# Newton's Laws and Coin Tossing

*Don Ornstein*

Abstract ergodic theory is the study of 1-1 (invertible) measure-preserving transformations $T$ on a measure space $X$ of total measure 1 or a one-parameter family of such transformations $T_t$ where $T_{t_2}(T_{t_1}(x)) = T_{t_1+t_2}(x)$ $(x \in X)$. (We call the latter a flow.)

There has been a recent explosion of interest in abstract ergodic theory, due mainly to its unexpected applications to number theory. It may therefore be of interest to revisit an early chapter in its history.

I am going to tell the story[1] of Bernoulli shifts, the abstract version of coin tossing. A central piece of this story is the existence of an abstract flow, the Bernoulli flow, or $B_t$, which pastes together the Bernoulli shifts.

$B_t$ is the "most random abstract flow possible". A priori, this statement has no real meaning, and to the extent that it does, one would expect a host of competitors for "most random". We will, however, state several properties of $B_t$ which, when taken together, can be interpreted as "$B_t$ is the most random abstract flow possible", even though we do not give a precise definition of "random".[2]

$B_t$ is the cause of all randomness in dynamical systems (for a precise statement, see "Positive Entropy and Factors").

There are classical dynamical systems, governed by Newton's Laws, that are the same as $B_t$ at our level of abstraction. We will discuss the concrete implications of this abstract equivalence and the relevance to chaos theory.

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[1] *We will also point out certain questions which at the time were overlooked.*

[2] *See the section "Positive Entropy and Factors".*

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This paper is narrowly focused and is not a survey. In particular, I do not discuss the first deep isomorphism theorem, which is due to Adler and Weiss [AW67] (or the directions it initiated: Markov partitions, finitary codings, etc.), even though it was an inspiration for so much later work. Nor do I discuss the Keane-Smorodinsky finitary isomorphism theorem for Bernoulli shifts.

The bibliography is very incomplete. One criterion I used is that the title should give some idea of the main result, so that reading the bibliography will add to the picture described in the text.

### Isomorphism

To pin down the objects we are talking about, we need to say when two abstract transformations or flows are the same or isomorphic.

$(T_t, X)$ and $(\hat{T}_t, \hat{X})$ are isomorphic if there is a 1-1 invertible measure-preserving map $\psi$ from $X$ to $\hat{X}$ such that $\hat{T}_t = \psi^{-1} T_t \psi$. (We have the same definition for transformations.)

For flows there is another weaker sense in which two flows could be the same. We say that $\hat{T}_t$ is "essentially the same" as $T_t$ if $\hat{T}_t = T_{ct}$ for some constant $c$.

## There Are Two Contexts In Which Abstract Ergodic Theory Arose: Concrete or Classical Dynamical Systems (Like the Time Evolution of a Gas Confined to a Box) and Stationary Processes

### Classical Dynamical Systems

Here, we try to abstract-out the statistical properties of the mechanism governing the time evolution. In particular, we ignore events of probability zero.

The state or configuration of the system is represented by a point $x$ in a smooth manifold $M$, the phase space. Newton's laws determine where $x$ in $M$ will be at time $t$; denote this by $T_t(x)$.

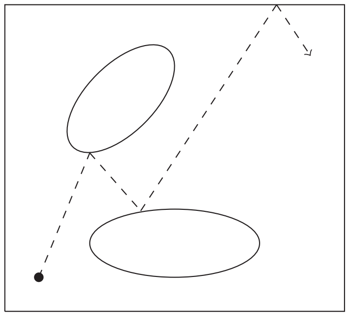

**Figure 1. Sinai billiards: the motion of a billiard ball on a frictionless table with convex obstacles.**

There is a smooth invariant probability measure $\mu$ with the same sets of measure 0 as Lebesgue measure (Liouville's theorem). $\mu(E)$ models the probability that the state of the system is in $E$. We need to introduce a probability structure, because Newton's laws do not respect the direction of time (and can't alone imply the increase of entropy or keep a pond from ejecting a stone).

At the level of abstract ergodic theory, we regard $(M, \mu)$ as an abstract measure space $(X, \mu)$ (i.e., we ignore sets of measure 0 and the geometry of $M$). We get back the geometry by a function that maps a point in $X$ to the corresponding point in $M$.

A good example is Sinai billiards. Here we fix a speed, and the points in $M$ are the position and direction of motion of the ball. $\mu$ is 3-dimensional Lebesgue measure. See Figure 1.

### Stationary Processes

A stationary process is a sequence or a one-parameter family of random variables, $X_n$ or $X_t$, defined on the same measure space $(\Omega, \mu)$. (For $\omega$ in $\Omega$, $X_n(\omega)$ or $X_t(\omega)$ is a realization of the process.)

We define a transformation in $\Omega$:

$$T\{X_n\} = \{X_{n+1}\} \quad \text{and} \quad T_t\{X_t\} = \{X_{t+\bar{t}}\}.$$

Stationarity means that the joint distribution of $\{X_0, X_1, \ldots, X_n\}$ is the same as the joint distribution $\{X_k, X_{1+k}, \ldots, X_{n+k}\}$ for all $k$, and the joint distribution of $\{X_{t_1}, \ldots, X_{t_n}\}$ is the same as the joint distribution of $\{X_{t_1+t}, \ldots, X_{t_n+t}\}$ for all $t$. This is the same as saying that $T$ (or $T_t$) is measure preserving.

We get the Kolmogorov model for a stationary process by starting with $(T, \Omega, \mu)$ or $(T_t, \Omega, \mu)$, and a function $F$ on $\Omega$, $F(\omega) = X_0(\omega)$, $\{F(T^n(\omega))\}_{-\infty}^{\infty}$ or $\{F(T_t(\omega))\}_{-\infty}^{\infty}$ are the realizations we started out with. We will denote this process by $(T, \Omega, F)$ or $(T_t, \Omega, F)$, or by $(T, X, F)$ or $(T_t, X, F)$.

### Bernoulli Shifts

*These are the transformations that come from independent processes.*

We will give an alternate description of how, in this case, we go from a stationary process to a measure-preserving transformation.

Let $\pi$ be a set with $k$ elements $e_i$ with probabilities $p_i$, $\sum_{i=1}^{k} p_i = 1$, and $x$ in $X$ is a doubly infinite sequence of the elements in $\pi$. $x$ is a realization of the process. A cylinder set is the set of $x$ where we fix the $e_i$ at a finite set of coordinates (or times). Its measure is the product of those $p_i$. This extends to a measure on $X$. $T$ is the transformation that shifts these sequences (the $n$ coordinates of $T(x)$ in the $n+1$ coordinate $x$). We will call the transformation above $B(p_1, \ldots, p_k)$.

We get the model for the independent process by a function $F$ on $X$ where $F(x)$ is the zero component of $x$. $F(T^i(x))$ is a realization of the process.

We will now describe a continuous time analog of the Bernoulli shifts. For simplicity, we will take $B(\frac{1}{2}, \frac{1}{2})$.

### The Coin Tossing Flow

We start with the Continuous Time Coin Tossing Process. Pick $t_1$ and $t_2$ such that $t_1/t_2$ is irrational. Toss a fair coin. If we get heads, output 0 for time $t_1$. If we get tails, output 1 for time $t_2$. Toss again.

We can visualize the flow $T_t$ that we get from the above process as follows: Start with $B(\frac{1}{2}, \frac{1}{2})$, defined on $X$. $T_t$ is defined on a subset, $Y$, of $X \times R$, pictured below:

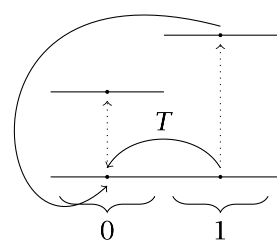


**Figure 2. Coin Tossing Flow**

A point $x$ in the base flows straight up for time $t_1$ (or $t_2$) and returns to the base at $B_{(\frac{1}{2}, \frac{1}{2})}(x)$.

Although this is a continuous time analog of a Bernoulli shift, the connection is much deeper. For example, the isomorphism theory will show that by discretizing time, we can get $B(\frac{1}{2}, \frac{1}{2})$.

We can visualize the model for continuous time coin tossing by a function, $F$, that labels the points above 1 by 1 and those above 0 by 0. $F(T_t(y))$ are the realization of our process.

*The Kolmogorov point of view puts dynamical systems and stationary processes into the same mathematical framework: a measure-preserving transformation or flow, together with a function that gives the concrete structure.*

## Entropy and Isomorphism

Our story of Bernoulli shifts begins with the long-standing question: Are all Bernoulli shifts isomorphic?

Kolmogorov [Kol59] was able to show that not all Bernoulli shifts were isomorphic by introducing an invariant, which he called "the entropy of the transformation". This revolutionized abstract ergodic theory.

To define the entropy of a transformation, we start with the Shannon entropy of a stationary process with a finite (or countable) number of outputs. Define the entropy to be the infimum of the $\alpha$ such that, given $\epsilon$, there is an $N(\epsilon)$ and the number of outputs of length $n$ ($n > N(\epsilon)$) is less than $2^{\alpha n}$ after removing a subset of these outputs of measure $\leq \epsilon$.

The entropy of a transformation $(T, X)$ is the sup of the entropies of the stationary process $(T, X, F)$ that come from $(T, X)$.

Kolmogorov's key result is that the entropy of a Bernoulli shift is the same as the entropy of the independent, identically distributed process used to define it. It is not hard to see that if the outputs of this process have probability $P_i$, then the entropy of this process is $-\sum P_i \log P_i$ (where $-\sum P_i \log P_i$ could be finite or infinite).

After Kolmogorov, the problem became: Are all Bernoulli shifts of the same entropy isomorphic? An answer was given twelve years later.

**Theorem** ([Orn70b], [Orn70d]). *Bernoulli shifts of the same entropy are isomorphic.*

Thus, when we go to the level of abstract ergodic theory, we get a very simple picture.

Independent processes are the least predictable and in this sense the most random possible, and Bernoulli shifts are in this sense the most random transformations.

The proof of the theorem above is the basis for the stronger and deeper theorem below.

**Bernoulli Isomorphism Theorem 1** ([Orn70b], [Orn70d], [Orn70a], [Orn73b]). *There exists an abstract finite entropy flow, $B_t$, with the following properties:*

- *(a) If we discretize time at multiples of $t_0$, then for all $t_0$ the resulting transformation, $B_{t_0}$, is a Bernoulli shift of finite entropy.*
- *(b) Every Bernoulli shift of finite entropy is isomorphic to $B_{t_0}$ for some $t_0$.*
- *(c) Uniqueness. For any flow $T_t$, if for some $t_0$ $T_{t_0}$ is isomorphic to a finite entropy Bernoulli shift, then $T_t$ as a flow is isomorphic to $B_{ct}$ for some constant $c$.*

We also have an analogous Bernoulli flow, $B_t^\infty$, of infinite entropy.

Three consequences of the theorem above are:

1. Bernoulli shifts of the same entropy are isomorphic. (The entropy of the transformation $T_{t_0}$ is proportional to $t_0$.)
2. We can complete our understanding of Bernoulli shifts by representing them as the discretizations of the Bernoulli flow, $B_t$, or the time of one transformation of the $B_{ct}$. *Thus, returning to Kolmogorov, even though the Bernoulli shifts are not all isomorphic, they are "essentially" the same in the sense that they differ only by a rescaling of time.*
3. A Bernoulli shift can be embedded in a flow and in particular it has roots of all order (it was not previously known that the Bernoulli shift $\frac{1}{2}, \frac{1}{2}$ had a square root).

*We can now return to some of the concrete systems that gave birth to abstract ergodic theory*

## An Important Aspect of the Proof of the Isomorphism Theorem Is Giving Criteria for Proving Bernoulli

The criteria are somewhat technical and we postpone their description. We will start with two applications of the Bernoulli criteria.

**Theorem.**[3] *The coin tossing flow and Sinai billiards are both isomorphic to $B_t$.*

This theorem links the two contexts in which abstract ergodic theory arose.

It came as a surprise that continuous time coin tossing and a classical dynamical system could be the same as abstract flows.

The isomorphism between Sinai billiards and the coin tossing flow implies that we can realize continuous time coin tossing by a fixed (nonrandom) function on the phase space of Sinai billiards and its time evolution could be derived from Newton's laws.

### Some Bernoulli Examples

- Our first example is a generalization of the continuous time coin tossing flow where we replace coin tossing by an $N$-step Markov process.

  We replace the continuous time coin tossing process by a process that we will call a *ct* (continuous time) Markovian process. This process has a finite number of outputs $O_i$. $O_i$ lasts for time $t_i$. The next output is determined by a roulette wheel[4] whose probabilities depend on $i$ and the previous $N$ outputs.

  We can visualize the flow as a "flow built under a function" where the transformation on the base is an $N$-step Markovian process.

  - Sinai billiards [GO74].

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[3] *Deep results of Sinai about Sinai billiards are needed to check our criteria.*

[4] *Or many coin tosses.*

- Geodesic flow on a manifold, $M$, of negative curvature (the flow is really on the tangent bundle of $M$) [OW73].
  This was the first concrete flow that was shown to be Bernoulli and showed how to use hyperbolic structure to verify our criteria (this was Weiss's insight and the beginning of a long collaboration). Almost all of the concrete flows that were shown to be Bernoulli follow this format.
- The following deep theorem of Pesin gives a general class of Bernoulli flows [Pes77]:
  Let $T_t$ be a smooth ergodic[5] flow in $C^{1+\varepsilon}$ on a compact 3-dimensional manifold with a smooth invariant measure. If $T_t$ has positive entropy, $T_t$ is either $B_{ct}$ or $B_{ct} \times R_t$ ($R_t$ is a rotation of the circle). (This also applies to the ergodic components of positive entropy. These have positive measure. A nice example is the double pendulum. It is not known if this has positive entropy.)
- We get an infinite entropy example by replacing the two shift by a Bernoulli shift of infinite entropy in "flow built under a function" (see Figure 2).
- Other infinite entropy examples are the flows we get from Brownian motion on the unit interval with reflecting barriers, Poisson processes, and diffusion processes.

Some discrete time examples are:

- Mixing Markov (or multistep Markov).
- Ergodic automorphisms of compact groups. This result is due to Lind [Lin77] and, independently, Miles and Thomas [MT78].
- One of the deepest results in abstract ergodic theory is due to Rudolph: mixing compact extensions of a Bernoulli shift are Bernoulli [Rud78a], [Rud78b].

The titles of some of the papers in the bibliography extend this sampling.

## Another Aspect of the Proof of the Isomorphism Theorem Is That in Certain Cases the Isomorphism Preserves the Concrete Structure

Proving that a dynamical system is Bernoulli identifies the system if we ignore the concrete structure given by a function on $X$.

### The Concrete Structures of All Bernoulli Systems Have a Certain Randomness Property

Recall the definition of a *ct* $N$-step Markovian process.

**Theorem** ([OW91]). *Fix $(F_t, M, F)$, where the range of $F$ is a finite set $(O_i)$. If $F_t$ is isomorphic to the Bernoulli flow, then given $\varepsilon$ we can change $F$ to $(\hat{F})$ on a set of measure $\varepsilon$; and if we regard $M$ as an abstract measure space, then $(T_t, M, \hat{F})$ is the model for a ct $N$-step Markovian process on the same $O_i$.*

A good example: display Sinai billiards on a TV screen. This is a $(B_t, M, F)$ where the range of $F$ is the states of the TV screen. If we fix a set of positions and directions of measure $< \varepsilon$ and force the TV to misread the position when in these states in a fixed nonrandom way, then the display on the TV screen becomes the model for a *ct* $N$-step Markovian process. *Thus, the evolution of the slightly altered TV movie that we get by looking at a fixed billiard ball orbit is determined by the outcome of a sequence of coin tosses.*

The discussion above holds for any classical system that is isomorphic to $B_t$ where $F$ is any coarse graining of the phase space.

*The theorems above have implications for chaos "theory", because they show that there are "super chaotic systems", which, at an abstract level, are the most random possible (see "Positive Entropy and Factors") and, at a concrete level, are essentially indistinguishable from systems that are driven by coin tossing and where the coin tossing, after many tosses, wipes out any memory of the past.*

The cornerstone of chaos "theory" is Lorenz's observation that for certain classical systems, a small change in the initial position grows exponentially fast. *(Sinai billiards are an example.)* This is usually referred to as "sensitivity to initial conditions" or the "butterfly effect". The conclusion is that the behavior of these systems is unpredictable or random. "Super chaos" is the ultimate strengthening of this conclusion. *In particular, the lack of predictability due to sensitivity to initial conditions is qualitatively different from the lack of predictability due to coin tossing.*

The results above also mean that the distinction between long-term determinism of Newtonian mechanics[6] and the long-term randomness introduced by coin flipping is not as sharp as one might expect.

### Statistical Stability

This is a more special case of isomorphisms between concrete systems where the proof of the isomorphism theorem allows us to preserve the geometry.

**Theorem** ([OW91]). *Let $F_t$ be geodesic flow on a manifold or surface $M$ of negative curvature (the flow is really on the tangent bundle $M^T$), and define $\hat{F}_t$ by a $C^2$ change in the Riemannian structure of $M$ that is small in the $C^2$-metric.*

[5] *Ergodic means that the only invariant sets have measure* 0 *or* 1.

[6] *It is critical that we are talking about the determinism of Newtonian mechanics, because any abstract system is deterministic in the sense that the starting point $x$ in $X$ determines the entire orbit.*

*Then, given $\varepsilon$, if the change is small enough, $\hat{F}_t$ will be isomorphic to $F_{ct}$, $(|c-1|<\varepsilon)$ by a map $\psi$ of $M^T$ to itself that moves all but $\varepsilon$ of the points by a distance $<\varepsilon$. (In particular, orbits map to orbits, preserving time.)*

We would get the same conclusion if, instead of changing the Riemannian structure, we made a small variable speed change along the orbits of $F_t$.

We would also get the same conclusion if we made a small change in the obstacles in Sinai billiards [Elo88].

The theorems above are examples of what we call statistical stability.

The isomorphisms above give a clean picture of the potentially messy long-term consequences of the small perturbations above.

### Comparison with the Structural Stability (of Peixoto, Anosov, Smale, Mané, etc.)

In the case of geodesic flow, if we define $\hat{F}_t$ as coming from the above change in Riemannian structure, then structural stability asserts that there is a homeomorphism of $\overline{\psi}$ of $M^T$ onto itself that takes the orbits of $F_t$ onto those of $\hat{F}_t$ (without preserving time) and moves all points by $<\varepsilon$.

In statistical stability, we lose the continuity of $\overline{\psi}$, and instead of moving all points by $<\epsilon$, there is a set of measure $\epsilon$ where points may be moved by a large distance.

*However, statistical stability corrects the following problems with structural stability in continuous time.*

1. *It is possible that $\overline{\psi}$ maps a set of probability one onto a set of probability zero.*
2. *Since $\overline{\psi}$ does not preserve the speed along orbits, sets that correspond under $\overline{\psi}$ at time 0 may not correspond to each other at time $t \neq 0$.*

If our perturbation were a variable speed change along the same orbits, $\overline{\psi}$ would be the identity, whereas $\psi$ must scramble orbits.

Sinai billiards is statistically stable (we make a small change in the curvature of the obstacles) but not structurally stable [Elo88].

The systems that are structurally stable are the axiom A attractors. These are Bernoulli with respect to the SBR measure, which is considered to be the measure that is physically relevant, and we can ask if they are statistically stable. The only obstruction comes from the existence of eigenfunctions. A precise statement can be found in [OW91].

## Positive Entropy and Factors and the Bernoulli Criterion

We now return to abstract ergodic theory.

An arbitrary $F$ defines a measure on the $\{F(T^n(x))\}_{-\infty}^{\infty}$ or $\{F(T_t(x))\}_{-\infty}^{\infty}$. This is the model for a stationary process, except that many $x$ can give the same $\{F(T^n(x))\}_{-\infty}^{\infty}$ or $\{F(T_t(x))\}_{-\infty}^{\infty}$. If we lump these points together, we get a new transformation or flow, which we call a *factor* of $(T,X,\mu)$ or $(T_t,X,\mu)$.

A factor can also be described as restricting the measurable sets to belong to a sub-$\sigma$-algebra, which is invariant under $T$ (or $T_t$).

**Bernoulli Isomorphism Theorem 2** ([Orn70b], [Orn70d], [Orn70c], [Orn70a], [Orn73b]).

(a) *The only factors of $B_{ct}$ are the $B_{\alpha t}$ for $\alpha \le c$.*
(b) *If $T_t$ has finite positive entropy, then $T_t$ has $B_{ct}$ as a factor of the same or smaller entropy.*

(There exists an abstract flow of infinite entropy, $B_t^{\infty}$, with the analogous properties.)

(a) and (b) are also true for Bernoulli shifts. Sinai proved (b) for Bernoulli shifts before the isomorphism theorems in this article. Even though Sinai's method was entirely different, knowledge of this result was of major importance.

The main consequences of the theorem above are:

- *There exists a unique abstract flow of finite (infinite) entropy that is the "most random" flow possible*:

  One meaning comes from characterizing the randomness of a flow, $(T_t,X)$, by the randomness of $T_t,X,F$, the stationary processes that come from the flow. (b) implies that any flow of positive (infinite) entropy that is not $B_t$ gives rise to more stationary processes and includes processes that are not VWB (see "Very Weak Bernoulli (VWB) Systems") and are therefore less unpredictable.

  Another sense of "most random" comes from its realization as the flow we get from continuous time coin tossing.

  Yet another sense comes from its discretizations being the most random transformations.
- (b) means that $T_t$ can be realized as a skew product with base $B_{ct}$, where $B_{ct}$ has full entropy and the action on the fibers does not contribute to the entropy. *In this sense, $B_t$ or $B_t^{\infty}$ is the cause of all randomness in flows* (flows of zero entropy are not considered random).
- (a) says that the flow we get from any $B_t,X,F$ is still $B_{ct}$; i.e., *the processes that give a Bernoulli flow are the same as the processes that we get from a Bernoulli flow*.

  We thus have a nice class of processes which we call Bernoulli.

## The Entropy Hierarchy and the Limits of the Isomorphism Theory

Our story began with Kolmogorov's introduction of entropy as an invariant for Bernoulli shifts. Soon after, Sinaĭ [Sin59] modified Kolmogorov's definition to give an invariant for any measure-preserving transformation or flow. This led to a hierarchy of transformations and flows and to the question: What is the role of Bernoulli shifts (and later $B_t$) in this hierarchy?

1. *Zero Entropy.* A discrete time process with a finite (or countable) number of outputs has zero entropy if the infinite past determines the infinite future.

   A transformation or flow has zero entropy if all of the discrete time processes that come from it have zero entropy.
2. *Positive Entropy.* A transformation or flow has positive entropy if some discrete time process that comes from it has nonzero entropy.
3. $K$. A transformation or flow is $K$ if no discrete time process that comes from it has zero entropy.

   This is equivalent to having no factors of zero entropy.
4. *The Bernoulli shifts in discrete time or $B_t$ ($B_t^\infty$) in continuous time* (Bernoulli implies $K$).

*The isomorphism theorem shows that Bernoulli shifts provide a simple top to the entropy hierarchy in discrete time.*

*The existence of $B_t$ ($B_t^\infty$) shows that there is a single top to the entropy hierarchy in continuous time.*

*Furthermore, all entropy comes from Bernoulli shifts or the Bernoulli flow.*

Kolmogorov introduced the $K$ property around the same time as the introduction of the Kolmogorov-Sinaĭ entropy. The two main questions about Bernoulli shifts at the time were: Are Bernoulli shifts of the same entropy isomorphic, and are $K$-transformations Bernoulli shifts? [KAT07]

The answer to the second question is no [Orn73a]. *In fact, there are uncountably many nonisomorphic $K$-automorphisms of the same entropy* [OS73] *and uncountably many nonisomorphic $K$-flows of the same entropy* [Smo73].

Sinai and his school are responsible for the first applications of hard abstract results to classical dynamical systems, proving the time one transformations to be $K$. The proofs rested on analyzing the hyperbolic structure of these systems, the structure that makes the distance between nearby points grow exponentially fast. Later, in some cases, this analysis also allowed the application of the Bernoulli criteria (see "Some Bernoulli Examples"), proving the time one transformation to be Bernoulli, thus narrowing the class to a single flow.

Pinsker [Pin60] conjectured that every transformation of positive entropy was the direct product of a transformation of zero entropy and a $K$-automorphism.

A theorem of Pesin shows that *for flows on 3-dimensional manifolds* (transformations on 2-dimensional surfaces), *a strong form of the Pinsker conjecture is true.* Every flow of positive entropy is the direct product of $B_t$ and a rotation (see item 4 above for an exact statement).

*In general, the Pinsker conjecture is false* [Orn73a], [Orn73c].

**Open Problem:** Is every transformation the direct product of a transformation of arbitrarily small entropy and a Bernoulli shift? These are called weak Pinsker transformations.

The first smooth non-Bernoulli $K$-automorphism is due to Katok [Kat80]. Rudolph gave the first example of a smooth non-Bernoulli $K$-flow. The first "natural" example of a non-Bernoulli $K$-automorphism is due to Kalikow [Kal82]. It's easy to describe—it is a random walk on a random environment—but difficult to show that it is non-Bernoulli.

The next question is: What kinds of non-Bernoulli behavior can we get without introducing any deterministic elements?

We now have a zoo of $K$-automorphisms with qualitatively different behaviors [Hof99b], [Hof99d], [Hof99c], [OS73], [Rud78a], [Rud76], [Rud77]. A small sampling is:

- $K$-automorphisms that are not isomorphic to their inverses.
- Two $K$-automorphisms that are not isomorphic, but all of their powers are isomorphic [Rud77].
- $K$-automorphisms that have no square root [Cla72].

Hoffman, using work of Rudolph, showed that the behaviors of finite permutations could be lifted to behaviors of $K$-automorphisms [Hof99c], [Hof99b], [Rud79]. Thus, instead of isolated examples, we have a systematic method for producing examples.

At the time these counterexamples were produced, the main focus was on the discrete time hierarchy, and the issue of embedding the examples in a flow was largely ignored, leaving a large gap in the theory.

## Factors of a Bernoulli Shift

We end by tying together the proofs and results of the two main strands of the abstract part of our story: the isomorphism theorem and the entropy hierarchy, together with the counterexamples.

The results concern the relationship of a factor to the whole Bernoulli shift. We say that two factors of $T$, $A$, and $B$ sit the same way if there is an automorphism of $T$ that takes $A$ onto $B$.

We say that $T$ is Bernoulli relative to a factor $A$ if $T = A \times B$.

### Thouvenot's Relative Isomorphism Theory

The starting point is Thouvenot's relative isomorphism theory, which tells us when $T$ is relatively Bernoulli with respect to $A$. (In Thouvenot's theory there is no restriction on $A$. In particular, $A$ does not have to be a factor of a Bernoulli shift.)

The theory gives the relative analog of the Bernoulli criteria in the appendix (relatively finitely determined, relatively VWB). Here is the analog of the theorem that factors of Bernoulli shifts are Bernoulli.

**Theorem** (Thouvenot)**.** *If $T = A \times B$ where $B$ is a Bernoulli shift, then any factor of $T$ that includes $A$ also has the form $A \times \overline{B}$, where $\overline{B}$ is a Bernoulli shift. (In general, $\overline{B}$ is not a factor of $B$.)*

Fieldsteel extended the relative isomorphism theorem to flows.

### The Relative Entropy Hierarchy

We now restrict to factors of a Bernoulli shift.

Recall that $T$ is Bernoulli relative to a factor $A$ if there is a factor $B$ such that $T = A \times B$. We say that $T$ is $K$ relative to a factor $A$ if any $B > A$ has greater entropy. There exists a factor with respect to which $T$ is relatively $K$ but not relatively Bernoulli [Orn75]. Hoffman [Hof99a], by a subtle modification of this example, produced uncountably many such factors that do not sit the same way.

$T$ has positive entropy relative to $A$ if the entropy of $T$ is greater than the entropy of $A$. $T$ has zero entropy relative to a factor of full entropy.

Rudolph proved that there are only a finite number of ways that a factor whose fibers have $k$ points can sit and that these are classified by an algebraic structure on the symmetric group on $k$ points [Rud78b].

Hoffman, using results of Rudolph, showed that the behaviors of permutations could be lifted to the ways that factors of Bernoulli shifts could sit [Hof99a].

**Proofs.** We get the factors by taking a skew product over a Bernoulli base with a counterexample. We use a criteria for Bernoulli to prove that the skew product is Bernoulli. The base is the factor we are interested in.

These results show how far we have come since the time of Kolmogorov. Before Kolmogorov, it was not known whether $B_{\frac{1}{3},\frac{2}{3}}$ had any nontrivial factors. After Kolmogorov, essentially all that was known was that a Bernoulli shift had a factor of any smaller entropy. It was not known whether or not a Bernoulli shift had a nontrivial factor of full entropy. We now know that the relative classification of factors of a Bernoulli shift mimics, to a large extent, the classification of general transformations.

Factors of a Bernoulli flow form another largely neglected area.

## Appendix: Criteria for Bernoulli

We will call a stationary process $(T, X, F)$, where $T$ is a Bernoulli shift, a $B$ process. We will now give characterizations of $B$ processes where $F$ is finite valued.

### The $\overline{d}$ Metric

We will describe a distance between $(T, X, F)$ and $(\overline{T}, \overline{X}, \overline{F})$ where $F$ and $\overline{F}$ have the same range. Both processes put a measure on sequences of length $n$ which can be realized by functions $f$ and $\overline{f}$ from nondiscrete measure spaces $Y$ and $\overline{Y}$ of total measure 1 to the sequences of length $n$. Denote these by $Y, f$, and $\overline{Y}, \overline{f}$. We say that these measures are closer than $\alpha$ in $\overline{d}$ if there is a 1-1 measure preserving map between $Y$ and $\overline{Y}$ where the corresponding sequences differ in less than $\alpha n$ places except for a set in $Y$ (or $\overline{Y}$) of measure $< \alpha$.

$\overline{d}((T, X, F), (\overline{T}, \overline{X}, \overline{F})) < \alpha$ if the measure they put on sequences of length $n$ for all $n$ are closer than $\alpha$.

### Finitely Determined (FD) Systems

$(T, F)$ is said to be finitely determined if, given $\varepsilon$, there is an $n$ and $\delta > 0$ such that if $\overline{T}, \overline{F}$ satisfies

$$\overline{d}\Big(\bigvee_{i=0}^{n} F(T^i(x)), \bigvee_{i=0}^{n} \overline{F}(\overline{t}^i(x))\Big) < \delta$$

and the entropies of $T, F$, and $\overline{T}, \overline{F}$ are closer than $\delta$, then $\overline{d}((T, F), (\overline{T}, \overline{F})) < \varepsilon$.

FD is the crucial idea behind our proofs, since it allows us to control infinite behavior by finite constructions. We prove:

**Theorem.** *FD processes of the same entropy are isomorphic.*

We then show that a specific process (e.g., an independent process) is FD.

Feldman made a subtle modification of FD that works for flows. He used this to give a proof of the technically more difficult isomorphism theorem for flows that parallel the discrete time proof [Fel80]. The original proof used a sequence of finer and finer discretizations.

### Very Weak Bernoulli (VWB) Systems

$T, F$ is said to be "very weak Bernoulli" (VWB) if, for every $\varepsilon > 0$, there is an $\overline{n}$ such that for all $m$ and $n \geq \overline{n}$ and all but $\varepsilon$ of the atoms in $\bigvee_{i=-m}^{-1} F(T^i(x))$, the conditional distribution of $\bigvee_{i=0}^{n} F(T^i(x))$ is closer than $\varepsilon$ in $\overline{d}$ to the unconditional distribution.

A process is FD if—*and only if*—it is VWB [OW74]. VWB is usually the easiest to check.

*The mixing hierarchy (discrete time).*

VWB allows us to view the relationship between $K$ and Bernoulli in terms of the dependence of the future on the past. VWB is such a condition; $K$ can be described by an analogous but weaker, condition, where we condition $F(T^n(x))$ rather than $\bigvee_{i=0}^{n} F(T^i(x))$ on the past.

*VWB completes the traditional mixing hierarchy: ergodicity, weak mixing, mild mixing, mixing, K, Bernoulli.*

## Postscript [7]

Most of the isomorphism theory of Bernoulli systems was carried over to actions of unimodular amenable groups in [OW87]. At that time we believed that there was no reasonable entropy-like theory for nonamenable groups and in particular for the free group on two generators where one can define a factor map from the 2-shift onto the 4-shift. It came as quite a surprise when Lewis Bowen in [Bow10b] proved the analogue of Kolmogorov's theorem for Bernoulli shifts defined over the free group, namely, that if they are isomorphic then they must have the same base entropy. He went on to extend this new theory to a very wide class of groups called sofic groups (no countable group has yet to be shown to be nonsofic) in [Bow10a]. This new theory has sparked a great deal of activity in the ergodic theory of actions of general groups. New phenomena appear here, as was shown a few years ago by Sorin Popa, who gave examples of factors of Bernoulli shifts over many nonamenable groups that were not isomorphic to Bernoulli shifts [Pop06].

Finally, one should mention Bowen's recent extension of the isomorphism theorem to all groups (if the distributions are not supported on just two elements) in [Bow12]. It is remarkable that the proof of this result makes essential use of the relative theory that was developed by Thouvenot for the classical Bernoulli shift.

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