1 Abridged notation, line formularium

Here I review the “abridged notation” of our ancestors following closely the beautiful “Treatise on Conics” [Sal17] of Salmon but drawing some more figures than he does (see also [Puc84, p.88], [Tod88, p.57]). This notation is, essentially, the representation of entities by equations involving the projective coordinates with respect to some “projective base” (see file Projective plane) whose coordinate axes are lines expressed through equations

\[ \mu = 0, \quad \rho = 0, \quad \lambda = 0. \]  

(1)

Somewhere in the plane, arbitrary but fixed, is the fourth “unit point” \( D(1,1,1) \) completing the coordinate base. The distinguishing characteristic of this method is to use the
equations describing the geometric entities rather than the coordinates entering in these equations. For example, the way we determine the location of a point in \(nr-2\) below is typical for this method. The following facts are considered within this framework:

1. \(a\mu + b\rho + c\lambda = 0\) represents an arbitrary line in the plane.

2. A point \(P\) is preferably represented as intersection of two lines:
   \[
   \mu - t\rho = 0 \quad \rho - t'\lambda = 0. \tag{2}
   \]
   We speak of the point \((t,t')\). Equation \(\mu - t\rho = 0\) represents a line through \(C\) and equation \(\rho - t'\lambda = 0\) represents a line through point \(A\).

3. The condition that this point belongs to line \(a\mu + b\rho + c\lambda = 0\) is
   \[
   att' + bt' + c = 0. \tag{3}
   \]

4. The coefficients \(\{a,b,c\}\) of the line passing through the points \(\{(t,t'),(s,s')\}\) satisfy the system of equations:
   \[
   att' + bt' + c = 0, \quad ass' + bs' + c = 0. \tag{4}
   \]

5. Thus, solving for \(\{a,b,c\}\), we find, up to multiplicative constant:
   \[
   a = t' - s', \quad b = ss' - tt', \quad c = t's'(t - s). \tag{5}
   \]

6. Replacing in \(nr-3\), we get the nice condition of “collinearity” of three points \((t,t'),(s,s'),(u,u')\) :
   \[
   (t' - s')uu' + (ss' - tt')u' + t's'(t - s) = 0. \tag{6}
   \]

7. For two lines represented by the equations \(att' + bt' + c = 0\) and \(a'tt' + b't' + c' = 0\), their intersection is given by:
   \[
   t = \frac{bc' - b'c}{a'c - ac'}, \quad t' = \frac{a'c - ac'}{ab' - a'b}. \tag{7}
   \]

8. Thus, the condition of “concurrency” of three lines results by replacing this into \(nr-3\) for a third line, which after simplification leads to:
   \[
   a''(bc' - b'c) + b''(a'c - ac') + c''(ab' - a'b) = 0. \tag{8}
   \]
   This is the determinant of the coefficients of the three lines.

9. The connection with the usual notation, which prefers to write a point in the coordinate base as \(P = xA + yB + zC\), results by replacing this into the system of equations \(2\), the correspondence being \(\{\mu = 0 \leftrightarrow x = 0, \quad \rho = 0 \leftrightarrow y = 0, \quad \lambda = 0 \leftrightarrow z = 0\}\) and getting:
   \[
   t = \frac{x}{y}, \quad t' = \frac{y}{z} \quad \text{and} \quad (x,y,z) = k(tt',t',1) \quad \text{with arbitrary} \quad k \neq 0. \tag{9}
   \]

10. The “cross ratio” of four lines \(\{\lambda_i = \mu - t_i\rho = 0, i = 1,2,3,4\}\) passing through the point \(C\) can be defined by the expression
    \[
    (\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \frac{t_1 - t_3}{t_2 - t_3}; \quad \frac{t_1 - t_4}{t_2 - t_4}. \tag{10}
    \]
    This is compatible with the definition of the cross ratio of four points on a line. This means that on every line \(\eta\) intersecting these four lines, the cross ratio of the four intersection points on \(\eta\) coincides with this one.
11. The equation of two “harmonic conjugate” lines w.r. to the lines $\mu = 0$ and $\rho = 0$ passing through $C$ is

$$\mu - t \rho = 0, \quad \mu + t \rho = 0.$$ (11)

2 Abridged notation, conics

Here the two lines are taken to be tangent to the conic and the third is the chord of contacts ([Car76, p.336]). Within this framework the following are valid properties:

![Diagram of bitangent conics with common chord AC](image)

Figure 2: Bitangent conics with common chord $AC$

1. The equation of the conic obtains the “bitangent conic” form

$$\lambda \mu - \rho^2 = 0.$$ (12)

2. More general

$$\lambda \mu - k \cdot \rho^2 = 0,$$ with variable $k$ (13)

represents all conics passing through $\{A, C\}$ and being tangent there to lines $\{\lambda, \mu\}$ correspondingly. The particular one of $nr-1$ is obtained by requiring from the conic to pass through $D$, which since $\lambda(D) = \mu(D) = \rho(D) = 1$, gives $k = 1$.

3. Every point $P$ on the conic can be determined as intersection of two lines passing through $A$ and $B$:

$$\rho - t \lambda = 0, \quad \mu - t^2 \lambda = 0 \quad \Rightarrow \quad \mu - t \rho = 0,$$ (14)

determine a point $P$ on the conic: intersection, for fixed $t$, of the lines $\rho - t \lambda = 0$

![Diagram of determination of a point on the conic](image)

Figure 3: Determination of a point on the conic

through $A$ and $\mu - t^2 \lambda = 0$ through $B$. Any pair out of these three lines can be used to determine the position of $P$ in terms of the position of the lines. The jargon is to identify points $P$ on the conic with the corresponding parameter $t$, thus, speaking of “the point $t$ on the conic”. One has to get used to this symbolism. Then it is quite handy and effective. Plus, it reflects the fact that conics are bijective images of projective lines.
4. For an arbitrary line $a\mu + b\rho + c\lambda = 0$, the intersection points with the conic are the “points $t$” found through the solution of the system:

$$a\mu + b\rho + c\lambda = 0, \quad \rho - t\lambda = 0, \quad \mu - t\rho = 0,$$

which leads to the quadratic equation:

$$at^2 + bt + c = 0.$$  

5. The equation of the line (chord) through points $P(t)$ and $P'(t')$ is

$$(tt')\lambda - (t + t')\rho + \mu = 0.$$  

This because it is satisfied by $P(t) : \rho - t\lambda = 0, \mu - t\rho = 0$ and also by $P'(t') : \rho - t'\lambda = 0, \mu - t'\rho = 0$.

6. If all these chords are to pass through a fixed point $Q$, then, replacing into this equation the coordinates of the point, we see that an equation of the form

$$(tt')p - (t + t')q + s = 0$$

must be satisfied with constants $(p,q,s)$. This represents the necessary and sufficient condition for “chords $tt'$” to pass through a fixed point.

7. Allowing complex numbers for $(t,t')$, the previous equation can represent any line of the plane, since every line in this case has two real or complex intersection points with the conic. For real conics $(t,t')$ are complex conjugate, hence $tt'$ and $t + t'$ are real.

8. The equation of the “tangent at $P(t)$” is

$$t^2\lambda - 2t\rho + \mu = 0,$$  

since it represents the limit position of the chord-line of nr-6, when $t'$ tends to coincide with $t$. The converse is also true, since one can reverse the arguments and show the following theorem.

**Theorem 1.** Any one-parameter line-equation, which can be put into the form of equation 19, represents a tangent to the conic $\lambda\mu - \rho^2 = 0$.

The theorem has many applications in finding a conic enveloping a one parameter family of lines.

9. The “cross ratio” of four points $P(t_i), \ i = 1,2,3,4$ on the conic can be defined using directly the values of $(t_i)$, as this was done in nr-10 of section 1 for the cross ratio of four lines:

$$\left(\begin{array}{c} t_1 t_2 t_3 t_4 \end{array} \right) = \left(\begin{array}{c} t_1 - t_3 \\ t_2 - t_3 \end{array} : \begin{array}{c} t_1 - t_4 \\ t_2 - t_4 \end{array} \right),$$

where $(t_i)$ are now “points on the conic”. It can be shown that this is independent of the representation of the lines and the conic and equal to the cross ratio of four points $(Q_i)$ on a line $e$, resulting by projecting perspective on $e$ the points $(P(t_i))$, using as perspectivity center an arbitrary fifth point $P_0$ on the conic (See Figure 4).

10. Requiring that the tangent of nr-8 passes through a point

$$P(s,s') : \mu = s\rho, \quad \rho = s'\lambda \quad \Rightarrow \quad t^2 - 2ts + ss' = 0.$$  

Point $t$ being on the conic implies $(t = \rho/\lambda, t^2 = \mu/\lambda)$ hence the equation

$$\mu - 2s\rho + ss'\lambda = 0,$$

which represents the “polar” of $P(s,s')$. 


3 Double tangency of two conics

We say that two conics are "doubly tangent", when they are different but have two common tangents. As an application of the abridged notation we prove the following theorem.

**Theorem 2.** Given three fixed chords \{AA', BB', CC'\} of a conic \(\kappa\), a fourth chord \(DD'\), for which the cross ratios are equal \((ABCD) = (A'B'C'D')\), is always tangent to a conic \(\kappa'\) having double tangency with \(\kappa\).

Figure 5 displays such an example. The chords \{AA', BB', CC'\} are fixed and point \(D\) is free to move on the conic \(\kappa\). Then, point \(D'\) is calculated so that the cross ratios are equal \((ABCD) = (A'B'C'D')\). These cross ratios are calculated using the line coordinates of the perspective projections of the points from an arbitrary point \(E \in \kappa\). In the figure the points of the conic are projected on two lines \(\{\varepsilon, \varepsilon'\}\) but the arguments below could use also one line only. The proof (after Salmon p. 253) is easy.

Assuming that \(A\) is given by a number \(a\) i.e. it is the intersection of lines

\[ A(a) : \quad \mu - a \rho = 0, \quad \rho - a \lambda = 0, \]

and using analogously small letters to represent the other points, the equality of the cross ratios translates, after nr-10 of section 2, to:

\[
\frac{a - c}{b - c} : \frac{a - d}{b - d} = \frac{a' - c'}{b' - c'} : \frac{a' - d'}{b' - d'}.
\]
This, considering \( \{d, d'\} \) as variables, obtains the form of a “homographic relation” (see file Homographic relation)

\[
\text{pdd'} + qd + rd' + s = 0, \quad \text{for constants } (p, q, r, s) \text{ depending on } a, b, c, \ldots
\]  

(21)

Solving for \( d' \) and substituting into equation 17 of the chord \( DD' \), gives the beautiful expression

\[
\lambda d(qd + s) + \rho(d(pd + r) - (qd + s)) - \mu(pd + r) = 0.
\]

(22)

As expected, this is a one-parameter family of lines w.r. to the parameter \( d \), which can be written in the form

\[
d^2(qλ + pρ) + d(sλ + (r - q)ρ - pμ) - (sρ + rμ) = 0.
\]

(23)

From theorem 1 follows that these lines, depending on the parameter \( d \), envelope the conic \( κ' \) with equation:

\[
(sλ + (r - q)ρ - pμ)^2 + 4(qλ + pρ)(rμ + sρ) = 0.
\]

(24)

The nice thing is that the equation of this conic can be put in the form:

\[
4(qr - ps)(λμ - ρ^2) + (sλ + (q + r)ρ + pμ)^2 = 0.
\]

(25)

Since \( λμ - ρ^2 = 0 \) is our conic \( κ \) and \( sλ + (q + r)ρ + pμ = 0 \) is a line, the conic 25 belongs to the family generated by the conic \( κ \) and a (double) line. This is a “bitangent family of conics”, all members of which are tangent to the conic \( κ : λμ - ρ^2 = 0 \) at the points where the line \( sλ + (q + r)ρ + pμ = 0 \) intersects \( κ \). Note that the intersection points can be imaginary as is for example the case with the family of concentric circles, which is also a “bitangent family” all members of which are tangent at the same two imaginary points at the same two imaginary lines.

**Remark-1**  The homographic relation becomes involutive (see file Homographic relation) when \( q = r \) and equation 21 takes the form

\[
\text{pdd'} + q(d + d') + s = 0,
\]

showing that all chords \( \{DD'\} \) pass through a common point (see equation 18). Thus, we should exclude this case from the beginning, since it shows a totally different behaviour.

**Remark-2**  Theorem 2 is equivalent to the well known fact, that a “homography” on a conic \( f : κ → κ \) is completely determined by prescribing the images \( \{A', B', C'\} \) at three arbitrary points \( \{A, B, C\} \) of it. Then, the chords \( \{DD', D' = f(D), D ∈ κ\} \) envelope a conic \( κ' \) bitangent to \( κ \).

From this point of view, remark 1 is equivalent with the condition that the three chords \( \{AA', BB', CC'\} \) have no common point. In the contrary case, we have still a homography but not a genuine enveloping conic \( κ' \). All chords \( \{DD'\} \) pass, in that case, through the common point of \( \{AA', BB', CC'\} \) and the homography becomes an “involution” of the conic \( κ \), characterized by the functional condition \( f^2 = e \), where \( e \) the identity transformation of the conic.

## 4 Inscribed polygons in conics

Here we discuss another application of the abridged method concerning polygons inscribed in a fixed conic and having their sides pass through fixed points.
Theorem 3. If a polygon of \( n \) sides is inscribed in a conic \( \kappa \) and its \( (n-1) \) sides pass through corresponding fixed points, then its \( n \)-th side envelopes another conic \( \kappa' \) bitangent to \( \kappa \).

Figure 6 displays such an example for triangles. Triangle \( ABC \) moves having all the time its vertices on the conic \( \kappa \) and two of its sides passing through two corresponding fixed points \( \{I,J\} \). Then the third side \( BC \) envelopes another conic \( \kappa' \) which is bitangent to \( \kappa \). The proof reduces to the theorem 2 by taking three different positions of the triangle and defining the corresponding chords \( \{B'C',B''C'',B'''C'''\} \). Then, for the moving fourth triangle \( ABC \) we have the preservation of cross ratio \( (B'B''B'''B) = (C'C''C'''C) \).

This, because the central correspondence (involution at) from \( I, f_I : B \rightarrow A \) preserves the cross ratio. This is due to the “homographic relation” (see equation 17)

\[
ptt' + (t + t')q + s = 0 \quad \text{with constants } p, q, s
\]

and the simple fact that such relations preserve the cross ratio. The same is true for the correspondence from \( J, f_J : A \rightarrow C \). Thus, composing the two correspondences we have the map \( f = f_J \circ f_I : B \rightarrow C \) preserving the cross ratio. Hence, theorem 2 applies for the moving side \( BC \). The proof is easily generalized for any \( n \).

Bibliography


Related material

1. Homographic relation
2. Projective plane