SOME OPEN PROBLEMS ON MULTIPLE ERGODIC AVERAGES

NIKOS FRANTZIKINAKIS

1. Problems related to random sequences

In this section we give a list of problems related to the study of multiple ergodic averages involving iterates given by random sequences of integers.

The random sequences that we work with are constructed by selecting a positive integer n to be a member of our sequence with probability $\sigma_n \in [0,1]$. More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent $0-1$ valued random variables with

$$\mathbb{P}(X_n = 1) := \sigma_n \quad \text{and} \quad \mathbb{P}(X_n = 0) := 1 - \sigma_n$$

where σ_n is a decreasing sequence of positive real numbers that satisfies $\sum_{n=1}^{\infty} \sigma_n = \infty$ (in which case $\sum_{n=1}^{\infty} X_n(\omega) = +\infty$ almost surely). The random sequence $(a_n(\omega))_{n\in\mathbb{N}}$ is constructed by taking the positive integers n for which $X_n(\omega) = 1$ in increasing order. Equivalently, $a_n(\omega)$ is the smallest $k \in \mathbb{N}$ such that $X_1(\omega) + \cdots + X_k(\omega) = n$. If $\sigma_n = n^{-a}$ for some $a \in (0,1)$, then one can show that almost surely $a_n(\omega)/n^{1/(1-a)}$ converges to a non-zero constant. On the other hand, if $\sigma_n = 1/n$, then almost surely there exists a subsequence (n_k) of the integers, of density arbitrarily close to one, such that the sequence $(a_{n_k}(\omega))$ is lacunary [5] (this is no longer the case if $n\sigma_n \to \infty$). So it makes some sense to call *random non-lacunary sequences* the random sequences one gets when σ_n satisfies $n\sigma_n \to \infty$.

We say that a certain property holds almost surely for the sequences $(a_n(\omega))_{n\in\mathbb{N}}$, if there exists a universal set $\Omega_0 \in \mathcal{F}$, such that $\mathbb{P}(\Omega_0) = 1$, and for every $\omega \in \Omega_0$ the sequence $(a_n(\omega))_{n\in\mathbb{N}}$ satisfies the given property.

The next result was proved by M. Boshernitzan [1] for mean convergence and by J. Bourgain [2] for pointwise convergence (see also [6] for a nice exposition of these results).

Theorem. *If $n\sigma_n \to \infty$, then almost surely the following holds: for every invertible measure preserving system $(X, \mathcal{X}, \mu, T)$ and function $f \in L^\infty(\mu)$, the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T^{a_n(\omega)} f \tag{1}$$

converge in the mean and their limit equals the $L^2(\mu)$-limit of the averages $\frac{1}{N}\sum_{n=1}^{N} T^n f$. Furthermore, if $n\sigma_n/(\log\log n)^{1+\delta} \to \infty$ for some $\delta > 0$, then the conclusion also holds pointwise.

It is known that the mean convergence result fails if $\sigma_n = 1/n$ and the pointwise convergence result fails if $\sigma_n = (\log\log n)^{1/3}/n$ (for both results see [5]). It is unclear whether the pointwise convergence result fails when say $\sigma_n = \log\log n/n$.

Date: March 2011.

One would naturally like to extend the previous convergence result to multiple ergodic averages:

Problem 1. *Suppose that $n\sigma_n \to \infty$. Show that almost surely the following holds: For every probability space $(X, \mathcal{X}, \mu)$, commuting invertible measure preserving transformations $T_1, \ldots, T_\ell \colon X \to X$, and functions $f_1, \ldots, f_\ell \in L^\infty(\mu)$, the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{a_n(\omega)} f_1 \cdot \ldots \cdot T_\ell^{a_n(\omega)} f_\ell$$

converge in the mean and their limit equals the $L^2(\mu)$-limit of the averages $\frac{1}{N}\sum_{n=1}^{N} T_1^n f_1 \cdot \ldots \cdot T_\ell^n f_\ell$.

A special case of this problem was mentioned in 2004 by M. Wierdl, it can be found here http://math.stanford.edu/yk70/ik70-op.pdf. Mean convergence is known when $\ell = 2$ and $\sigma_n = n^{-a}$ where $a \in (0, 1/2)$ [4]. The methods of [4] also work for mean convergence when $\ell > 2$, but for a smaller range of eligible exponents a that depends on ℓ. Under the assumption that $n\sigma_n \to \infty$, pointwise convergence is known when the transformations are given by powers of the same nilrotation [3] and the functions are continuous. On the other hand, when $\sigma_n = n^{-a}$ for some $a > 1/2$, $\ell = 2$, and $T_2 = T_1^2$, mean convergence is not even known when we restrict ourselves to the class of weak mixing systems.

Next we mention a problem where one combines deterministic and random iterates.

Problem 2. *Suppose that $n\sigma_n \to \infty$. Show that almost surely the following holds: For every probability space $(X, \mathcal{X}, \mu)$, commuting invertible measure preserving transformations $T, S \colon X \to X$, and functions $f, g \in L^\infty(\mu)$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n f \cdot S^{a_n(\omega)} g = \mathbb{E}(f|\mathcal{I}_T) \cdot \mathbb{E}(g|\mathcal{I}_S)$$

where the limit is taken in $L^2(\mu)$. Furthermore, if $\sigma_n = n^{-a}$ for some $a \in (0, 1)$, show that the convergence also holds pointwise.

The desired convergence (in the mean and pointwise) is only known when $a \in (0, /14)$ [4]. The problem does not appear to be much easier to solve when $T = S$ is a weak mixing transformation and we are only interested in mean convergence.

In the next problem we are going to work with two random sequences with different growth rates that are chosen independently of each other. More precisely, let $X_1, Y_1, X_2, Y_2, \ldots$ be a sequence of independent 0-1 valued random variables with $\mathbb{P}(X_n = 1) := n^{-a}$ and $\mathbb{P}(Y_n = 1) := n^{-b}$ for some $a, b \in (0, 1)$. We construct the random sequence $(a_n(\omega))$ by taking the positive integers n for which $X_n(\omega) = 1$ in increasing order, and the random sequence $(b_n(\omega))$ by taking the positive integers n for which $Y_n(\omega) = 1$ in increasing order.

Problem 3. *Suppose that $a \neq b$. Show that almost surely the following holds: For every probability space $(X, \mathcal{X}, \mu)$, commuting invertible measure preserving transformations $T, S \colon X \to X$, and functions $f, g \in L^\infty(\mu)$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{a_n(\omega)} f \cdot S^{b_n(\omega)} g = \mathbb{E}(f|\mathcal{I}_T) \cdot \mathbb{E}(g|\mathcal{I}_S)$$

where the limit is taken in $L^2(\mu)$ or pointwise.

The problem seems non-trivial even when $T = S$ is a weak mixing transformation.

References

[1] M. Boshernitzan. Homogeneously distributed sequences and Poincaré sequences of integers of sublacunary growth. *Monatsh. Math.* **96** (1983), no. 3, 173–181.

[2] J. Bourgain. On the maximal ergodic theorem for certain subsets of the positive integers. *Israel J. Math.* **61** (1988), 39–72.

[3] N. Frantzikinakis. Equidistribution of sparse sequences on nilmanifolds. *J. Analyse Math.* **109** (2009), 1–43.

[4] N. Frantzikinakis, E. Lesigne, M. Wierdl. Random sequences and pointwise convergence of multiple ergodic averages. Preprint, `arXiv:1012.1130`.

[5] R. Jones, M. Lacey, M. Wierdl. Integer sequences with big gaps and the pointwise ergodic theorem. *Ergodic Theory Dynam. Systems* **19** (1999), no.5, 1295–1308.

[6] J. Rosenblatt, M. Wierdl. Pointwise ergodic theorems via harmonic analysis. Ergodic theory and its connections with harmonic analysis (Alexandria, 1993). *London Math. Soc. Lecture Note Ser.*, **205**, Cambridge Univ. Press, Cambridge, (1995), 3–151.

(Nikos Frantzikinakis) University of Crete, Department of mathematics, Knossos Avenue, Heraklion 71409, Greece

E-mail address: `frantzikinakis@gmail.com`