Droz-Farny, an inverse view

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Abstract

In this article we study some circle constructions related to the orthocenter of a triangle. By inverting these circles with respect to circles centerred at the orthocenter we obtain a pencil of circles, whose line of centers is the Droz-Farny line of a proper right angle centerred at the orthocenter of a related triangle. The study of this configuration leads to a simple proof of the well known theorem on the Droz-Farny lines.

1 A coincidence lemma

Our basic configuration is a triangle ABC and the three circles α, β, γ centerred, correspondingly, at the vertices of the triangle, passing through the orthocenter H of the triangle and intersecting at the points $\{A', B', C'\}$, which lie on the circumcircle κ of ABC

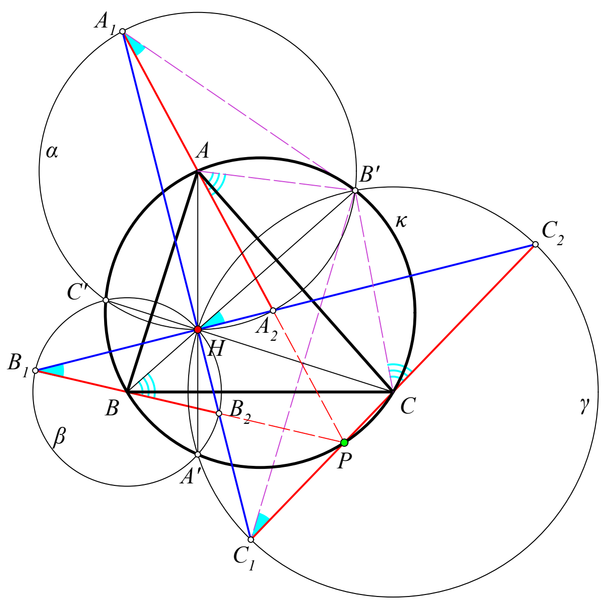

Figure 1: The basic configuration

(see Figure 1). Next lemma formulates a basic fact for this configuration concerning three diameters, which in the sequel will be called the *diameters of point P*.

Lemma 1. *For every point P on the circumcircle κ of the triangle, let A_1A_2, B_1B_2, C_1C_2 be the diameters, respectively of α, β, γ passing through P. The endpoints of these diameters lie by three on two orthogonal lines through the orthocenter.*

Proof. We show first that $\widehat{A_2HB'} = \widehat{C_2HB'}$. This will prove that points H, A_2 and C_2 are collinear. A similar argument, then, will show that also B_1, H and A_2 are collinear. In fact, $\widehat{A_2HB'}$ is half the angle $\widehat{A_2AB'}$, which, because of the inscribed quadrilateral $B'APC$ equals angle $\widehat{B'CC_2}$, which is double the angle $B'C_1C_2$, which in turn is equal to $\widehat{B'HC_2}$, as claimed. □

Remark Obviously, if we start with a right angle centerred at H, we obtain a point P on the circumcircle κ, for example, by locating first the intersection points C_1, C_2 of κ with the legs of the right angle, and then intersecting κ with the line C_1C_2. Thus, right angles at H correspond uniquely to points P on κ, constructed by the above procedure, and vice versa. In the sequel we call the sides of these right angles at H the *orthogonals of P*.

2 Anti-inverting on a proper circle

The next few figures show what happens to the basic configuration, when anti-inverted w.r. to a circle δ, centerred at H. It doesn't really matter which is the radius of δ, but for convenience we set this radius $r = \sqrt{p}$, where $p = |HA||HA_0| = |HB||HB_0| = |HC||HC_0|$, points $\{A_0, B_0, C_0\}$ being the feet of the altitudes of triangle BC ([Joh60, 164]) (see Figure 2).

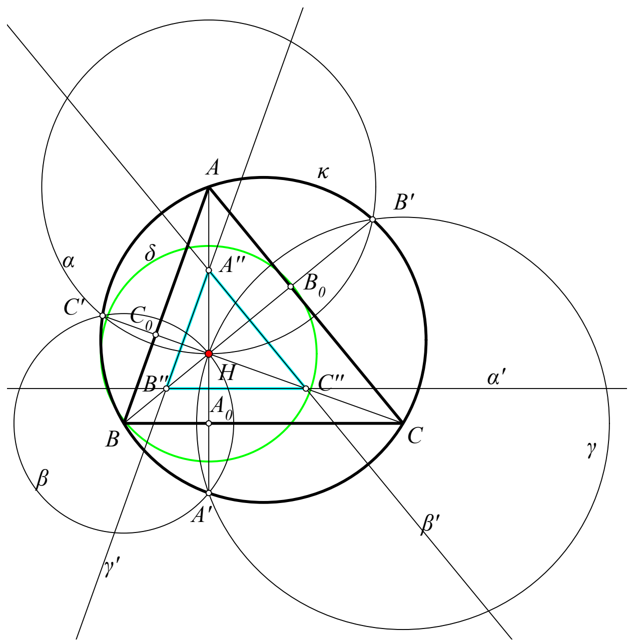

Figure 2: The inverting circle δ

By definition, the anti-inversion w.r. to a circle $\delta(H, r)$ is the composition of the ordinary inversion w.r. to the circle and the symmetry w.r. to its center H ([Ped90, 75]). From this follows that anti-inversions have properties similar to those of the ordinary inversions, for example preserving the set of circles and lines of the plane and the angles at their intersection points. In contrast to the inversion, though, which fixes the points of δ, the anti-inversion maps a point of δ to its antipodal.

Comming now to our configuration and applying the anti-inversion F w.r. to the circle $\delta(H, r)$, defined above, we see that the circles α, β, γ map, correspondingly, to lines α', β', γ', which define the sides of a triangle $A''B''C''$. The following lemma results trivially from fundamental properties of the inversion ([Ped90, 75]) and I omit its proof (see

Figure 2).

Lemma 2. *The following properties for the triangle $A''B''C''$ are valid*

1. *Lines α', β', γ' are orthogonal, correspondingly to the altitudes HA, HB, HC of ABC.*
2. *The vertices A'', B'', C'' are the anti-inverses, correspondingly, of A', B', C'.*
3. *The circumcircle ε of $A''B''C''$ is the anti-inverse of the circumcircle κ of ABC.*
4. *Triangles ABC and $A''B''C''$ are homothetic w.r.t. to H, the homothety ratio being $\lambda = 2$.*

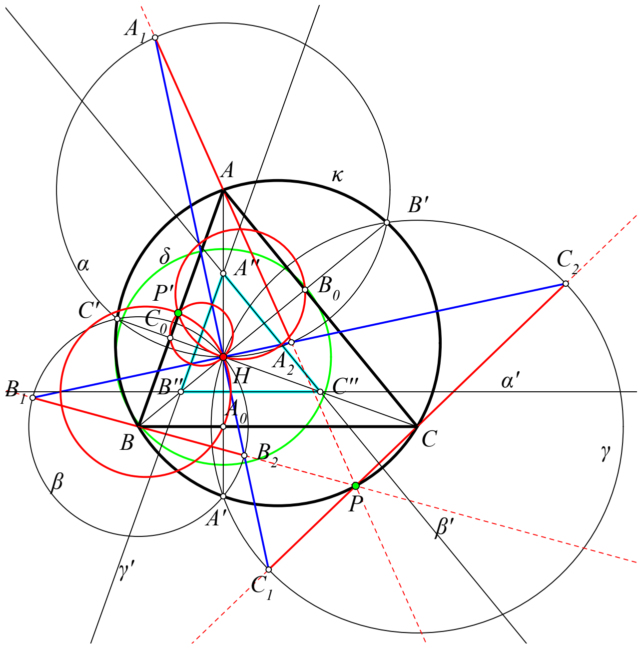

Figure 3: The pencil of the anti-inverses of the diameters of P

Next lemma, which is also an immediate consequence of the properties of inversions, formulates a fact for the anti-inverses of the *diameters of P*.

Lemma 3. *The following properties for the triangle $A''B''C''$ are valid*

1. *The anti-inverses, by F, of the three lines carrying the diameters of P, are correspondingly three circles $\alpha'', \beta'', \gamma''$ passing through H and the anti-inverse $P' = F(P)$ of P.*
2. *The circles circles $\alpha'', \beta'', \gamma''$ intersect the orthogonals of P at the intersection points of these orthogonals with the sides of the triangle $A''B''C''$.*

The two lemmas imply an easy proof of the well known Droz-Farny theorem ([Aym04], [TA12]).

Theorem 1. *For every pair of orthogonal lines through the orthocenter H of a triangle ABC, the intersections of the lines with the sides of the triangle define three segments, whose middles are collinear.*

Proof. In fact, by the above analysis, the orthogonal lines are the orthogonals of a unique point P of the circumcircle κ of ABC. Carrying out the above anti-inversion F, and composing it with the homothety $G(H, \lambda)$ we obtain a pencil of circles having the three stated segments as diameters. Hence their centers are on the line of centers of the pencil. □

Remark The line of centers of the above pencil is often called a Droz-Farny line of the triangle. The proof shows that the Droz-Farny lines of the triangle $A''B''C''$ are the medial lines of HP', where P' is an arbitrary point of the circumcircle κ of the triangle. Thus, they are the tangents of the inscribed conic with focal points at H and its isogonal conjugate, which is the circumcenter O of the triangle ([Yiu13, p.131]). Figure 4 displays this

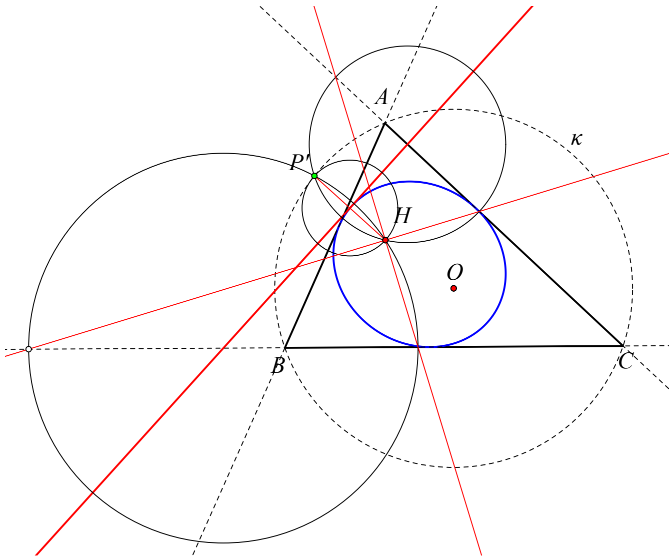

Figure 4: The envelope of the Droz-Farny lines

configuration.

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