^2+\cdots+a_n(x-\xi)^n+\cdots$$

ξ $a_0,a_1,a_2,\ldots$. x . , : x , , x ( ) . x : x , $+\infty$ $-\infty$ x $\pm\infty$.
’ x − , , x. .

: (1) $0+\sum_{n=1}^{+\infty}0(x-\xi)^n$ 0 , , x 0.

(2) $1+\sum_{n=1}^{+\infty}1(x-\xi)^n$ 1 $1+\sum_{n=1}^{+\infty}a^n$, $a=x-\xi$. $-1<x-\xi<1$, , $\xi-1<x<\xi+1$ $\frac{1}{1-a}=\frac{1}{1-(x-\xi)}$. ,

$$\boxed{1+\sum_{n=1}^{+\infty}(x-\xi)^n=\frac{1}{1-(x-\xi)}\qquad(\xi-1<x<\xi+1).}$$

x . .

10.15 $a_0 + \sum_{n=1}^{+\infty} a_n(x-\xi)^n$. :
(i) $(-\infty,+\infty)$,
(ii) $\{\xi\}$,
(iii) $R>0$ $(\xi-R,\xi+R)$ $(\xi-R,\xi+R]$ $[\xi-R,\xi+R)$ $[\xi-R,\xi+R]$.
, x $(-\infty,+\infty)$ (i), x $\{\xi\}$ ($x=\xi$) (ii) x $(\xi-R,\xi+R)$ (iii).

. , , . , $x=\xi$ $a_0 + \sum_{n=1}^{+\infty} a_n(x-\xi)^n$ $a_0 + \sum_{n=1}^{+\infty} a_n 0^n = a_0$, , . (i) (ii) 10.15 (iii). $(-\infty,+\infty) = (\xi-(+\infty), \xi+(+\infty))$ $(\xi-R,\xi+R)$ $R=+\infty$. $\{\xi\}$ $[\xi-R,\xi+R]$ $R=0$. . R $0\le R\le+\infty$.

: (1) x, $(-\infty,+\infty)$ $R=+\infty$.

(2) $1+\sum_{n=1}^{+\infty}(x-\xi)^n$ x $(\xi-1,\xi+1)$ x. $(\xi-1,\xi+1)$ $R=1$.

(3) $1+\sum_{n=1}^{+\infty} n^n(x-\xi)^n$. $x=x_1$. $\lim_{n\to+\infty} n^n(x_1-\xi)^n=0$, M $n^n|x_1-\xi|^n\le M$ n. $|x_1-\xi|\le\frac{\sqrt[n]{M}}{n}$ n , $n\to+\infty$, $|x_1-\xi|\le 0$, , $x_1=\xi$. $\{\xi\}$ $R=0$.

, ′ . , , .

10.16 $\mu=\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|$. $a_0+\sum_{n=1}^{+\infty} a_n(x-\xi)^n$ $\frac{1}{\mu}$, $\frac{1}{\mu}$ 0, $\mu=+\infty$, $+\infty$, $\mu=0$.

: , . $0<\mu<+\infty$, $\lim_{n\to+\infty}\left|\frac{a_{n+1}(x-\xi)^{n+1}}{a_n(x-\xi)^n}\right| = |x-\xi|\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-\xi|\mu$, : $|x-\xi|<\frac{1}{\mu}$, , $|x-\xi|>\frac{1}{\mu}$, . $\frac{1}{\mu}$. $\mu=0$, $\lim_{n\to+\infty}\left|\frac{a_{n+1}(x-\xi)^{n+1}}{a_n(x-\xi)^n}\right| = |x-\xi|\cdot 0 = 0 < 1$, , x , , $+\infty=\frac{1}{\mu}$. , $\mu=+\infty$, $x\ne\xi$ $|x-\xi|>0$, , $\lim_{n\to+\infty}\left|\frac{a_{n+1}(x-\xi)^{n+1}}{a_n(x-\xi)^n}\right| = |x-\xi|(+\infty) = +\infty > 1$. $x\ne\xi$, $0=\frac{1}{\mu}$.

10.17 $\mu=\lim_{n\to+\infty}\sqrt[n]{|a_n|}$. $a_0+\sum_{n=1}^{+\infty} a_n(x-\xi)^n$ $\frac{1}{\mu}$, $\frac{1}{\mu}$ 0, $\mu=+\infty$, $+\infty$, $\mu=0$.

: , . $0<\mu<+\infty$, $\lim_{n\to+\infty}\sqrt[n]{|a_n(x-\xi)^n|} = |x-\xi|\lim_{n\to+\infty}\sqrt[n]{|a_n|} = |x-\xi|\mu$, : $|x-\xi|<\frac{1}{\mu}$, , $|x-\xi|>\frac{1}{\mu}$, . $\frac{1}{\mu}$. $\mu=0$, $\lim_{n\to+\infty}\sqrt[n]{|a_n(x-\xi)^n|} = |x-\xi|\cdot 0 = 0<1$, x $+\infty=\frac{1}{\mu}$. , $\mu=+\infty$, $x\ne\xi$ $|x-\xi|>0$, , $\lim_{n\to+\infty}\sqrt[n]{|a_n(x-\xi)^n|} = |x-\xi|(+\infty) = +\infty>1$. $x\ne\xi$, $0=\frac{1}{\mu}$.

: (1) $\sum_{n=1}^{+\infty}\frac{1}{n}x^n$: $\lim_{n\to+\infty}\left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to+\infty}\frac{n}{n+1} = 1$ $\lim_{n\to+\infty}\sqrt[n]{|\frac{1}{n}|} = \lim_{n\to+\infty}\frac{1}{\sqrt[n]{n}} = 1$. 1 , , 0. $x=1$ $\sum_{n=1}^{+\infty}\frac{1}{n}$ $x=-1$ $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}$. $[-1,1)$.

(2) $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}x^n$, . , , . $t=-x$ $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}x^n = \sum_{n=1}^{+\infty}\frac{1}{n}t^n$. t $[-1,1)$, x $(-1,1]$. $(-1,1]$.

(3) $\sum_{n=1}^{+\infty}\frac{1}{n^2}x^n$. : $\lim_{n\to+\infty}\left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| = \lim_{n\to+\infty}\frac{n^2}{(n+1)^2} = 1$ $\lim_{n\to+\infty}\sqrt[n]{|\frac{1}{n^2}|} = \lim_{n\to+\infty}\frac{1}{(\sqrt[n]{n})^2} = 1$. 1 0. $x=1$ $\sum_{n=1}^{+\infty}\frac{1}{n^2}$ $x=-1$ $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}$, , . $[-1,1]$.

(4) . $1+\sum_{n=1}^{+\infty} x^n$ $(-1,1)$.

.

: (1)

$$\boxed{1+\sum_{n=1}^{+\infty}\frac{1}{n!}x^n = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots.}$$

: $\lim_{n\to+\infty}\left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to+\infty}\frac{n!}{(n+1)!} = \lim_{n\to+\infty}\frac{1}{n+1} = 0.$ - $+\infty$ $(-\infty,+\infty)$. , $\lim_{n\to+\infty}\sqrt[n]{|\frac{1}{n!}|} = \lim_{n\to+\infty}\frac{1}{\sqrt[n]{n!}}$, $\lim_{n\to+\infty}\sqrt[n]{n!}$. . , $\lim_{n\to+\infty}\sqrt[n]{n!}$. , :

$$\boxed{\lim_{n\to+\infty}\sqrt[n]{n!} = +\infty.}$$

: n , $n! = 1\cdots\frac{n}{2}(\frac{n}{2}+1)\cdots n \geq (\frac{n}{2}+1)\cdots n \geq (\frac{n}{2}+1)^{\frac{n}{2}} \geq (\frac{n+1}{2})^{\frac{n}{2}}$, $\sqrt[n]{n!}\geq\sqrt{\frac{n+1}{2}}$. n , $n! = 1\cdots\frac{n-1}{2}\frac{n+1}{2}\cdots n \geq \frac{n+1}{2}\cdots n \geq (\frac{n+1}{2})^{\frac{n+1}{2}} \geq (\frac{n+1}{2})^{\frac{n}{2}}$, $\sqrt[n]{n!}\geq\sqrt{\frac{n+1}{2}}$. , $\sqrt[n]{n!}\geq\sqrt{\frac{n+1}{2}}$, $\lim_{n\to+\infty}\sqrt{\frac{n+1}{2}} = +\infty$, $\lim_{n\to+\infty}\sqrt[n]{n!} = +\infty$.

$+\infty$.

(2)

$$\boxed{1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{(2k)!}x^{2k} = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots.}$$

$t = x^2$, $1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{(2k)!}t^k = 1-\frac{t}{2!}+\frac{t^2}{4!}-\frac{t^3}{6!}+\cdots$. . $\lim_{k\to+\infty}\left|\frac{\frac{(-1)^{k+1}}{(2k+2)!}}{\frac{(-1)^k}{(2k)!}}\right| = \lim_{k\to+\infty}\frac{(2k)!}{(2k+2)!} = \lim_{k\to+\infty}\frac{1}{(2k+1)(2k+2)} = 0.$, $\lim_{k\to+\infty}\sqrt[k]{|\frac{(-1)^k}{(2k)!}|} = \lim_{k\to+\infty}\frac{1}{\sqrt[k]{(2k)!}} = 0$, $0\leq\frac{1}{\sqrt[k]{(2k)!}}\leq\frac{1}{\sqrt[k]{k!}}$ $\lim_{k\to+\infty}\frac{1}{\sqrt[k]{k!}} = 0$. $+\infty$. t, x , , $(-\infty,+\infty)$.

. : x^n $a_n = 0$, n , $a_n = \frac{(-1)^{\frac{n}{2}}}{n!}$, n . , $\left(|\frac{a_{n+1}}{a_n}|\right)$. $\sqrt[n]{|a_n|} = \frac{1}{\sqrt[n]{n!}}$, n , $\sqrt[n]{|a_n|} = 0$, n . $0\leq\sqrt[n]{|a_n|}\leq\frac{1}{\sqrt[n]{n!}}$ n , $\lim_{n\to+\infty}\frac{1}{\sqrt[n]{n!}} = 0$, $\lim_{n\to+\infty}\sqrt[n]{|a_n|} = 0$ $+\infty$.

, , ′ ! $\lim_{k\to+\infty}\left|\frac{\frac{(-1)^{k+1}}{(2k+2)!}x^{2k+2}}{\frac{(-1)^k}{(2k)!}x^{2k}}\right| = \lim_{k\to+\infty}\frac{x^2}{(2k+1)(2k+2)} = 0 < 1$ $\lim_{k\to+\infty}\sqrt[k]{|\frac{(-1)^k}{(2k)!}x^{2k}|} = \lim_{k\to+\infty}\frac{x^2}{\sqrt[k]{(2k)!}} = 0 < 1$ x. ($0\leq\frac{x^2}{\sqrt[k]{(2k)!}}\leq\frac{x^2}{\sqrt[k]{k!}}$ k.) x $(-\infty,+\infty)$.

(3)

$$\boxed{\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{(2k-1)!}x^{2k-1}=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots.}$$

$+\infty$ $(-\infty,+\infty)$.

(4)

$$\boxed{\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{2k-1}x^{2k-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots.}$$

x, , , $1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{2k+1}x^{2k}$, $t=x^2$, $1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{2k+1}t^k$. $\lim_{k\to+\infty}\left|\frac{\frac{(-1)^{k+1}}{2k+3}}{\frac{(-1)^k}{2k+1}}\right|=\lim_{k\to+\infty}\frac{2k+1}{2k+3}=1$ $\lim_{k\to+\infty}\sqrt[k]{|\frac{(-1)^k}{2k+1}|}=\lim_{k\to+\infty}\frac{1}{\sqrt[k]{2k+1}}=1$, 1. $t=1$, $1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{2k+1}$. , $t=-1$, $1+\sum_{k=1}^{+\infty}\frac{1}{2k+1}$ $1+\sum_{k=1}^{+\infty}\frac{1}{k}$. $1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{2k+1}t^k$ t $(-1,1]$. $t=x^2$, $1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{2k+1}x^{2k}$ x $[-1,1]$, x $[-1,1]$.

, , ' . $\lim_{k\to+\infty}\left|\frac{\frac{(-1)^k}{2k+1}x^{2k+1}}{\frac{(-1)^{k-1}}{2k-1}x^{2k-1}}\right|=\lim_{k\to+\infty}\frac{2k-1}{2k+1}x^2=x^2$ $\lim_{k\to+\infty}\sqrt[k]{|\frac{(-1)^{k-1}}{2k-1}x^{2k-1}|}=\lim_{k\to+\infty}\frac{|x|^{2-\frac{1}{k}}}{\sqrt[k]{2k-1}}=x^2$. , $x^2<1$, , $x^2>1$. $x=1$, $\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{2k-1}$. $x=-1$, $-\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{2k-1}$, . , $[-1,1]$.

! x^n $a_n=\frac{(-1)^{\frac{n-1}{2}}}{n}$, n , $a_n=0$, n . $\left(|\frac{a_{n+1}}{a_n}|\right)$. , $\sqrt[n]{|a_n|}=\frac{1}{\sqrt[n]{n}}$, n , $\sqrt[n]{|a_n|}=0$, n . $\lim_{n\to+\infty}\sqrt[n]{|a_n|}$, .

(5)

$$\boxed{1+\sum_{n=1}^{+\infty}\binom{\alpha}{n}x^n=1+\binom{\alpha}{1}x+\binom{\alpha}{2}x^2+\cdots,}$$

$\binom{\alpha}{n}$

$$\binom{\alpha}{n}=\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$

n α

$$\binom{\alpha}{0}=1.$$

$\binom{\alpha}{n}$ $\binom{m}{n}$, n m $0\le n\le m$. , α , $\binom{\alpha}{n}=0$ $n\ge\alpha+1$, $1+\binom{\alpha}{1}x+\binom{\alpha}{2}x^2+\cdots+\binom{\alpha}{\alpha-1}x^{\alpha-1}+\binom{\alpha}{\alpha}x^\alpha=(1+x)^\alpha$, Newton. , α x $(-\infty,+\infty)$.

α , $\lim_{n\to+\infty}\left|\frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}}\right|=\lim_{n\to+\infty}|\frac{\alpha-n}{n+1}|=1$, 1. : $(-1,1)$, $(-1,1]$, $[-1,1)$, $[-1,1]$. (i) $\alpha\le-1$, $(-1,1)$, (ii) $-1<\alpha<0$, $(-1,1]$ (iii) $\alpha\ge0$ (α), $[-1,1]$.

±1 .

10.1 $\mu,\nu>-1,$ $c_1,c_2>0,$ $\mu,\nu,$

$$c_1 n^{\mu-\nu}\le \frac{(\mu+1)(\mu+2)\cdots(\mu+n)}{(\nu+1)(\nu+2)\cdots(\nu+n)}\le c_2 n^{\mu-\nu}$$

$n.$

. $\frac{(\mu+1)(\mu+2)\cdots(\mu+n)}{(\nu+1)(\nu+2)\cdots(\nu+n)} = (1+\frac{\mu-\nu}{\nu+1})(1+\frac{\mu-\nu}{\nu+2})\cdots(1+\frac{\mu-\nu}{\nu+n}) \le e^{\frac{\mu-\nu}{\nu+1}+\frac{\mu-\nu}{\nu+2}+\cdots+\frac{\mu-\nu}{\nu+n}} = e^{(\mu-\nu)(\frac{1}{\nu+1}+\frac{1}{\nu+2}+\cdots+\frac{1}{\nu+n})}$. , $\nu\le\mu,$ $\frac{(\mu+1)(\mu+2)\cdots(\mu+n)}{(\nu+1)(\nu+2)\cdots(\nu+n)} \le e^{(\mu-\nu)\left(\frac{1}{\nu+1}+\int_1^n \frac{1}{\nu+x}\,dx\right)} = e^{\frac{\mu-\nu}{\nu+1}+(\mu-\nu)\log\frac{\nu+n}{\nu+1}} = e^{\frac{\mu-\nu}{\nu+1}}(\frac{\nu+n}{\nu+1})^{\mu-\nu}$. $\nu+n\le(\nu+2)n,$ $\frac{(\mu+1)(\mu+2)\cdots(\mu+n)}{(\nu+1)(\nu+2)\cdots(\nu+n)} \le e^{\frac{\mu-\nu}{\nu+1}}(\frac{\nu+2}{\nu+1})^{\mu-\nu}n^{\mu-\nu}$. $\mu\le\nu$ $\frac{(\mu+1)(\mu+2)\cdots(\mu+n)}{(\nu+1)(\nu+2)\cdots(\nu+n)} \le e^{(\mu-\nu)\int_1^{n+1}\frac{1}{\nu+x}\,dx} = e^{(\mu-\nu)\log\frac{\nu+n+1}{\nu+1}} = (\frac{\nu+n+1}{\nu+1})^{\mu-\nu}$, $\nu+n+1\ge n,$ $\frac{(\mu+1)(\mu+2)\cdots(\mu+n)}{(\nu+1)(\nu+2)\cdots(\nu+n)} \le \frac{1}{(\nu+1)^{\mu-\nu}}n^{\mu-\nu}$. , $\frac{(\mu+1)(\mu+2)\cdots(\mu+n)}{(\nu+1)(\nu+2)\cdots(\nu+n)} \le c_2 n^{\mu-\nu}$, c_2 $e^{\frac{\mu-\nu}{\nu+1}}(\frac{\nu+2}{\nu+1})^{\mu-\nu}$ $\frac{1}{(\nu+1)^{\mu-\nu}}$ $-1<\nu\le\mu$ $-1<\mu\le\nu$, . ($c_1=\frac{1}{c_2}$), μ ν.

$1+\sum_{n=1}^{+\infty}\binom{\alpha}{n}x^n$ $x=\pm1.$

(i) $x=1,$ $1+\sum_{n=1}^{+\infty}\binom{\alpha}{n}.$

$\alpha<0,$ $\binom{\alpha}{n}=(-1)^n\frac{|\alpha|(|\alpha|+1)\cdots(|\alpha|+n-1)}{n!}$. $\alpha\le-1,$ $|\binom{\alpha}{n}|\ge1,$. $-1<\alpha<0,$ $\frac{|\alpha|(|\alpha|+1)\cdots(|\alpha|+n-1)}{n!}$, n , $\frac{|\alpha|(|\alpha|+1)\cdots(|\alpha|+n-1)}{n!}\le c_2 n^{|\alpha|-1}$, $\lim_{n\to+\infty}\frac{|\alpha|(|\alpha|+1)\cdots(|\alpha|+n-1)}{n!}=0$. , $-1<\alpha<0,$. , $|\binom{\alpha}{n}|=\frac{|\alpha|(|\alpha|+1)\cdots(|\alpha|+n-1)}{n!}\ge c_1 n^{|\alpha|-1}$, .

$\alpha\ge0,$ α , $m<\alpha<m+1,$ $m=[\alpha]$. $n\ge m+1$ $|\binom{\alpha}{n}| = \frac{\alpha\cdots(\alpha-m)(m+1-\alpha)\cdots(n-1-\alpha)}{n!} = \frac{\alpha\cdots(\alpha-m)}{1\cdots(m+1)}\frac{(m+1-\alpha)\cdots(n-1-\alpha)}{(m+2)\cdots n} \le c_2\frac{\alpha\cdots(\alpha-m)}{1\cdots(m+1)}\frac{1}{n^{1+\alpha}}$. .

(ii) $x=-1,$ $1+\sum_{n=1}^{+\infty}\binom{\alpha}{n}(-1)^n$.

$\alpha<0,$ $\binom{\alpha}{n}(-1)^n=\frac{|\alpha|(|\alpha|+1)\cdots(|\alpha|+n-1)}{n!}\ge c_1\frac{1}{n^{1-|\alpha|}}$, .

$\alpha\ge0,$ $m<\alpha<m+1,$ $m=[\alpha]$. $n\ge m+1$ $|\binom{\alpha}{n}(-1)^n| = \frac{\alpha\cdots(\alpha-m)(m+1-\alpha)\cdots(n-1-\alpha)}{n!} = \frac{\alpha\cdots(\alpha-m)}{1\cdots(m+1)}\frac{(m+1-\alpha)\cdots(n-1-\alpha)}{(m+2)\cdots n} \le c_2\frac{\alpha\cdots(\alpha-m)}{1\cdots(m+1)}\frac{1}{n^{1+\alpha}}$.

•

1. . .

$$1+\sum_{n=1}^{+\infty}2^n x^n,\quad 1+\sum_{n=1}^{+\infty}\frac{1}{2^n}x^n,\quad \sum_{n=1}^{+\infty}\frac{x^n}{n^2},\quad \sum_{n=1}^{+\infty}n^3x^n,\quad \sum_{n=1}^{+\infty}\frac{2^nx^n}{n^2},$$

$$\sum_{n=1}^{+\infty}\frac{n^3}{3^n}x^n,\quad \sum_{n=1}^{+\infty}\frac{1}{n2^n}x^n,\quad \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}n}{\sqrt{n^3+1}}x^n,\quad \sum_{n=1}^{+\infty}n^nx^n,\quad \sum_{n=1}^{+\infty}\frac{1}{n^n}x^n,$$

$$1+\sum_{n=1}^{+\infty}n!x^n,\quad 1+\sum_{n=1}^{+\infty}\frac{2^n}{n!}x^n,\quad 1+\sum_{n=1}^{+\infty}\left(2^n+\frac{3^n}{n}\right)x^n,$$

$$\sum_{n=1}^{+\infty}\frac{1}{2^n+3^n}x^n,\quad \sum_{n=1}^{+\infty}\frac{n^n}{(n+1)^n}x^n,\quad \sum_{n=1}^{+\infty}\frac{n^n}{(n+1)^{n+1}}x^n.$$

2. . ′ . ′ ;

$$\sum_{n=1}^{+\infty}\frac{1}{2n-1}x^{2n-1},\quad \sum_{n=1}^{+\infty}\frac{1}{2n-1}x^{2n},\quad \sum_{n=1}^{+\infty}\frac{2^n}{4n-3}x^{3n},\quad \sum_{n=1}^{+\infty}\frac{3^{n^2}}{\sqrt{n}}x^{n^2}.$$

3. .

$$\sum_{n=1}^{+\infty} \frac{(2n)!}{(n!)^2} x^n\,, \quad \sum_{n=1}^{+\infty} \frac{1}{2^n}\Big(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\Big)^3 x^n\,.$$

– – 10.1.

4. $p, q,$ 0, $a_0 + \sum_{n=1}^{+\infty} a_n x^n$ $a_{n+2} = pa_{n+1} + qa_n$ $n \geq 0$.

(1) $f(x)$ x – – $(1 - px - qx^2)f(x) = a_0 + (a_1 - pa_0)x$.

(: $a_{n+2} = pa_{n+1} + qa_n$ x^{n+2}, $\sum_{n=0}^{+\infty}$.)

(*) (2) .

(: 5 2.1.)

5. (*)

$$1 + \frac{ab}{1\cdot c}x + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}x^3 + \cdots\,,$$

$$y = F(a, b, c; x).$$

a, b c.

(: 10.1.)

10.6 Taylor.

$a_0 + \sum_{n=1}^{+\infty} a_n(x-\xi)^n$. R I, $\{\xi\}$, $R = 0$, $(-\infty, +\infty)$, $R = +\infty$, $\xi \pm R$, $0 < R < +\infty$. $R > 0$, I ξ. x I $f(x)$, $a_0 + \sum_{n=1}^{+\infty} a_n(x-\xi)^n$, I

$$y = f(x) = a_0 + \sum_{n=1}^{+\infty} a_n(x-\xi)^n \qquad (x \ \ I)$$

$y = f(x)$ $a_0 + \sum_{n=1}^{+\infty} a_n(x-\xi)^n$.

: $1 + \sum_{n=1}^{+\infty} x^n$ $(-1, 1)$ x $1 + \sum_{n=1}^{+\infty} x^n = \frac{1}{1-x}$. $y = \frac{1}{1-x}$ $(-1, 1)$ $1 + \sum_{n=1}^{+\infty} x^n$ $(-1, 1)$.

: $y = \frac{1}{1-x}$ ($1 + \sum_{n=1}^{+\infty} x^n$) $(-\infty, 1) \cup (1, +\infty)$ $(-1, 1)$, .

. $y = f(x)$ ξ . I ξ, ξ, $y = f(x)$ $a_0 + \sum_{n=1}^{+\infty} a_n(x-\xi)^n$ x I $f(x) = a_0 + \sum_{n=1}^{+\infty} a_n(x-\xi)^n$ x I, $a_0 + \sum_{n=1}^{+\infty} a_n(x-\xi)^n$ **Taylor** $y = f(x)$ ξ.

: $y = \frac{1}{1-x}$ $(-\infty, 1) \cup (1, +\infty)$. $(-1, 1)$ 0 $1 + \sum_{n=1}^{+\infty} x^n$ x $(-1, 1)$ $1 + \sum_{n=1}^{+\infty} x^n = \frac{1}{1-x}$ x $(-1, 1)$. , $1 + \sum_{n=1}^{+\infty} x^n$ Taylor $y = \frac{1}{1-x}$ 0.

.

. $a_0+\sum_{n=1}^{+\infty} a_n(x-\xi)^n$ I. , $y=f(x)=a_0+\sum_{n=1}^{+\infty} a_n(x-\xi)^n$ x I. ; , $1+\sum_{n=1}^{+\infty} x^n$ $(-1,1)$ $y=f(x)=1+\sum_{n=1}^{+\infty} x^n$ $y=f(x)=\frac{1}{1-x}$. ′ . $a_0+\sum_{k=1}^{n} a_k(x-\xi)^k$ $n\to+\infty$. $1+x+x^2+\cdots+x^n=\frac{1-x^{n+1}}{1-x}$, , $\frac{1}{1-x}$ $-1<x<1$. , , , , , $1+\sum_{n=1}^{+\infty}\frac{1}{n!}x^n$.

. $y=f(x)$ ξ . Taylor ξ ( ) ; , .

10.18 $y=f(x)$ I, ξ ξ. x I n

$$f(x)=f(\xi)+\frac{f'(\xi)}{1!}(x-\xi)+\cdots+\frac{f^{(n)}(\xi)}{n!}(x-\xi)^n+R_n(x;\xi;f),$$

$R_n(x;\xi;f)$ *Lagrange* . $\lim_{n\to+\infty} R_n(x;\xi;f)=0$ x I, *Taylor* ξ $f(\xi)+\sum_{n=1}^{+\infty}\frac{f^{(n)}(\xi)}{n!}(x-\xi)^n$.

$$\boxed{f(x)=f(\xi)+\sum_{n=1}^{+\infty}\frac{f^{(n)}(\xi)}{n!}(x-\xi)^n \qquad (x\ \ I).}$$

, Taylor $y=f(x)$ ξ $a_0=f(\xi)$ $a_n=\frac{f^{(n)}(\xi)}{n!}$ $(n\geq 1)$.

10.18 9.1 9.2. : $\lim_{n\to+\infty} R_n(x;\xi;f)=0$ x I, $\lim_{n\to+\infty}\big(f(\xi)+\frac{f'(\xi)}{1!}(x-\xi)+\cdots+\frac{f^{(n)}(\xi)}{n!}(x-\xi)^n\big)=f(x)$, , $f(\xi)+\sum_{n=1}^{+\infty}\frac{f^{(n)}(\xi)}{n!}(x-\xi)^n=f(x)$ x I.

: (1) $p(x)=a_0+a_1x+\cdots+a_Nx^N$ N ξ.

x $n\geq N$ $p^{(n+1)}(x)=0$, Lagrange $R_n(x;\xi;p)=\frac{p^{(n+1)}(\eta)}{(n+1)!}(x-\xi)^{n+1}=0$. $\lim_{n\to+\infty} R_n(x;\xi;p)=0$ x Taylor $y=p(x)$ ξ $p(\xi)+\sum_{n=1}^{+\infty}\frac{p^{(n)}(\xi)}{n!}(x-\xi)^n=p(\xi)+\frac{p'(\xi)}{1!}(x-\xi)+\cdots+\frac{p^{(N)}(\xi)}{N!}(x-\xi)^N$.

$$\boxed{p(x)=p(\xi)+\frac{p'(\xi)}{1!}(x-\xi)+\cdots+\frac{p^{(N)}(\xi)}{N!}(x-\xi)^N\,.}$$

$x-\xi$ (x). , $\xi=0$ $p(x)=p(0)+\frac{p'(0)}{1!}x+\cdots+\frac{p^{(N)}(0)}{N!}x^N$.

(2) $y=e^x$, $(-\infty,+\infty)$, , $\frac{d^n e^x}{dx^n}=e^x$ n x. , $\frac{d^n e^x}{dx^n}\big|_{x=0}=1$ n, Taylor $y=e^x$ 0 $1+\sum_{n=1}^{+\infty}\frac{1}{n!}x^n$.

$\xi=0$ Lagrange $R_n(x;0;e^x)=\frac{e^\eta}{(n+1)!}x^{n+1}$, η 0 x. $x\geq 0$, $|R_n(x;0;e^x)|=\frac{e^\eta}{(n+1)!}|x|^{n+1}\leq\frac{e^x}{(n+1)!}|x|^{n+1}$, $x\leq 0$, $|R_n(x;0;e^x)|=\frac{e^\eta}{(n+1)!}|x|^{n+1}\leq\frac{1}{(n+1)!}|x|^{n+1}$.

a

$$\boxed{\lim_{n\to+\infty}\frac{a^n}{n!}=0.}$$

. $1+\sum_{n=1}^{+\infty}\frac{1}{n!}x^n$ x, $x=a$. $1+\sum_{n=1}^{+\infty}\frac{1}{n!}a^n$.

$\lim_{n\to+\infty}\frac{a^n}{n!}=0$. $m=[|a|]$, $n\geq m+1$ $0\leq\frac{|a|^n}{n!}=\frac{|a|^m}{m!}\frac{|a|}{m+1}\cdots\frac{|a|}{n}\leq\frac{|a|^m}{m!}\frac{|a|}{m+1}\cdots\frac{|a|}{m+1}=\frac{(m+1)^m}{m!}(\frac{|a|}{m+1})^n$. $0\leq\frac{|a|}{m+1}<1$, $\lim_{n\to+\infty}(\frac{|a|}{m+1})^n=0$. $\lim_{n\to+\infty}\left|\frac{a^n}{n!}\right|=\lim_{n\to+\infty}\frac{|a|^n}{n!}=0$, , $\lim_{n\to+\infty}\frac{a^n}{n!}=0$.

, $\lim_{n\to+\infty}|R_n(x;0;e^x)|=0$, , $\lim_{n\to+\infty}R_n(x;0;e^x)=0$ x. Taylor $y=e^x$ 0 $1+\sum_{n=1}^{+\infty}\frac{1}{n!}x^n$ $(-\infty,+\infty)$.

$$e^x=1+\sum_{n=1}^{+\infty}\frac{1}{n!}x^n=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\quad(-\infty<x<+\infty).$$

$1+\sum_{n=1}^{+\infty}\frac{1}{n!}x^n$ $(-\infty,+\infty)$.
$1+\sum_{n=1}^{+\infty}\frac{1}{n!}x^n$.

(3) $y=\cos x$ $(-\infty,+\infty)$ $\frac{d^n\cos x}{dx^n}=\pm\cos x$ $\pm\sin x$. , $\frac{d^n\cos x}{dx^n}\Big|_{x=0}=1$ 0 -1 0 n 4 0 1 2 3, . Taylor $y=\cos x$ 0 $1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{(2k)!}x^{2k}$.
$\xi=0$ Lagrange $R_n(x;0;\cos x)=\frac{\pm\cos\eta}{(n+1)!}x^{n+1}$ $\frac{\pm\sin\eta}{(n+1)!}x^{n+1}$, η 0 x. $|R_n(x;0;\cos x)|\leq\frac{|x|^{n+1}}{(n+1)!}$, $\lim_{n\to+\infty}R_n(x;0;\cos x)=0$ x. Taylor $y=\cos x$ 0 $1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{(2k)!}x^{2k}$ $(-\infty,+\infty)$.

$$\cos x=1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{(2k)!}x^{2k}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\quad(-\infty<x<+\infty).$$

, $1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{(2k)!}x^{2k}$ $(-\infty,+\infty)$ $y=\cos x$.
Taylor $y=\sin x$ 0:

$$\sin x=\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{(2k-1)!}x^{2k-1}=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\quad(-\infty<x<+\infty).$$

$1+\sum_{k=1}^{+\infty}\frac{(-1)^k}{(2k)!}x^{2k}$ $\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{(2k-1)!}x^{2k-1}$.

(4) $y=\log(1+x)$ $(-1,+\infty)$ $\frac{d^n\log(1+x)}{dx^n}=\frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ n x $(-1,+\infty)$. , $\frac{d^n\log(1+x)}{dx^n}\Big|_{x=0}=(-1)^{n-1}(n-1)!$, Taylor $y=\log(1+x)$ 0 $\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}(n-1)!}{n!}x^n=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}x^n$. , , $(-1,1]$.

$$\log(1+x)=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}x^n=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\quad(-1<x\leq1).$$

Lagrange $y=\log(1+x)$ $\xi=0$ $R_n(x;0;\log(1+x))=\frac{(-1)^n n!}{(1+\eta)^{n+1}(n+1)!}x^{n+1}=\frac{(-1)^n}{(1+\eta)^{n+1}(n+1)}x^{n+1}$, η 0 x. $0<x\leq1$, $|R_n(x;0;\log(1+x))|=\frac{x^{n+1}}{(1+\eta)^{n+1}(n+1)}\leq\frac{1}{n+1}$, $\lim_{n\to+\infty}R_n(x;0;\log(1+x))=0$.

$R_n(x;0;\log(1+x)) = \frac{1}{n!}\int_0^x \frac{(-1)^n n!}{(1+t)^{n+1}}(x-t)^n\,dt = (-1)^n\int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,dt$. $-1 < x \le 0$, $R_n(x;0;\log(1+x)) = -\int_x^0 \frac{(t-x)^n}{(1+t)^{n+1}}\,dt$, $|R_n(x;0;\log(1+x))| = \int_x^0 \frac{(t-x)^n}{(1+t)^{n+1}}\,dt$. $(\frac{t-x}{1+t})^n \le |x|^n$ t $[x,0]$, $|R_n(x;0;\log(1+x))| \le |x|^n \int_x^0 \frac{1}{1+t}\,dt = |x|^n \log\frac{1}{1+x}$. $\lim_{n\to+\infty} R_n(x;0;\log(1+x)) = 0$.

, , $\lim_{n\to+\infty} R_n(x;0;\log(1+x)) = 0$ x $-1 < x \le 1$.

x $-x$,

$$\log\frac{1}{1-x} = \sum_{n=1}^{+\infty}\frac{1}{n}x^n = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \quad (-1 \le x < 1).$$

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

$\sum_{n=1}^{+\infty}\frac{1}{n}x^n$ $[-1,1)$ $[-1,1)$ $y = \log\frac{1}{1-x}$.

Taylor $y = \log(1+x)$ $(-1,1]$ 10.18. , 10.18.

$\frac{1-(-t)^n}{1+t} = 1 + (-t) + \cdots + (-t)^{n-1}$, $t \ne -1$, $\frac{1}{1+t} = 1 - t + \cdots + (-1)^{n-1}t^{n-1} + (-1)^n\frac{t^n}{1+t}$. , $\int_0^x \frac{1}{1+t}\,dt = \int_0^x 1\,dt - \int_0^x t\,dt + \cdots + (-1)^{n-1}\int_0^x t^{n-1}\,dt + (-1)^n\int_0^x \frac{t^n}{1+t}\,dt$ $x > -1$,

$$\log(1+x) = x - \frac{1}{2}x^2 + \cdots + \frac{(-1)^{n-1}}{n}x^n + (-1)^n\int_0^x \frac{t^n}{1+t}\,dt$$

$x > -1$.

$0 \le x \le 1$, $\left|(-1)^n\int_0^x \frac{t^n}{1+t}\,dt\right| = \int_0^x \frac{t^n}{1+t}\,dt \le \int_0^x t^n\,dt = \frac{x^{n+1}}{n+1} \le \frac{1}{n+1}$, , $\lim_{n\to+\infty}(-1)^n\int_0^x \frac{t^n}{1+t}\,dt = 0$.

$-1 < x \le 0$, $\left|(-1)^n\int_0^x \frac{t^n}{1+t}\,dt\right| = \int_x^0 \frac{|t|^n}{1+t}\,dt \le \frac{1}{1+x}\int_x^0 |t|^n\,dt = \frac{|x|^{n+1}}{(n+1)(1+x)} \le \frac{1}{(n+1)(1+x)}$, $\lim_{n\to+\infty}(-1)^n\int_0^x \frac{t^n}{1+t}\,dt = 0$.

x $(-1,1]$ $\lim_{n\to+\infty}(-1)^n\int_0^x \frac{t^n}{1+t}\,dt = 0$, , $\lim_{n\to+\infty}\left(x - \frac{1}{2}x^2 + \cdots + \frac{(-1)^{n-1}}{n}x^n\right) = \log(1+x)$. $\log(1+x) = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}x^n$ x $(-1,1]$.

(5) Taylor $y = \arctan x$ 0 $\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{2n-1}x^{2n-1}$ $[-1,1]$.

$$\arctan x = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{2n-1}x^{2n-1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots \quad (-1 \le x \le 1).$$

$[-1,1]$.

$x = 1$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

$y = \arctan x$ $\frac{1}{x^2+1}$ 10.18. .

$\frac{1-(-t^2)^n}{1+t^2} = 1 - t^2 + t^4 + \cdots + (-1)^{n-1}t^{2n-2}$, $\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots + (-1)^{n-1}t^{2n-2} + (-1)^n\frac{t^{2n}}{1+t^2}$ t. , $\int_0^x \frac{1}{1+t^2}\,dt = \int_0^x 1\,dt - \int_0^x t^2\,dt + \int_0^x t^4\,dt - \cdots + (-1)^{n-1}\int_0^x t^{2n-2}\,dt + (-1)^n\int_0^x \frac{t^{2n}}{1+t^2}\,dt$.

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots + \frac{(-1)^{n-1}}{2n-1}x^{2n-1} + (-1)^n\int_0^x \frac{t^{2n}}{1+t^2}\,dt.$$

$|x| \le 1$, $\left|(-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt\right| = \int_0^{|x|} \frac{t^{2n}}{1+t^2}\,dt \le \int_0^{|x|} t^{2n}\,dt = \frac{|x|^{2n+1}}{2n+1} \le \frac{1}{2n+1}$, $\lim_{n\to+\infty}(-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt = 0$.

$\lim_{n\to+\infty}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots + \frac{(-1)^{n-1}}{2n-1}x^{2n-1}\right) = \arctan x$ x $[-1,1]$, $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{2n-1}x^{2n-1} = \arctan x$ x $[-1,1]$.

(6) $y = (1+x)^\alpha$. $y = (1+x)^\alpha$ $\frac{d^n\,(1+x)^\alpha}{dx^n} = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$, , $\frac{d^n\,(1+x)^\alpha}{dx^n}\Big|_{x=0} = \alpha(\alpha-1)\cdots(\alpha-n+1)$. , Taylor $y = (1+x)^\alpha$ 0 $1 + \sum_{n=1}^{+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n = 1 + \sum_{n=1}^{+\infty}\binom{\alpha}{n}x^n$.

α , ′ $y = (1+x)^\alpha$ α $(-\infty,+\infty)$ ′ – – $1 + \binom{\alpha}{1}x + \cdots + \binom{\alpha}{\alpha-1}x^{\alpha-1} + \binom{\alpha}{\alpha}x^\alpha$.

$$(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \cdots + \binom{\alpha}{\alpha-1}x^{\alpha-1} + \binom{\alpha}{\alpha}x^\alpha$$

Newton , , , , Taylor $y = (1+x)^\alpha$ 0 $(-\infty,+\infty)$.

α , Taylor $y = (1+x)^\alpha$ 0 $1 + \sum_{n=1}^{+\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n = 1 + \sum_{n=1}^{+\infty}\binom{\alpha}{n}x^n$. $y = (1+x)^\alpha$ (i) $(-1,1)$, $\alpha \le -1$, (ii) $(-1,1]$, $-1 < \alpha < 0$, (iii) $[-1,1]$, $\alpha \ge 0$ (α).

$$(1+x)^\alpha = 1 + \sum_{n=1}^{+\infty}\binom{\alpha}{n}x^n \qquad \begin{pmatrix} -1 < x < 1, & \alpha \le -1, \\ -1 < x \le 1, & -1 < \alpha < 0, \\ -1 \le x \le 1, & \alpha \ge 0 \ \ \alpha \ \ \mathbf{Z}. \end{pmatrix}$$

Lagrange $R_n(x;0;(1+x)^\alpha) = \frac{\alpha(\alpha-1)\cdots(\alpha-n)(1+\eta)^{\alpha-n-1}}{(n+1)!}x^{n+1} = \binom{\alpha}{n+1}(1+\eta)^{\alpha-n-1}x^{n+1}$ η 0 x.

$0 \le x \le 1$, $x \le 1 \le 1+\eta \le 2$, , $|R_n(x;0;(1+x)^\alpha)| = \left|\binom{\alpha}{n+1}\right|(1+\eta)^\alpha(\frac{x}{1+\eta})^{n+1}$. , $a > 0$, 10.1, $n > [\alpha]$ $|R_n(x;0;(1+x)^\alpha)| \le c_2\binom{\alpha}{[\alpha]+1}(n-[\alpha])^{-\alpha-1}2^\alpha$, $\lim_{n\to+\infty} R_n(x;0;(1+x)^\alpha) = 0$. $-1 < \alpha < 0$, 10.1 $|R_n(x;0;(1+x)^\alpha)| \le c_2(n+1)^{-\alpha-1}$, $\lim_{n\to+\infty} R_n(x;0;(1+x)^\alpha) = 0$.

$0 \le x < 1$ $\alpha \le -1$, 10.1 $|R_n(x;0;(1+x)^\alpha)| \le \left|\binom{\alpha}{n+1}\right| x^{n+1} \le c_2(n+1)^{-\alpha-1}x^{n+1}$, $\lim_{n\to+\infty} R_n(x;0;(1+x)^\alpha) = 0$.

$-1 < x \le 0$, $R_n(x;0;(1+x)^\alpha) = \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}\int_0^x (1+t)^{\alpha-n-1}(x-t)^n\,dt$, , $|R_n(x;0;(1+x)^\alpha)| = \left|\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}\right| \int_x^0 (1+t)^{\alpha-n-1}(t-x)^n\,dt$. $\frac{t-x}{1+t} \le -x$ $x \le t \le 0$, $|R_n(x;0;(1+x)^\alpha)| \le \left|\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}\right||x|^n \int_x^0 (1+t)^{\alpha-1}\,dt = \left|\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}\right||x|^n \frac{1-(1+x)^\alpha}{\alpha} = (n+1)\left|\binom{\alpha}{n+1}\right||x|^n \frac{1-(1+x)^\alpha}{\alpha}$. $\alpha > 0$, $n > [\alpha]$ $|R_n(x;0;(1+x)^\alpha)| \le c_2(n+1)\binom{\alpha}{[\alpha]+1}(n-[\alpha])^{-\alpha-1}|x|^n \frac{1-(1+x)^\alpha}{\alpha}$, $\lim_{n\to+\infty} R_n(x;0;(1+x)^\alpha) = 0$. $\alpha < 0$, $|R_n(x;0;(1+x)^\alpha)| \le c_2(n+1)^{-\alpha}|x|^n \frac{(1+x)^\alpha - 1}{|\alpha|}$, $\lim_{n\to+\infty} R_n(x;0;(1+x)^\alpha) = 0$.

: $\lim_{n\to+\infty}\left(1 + \binom{\alpha}{1}x + \cdots + \binom{\alpha}{n}x^n\right) = (1+x)^\alpha$, , $1 + \sum_{n=1}^{+\infty}\binom{\alpha}{n}x^n = (1+x)^\alpha$.

$x = -1$ $\alpha > 0$. , , 9.2, . $-1 < x \le 0$ $\left|(1+x)^\alpha - 1 - \binom{\alpha}{1}x - \cdots - \binom{\alpha}{n}x^n\right| = |R_n(x;0;(1+x)^\alpha)| \le c_2(n+1)\binom{\alpha}{[\alpha]+1}(n-[\alpha])^{-\alpha-1}|x|^n \frac{1-(1+x)^\alpha}{\alpha}$. $x \to -1+$, $\left|0 - 1 - \binom{\alpha}{1}(-1) - \cdots - \binom{\alpha}{n}(-1)^n\right| \le c_2(n+1)\binom{\alpha}{[\alpha]+1}(n-[\alpha])^{-\alpha-1}\frac{1}{\alpha}$. $\lim_{n\to+\infty}\left(1 + \binom{\alpha}{1}(-1) + \cdots + \binom{\alpha}{n}(-1)^n\right) = 0$. – $x = -1$ $\alpha > 0$ – $1 + \sum_{n=1}^{+\infty}\binom{\alpha}{n}x^n = (1+x)^\alpha$.

. $\alpha = \frac{1}{2}$

$$\sqrt{1+x} = 1 + \sum_{n=1}^{+\infty} \binom{\frac{1}{2}}{n} x^n = 1 + \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} \frac{x^n}{2n-1}$$

$$= 1 + \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \cdots$$

x $[-1,1]$. $\alpha = -\frac{1}{2}$

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{+\infty} \binom{-\frac{1}{2}}{n} x^n = 1 + \sum_{n=1}^{+\infty} (-1)^{n} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)} x^n$$

$$= 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 - \cdots$$

x $(-1,1]$.

.

1. $1+2x-x^2-4x^3+x^4$ $x-1$.
2. . , $1+x+\cdots+x^n = \frac{1-x^{n+1}}{1-x}$.

$$\sum_{n=1}^{+\infty} nx^n\,, \qquad \sum_{n=1}^{+\infty} n^2x^n\,, \qquad \sum_{n=1}^{+\infty} n^3x^n\,.$$

3. Taylor .

$$\sum_{n=1}^{+\infty} \big(1-(-2)^n\big)x^n\,, \quad \sum_{n=1}^{+\infty} \frac{1}{2n-1}x^{2n-1}\,, \quad \sum_{n=1}^{+\infty} \frac{1}{2n}x^{2n}\,, \quad 1+\sum_{n=1}^{+\infty} \frac{(-1)^n}{n!}x^{2n}\,,$$

$$\sum_{n=1}^{+\infty} \frac{n-1}{n!}x^n\,, \quad \sum_{n=1}^{+\infty} \frac{(n+1)^2}{n!}x^n\,, \quad \sum_{n=1}^{+\infty} \frac{1}{(2n-1)!}x^{2n-1}\,, \quad 1+\sum_{n=1}^{+\infty} \frac{1}{(2n)!}x^{2n}\,,$$

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}2^{2n-1}}{(2n)!}x^{2n}\,, \quad 1+\sum_{n=1}^{+\infty} \frac{(\log a)^n}{n!}x^n\,, \quad \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{2n-1}\Big(\frac{x-1}{x+1}\Big)^{2n-1}\,,$$

$$1+\sum_{n=1}^{+\infty} (-1)^n \frac{1\cdot 4\cdot 7\cdots(3n-2)}{3\cdot 6\cdot 9\cdots(3n)}x^n\,, \quad 1+\sum_{n=1}^{+\infty} \frac{(2n)!}{2^n(n!)^2}x^n\,.$$

4. Taylor $y=\sin x$ $y=\cos x$ ξ.
   Taylor,
   (i) $\sin x = \sin\xi\cos(x-\xi) + \cos\xi\sin(x-\xi)$
   (ii) $\cos x = \cos\xi\cos(x-\xi) - \sin\xi\sin(x-\xi)$
   .

5. Taylor 0 .

$$y = \cosh x, \qquad y = \sinh x.$$

6. Taylor 0 .

$$y = \tan x, \qquad y = \frac{1}{\cos x}, \qquad y = \arcsin x, \qquad y = \arccos x.$$

7. .

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{(2n)^2 - 1}, \qquad \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{(2n+1)^2 - 1}.$$

8. 18 6.12 $y = h(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0, \\ 0, & x \le 0, \end{cases}$ $(-\infty, +\infty)$ $h^{(n)}(0) = 0$ n. $h(0) + \sum_{n=1}^{+\infty} \frac{h^{(n)}(0)}{n!} x^n$ Taylor $y = h(x)$ 0;

9. $f(0) + \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n$ Taylor $y = f(x)$ 0. $y = f(x)$ 0 Taylor x , , .

10.7 , .

. .

10.5 $1 + \sum_{n=1}^{+\infty} \frac{1}{n!} x^n$ x $(-\infty, +\infty)$ 10.6 $y = e^x$. , , ! , , $y = e^x$ $(-\infty, +\infty)$

$$e^x = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} x^n \qquad (-\infty < x < +\infty).$$

$e^0 = 1$. , , $e^x \ge 1$ $x > 0$. $y = e^x$ $\frac{d\,e^x}{dx} = e^x$ $(-\infty, +\infty)$.

: .

10.2 x $h \neq 0$ $|h| \le 1$ $n \ge 2$

$$\Big| \frac{(x+h)^n - x^n}{h} - nx^{n-1} \Big| \le n(n-1)(|x|+1)^{n-2}|h|.$$

: , $\frac{(x+h)^n - x^n}{h} = n(x+\xi)^{n-1}$ ξ 0 h. $(x+\xi)^{n-1} - x^{n-1} = (n-1)(x+\eta)^{n-2}\xi$ η 0 ξ , , 0 h. $\Big|\frac{(x+h)^n - x^n}{h} - nx^{n-1}\Big| = |n(x+\xi)^{n-1} - nx^{n-1}| = n(n-1)|x+\eta|^{n-2}|\xi| \le n(n-1)(|x|+1)^{n-2}|h|$.

$\frac{d\,e^x}{dx} = e^x$: $h \neq 0$ $|h| \le 1$

$$\begin{aligned} \Big| \frac{e^{x+h} - e^x}{h} - e^x \Big| &= \Big| \sum_{n=2}^{+\infty} \frac{1}{n!} \frac{(x+h)^n - x^n}{h} - \sum_{n=2}^{+\infty} \frac{1}{(n-1)!} x^{n-1} \Big| \\ &= \Big| \sum_{n=2}^{+\infty} \frac{1}{n!} \Big(\frac{(x+h)^n - x^n}{h} - nx^{n-1} \Big) \Big| \end{aligned}$$

$$\le \sum_{n=2}^{+\infty} \frac{1}{n!}\left|\frac{(x+h)^n - x^n}{h} - nx^{n-1}\right|$$
$$\le \sum_{n=2}^{+\infty} \frac{1}{n!} n(n-1)(|x|+1)^{n-2}|h|$$
$$= e^{|x|+1}|h|.$$

$\lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = e^x$, , $\frac{d\,e^x}{dx} = e^x$.

, $e^{a+b} = e^a e^b$ a, b. $y = h(x) = e^{a+b-x}e^x$, $h'(x) = -e^{a+b-x}e^x + e^{a+b-x}e^x = 0$, , $y = h(x)$ $(-\infty, +\infty)$. $e^{a+b-x}e^x = h(x) = h(0) = e^{a+b-0}e^0 = e^{a+b}$ x $x = b$ $e^{a+b} = e^a e^b$. , $e^x e^{-x} = e^0 = 1$, $e^x \neq 0$ x. $e^0 = 1 > 0$, $e^x > 0$ x. $\frac{d\,e^x}{dx} = e^x$ $y = e^x$ $(-\infty, +\infty)$.

$y = e^x$ $e^x \ge x$ $x \ge 0$, $\lim_{x\to+\infty} e^x = +\infty$. , $\lim_{x\to-\infty} e^x = \lim_{t\to+\infty} e^{-t} = \lim_{t\to+\infty} \frac{1}{e^t} = 0$. $y = e^x$ $(-\infty, +\infty)$ $(0, +\infty)$.

, e

$$e = e^1 = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!}\,.$$

, $y = \log x$ $x = e^y$. ,

$$y = \log x \qquad\qquad x = e^y\,.$$

$y = \log x$ $(0, +\infty)$ $(-\infty, +\infty)$, $\frac{d\,\log x}{dx} = \frac{1}{\frac{d\,e^y}{dy}\big|_{y=\log x}} = \frac{1}{x}$ $(0, +\infty)$. $\log(ab) = \log a + \log b$. , $c = \log a$ $d = \log b$ $ab = e^c e^d = e^{c+d}$. $\log(ab) = c + d = \log a + \log b$. , $y = h(x) = \log(xb) - \log x$ $h'(x) = b\frac{1}{xb} - \frac{1}{x} = 0$ x. $y = h(x)$, $\log(xb) - \log x = h(x) = h(1) = \log b - \log 1 = \log b$ x. $x = a$ $\log(ab) = \log a + \log b$.

, $a > 0$ x

$$a^x = e^{x\log a}\,.$$

$a^1 = e^{1\cdot\log a} = e^{\log a} = a$ $a^0 = e^{0\cdot\log a} = e^0 = 1$. a^x .

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, «», a^x x, ¨, , .

$a > 1$ $x > 0$. (s_n) x, (s_n) , $s_n < x$ n $\lim_{n\to+\infty} s_n = x$. $s > x$, $s = [x] + 1$, $a^{s_n} < a^s$ n. (a^{s_n}) , , . a^x ,

$$a^x = \lim_{n\to+\infty} a^{s_n}\,.$$

, $x < 0$,

$$a^x = \frac{1}{a^{-x}}$$

, , (s_n) $-x > 0$, $\lim_{n\to+\infty}(-s_n) = -\lim_{n\to+\infty} s_n = -(-x) = x$ $\lim_{n\to+\infty} a^{-s_n} = \lim_{n\to+\infty} \frac{1}{a^{s_n}} = \frac{1}{a^{-x}} = a^x$.

, , x (t_n) $\lim_{n\to+\infty} t_n = x$ $\lim_{n\to+\infty} a^{t_n} = a^x$. , , x. , (t_n) $t_n = x$ n , , $\lim_{n\to+\infty} t_n = x$ $\lim_{n\to+\infty} a^{t_n} = a^x$.

: x (t_n) $\lim_{n\to+\infty} t_n = x$ $\lim_{n\to+\infty} a^{t_n} = a^x$.

(r_n) $\lim_{n\to+\infty} r_n = 0$ $\lim_{n\to+\infty} a^{r_n} = 1$. 2.1. ϵ $0 < \epsilon < 1$ $N = [\frac{a-1}{\epsilon}] + 1$. $\lim_{n\to+\infty} r_n = 0$, n_0 $|r_n| < \frac{1}{N}$ $n \geq n_0$. $n \geq n_0$ $(1+\epsilon)^N \geq 1 + N\epsilon > 1 + \frac{a-1}{\epsilon}\epsilon = a$, , $a^{r_n} < a^{\frac{1}{N}} < 1+\epsilon$. , $n \geq n_0$ $1-\epsilon < \frac{1}{1+\epsilon} < a^{-\frac{1}{N}} < a^{r_n}$. , $n \geq n_0$ $1-\epsilon < a^{r_n} < 1+\epsilon$.

(r_n) $\lim_{n\to+\infty} r_n = x$ $\lim_{n\to+\infty} a^{r_n} = a^x$. , $r_n - x$ $\lim_{n\to+\infty}(r_n - x) = 0$, $\lim_{n\to+\infty} a^{r_n} = \lim_{n\to+\infty} a^{r_n - x} a^x = 1 \cdot a^x = a^x$.

(r_n) $\lim_{n\to+\infty} r_n = x$ $\lim_{n\to+\infty} a^{r_n} = a^x$. , (t_n) – – $\lim_{n\to+\infty} t_n = x$ $\lim_{n\to+\infty} a^{t_n} = a^x$. $r_n - t_n$ $\lim_{n\to+\infty}(r_n - t_n) = x - x = 0$, $\lim_{n\to+\infty} a^{r_n} = \lim_{n\to+\infty} a^{r_n - t_n} a^{t_n} = 1 \cdot a^x = a^x$.

: x (r_n) $\lim_{n\to+\infty} r_n = x$ $\lim_{n\to+\infty} a^{r_n} = a^x$.

$a > 1$. $1^x = 1$ x , $0 < a < 1$, $a^x = (\frac{1}{a})^{-x}$ x. , , x $a = 1$ $0 < a < 1$.

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x, y $a, b > 0$. (r_n) (u_n) $\lim_{n\to+\infty} r_n = x$ $\lim_{n\to+\infty} u_n = y$. $(r_n + u_n)$ $\lim_{n\to+\infty}(r_n + u_n) = x + y$. , $(r_n u_n)$ $\lim_{n\to+\infty}(r_n u_n) = xy$.

$$a^x a^y = \lim_{n\to+\infty} a^{r_n} \lim_{n\to+\infty} a^{u_n} = \lim_{n\to+\infty} a^{r_n} a^{u_n} = \lim_{n\to+\infty} a^{r_n + u_n} = a^{x+y} .$$

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$$a^x b^x = \lim_{n\to+\infty} a^{r_n} \lim_{n\to+\infty} b^{r_n} = \lim_{n\to+\infty} a^{r_n} b^{r_n} = \lim_{n\to+\infty} (ab)^{r_n} = (ab)^x .$$

$(a^x)^y = a^{xy}$. , , : $x_n > 0$ n, $\lim_{n\to+\infty} x_n = 1$ (y_n) , $\lim_{n\to+\infty} {x_n}^{y_n} = 1$. n ${x_n}' = \min\{x_n, \frac{1}{x_n}\} = \frac{x_n + \frac{1}{x_n} - |x_n - \frac{1}{x_n}|}{2}$ ${x_n}'' = \max\{x_n, \frac{1}{x_n}\} = \frac{x_n + \frac{1}{x_n} + |x_n - \frac{1}{x_n}|}{2}$. ${x_n}' \leq x_n \leq {x_n}''$ ${x_n}' \leq \frac{1}{x_n} \leq {x_n}''$ n , , $\lim_{n\to+\infty} {x_n}' = \lim_{n\to+\infty} {x_n}'' = 1$. , (y_n) , $-N \leq y_n \leq N$ n. $({x_n}')^{-N} \leq {x_n}^{y_n} \leq ({x_n}'')^N$ n $\lim_{n\to+\infty} {x_n}^{y_n} = 1$. , !

$$\begin{aligned} a^{xy} &= \lim_{n\to+\infty} a^{r_n u_n} = \lim_{n\to+\infty} (a^{r_n})^{u_n} = \lim_{n\to+\infty} \left(\frac{a^{r_n}}{a^x} a^x\right)^{u_n} \\ &= \lim_{n\to+\infty} \left(\frac{a^{r_n}}{a^x}\right)^{u_n} \lim_{n\to+\infty} (a^x)^{u_n} = 1 \cdot (a^x)^y = (a^x)^y . \end{aligned}$$

, $a > 1$ $x > 0$. x , $a^x > 1$. x . , , (s_n) x. $\lim_{n\to+\infty} s_n = x$, n_0 $s_{n_0} > 0$. $a^{s_{n_0}} > 1$. $a^x = \lim_{n\to+\infty} a^{s_n}$ (a^{s_n}) , $a^x \geq a^{s_n}$ n. $n = n_0$ $a^x > 1$. .

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, , $y=\cos x$ $y=\sin x$. :

$$\cos x = 1 + \sum_{k=1}^{+\infty} \frac{(-1)^k}{(2k)!} x^{2k}\,, \qquad \sin x = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{(2k-1)!} x^{2k-1}\,.$$

10.5 x, $\cos x$ $\sin x$ x. $\frac{d\,e^x}{dx} = e^x$, , $\frac{d\,\cos x}{dx} = -\sin x$ $\frac{d\,\sin x}{dx} = \cos x$. $\cos x$ x , $y=\cos x$. , $y=\sin x$. , $\cos 0 = 1$ $\sin 0 = 0$. $y = h(x) = \cos(a+b-x)\cos x - \sin(a+b-x)\sin x$, $(-\infty,+\infty)$, $\cos(a+b-x)\cos x - \sin(a+b-x)\sin x = h(x) = h(0) = \cos(a+b)$ x. $x=b$ $\cos a\cos b - \sin a \sin b = \cos(a+b)$. , $b=-a$, $(\cos a)^2 + (\sin a)^2 = 1$ $y=\cos x$ $y=\sin x$: $|\cos x| \leq 1$ $|\sin x| \leq 1$ x.

$\cos x \neq 0$ x, $\cos 0 = 1$, $\cos x > 0$ x. $y=\sin x$, $\sin x \geq \sin 1 > \sin 0 = 0$ $x \geq 1$. , , $x > 1$ ξ $[1,x]$ $\cos x - \cos 1 = -(x-1)\sin\xi \leq -(x-1)\sin 1$. $\cos x \leq -\sin 1 \cdot x + \cos 1 + \sin 1$, , $\lim_{x\to+\infty} \cos x = -\infty$, $y=\cos x$. x $\cos x = 0$. $y=\cos x$, $x>0$ $\cos x = 0$. , − , , − $x>0$ $\cos x = 0$. π.

$$\pi = 2 \cdot \qquad \cos x = 0.$$

, , $\cos\frac{\pi}{2} = 0$ $\cos x > 0$ x $[0,\frac{\pi}{2})$. $y=\sin x$ $[0,\frac{\pi}{2}]$, $\sin\frac{\pi}{2} > \sin 0 = 0$ $(\sin\frac{\pi}{2})^2 + (\cos\frac{\pi}{2})^2 = 1$ $\sin\frac{\pi}{2} = 1$. $y=\sin x$ $[0,\frac{\pi}{2}]$ $[0,1]$. o $\sin x > 0$ $(0,\frac{\pi}{2})$ $y=\cos x$ $[0,\frac{\pi}{2}]$, $y=\cos x$ $[0,\frac{\pi}{2}]$ $[0,1]$. $\cos x = -\sin(x-\frac{\pi}{2})$ $\sin x = \cos(x-\frac{\pi}{2})$, $\cos a\cos b - \sin a\sin b = \cos(a+b)$, $y=\cos x$ $y=\sin x$ $[\frac{\pi}{2},\pi]$, $[\pi,\frac{3\pi}{2}]$ $[\frac{3\pi}{2},2\pi]$, , $[0,2\pi]$. , , $\cos(x+2\pi) = \cos x$ $\sin(x+2\pi) = \sin x$ x, 2π.

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