Apollonius problem

by Paris Pamfilos

Meanwhile, there developed among the Greeks a quite special, one might say instinctive, sense of beauty which was peculiar to them alone of all the nations that have ever existed on earth; a sense that was fine and correct.

A. Schopenhauer, On Religion

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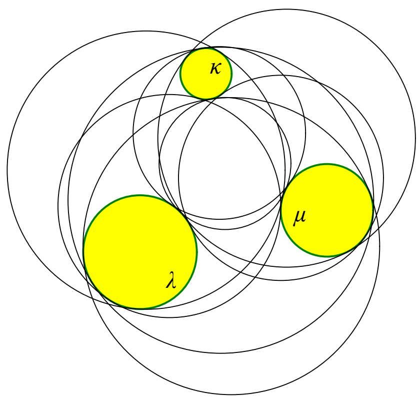


Figure 1: 8 Circles simultaneously tangent to three given circles

1 Apollonius problem

The problem of tangent circles of Apollonius consists in finding the circles which are simultaneously tangent to three given circles. Figure 1 shows the eight circles which solve the problem, in the case the given circles $\{\kappa, \lambda, \mu\}$ are mutually external.

Apollonius in his treatise *"contacts,"* which was lost, formulated more generally the problem of finding circles which are tangent to three *"things."* The *"things"* may be circles, lines and points. The wanted circles therefore may be tangent to given circles and/or lines and/or pass through given points. This results in 10 main categories for the problem where the *"things"* are (table on the right):

circles	lines	points
3	0	0
2	1	0
1	2	0
0	3	0
2	0	1
1	1	1
0	2	1
1	0	2
0	1	2
0	0	3

In each category there are different cases and the individual problems have drawn the attention of many Mathematicians throughout the centuries ([Mui95]). The first category

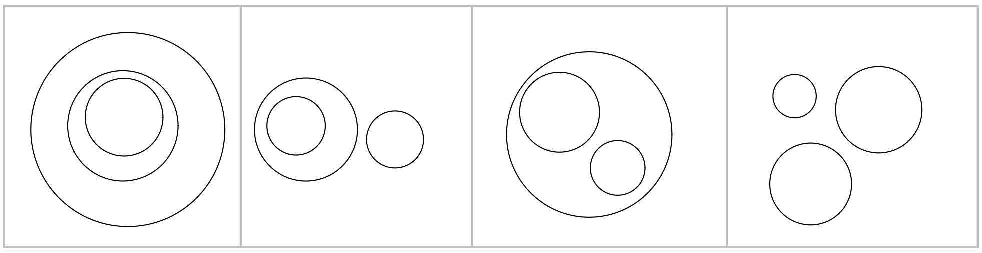


Figure 2: Possible positions of three non intersecting circles

contains the greatest number of special cases, depending on the position of the circles. This way, for example, for *non intersecting circles* we have figuratively the four possibilities shown in figure 2, from which the first two do not admit a solution. The category $(0,0,3)$ concerns the circles passing through three points $\{A, B, C\}$ and has precisely one solution: the circumcircle of triangle ABC, if the points are *"in general position"* i.e. we have no coincidences or collinearities. The category $(0,3,0)$ concerns the circles simultaneously tangent to three lines $\{\alpha, \beta, \gamma\}$ i.e. the *"tritangent circles"* of the triangle formed by these three lines (see file **Tritangent circles of the triangle**), again if the three circles are in *"general position"*.

The solution of Apollonius' problem has been approached using various different methods ([Cox68], [GR04], [Hel56], [Kun07]), the one adopted here being the use of *"inversion transformations"* (see file **Inversion**), by which the more complex cases are gradually reduced to simpler ones.

2 Two points and a line, category (0,1,2)

One of the easiest cases is the category (0,1,2) i.e. the search for a circle κ tangent to a line ε and passing through two points $\{A, B\}$. Figure 3 shows the solution in the case there

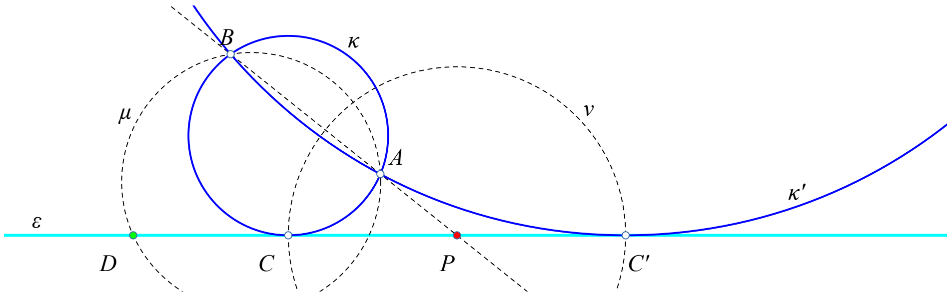


Figure 3: Two points, one line. Main case of category (0,1,2)

is one, i.e. the two points are on the same side of the line ε. A simple recipe for this is: (1) to find the intersection point $P = AB \cap \varepsilon$, and (2) use the *"power"* w.r. to the circle $PC^2 = PA \cdot PB$.

Using these two steps we find the location of the contact point C on line ε. The problem admits, in general, two solutions $\{\kappa, \kappa'\}$ corresponding to the two intersections of line ε with the circle $\nu(P, |PC|)$. In the case line AB is parallel to ε, the problem admits only one solution, point C being in that case the intersection of ε with the *"medial line"* of AB.

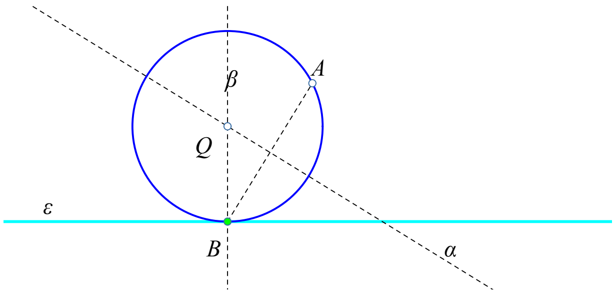


Figure 4: Two points, one line, special case

Figure 4 shows the last case that may occur in this category, namely the case in which one of the points (B) is *"on"* the line ε. The proof can be read from this figure. The center Q of the requested circle is the intersection of the orthogonal β to ε at B and the medial line α of AB.

3 Two points and a circle, category (1,0,2)

This case, concerning the construction of a circle tangent to a given circle κ and passing through two given points $\{A, B\}$, can be reduced to the previous one using an *"inversion transformation"* w.r.t. a circle with center an arbitrary point Q of the circle κ (see file **Inversion**). Such an inversion f transforms the circle κ to a line $\kappa' = f(\kappa)$ and the points $\{A, B\}$ to the points $\{A', B'\}$. After applying the recipe of the previous section and constructing circles α, β passing through $\{A', B'\}$ and tangent to the line κ', we apply again the inversion f and obtain the solutions $\{\alpha' = f(\alpha), \beta' = f(\beta)\}$.
Figure 5 illustrates a, in a sense, more direct approach of the problem, using the *"pencil of circles"* through the points $\{A, B\}$ (see file **Circle Pencils**). It can be proved easily that the radical axes of κ and the circle-members of the pencil pass all through a common point C on the line AB, which is the *"radical axis of the pencil"*. The wanted circles are the special members of this pencil, which are tangent to the given circle κ. They are found

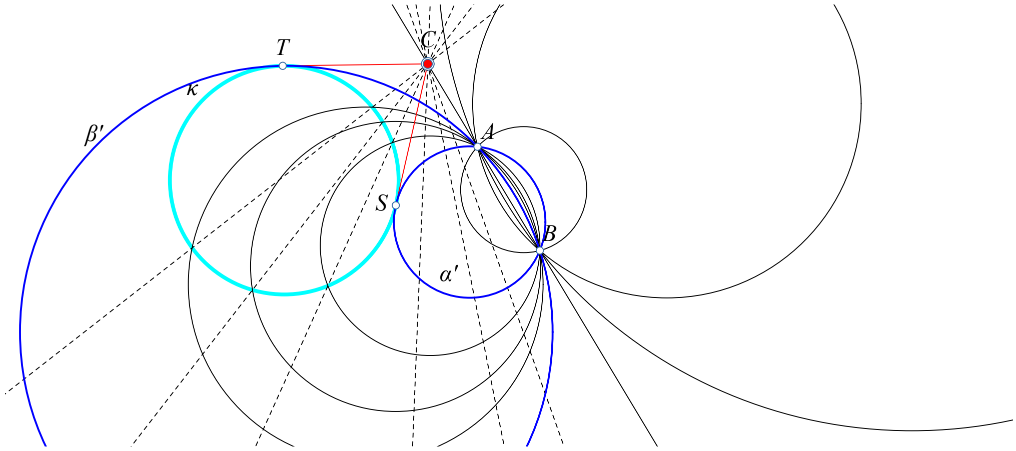


Figure 5: Two points, one circle, category (1,0,2)

by drawing the tangents to κ from C and taking the circles $\{\alpha', \beta'\}$ through $\{A, B\}$ and the contact points $\{S, T\}$ of these two tangents.

In the case the two points $\{A, B\}$ are *inside* the circle κ, we can apply the inversion g with respect to κ and find their inverses $\{A' = g(A), B' = g(B)\}$. Then, solve the problem with data $\{\kappa, A', B'\}$ and find the circles-solutions $\{\alpha', \beta'\}$. Applying again the inversion g we find then the solutions $\{\alpha = g(\alpha'), \beta = g(\beta')\}$ of the original problem.

4 Two lines a point, category (0,2,1)

Another from the easiest cases is the category (0,2,1) i.e. the search for a circle κ tangent to two lines $\{\alpha, \beta\}$

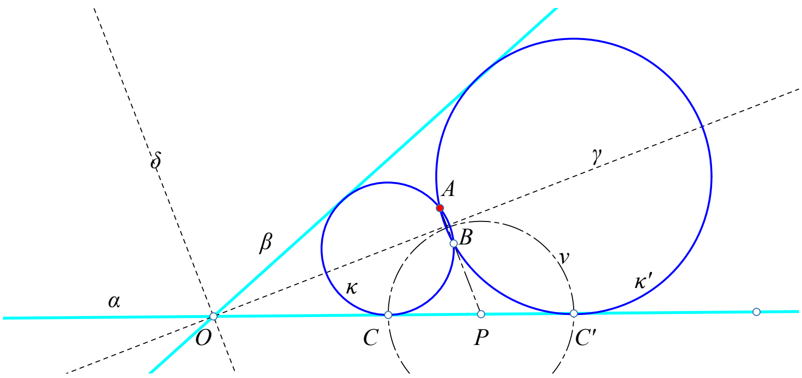


Figure 6: Two lines one point. Category (0,2,1)

and passing through a point A (See Figure 6). This can be reduced to the previous one, by considering the point symmetric B to A w.r. to one of the bisectors $\{\gamma, \delta\}$ of the two given lines $\{\alpha, \beta.\}$ Then we apply the previous recipe for the case of two points $\{A, B\}$ and a line α.

Of the two symmetrics of A relative to the bisectors $\{\alpha, \beta\}$ one only leads to a solution, since one of them has necessarily its symmetric on different sides of these lines. Figure 6 shows the two solutions $\{\kappa, \kappa'\}$ in the general case. In the case A falls on one of the given lines $\{\alpha, \beta\}$ or one of the bisectors $\{\gamma, \delta\}$ we have again two solutions.

A particular case occurs also when the two given lines $\{\alpha, \beta\}$ are parallel. Then γ must be replaced with the mean-parallel of the two parallels and the problem has again two solutions if A lies inside the strip defined by the two parallels. The two circles in this case are equal.

5 One circle one line one point, category (1,1,1)

The case of category (1,1,1) i.e. the construction of circles tangent to a given circle κ, a given line ε and passing through a given point P, can be reduced again to a simpler problem using properties of the *"inversion transformation"* w.r. to a given circle (see file **Inversion**).

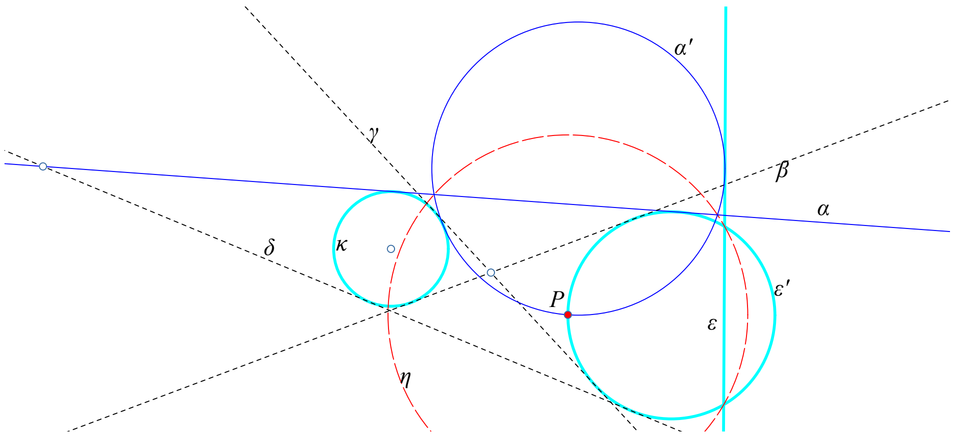


Figure 7: One circle one line one point. Category (1,1,1)

Figure 7 shows such a case. In this the, constructible from the given data, auxiliary circle η is taken to be *"orthogonal"* to the given circle κ and centered at the given point P. The inversion f w.r. to this circle, transforms the given line ε to a circle ε' through P and maps the circle κ to itself. This follows by applying standard properties of the inversion transformation. By the same properties of f, the wanted circles-solutions through P map to lines, which must be simultaneously tangent to the circles $\{\kappa, \varepsilon'\}$ i.e. they are the *"common tangent lines"* of these two circles. Thus, the problem is reduced to the construction of the common tangents to two given circles. Figure 7 shows the common tangents $\{\alpha, \beta, \gamma, \delta\}$ to the circles $\{\kappa, \varepsilon'\}$ and the transformed $\alpha' = f(\alpha)$ of α, which is one of the solutions of the original problem. Analogously the other solutions are the circles $\{f(\beta), f(\gamma), f(\delta)\}$ through P.

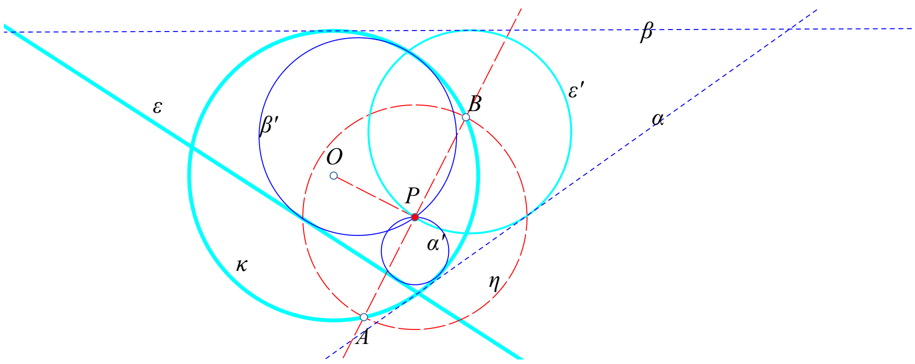


Figure 8: One circle one line one point, special case

Depending on the relative location of the data $\{\kappa, \varepsilon, P\}$, the problem can have 4,3,2,1 or none solution. Latter, for example, when P is inside κ and the line ε has no intersection with κ. There are various particular cases leading to interesting configurations. Figure 8 shows the special case for which the point P is inside the circle $\kappa(O)$. The recipe in this case is slightly modified by taking the *"anti-inversion"* f w.r. to an appropriate circle η that preserves the given circle κ (see file **Inversion**). The circle η has for diameter the chord AB of κ which is orthogonal to OP. The properties of *anti-inversions* are similar to those of *inversions* and imply that the wanted circles $\{\alpha', \beta'\}$ are anti-inverted to lines $\{\alpha, \beta\}$ simulteneously tangent to $\{\kappa, \varepsilon' = f(\varepsilon)\}$.

6 Two circles one point, category (2,0,1)

The case of category (2,0,1) i.e. the construction of circles tangent to two given circles $\{\kappa, \lambda\}$ and passing through a given point P, can be reduced to the category (1,1,1), handled in the previous section. This, using again an inversion f, w.r. to a circle $\eta(Q)$, whose

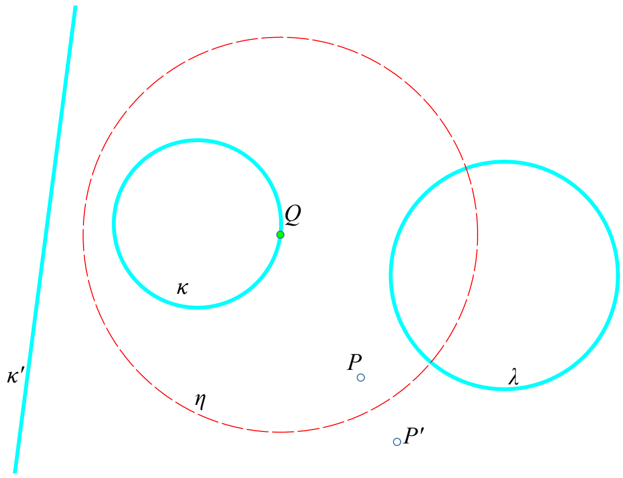


Figure 9: Two circles one point. Category (2,0,1)

center is an arbitrary point of one of the circles, $Q \in \kappa$ say (See Figure 9). If $P' = f(P)$ is the inverted of point P, the problem reduces to the construction of circles tangent to the line $\kappa' = f(\kappa)$, to the circle $\lambda' = f(\lambda)$ and passing through the point P'. Having these circles $\{\alpha, \beta, ...\}$, we obtain the solutions of the original problem by applying to these the inversion f. Figure 9 illustrates the case in which the two circles are exterior to each other and the point P is exterior to both circles. The circle η, defining the inversion f, is taken to be orthogonal to λ, hence f leaves λ invariant $(f(\lambda) = \lambda)$.
Figure 10 illustrates another approach of the problem of this category. The method uses the *"similarity centers"* $\{C, D\}$ of two circles. The crucial fact here is the property:

Property. *If a circle α touches two other circles $\{\kappa, \lambda\}$, then the line XY of the contact points passes through a similarity center of the circles $\{\kappa, \lambda\}$ and α is invariant under the inversion/anti-inversion interchanging the circles $\{\kappa, \lambda\}$.*

The inversion/anti-inversion f interchanging the two circles is easily constructible and by applying such a map to P we find a second point $Q = f(P)$ through which also passes the wanted circle. Thus, the problem reduces to one of the category (1,0,2) handled in section 3.

Figure 10 shows the four solutions $\{\alpha, \beta, \gamma, \delta\}$ for the displayed relative positions of the two circles and the point $\{\kappa, \lambda, P\}$. The circle α passes through $\{P, Q\}$ and the line

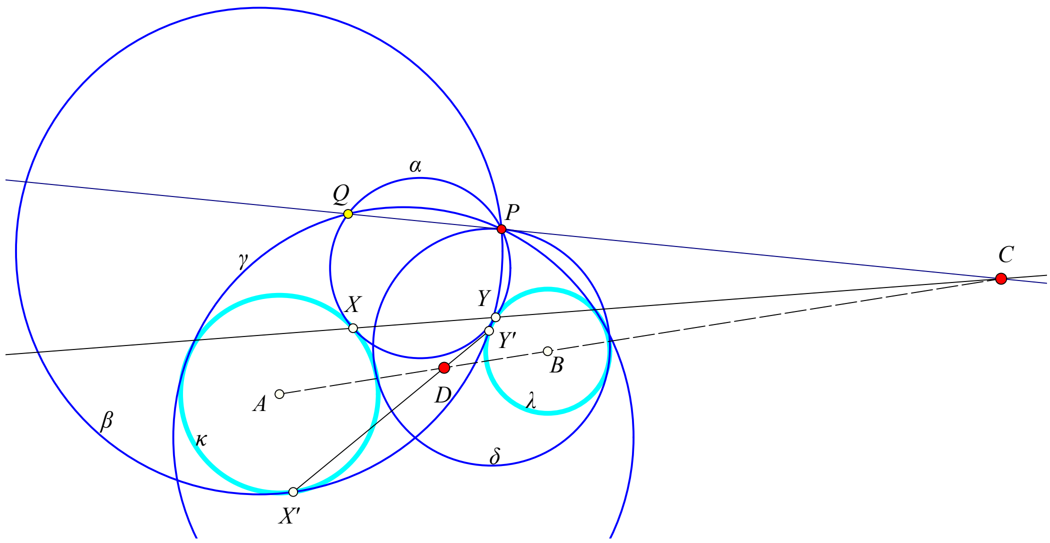


Figure 10: Two circles one point. Category (2,0,1), other approach

of contacts XY passes through the *"external similarity center"* C of the two circles. The solution-circle β passes analogously through P and its anti-inverted Q' (not shown in the figure) with respect to the anti-inversion with center at the *"inner similarity center"* D of the two circles. The circle γ is again invariant w.r. to the inversion and the corresponding line of contacts passes through the external similarity center C. The circle δ is invariant w.r. to the anti-inversion and the line of contacts passes through the inner similarity center D of the two circles $\{\kappa, \lambda\}$.

For this category, as is the case with the others, there is also a mutlitude of particular configurations and special cases. Figure 11 shows the case of two non intersecting equal

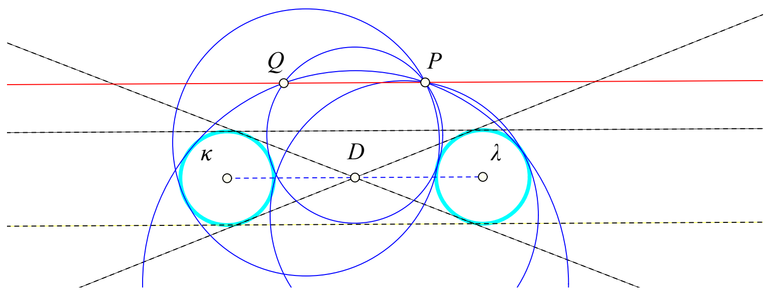


Figure 11: Two circles one point, case of equal circles

circles. The *"inversion"* of the previous case is now the *"reflection"* on the medial line of the segment of the centers of the two given circles. The *"anti-inversion"* in this case is the *"symmetry"* w.r. to the middle D of this segment.

7 Three circles, category (3,0,0)

The case of category (3,0,0) i.e. the construction of circles tangent to three given circles can be reduced to the category (2,0,1) of the previous section. There are here also several cases, a typical one being the case of three external to each other circles.

We handle here the general case, in which the radii of the circles are pairwise different and

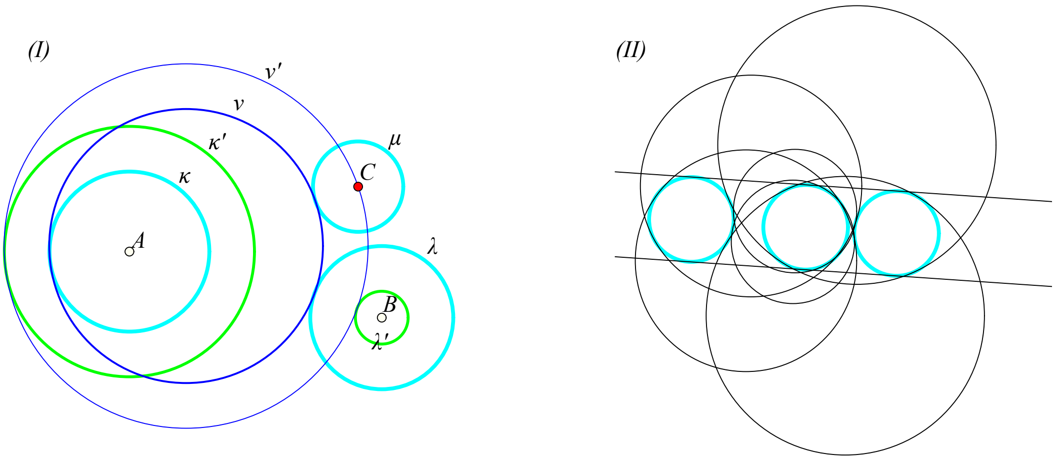


Figure 12: Three circles. Category (3,0,0)

satisfy $r_\kappa > r_\lambda > r_\mu$. We will discuss the special cases later. For the position of the wanted circle ν there are eight cases, which can be denoted with one symbol (***), consisting of three signs. In each case we put a + or a - depending on whether a solution-circle ν has the given circle in its exterior or its interior. This way, (+++) means that ν has all circles in its exterior, (- - -) that it has all circles in its interior, (-++) that it has κ in its interior and λ and μ in its exterior, as in figure 12-I. In each of the eight cases we reduce the problem into one of category (2,0,1) using the following trick:

1. We replace the smaller circle (μ) with its center C.
2. We replace each of the two other circles $\{\kappa, \lambda\}$ with a circle of the same center but radii respectively $r_\kappa \pm r_\mu$ and $r_\lambda \pm r_\mu$.
3. This results in two new circles $\{\kappa', \lambda'\}$ and instead of ν we determine the concentric of it ν', which passes through point C and is tangent to κ', λ'.

According to the discussion in the previous section, there exists exactly one circle $\nu'(\rho)$, constructible from the given data. The wanted ν will have the same center with the constructed one and radius $\rho - r_\mu$. Solving therefore the problem for each one of the eight cases we also get the general solution to the problem.

In the cases where the two circles have the same radius, the solution is simplified a bit, since the initial problem is reduced to the simpler constructions. This way, for example in case $r_\kappa > r_\lambda = r_\mu$, the previous process, in the case (-++), is reduced to the one of finding a circle ν' which is tangent to the concentric κ' to κ with radius $r_\kappa + r_\lambda$ and passes through the two points $\{B, C\}$ (section 3).

Another example is the one where the three circles have the same radius and their centers are collinear (See Figure 12-II). Then the solutions include two lines symmetrically lying relative to the center line. Also the rest of the solutions consists of three pairs of circles symmetrically lying relative to the center line of the three circles. The work therefore is reduced to the half.

Figure 13 shows the eight solutions for three mutually external circles with radii $\alpha > \beta > \gamma$. Noted on them is also their kind, as it is described in the previous discussion. I note that a detailed solution of the problem of Apollonius for all possible categories and all possible sub-cases is, if not difficult, rather painful (a list of all the cases is contained in [Mui95]).

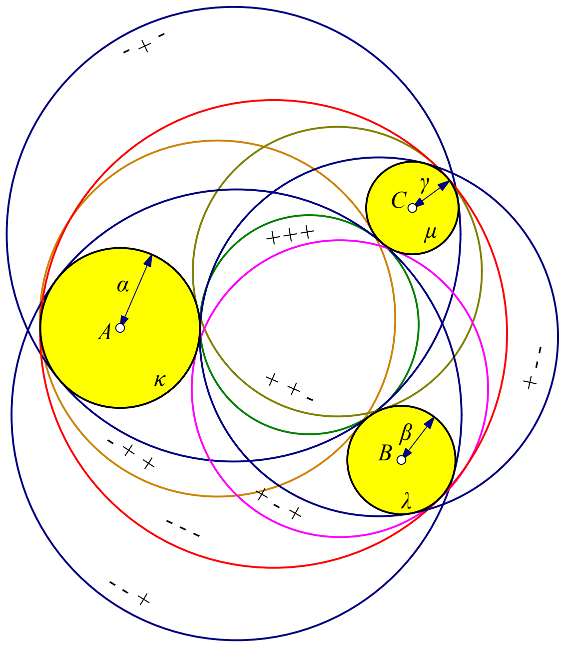


Figure 13: Counting eight solutions in category (3,0,0)

8 Soddy circles

The *"Soddy circles"* of three externally pairwise tangent circles $\{\kappa, \lambda, \mu\}$ are the circles tan-

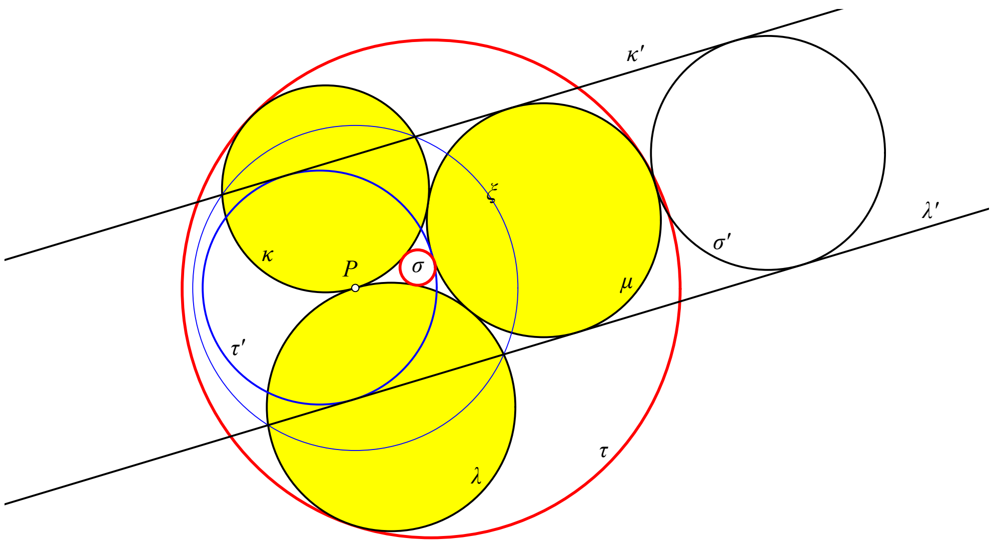


Figure 14: The two Soddy circles $\{\tau, \sigma\}$

gent to all these three circles. This particular Apollonius problem has two solutions only: the internal (σ), and external (τ) (See Figure 14).

The problem of constructing the Soddy circles σ and τ is easily reducible to a simpler one through the inversion f relative to a suitable circle $\xi(P, r)$. The center of this circle is taken to be one (of the three) contact points of the given circles, for example the contact point P of $\{\kappa, \lambda.\}$ The radius r of ξ is taken so, that this circle is orthogonal to the third circle μ. The inverses relative to ξ circles $\{\kappa' = f(\kappa), \lambda' = f(\lambda)\}$ are then two parallel lines tangent to μ, while μ is the inverse of itself . The inverses σ', τ' of the wanted circles are circles tangent to the two parallels and tangent also to μ, hence *"congruent"* to μ and tangent to it. Thus $\{\sigma', \tau'\}$ are directly constructible and their inverses are the wanted circles $\sigma = f(\sigma')$ and $\tau = f(\tau')$.

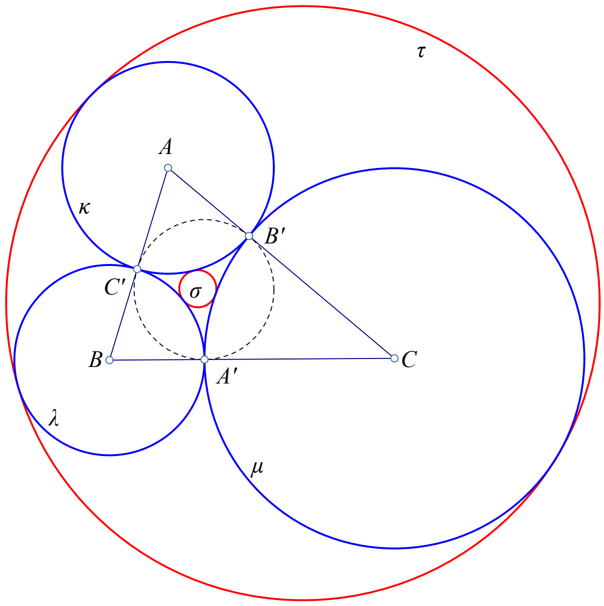


Figure 15: The two Soddy circles of a triangle

Every triangle defines, through the contact points of its *"incircle"* with the sides, three external to each other mutually tangent circles $\{\kappa, \lambda, \mu\}$ (see figure 15). Consequently one can speak of the *"Soddy circles of a triangle"*.

9 Pictures of various cases

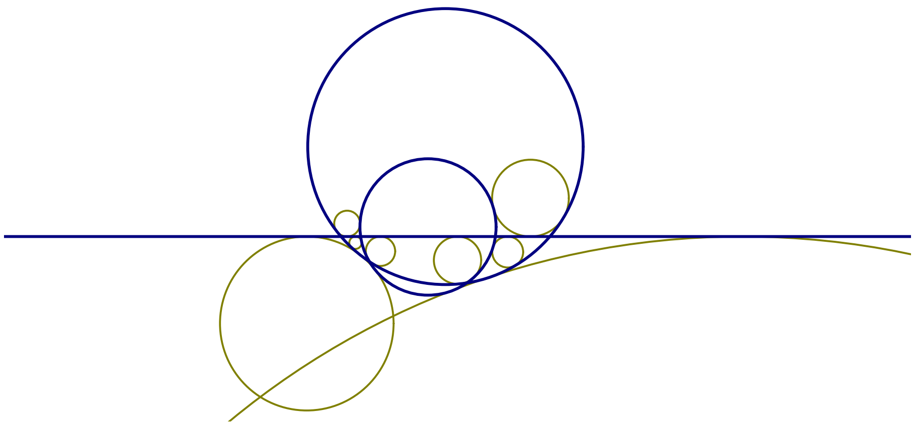


Figure 16: Two circles and a line intersecting

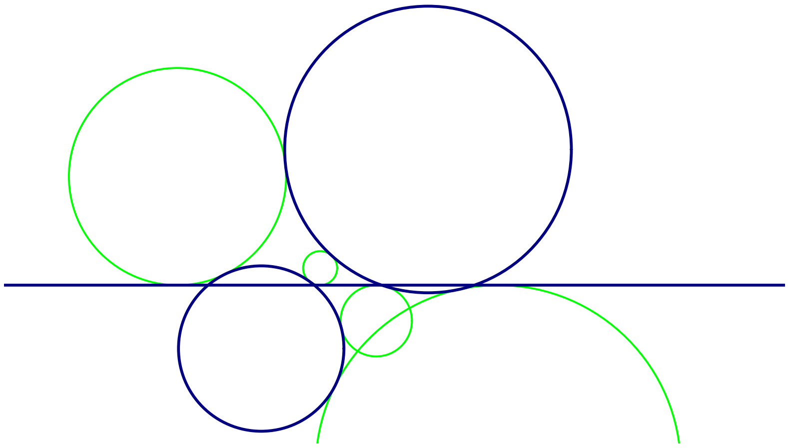


Figure 17: Two circles and a line intersecting, different sides

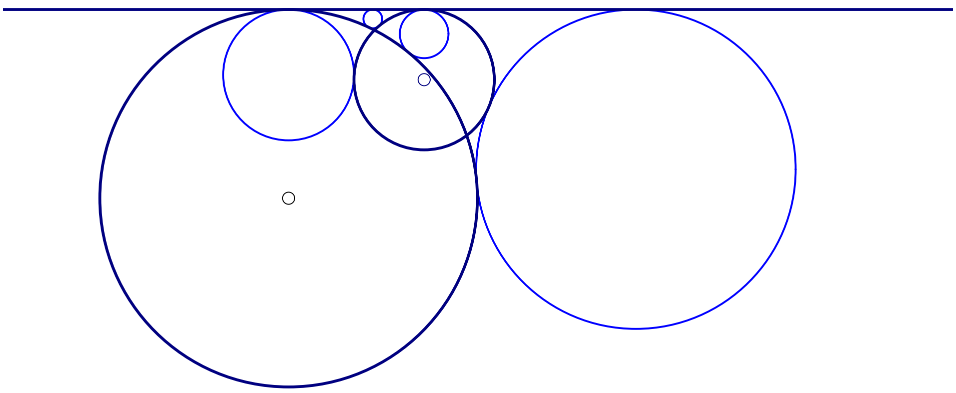


Figure 18: Two circles intersecting tangent to a line

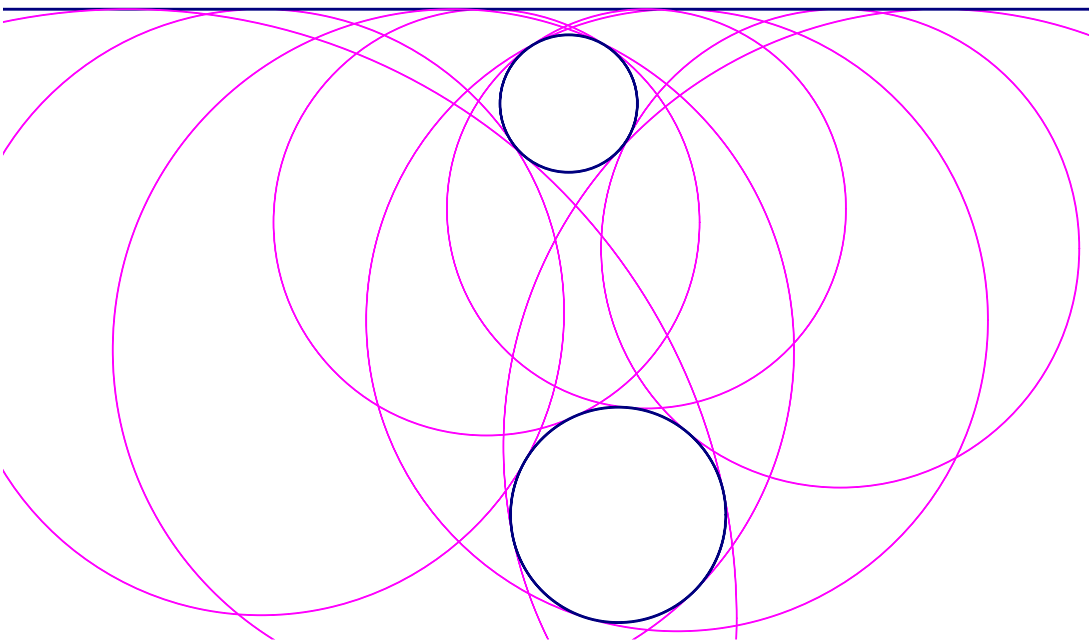


Figure 19: Two circles and a line non-intersecting

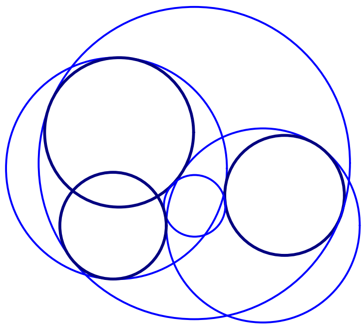
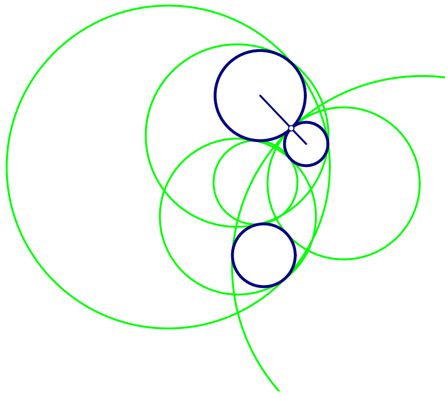


Figure 20: Two intersecting one exterior



Two tangent one exterior

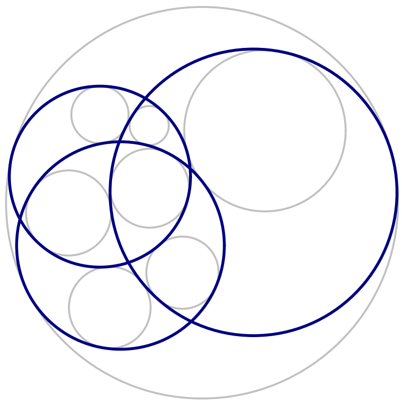
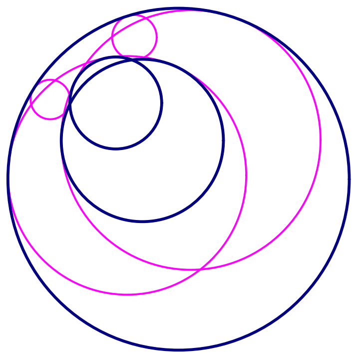


Figure 21: Three intersecting



Two intersecting interior

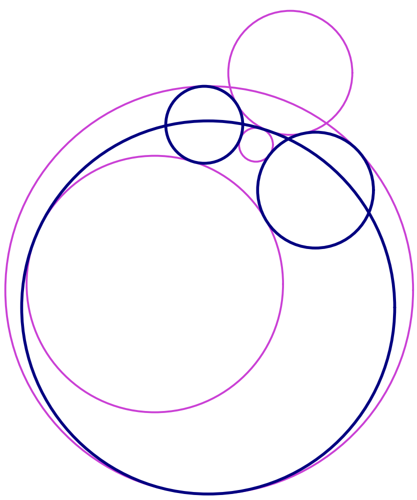
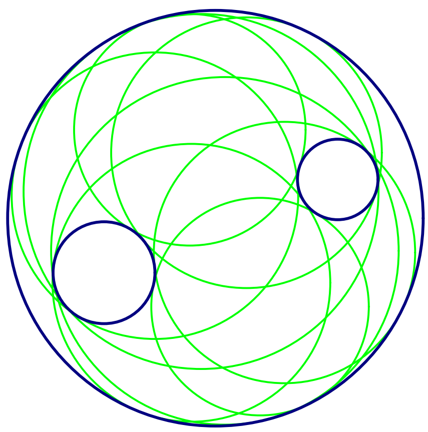


Figure 22: Three intersecting



Two non-intersecting interior

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Related material

1. **Circle Pencils**
2. **Inversion**
3. **Isodynamic points of the triangle**
4. **Pedals**
5. **Tritangent circles of the triangle**

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