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GR-87.0006-88.0016- Coordinator: S.N.Pnevmatikos -89.0021-90.0006-GR

Series of Scientific Monographies
“Mathematics and Fundamental Applications”

LECTURES
ON THE CLASSIFICATION OF
COMPLEX AND REAL LIE ALGEBRAS

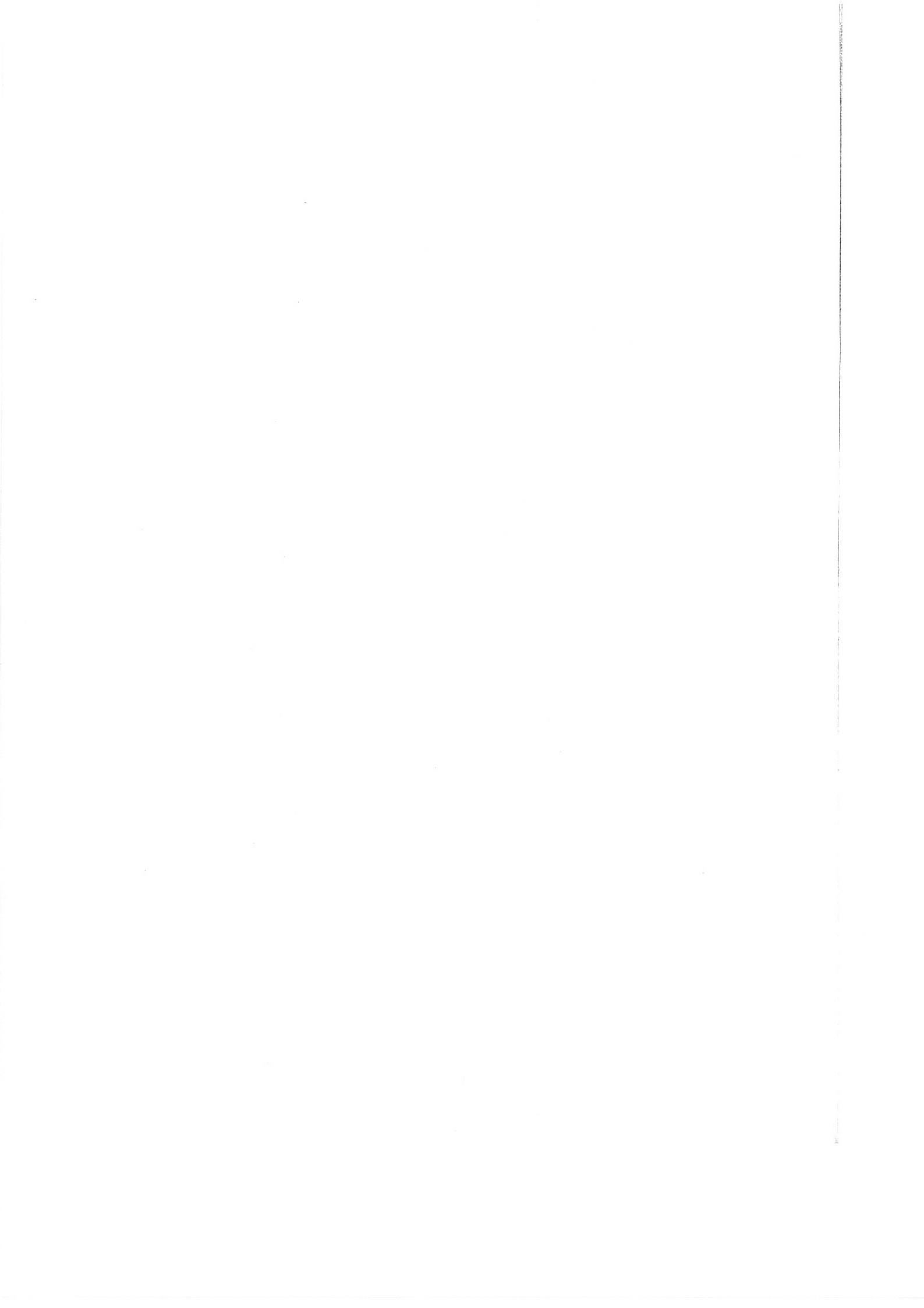
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*Lectures given in the framework of the
Erasmus Interuniversity Cooperation Programme
“Mathematics and Fundamental Applications”
coordinated by the
Aristotle University of Thessaloniki*

*with the support of the
Commission of the European Communities*

1991



ERASMUS

ΕΥΡΩΠΑΪΚΟ ΔΙΑΠΑΝΕΠΙΣΤΗΜΙΑΚΟ ΠΡΟΓΡΑΜΜΑ ΣΥΝΕΡΓΑΣΙΑΣ
EUROPEAN INTERUNIVERSITY COOPERATION PROGRAMME

“ΜΑΘΗΜΑΤΙΚΑ ΚΑΙ ΘΕΜΕΛΙΩΔΕΙΣ ΕΦΑΡΜΟΓΕΣ”
“MATHEMATICS AND FUNDAMENTAL APPLICATIONS”

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1990-1991

Preparing the "Series of Scientific Monographs" has been one of the activities of the Erasmus Inter-university Cooperation Programme (ICP) "Mathematics and Fundamental Applications". The ICP is based on a network first established in 1987 when the coordinating institution was the Aristotle University of Thessaloniki. The monographs are written by Professors at the partner universities and consist of common teaching material for all ICP student participants. Their content corresponds generally to the needs of advanced undergraduate level or to those of early postgraduate studies. We are most grateful to all colleagues who freely gave up their time to prepare and then teach this coursework to the students. We are also indebted to the major contribution of the Foundation for Research and Technology of Crete, in allowing us to make use of its facilities for the production of this series.

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Lectures
on the
Classification
of
Complex
and
Real
Lie Algebras

Preface

The following notes grew out from a seminar on the classification of real Lie algebras, held at the University of Crete in the WS 1988/89. This seminar was considered as a continuation to the excellent lectures on Lie groups and Lie algebras, given by my friend Hans Samelson, one semester before. The notes were prepared by a non-expert in the field (that's me), to serve as a help, for those which wanted to learn quickly, how the simple real Lie algebras are classified. This, of course, is not an excuse for the errors they contain *, but a warning to the innocent reader.

The reader, which, among other things, has to be patient with the different verses and motifs from poets and philosophers figuring everywhere in the text. Incidentally, 1988 was the 200-th anniversary of Byron's birth, so it was reasonable to honor this great poet (and some others as well) and remember his immortal verses not less than the details of the structure of the Lie algebras.

The list on the next page indicates, I hope, the contents and the way we get to our aim.

Paris Pamfilos, Paleochora, Crete, the 23 February 1991

* I would be very grateful to those readers (if any) which would point out to me errors and suggestions, to make the text better. They can contact me, writing to: University of Crete, Department of Mathematics, Iraklion-Crete-Greece, P.O. Box 1470. (or call 0030-81-246428)

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Relations, between real and complex vector spaces. Additional material in the books of Nomizu[1] and Samelson[2].
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Same content, same reference.
3. **Complex structures**
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4. **Bilinear forms**
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5. **Complex quadratic forms**
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6. **Pairs of quadratic forms and Jordan normal form**
Some reference on simultaneous diagonalization.
7. **Schur's lemma**
Following Nomizu's exposition.
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The standard basis, the corresponding (Lie bracket) multiplication table, the center. A first encounter with Lie algebras.
9. **Ad and ad of $\mathfrak{gl}(n; \mathbb{C})$**
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II. Lie algebras, general facts.

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Definition, some examples, structure constants, homomorphism, normalizer, centralizer, quotient, direct product, semi-direct product, the two-dimensional non-abelian example, some interesting exercises.
11. **Adjoint representation and Killing form**
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Simplicity of -, Killing form of -, study of diagonal subalgebra.
13. **Irreducible representations of $\mathfrak{sl}(2; \mathbb{C})$**
A key for many details later. Following closely Samelson.

[1] K. Nomizu . Linear algebra, Academic Press 1978

[2] H. Samelson. Notes on Lie algebras, Springer 1990

14. **Solvable Lie algebras**
Derived series, solvability, upper triangular matrices, radical, the most general solvable subalgebras of $gl(n;C)$, interesting exercises.
15. **Nilpotent Lie algebras**
Lower central series, nil-radical, nilpotent subalgebras of $gl(n;C)$.
16. **Cartan's first criterion**
Used to prove its second. Uses the Jordan-Chevalley decomposition theorem of linear algebra (Jordan normal form).
17. **Semi-simplicity, Cartan's second criterion**
Definition, non-degeneracy of Killing form, following Samelson.
18. **The Casimir element**
Needed only for a short proof in the next §, following Humphreys[3].
19. **Complete reducibility and semi simplicity**
Following Humphreys and Samelson. Mainly interested in the representations of $sl(2;C)$.
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III. Structure of complex semi simple Lie algebras

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23. **Strings of roots, coroots**
Main properties of roots and their corresponding root vectors, same reference.
24. **Cartan integers and Weyl group**
Two-dimensional root systems and definition of the Weyl group, same refer.
25. **Coxeter-Dynkin diagrams**
By which the root systems are classified, which in turn classify all semi simple complex Lie algebras, same reference.
26. **Weyl group and Weyl chambers**
Study of root systems in their abstract setting, following Samelson and Humphreys. The rest of the chapter deals with construction of models for the different classes of simple complex Lie algebras.
27. **The structure of $sl(n+1;C)$** A_n
Including an important Lemma.
28. **The structure of $sp(n;C)$** C_n
29. **The structure of $o(2n;C)$** D_n
30. **The structure of $o(2n+1;C)$** B_n
31. **Freudenthal's construction**
Following Hausner Schwarz[4] and Chow[5].
32. **The structure of G_2**
33. **The structure of E_8**
With a table of all its roots .
34. **The structure of E_7 and E_6**

[3] Humphreys. Introduction to Lie algebras and Representation Theory, Springer 1972

[4] M. Hausner and J. Schwarz. Lie groups, Lie algebras, Gordon and Breach 1968

[5] Chow. Lie groups and Lie algebras. 2 Vols.

- 35. **The structure of F_4**
A table of all roots of this Lie algebra.
- 36. **The order of the Weyl group**
The order of the Weyl groups of the exceptional Lie algebras.
- 37. **Weyl-Chevalley normal form**
Final normalization of the most simple bases of semi simple complex Lie algebras, following Samelson. Important exercises stated as theorems in Jacobson[6]
- 38. **Existence and conjugacy of Cartan algebras**
Regular, singular elements, following Seminaire Sophus Lie[7]

IV. Structure of real semi simple Lie algebras

- 39. **Automorphisms**
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- 40. **Real forms, Cartan decomposition**
How to find all real forms of a simple complex Lie algebra.
- 41. **Real semi simple Lie algebras**
Intensive study of the relations to their complex brothers.
- 42. **Compact Lie algebras**
Structure and roots, diagram, various lattices, all real.
- 43. **Automorphisms of compact Lie algebras**
Copying and adapting §39 to our real needs.
- 44. **Diagram and lattices**
With three nice drawings.
- 45. **Inner involutions of simple compact Lie algebras**
Following Borel [9] and Murakami [10,11]. Table of maximal roots and extended Dynkin diagrams of the 9 types of simple complex Lie algebras.
- 46. **Simple real Lie algebras of inner type**
Following Borel, de Siebenthal and Murakami [12]
- 47. **Canonical representation of automorphisms [12]**
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Following Murakami.
- 49. **Real forms of A_d**
- 50. **Real forms of B_d**
- 51. **Real forms of D_d**
- 52. **Real forms of C_d**
- 53. **Signature and normal real form**
- 54. **Catalog of simple real Lie algebras**
Table must be completed by realifications of simple complex Lie algebras.
End of the tale.

[6] N. Jacobson. Lie algebras, Dover 1979

[7] Seminaire Sophus Lie, Ecole Norm. Sup. 1955

[8] Z. X. Wan. Lie algebras, Pergammon Press 1975

[9] A. Borel and J. de Siebenthal. Les sous groupes fermés de rang maximum Comm. M. Hel. 23(1949)

[10] S. Murakami. On the automorphisms of a real semi simple Lie algebra. J. Math. Soc. Japan 4(1952)

[11] S. Murakami. Supplements and corrections to [10]. J. Math. Soc. Japan 5(1953)

[12] S. Murakami. Sur la classification des algèbres de Lie réelles et simples. Osaka J. Math. 2(1965)

I.

Preliminary

Linear

Algebra

That is the usual method, but not mine-
 My way is to begin with the beginning;
 The regularity of my design
 Forbids all wandering as the worst of sinning,
 And therefore I shall open with a line
 (Although it cost me half an hour in spinning)
 Narrating somewhat of Don Juan's father,
 And also of his mother, if you'd rather.
 Byron, Don Juan, Canto I, 7

1. Complexification, real forms

The vector spaces, we are concerned here, are real (\mathbf{R}) or complex (\mathbf{C}). For a real vector space V , the complexification $V_{\mathbf{C}}$ is a complex vector space defined as the set of formal sums

$$V_{\mathbf{C}} = \{X+iY \mid X, Y \in V, i \cdot i = -1\}. \quad (1)$$

In $V_{\mathbf{C}}$ one defines addition and multiplication by complex numbers through the formulas

$$(X+iY) + (X'+iY') = (X+X') + i(Y+Y'), \quad (2)$$

$$(u+iv) \cdot (X+iY) = (uX-vY) + i(vX+uY). \quad (3)$$

Exercise-1 Show that $V_{\mathbf{C}}$ is a complex vector space, with respect to these operations and the \mathbf{C} -dimension of this vector space is equal with the \mathbf{R} -dimension of V . More precisely, show that a \mathbf{R} -basis of V is also a \mathbf{C} -basis of $V_{\mathbf{C}}$.

Exercise-2 Show that a \mathbf{R} -linear map $F: V \rightarrow V$ can be extended to a \mathbf{C} -linear

$F_{\mathbf{C}}: V_{\mathbf{C}} \rightarrow V_{\mathbf{C}}$, through the definition:

$$F_{\mathbf{C}}(X+iY) = F(X) + iF(Y). \quad (4)$$

Exercise-3 Each basis of V is a basis of $V_{\mathbf{C}}$ (identify X with $X+i0$). Show that the matrices of F and $F_{\mathbf{C}}$, with respect to such a basis, are identical.

Given a complex vector space W , there are many real vector subspaces V of W , whose complexification $V_{\mathbf{C}} = W$. These are called **real forms** of W and are constructed as follows: Take a \mathbf{C} -basis $e = \{e_1, \dots, e_n\}$ of W and consider the real linear span V_e

$$V_e = \langle e_1, \dots, e_n \rangle = \text{set of all } \mathbf{R}\text{-linear combinations of } \{e_1, \dots, e_n\}. \quad (5)$$

Exercise-4 Show that V_e is a real vector subspace of the same dimension as W . Show further that $(V_e)_{\mathbf{C}} = W$.

Obviously, different bases of W give different real forms V_e . When, however, the \mathbf{C} bases of W $\{e_1, \dots, e_n\}$, $\{e'_1, \dots, e'_n\}$ have a real change-basis matrix T , then the corresponding real forms are identical.

Question-1 Given a \mathbf{C} -vector space W and some real form V of W , when is a complex linear map $F: W \rightarrow W$, the complex extension of a real linear $F_0: V \rightarrow V$?

Question-2 Fix a real form V of W . When is the complex linear $F: W \rightarrow W$, the complex extension of a real linear $F_0: V \rightarrow V$?

Exercise-5 Show that a linear $F : W \rightarrow W$ is extension of the real $F_0 : V \rightarrow V$, if and only if $F(V) \subset V$. [When this happens, then $F|_V = F_0$ is the \mathbf{R} -linear $V \rightarrow V$.] Equivalently, there is a basis of V , with respect to which, F is represented by a real matrix.

Question-3 Find the structure of the set of all linear $F : W \rightarrow W$, which are extensions of some \mathbf{R} -linear $F_0 : V \rightarrow V$, for some real form V of W .

Fix a basis $e = \{e_1, \dots, e_n\}$ of W . Then, all other bases of W are given by bases e' resulting from e , by right multiplication with a matrix $g \in GL(n, \mathbf{C})$:

$$e' = e \cdot g, \quad g \in GL(n, \mathbf{C}). \quad (6)$$

The matrix of the \mathbf{C} -linear F , with respect to this basis, is given by $h \in GL(n, \mathbf{R})$, if F is extension of some real linear map on the real form defined by e' .

$$F(e') = e' \cdot h.$$

Thus, such an extension is defined by a pair of two matrices $(g, h) \in GL(n, \mathbf{C}) \times GL(n, \mathbf{R})$. Two pairs (g, h) and (g', h') define the same extension if and only if, the matrix $t = g^{-1} \cdot g'$ is real and $h' = t^{-1} \cdot h \cdot t$.

$$(g, h) \approx (g', h') \iff t = g^{-1} \cdot g' \text{ is real, and } h' = t^{-1} \cdot h \cdot t, \quad (7)$$

defines an equivalence relation in the set of pairs of matrices and shows that each extension of a real linear map on some real form of W , corresponds to some orbit of $GL(n, \mathbf{C}) \times GL(n, \mathbf{R})$, under the action of $GL(n, \mathbf{R})$:

$$(t, (g, h)) \rightarrow (g' = g \cdot t, h' = t^{-1} \cdot h \cdot t). \quad (8)$$

Thus, the set of complex extensions of real linear maps on real forms of W , is in one-to-one correspondence with the space of orbits of this action:

$$GL(n, \mathbf{C}) \times GL(n, \mathbf{R}) / GL(n, \mathbf{R}). \quad (9)$$

Real forms are in 1-1 correspondence with conjugations i.e. involutive conjugate-linear maps of the complex vector space V :

$$\begin{aligned} \Phi: V &\rightarrow V, \\ \Phi(X + aY) &= \Phi(X) + \bar{a}\Phi(Y), \text{ for every } a \in \mathbf{C}, \\ \Phi \circ \Phi &= I. \end{aligned} \quad (10)$$

Given Φ , the corresponding real form V is the (real) vector space of fixed points of Φ :

$$V_\Phi = \{X \in V \mid \Phi(X) = X\}.$$

Inversely, given the real form W , one defines the corresponding conjugation by:

$$F(X + iY) = X - iY, \text{ for every } X, Y \in W.$$

Notice that the composition $\Phi \circ \Phi'$ of two conjugations is a complex isomorphism of the vector space V . Consequently, every conjugation Φ' of V is of the form $\Phi' = \Phi \circ g$, where g is an isomorphism of V and Φ is some fixed conjugation of V .

Nothing so difficult as a beginning
 In poesy, unless perhaps the end;
 For oftentimes when Pegasus seems winning
 The race, he sprains a wing, and down we tend,
 Like Lucifer when hurl'd from heaven for sinning;
 Our sin the same, and hard as his to mend,
 Being pride, which leads the mind to soar too far,
 Till our own weakness shows us what we are.

Byron, Don Juan, Canto IV

2. Realification

Each complex vector space W is at the same time a real vector space, with respect to the same addition and multiplication by scalars (the only difference is that we restrict ourselves to real scalars). When $e = \{e_1, \dots, e_n\}$ is a basis of W , then

$$\{e_1, \dots, e_n, ie_1, \dots, ie_n\} \quad (1)$$

is a basis of W , considered as a real vector space. We denote this vectorspace by $W_{\mathbf{R}}$ and call it the **realification** of W . Obviously

$$\dim_{\mathbf{R}} W_{\mathbf{R}} = 2 \dim_{\mathbf{C}} W. \quad (2)$$

Exercise-1 Show that "multiplication by i " in $W_{\mathbf{R}}$, defines a \mathbf{R} -linear map

$$J : W_{\mathbf{R}} \rightarrow W_{\mathbf{R}}, \text{ with } J^2 = -I. \quad (3)$$

The matrix of this linear map, with respect to the basis (1) is

$$J = \left(\begin{array}{ccc|ccc} 0 & \dots & 0 & -1 & \dots & 0 \\ \dots & & & & & \\ 0 & \dots & 0 & 0 & \dots & -1 \\ \dots & & & & & \\ 1 & \dots & 0 & 0 & \dots & 0 \\ \dots & & & & & \\ 0 & \dots & 1 & 0 & \dots & 0 \end{array} \right) \quad (4)$$

Each \mathbf{C} -linear map $F : W \rightarrow W$, may be considered as an \mathbf{R} -linear $F_{\mathbf{R}} : W_{\mathbf{R}} \rightarrow W_{\mathbf{R}}$, which satisfies

$$F_{\mathbf{R}}(i \cdot X) = i \cdot F_{\mathbf{R}}(X), \text{ for all } X \in W, \quad (5)$$

which, by the definition of J , is equivalent with

$$F_{\mathbf{R}} \cdot J = J \cdot F_{\mathbf{R}}. \quad (6)$$

Exercise-2 Let $g = X + iY \in GL(n, \mathbf{C})$ be the matrix of the linear map $F : W \rightarrow W$, with respect to the basis e (X, Y are real matrices, but not necessarily in $GL(n, \mathbf{R})$). Then the corresponding $F_{\mathbf{R}} : W_{\mathbf{R}} \rightarrow W_{\mathbf{R}}$, has with respect to the basis (1), the matrix

$$g_{\mathbf{R}} = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}. \quad (7)$$

Exercise-3 Show that the matrices of the form (7) are precisely the real $2n \times 2n$ matrices, which commute with J , defined by (4).

Exercise-4 Show that each matrix of the form (7) defines a \mathbf{C} -linear map $F : W \rightarrow W$,