Discovery is the ability to be puzzled by simple things.
Noam Chomsky, Chicago Tribune 1/1/1993

Contents

1 Apollonian circles of a segment 1
2 The Apollonian pencil of a segment 3
3 Related exercises 4

1 Apollonian circles of a segment

Given a segment $AB$ and a positive real number $k$, the “Apollonian circle of the segment $AB$ for the ratio $k$” is defined to be the geometric locus of points $P$, such that $|PA|/|PB| = k$. If $\{PC, PD\}$ are the internal and external bissectors of the angle $\alpha_{APB}$, it is a well known elementary theorem ([Cou80, p.15]), that $|CA|/|CB| = |DA|/|DB| = k$, hence $\{C, D\}$ which are well defined points on the line $AB$, belong to the locus (See Figure 1). But the angle $\angle CPD$ is a right one. Thus, every point $P$ of the locus is viewing the segment $CD$ under a right angle. Thus, it belongs to the circle with diameter $CD$.

Theorem 1. Some further properties of figure 1 are the following:

1. The angles $\angle OPA$ and $\angle PBO$ are equal.
2. The triangles $\{OPA, OPB\}$ are similar.
3. It is $OP^2 = OA \cdot OB$ and the circles $\{\alpha, \gamma\}$ are orthogonal and $\{D, C\}$ are harmonic conjugate relative to $\{A, B\}$. 
4. For a constant $AB$ and variable $k$ the “Apollonian circles $\alpha$ for $k$” form a “non-intersecting pencil of circles”

**Proof.** Nr-1. To see this, add to $\overline{OPA}$ the angle $\overline{APC}$ and get $\overline{OPC}$. Add to $\overline{PBO}$ the angle $\overline{CPB} = \overline{APC}$ (because $PC$ bisector) to get $\overline{OCP} = \overline{OPC}$ from the isosceles $OPC$.

Nr-2 follows from nr-1. To prove the remaining properties see that triangles $\{OPA,OPB\}$ being similar implies $|OP||OP| = |OA||OB|$. This means: (i) that $OP$ is tangent to the circumcircle $\gamma$ of $ABP$, (ii) that the Appolonius circle $\alpha$ is orthogonal to the circumcircle $\gamma$ of $ABP$, hence also to the circle $\delta$ on diameter $AB$. Thus, all these Apollonian circles form the pencil of circles orthogonal to $\delta$ and line $AB$. □

**Remark 1.** The Apollonian circles of the segment $AB$ form one of the main categories of “circle pencils”, namely that of “non-intersecting type”. All types of circle pencils are studied in the file **Circle Pencils**.

**Exercise 1.** Given a circle $\alpha(O)$ and a point $A \neq O$ show that there is precisely one segment $AB$ for which $\alpha$ is an Apollonian circle (See Figure 1).

**Exercise 2.** Given two consecutive segments $\{AC,CB\}$ on a line, show that the geometric locus of the points $X$ viewing the segments under equal angles is an Apollonian circle of the segment $AB$ (See Figure 1).

![Figure 2: Point X viewing the segments {AB,BC,CD} under equal angles](image)

**Exercise 3.** Given three consecutive segments $\{AB,BC,CD\}$ on a line, find a point $X$ viewing these segments under equal angles (See Figure 2). Find conditions of existence of such a point.

![Figure 3: Apollonian pencil of segment AB](image)
2 The Apollonian pencil of a segment

Consider a segment $AB$ and the Apollonian circles dividing it in various ratios. All these circles are members of a (coaxal) “hyperbolic pencil” of circles, which I call the “Apollonian pencil” of the segment. This is a pencil of “non-intersecting type”, i.e. two member-circles do not intersect. The common radical axis of the member-circles of the pencil is the perpendicular bisector line of the segment. As is with every pencil, the Apollonian pencil is generated, i.e. completely defined, from two particular members, one of which can be the perpendicular bisector $e$ of the segment. The points $\{A,B\}$, called the “limit points” of the pencil are considered as member-circles with radius zero. Figure 3 shows some circles of such a pencil labeled by their characteristic ratio satisfying $\frac{|XA|}{|XB|} = k$ for all points of the corresponding circle. The perpendicular bisector line is the particular case for $k = 1$.

![Figure 4: Symmetric lying Apollonian circles of AB](image)

**Theorem 2.** In the following formularium are listed the formulas for various distances concerning the Apollonian circle $\alpha$ of $AB$ relative to the ratio $k > 0$ (See Figure 4).

1. $\frac{|XA|}{|XB|} = k$, for $X \in \alpha$ implies the following: $CA = \frac{k}{1+k}AB$, $CB = \frac{1}{1+k}AB$,

2. $DA = \frac{k}{1-k}AB$, $DB = \frac{1}{1-k}AB$, $CM = \frac{1+k}{2(1+k)}AB$, $DM = \frac{1+k}{2(1-k)}AB$,

3. $OA = \frac{k^2}{1-k^2}AB$, $OB = \frac{1}{1-k^2}AB$, $OM = \frac{1+k^2}{2(1-k^2)}AB$, $|OZ| = \frac{k}{|1-k^2|}|AB|$.

The Apollonian circles of $AB$ relative to the ratios $k$ and $\frac{1}{k}$ have equal radii and lie symmetrically relative to the medial line of $AB$ (See Figure 4).

**Proof.** The formulas are easily computed. The geometric property by which the circles corresponding to ratios $k$ and $\frac{1}{k}$ lie symmetrically relative to the orthogonal bisector of the segment $AB$, results from the fact that for every point $X$ which satisfies $\frac{|XA|}{|XB|} = k$ its symmetric $Y$ relative to the perpendicular bisector of $AB$ will satisfy $\frac{|YA|}{|YB|} = \frac{1}{k}$.

A half-circle $\lambda$ with diameter the segment $AB$ gives another parametrization of the pencil, in the sense, that every member-circle $\kappa$ cuts $\lambda$ in exactly one point $Z$ and $k = \frac{|ZA|}{|ZB|} = \tan(\phi)$. This leads to another way to construct the member-circle corresponding to ratio $k$:

1. Find $\phi = \arctan(k)$ and locate point $Z$ on the half-circle.
2. Draw $ZO$ orthogonal to $ZM$ at $Z$. This determines the center $O$ and the radius $|OZ| = |AB|\tan(2\phi)/2$ of the $k$–ratio member-circle.
Notice that $\lambda$ is part of the minimal member-circle of the pencil $\mathcal{P}$ whose member-circles are orthogonal to the Apollonian circles of $AB$. Figure 5 shows also the member-circle $\kappa'$ passing through the center $O$ of $\kappa$, for which the following property holds.

**Exercise 4.** If $\kappa(O)$ is an Apollonian circle of the segment $AB$ characterized by the ratio $k$, then the Apollonian circle passing through the center $O$ of $\kappa$ is characterized by the ratio $k^2$.

### 3 Related exercises

**Exercise 5.** Construct a triangle $ABC$ from its sides $\{b = |A\Gamma|, c = |AB|\}$ and its bisector $d = |AD|$ from $A$.

*Hint:* Consider an arbitrary segment $B'C'$ of length $a'$. Construct its Apollonian circle $\kappa_1$, for the ratio $\lambda = b/c$, which intersects $B'C'$ at its inner point $D'$. Construct then the Apollonian circle $\kappa_2$ of the segment $D'C'$ for the ratio $\mu = d/b$. If $A'$ is an intersection point of the circles $\{\kappa_1, \kappa_2\}$, then the triangle $A'B'C'$ is similar to the wanted, with a known similarity ratio.

**Exercise 6.** Given the triangle $ABC$, to determine a point $X$, which has the two ratios of distances $|XA|/|XB| = \kappa$ and $|XB|/|XC| = \lambda$. Show that for given $\{\kappa, \lambda\}$, there exist in general two points $\{X, X'\}$, whose line $XX'$ passes through the center $K$ of the circumcircle $\mu(K, r\mu)$ of the triangle $ABC$. Show also that if $\kappa\lambda = \sigma$ is constant, then the points $\{X\}$ are on a fixed circle.

*Hint:* $X$ is one of the intersection points of the two Apollonian circles $\{\alpha\beta, \beta\gamma\}$, whose points satisfy respectively $|XA|/|XB| = \kappa$ and $|XB|/|XC| = \lambda$ (See Figure 6). Then it will also hold $|XA|/|XC| = \kappa \cdot \lambda$ and the corresponding Apollonian circle for $AC$ relative
to the ratio $\kappa \lambda$ will also pass through points $\{X, X'\}$. The three circles are orthogonal to the circumcircle $\mu$, hence the center $K$ of $\mu$ is contained in the common radical axis of $\{\alpha \beta, \beta \gamma, \gamma \alpha\}$. From the fact that the power of $K$ relative to $\alpha \beta$ is equal to $r_{\mu}^2$, follows $|KX| \cdot |KX'| = r_{\mu}^2$. This shows that the two points $\{X, X'\}$ are coincident only in the case when the circles $\{\alpha \beta, \beta \gamma, \gamma \alpha\}$ are tangent at a point $X$ contained in the circumcircle $\mu$ of $ABC$ and then the common tangent to these circles coincides with $XK$.

**Remark 2.** The previous exercise generalizes the construction of the “three Apollonian circles of a triangle” (see file *Isodynamic points of the triangle*).

**Exercise 7.** Given is a line $\eta$, a point $H$ not lying on it and a constant $\kappa > 0$. Point $A$ moves along a given line $\alpha$. There exist at most two positions of $A$ on line $\alpha$, for which holds $|AH| = \kappa |AX|$, where $X$ is the projection of $A$ onto $\eta$.

![Figure 7: Intersections of a line and a conic](image)

**Hint:** Assume that the line $\alpha$ intersects line $\eta$ at point $C$ (See Figure 7-I). Then, for every point $A$ of $\alpha$ the ratio $\lambda = \frac{|AX|}{|AC|}$ will be fixed. Consequently, for the wanted positions of $A$, the ratio $\mu = \frac{|AH|}{|AC|} = \frac{|AH|}{|AX|} \cdot \frac{|AX|}{|AC|} = \kappa \lambda$ will be known. It follows that the wanted points $A$ coincide with the intersection points of line $\alpha$ and the Apollonian circle of segment $HC$ relative to ratio $\mu$. In the case where $\alpha$ is parallel to $\eta$, segment $|AX|$ is a known fixed length and consequently segment $|AH| = \kappa |AX|$ is also known (See Figure 7-II). In this case then, the wanted points are again intersections of a circle and a line.

**Remark 3.** The last proposition gives the geometric proof of the fact that a line ($\alpha$) intersects a conic in at most two points ([Sup01, p.42]).

**Exercise 8.** Construct a triangle, for which are given the side $a = |BC|$, the altitude from the opposite side $\nu_A$ and the trace of the bisector (internal or external) from $A$.

**Exercise 9.** Construct a triangle, for which are given the side $a = |BC|$, the angle $\alpha = \angle BAC$ of the opposite vertex and the trace of the bisector (internal or external) from $A$.

**Exercise 10.** Construct a triangle, for which are given the side $a = |BC|$, and the traces of the altitude and the bisector on that side.

**Exercise 11.** Find points $\{I, K\}$ on the sides of a triangle $ABC$, such that segments $\{CI, IK, KB\}$ are equal in length.

**Hint:** ([Yag62, I,p.133]) Since $\{IC, KB\}$ are assumed to be equal, there is a rotation about some point $D$ that brings the one to the other. Point $D$ can be determined without knowing the exact lengths of $\{CI, IK, KB\}$. In fact, the center of the rotation has to be on the medial line of $BC$ and triangles $\{DCI, DBK\}$ must be equal, the angles at $D$ being $\angle CDB = \angle DKB = \angle CAB$. Thus $D$ is one of the intersection points of the circumcircle $\kappa$ of
Section 3: Related exercises

**Figure 8:** Construct equal segments \( CI = IK = KB \)

\( ABC \) with the perpendicular bisector of \( BC \). Now the isosceles \( DCB \) with a known angle at \( D \) has a known ratio of side-lengths (depending on \( C \hat{A} B \))

\[ k = CI/ID = IK/ID = CB/DC = CB/DB. \]

Hence \( I \) lies on the Apollonian circle \( \mu \) of the segment \( CD \), relative to the ratio \( k \).

**Figure 9:** Construct equal segments \( CI = IK = KB \) (II)

Figure 9 suggests another way to solve the problem. In fact, start by assuming that \( \{I,K\} \) have been constructed, \( AC \) is shorter than \( AB \) and the ratio \( r = CI/CA \) is known. Define \( D \) on \( AB \), so that \( AD = AC \). The homothety \( f_{C,r} \) (center at \( C \), ratio \( r \)) maps \( D \) to a point \( G \), so that \( IG = IC \). Thus, \( BGIK \) is a parallelogram and \( IK = IC = IG \), implying that it is a rhombus. Draw a parallel to \( BG \) through \( D \) intersecting \( BC \) at \( E \). Because \( DA = DE \), this is a constructible point and maps through \( f_{C,r} \) on \( B \). Thus, \( r = BC/BD \) can be determined from the data. Having the homothety \( f_{C,r} \) the construction is obvious.

**Figure 10:** Triangle construction from \( \{\hat{A}, a + b, a + c\} \)
Exercise 12. To construct a triangle $ABC$ from the angle $A$ and the sums of its side-lengths $a + c$ and $a + b$.

Hint: Use the previous exercise starting from the constructible $AB'C'$ and locating $\{B,C\}$.

Bibliography


Related material

1. Circle Pencils
2. Isodynamic points of the triangle.