Barycentric coordinates or Barycentrics

A file of the Geometrikon gallery by Paris Pamfilos

The artist finds a greater pleasure in painting than in having completed the picture.

Seneca, Letter to Lucilius

Contents

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1 Preliminaries and definition

"Barycentric coordinates" or *"barycentrics"* do not use distances of points, but only ratios of lengths and areas. Thus, they belong to the geometry of the *"affine plane"* ([2, p.191]), which deals with parallels and ratios of lengths of collinear segments without to use a notion of distance between two points, as it does euclidean geometry.

In the subsequent discussion points $\{A, B, ...\}$ of the plane are identified with two dimensional vectors $\{(a_1, a_2), (b_1, b_2) ...\}$. Next properties are easily verified ([8, p.30]):

1. Three points of the plane $\{A, B, C\}$ are collinear, if and only, there are three numbers $\{x, y, z\}$ not all zero, such that
$$xA + yB + cZ = 0 \quad \text{and} \quad x + y + z = 0.$$
2. Let the points of the plane $\{A, B, C\}$ be non-collinear. Then, for every other point D of the plane there are unique numbers $\{x, y, z\}$ with
$$x + y + z = 1 \quad \text{and} \quad D = xA + yB + zC.$$
3. For non-collinear points $\{A, B, C\}$ the point D, defined by the equation
$$OD = x \cdot OA + y \cdot OB + z \cdot OC, \quad \text{with} \quad x + y + z = 1, \tag{1}$$
is independent of the choice of origin O and depends only on $\{x, y, z\}$.
4. The numbers $\{x, y, z\}$ with $x + y + z = 1$ expressing D in the preceding equation are equal to the quotients of signed areas:

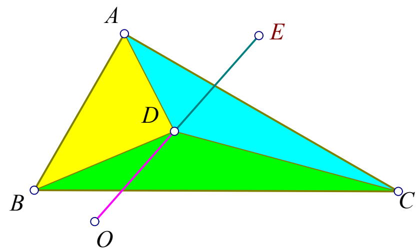

Figure 1: Barycentric coordinates (x, y, z) of E

$$x = \frac{Area(DBC)}{Area(ABC)}, \qquad y = \frac{Area(DCA)}{Area(ABC)}, \qquad z = \frac{Area(DAB)}{Area(ABC)}. \tag{2}$$

Nr-1. If $x + y + z = 0$ and $xA + yB + zC = 0$, then
$$(-y - z)A + yB + zC = 0 \quad \Rightarrow \quad y(B - A) + z(C - A) = 0$$
i.e. $\{A, B, C\}$ are collinear. For the converse, simply reverse the preceding implications.
Nr-2. Extend the two-dimensional vectors $\{(x, y)\}$ to three dimensional $\{(x, y, 1)\}$ and denote the corresponding space-points by $\{A', B', C', D'\}$. Apply first *nr-1* to see that $\{A', B', C'\}$ are independent. Then express D' in the basis $\{A', B', C'\}$ to find the uniquely defined (x, y, z) as required.
Nr-3. Using another point O' for origin, the same equation and the analogous numbers (x', y', z') to express
$$D : O'D = x' \cdot O'A + y' \cdot O'B + z' \cdot O'C$$

and substracting we get:

$$\begin{aligned} OD - O'D &= (x \cdot OA + y \cdot OB + z \cdot OC) - (x' \cdot O'A + y' \cdot O'B + z' \cdot O'C) \\ = (x-x')A + (y-y')B + (z-z')C &- (xO - x'O' + yO - y'O' + zO - z'O') \quad \Rightarrow \\ OO' \ = \ O' - O &= (x-x')A + (y-y')B + (z-z')C - (O - O') \quad \Rightarrow \\ (x-x')A + (y-y')B &+ (z-z')C \ = \ 0\,. \end{aligned}$$

By *nr-1* this implies $\{(x-x') = 0, (y-y') = 0, (z-z') = 0\}$, as desired.
Nr-4. Using the independence of (x,y,z) from the position of O we select $O = D$ and use the relation $x = 1 - y - z$ implying

$$\begin{aligned} &(1-y-z)DA + yDB + zDC = 0 \quad \Rightarrow \\ &DA + y(DB - DA) + z(DC - DA) = 0 \quad \Rightarrow \\ &AD = y(AB) + z'(AC). \end{aligned}$$

Multiplying externally by AB we get

$$AD \times AB = z(AC \times AB) \quad \Leftrightarrow \quad z(AB \times AC) = AB \times AD.$$

Analogous equations result also for y and x. Taking the positive orientation in direction $AB \times AD$ we have the stated result.

The numbers $\{x,y,z\}$ in (2) are called *"absolute barycentric homogeneous coordinates"* of the point D, relative to the triangle ABC. The multiples $(k \cdot x, k \cdot y, k \cdot z)$ by a constant $k \neq 0$ are called *"barycentric homogeneous coordinates"* or *"barycentrics"* of the point D ([1, p.64], [9, p.25], [5]). The signs of the numbers result from the orientations of the corresponding triangles e.g. if $\{D, A\}$ are on the same side of BC then x is positive. Otherwise it is negative and for D on BC it is zero $x = 0$. In fact $x = 0$ characterizes line BC. Analogous properties are valid also for y and z.

Remark 1. Figure 1 shows a point E defined by an equation as in *nr-3*, but without the restriction $\sigma = x + y + z = 1$. Point D is then obtained by dividing with σ. Thus, the points are collinear with O and defined by the equations

$$OE = xOA + yOB + zOC \quad \text{and} \quad OD = \frac{x}{\sigma}OA + \frac{y}{\sigma}OB + \frac{z}{\sigma}OC.$$

2 Traces, ratios, harmonic conjugation

Changing only the variable x, causes D to move along a fixed line through A. For $x = 0$, we get a point on the line BC : $D_A = yB + zC$, called *"trace"* of D on the side BC of $\triangle ABC$ (see figure 2). Analogously are defined the traces D_B and D_C. The oriented ratio $r = D_AB/D_AC$, calculated using $y + z = 1$, gives the value $r = -(z/y)$.

Theorem 1. *Given a point $P = yB + zC$ on the line BC the "harmonic conjugate" of P w.r. (B,C) is $P' = yB - zC$.*

This follows from the expression of the ratio $PB/PC = -z/y$ and correspondingly for the conjugate P' : $P'B/P'C = z/y$, from which follows that the *"cross ratio"* has the value

$$(BC, PP') = (PB/PC) : (P'B/P'C) = -1,$$

proving that $\{P, P'\}$ are harmonic conjugate w.r.t. $\{B, C\}$.

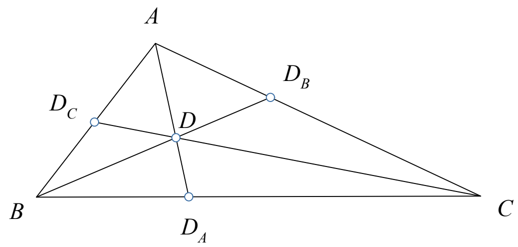

Figure 2: Traces D_A, D_B, D_C of D on the sides of ABC

Remark 2. Since any line can be described in *"parametric form"* $P = yB + zC$ and points $\{B, C\}$ can be complemented by a point A non-collinear with $\{B, C\}$ to a triangle of reference, the preceding property is generally valid for any two points on a line.

Remark 3. The *"centroid"* G of the triangle ABC, i.e. the intersection-point of its *"medians,"* defines triangles $\{GBC, GCA, GAB\}$ with equal areas. Thus it has *"barycentric homogeneous coordinates"* equal to $(1, 1, 1)$. This identifies these coordinates with the projective homogeneous coordinates w.r.t. the *"projective base"* $\{A, B, C, G\}$ with G the *"unit"* point (see file **Projective plane**).

3 Lines in barycentric coordinates

The following facts concerning the representation of lines with barycentric coordinates are easily verifiable:

1. A line ε is represented by an equation of the form
$$\varepsilon : \quad ax + by + cz = 0. \tag{3}$$
2. The intersection point of line ε with the line $\varepsilon' : a'x + b'y + c'z = 0$ is given by the *"vector product"* of the vectors of the coefficients:
$$\varepsilon \cap \varepsilon' = (bc' - cb',\ ca' - ac',\ ab' - ba'). \tag{4}$$
3. The *"line at infinity"* is represented by the equation (see equation 19)
$$\varepsilon_\infty : \quad x + y + z = 0. \tag{5}$$
4. The point at infinity of the line $\varepsilon : ax + by + cz = 0$ i.e. its intersection with the line at infinity, identified with its *"direction"*, is given by
$$\varepsilon \cap \varepsilon_\infty = (b - c,\ c - a,\ a - b). \tag{6}$$
5. Two lines $\{\varepsilon, \varepsilon'\}$ are parallel if they meet at a point at infinity, hence their coefficients satisfy:
$$bc' - cb' + ca' - ac' + ab' - ba' = 0. \tag{7}$$
6. The line passing through two given points $\{P(a, b, c), P'(a', b', c')\}$ has coefficients the coordinates of the vector product and can be expressed by a determinant:
$$\text{line} \quad PP' : \quad (bc' - cb',\ ca' - ac',\ ab' - ba'), \quad \text{equation:} \begin{vmatrix} a & b & c \\ a' & b' & c' \\ x & y & z \end{vmatrix} = 0. \tag{8}$$

7. The direction of the line passing through the points $\{A = (a, b, c),\ B = (a', b', c')\}$, i.e. its point at infinity is given by the weighted difference of coordinates

$$\sigma_A \cdot (a', b', c') - \sigma_B \cdot (a, b, c) \quad \text{with} \quad \sigma_A = a + b + c, \quad \sigma_B = a' + b' + c'. \tag{9}$$

8. The middle M of two points $A(a, b, c)$ and $B(a', b', c')$ is found as the harmonic conjugate of the point at infinity of line AB. Applying the preceding calculations it is found to be (the equality being for the tripples of coordinates)

$$M = (a + b + c)B + (a' + b' + c')A. \tag{10}$$

9. The line η parallel from $A(a, b, c)$ to the given line $\varepsilon :\ a'x + b'y + c'z = 0$ is the line joining A with the point at infinity of the line ε, which is $(b' - c', c' - a', a' - b')$. Thus the equation of this line is

$$\begin{vmatrix} b' - c' & c' - a' & a' - b' \\ a & b & c \\ x & y & z \end{vmatrix} = 0,$$

and the coefficients of this line, written in the form $px + qy + rz = 0$, are given by:

$$(p, q, r) = (a + b + c)(a', b', c') - (aa' + bb' + cc')(1, 1, 1). \tag{11}$$

Thus, the line η can be written as a linear combination of the given line ε and the line at infinity ε_∞, in the form

$$\begin{aligned} \eta :\quad (a + b + c)\varepsilon - (aa' + bb' + cc')\varepsilon_\infty &= 0 \quad \Leftrightarrow \\ \eta :\ (a + b + c)(a'x + b'y + c'z) - (aa' + bb' + cc')(x + y + z) &= 0. \end{aligned} \tag{12}$$

4 Ceva's and Menelaus' theorems in barycentrics

Three points $\{A', B', C'\}$ on the sides of the triangle of reference ABC are the *"traces"* of a point $P(x, y, z)$ if and only if they can be written in the form:

$$A'(0, y, z), \quad B'(x, 0, z), \quad C'(x, y, 0). \tag{13}$$

Their ratios on the triangle's sides satisfy then the equation:

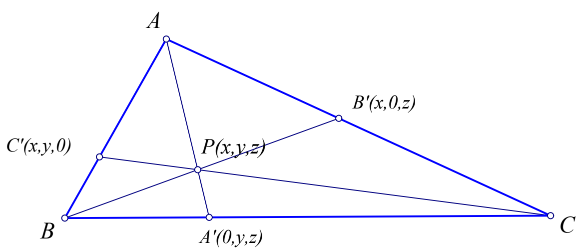

Figure 3: Ceva's theorem

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = -1. \tag{14}$$

As in the preceding section, this follows from the expressions of the ratios

$$\frac{A'B}{A'C} = -\frac{z}{y}, \quad \frac{B'C}{B'A} = -\frac{x}{z}, \quad \frac{C'A}{C'B} = -\frac{y}{x}.$$

Multiplying the three ratios gives the value -1. The inverse is also obvious. If the product of the three ratios is -1, then $\{x,y,z\}$ can be found such that $\{A',B',C'\}$ have the shown coordinates. This defines uniquely $P(x,y,z)$. For another more conventional view of Ceva's theorem look at the file **Ceva's theorem**.

It is obvious, that the points $\{A',B',C'\}$ on the sides of the triangle of reference ABC are on a line $px+qy+rz=0$, if and only if they can be written in the form:

$$(0,-r,q), \quad (r,0,-p), \quad (-q,p,0).$$

Their ratios satisfy then the equation:

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1. \tag{15}$$

This follows from the expression of the ratio $DB/DC = -z/y$, for $D = yB + zC$, seen in

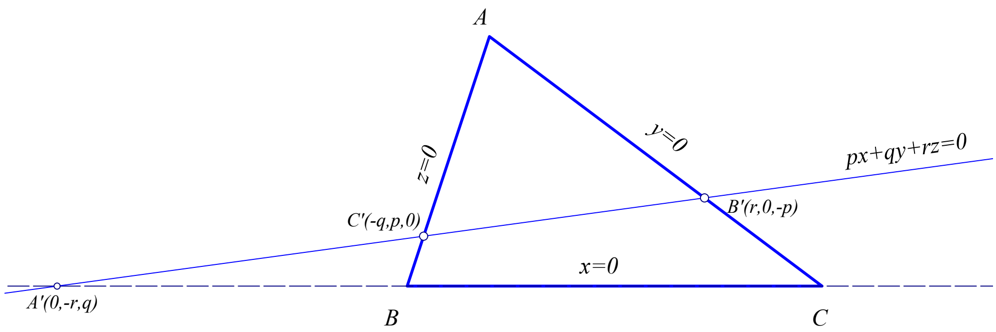

Figure 4: Menelaus' theorem

section 2. By this $A' = -rB + qC \quad \Rightarrow \quad A'B/A'C = q/r$ etc. (See Figure 4). For another more conventional view of Menelaus' theorem look at the file **Menelaus' theorem**.

5 Trilinear polar in barycentrics

In this section the notion of *"cross ratio"* $(AB,CD) = \frac{CA}{CB} : \frac{DA}{DB}$ of four collinear points is needed and the associated notion of harmonic conjugate points $\{C,D\}$ w.r.t. $\{A,B\}$, defined by the condition $(AB,CD) = -1$. When four points satisfy the last condition we say D is *"harmonic conjugate"* to C w.r.t $\{A,B\}$ and write $D = C(AB)$. More on this can be found in the file **Cross Ratio**.

The *"trilinear polar"* or *"tripolar"* ε_P of the point P w.r.t. the triangle ABC is the line containing the *"harmonic conjugates"* of the traces $\{A',B',C'\}$ of P on the sides of ABC :

$$A'' = A'(BC), \quad B'' = B'(AC), \quad C'' = C'(BA).$$

That these three points $\{A'',B'',C''\}$ are on a line follows by writing $P = uA + vB + wC$. Then the traces are given by $\{A' = vB + wC, P_B = wC + uA, P_C = uA + vB\}$. By section 2 the harmonic conjugates of the traces are then

$$A'' = vB - wC\,, \;\; B'' = wC - uA\,, \;\; C'' = uA - vB$$

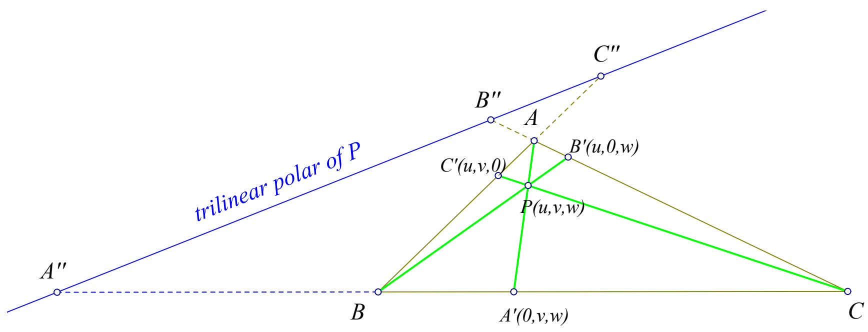

Figure 5: Trilinear polar of P

and satisfy $A'' + B'' + C'' = 0$ showing that C'' is on the line of $A''B''$ (see figure 5).

Alternatively, the barycentrics of $\{A'', B'', C''\}$ are respectively

$$A''(0, v, -w), \quad B''(-u, 0, w), \quad C''(u, -v, 0),$$

which up to a multiplicative factor can be written in the form

$$A''(0, 1/w, -1/v), \quad B''(-1/w, 0, 1/u), \quad C''(1/v, -1/u, 0),$$

implying that they are points of the line described by the next theorem:

Theorem 2. *For every point $P(u, v, w)$ of the plane, different from the vertices of the triangle of reference ABC, the corresponding "trilinear polar" $tr(P)$ is described by the equation:*

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0. \tag{16}$$

Conversely, every line $\varepsilon : \; pu + qv + rw = 0$ of the plane, not passing through a vertex of ABC is the trilinear polar $tr(P)$ of the point

$$P\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\right), \tag{17}$$

which is called "tripole" of the line ε.

Remark 4. The map $P \mapsto tr(P)$ is one-to-one for all points of the plane except those lying on the side-lines of the triangle of reference ABC. For these points P, if they are different from a vertex and lie on a side-line ε, the corresponding line $tr(P)$ coincides with ε. For the vertices of ABC the trilinear polar cannot be uniquely defined.

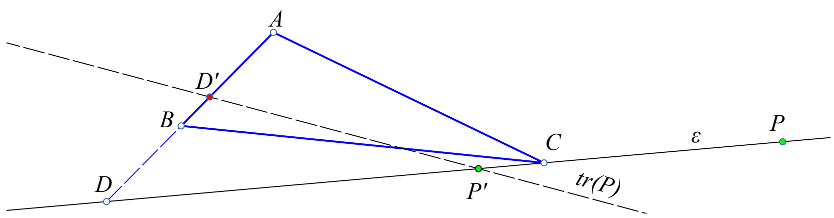

Figure 6: Trilinear polars for $P \in \varepsilon \ni C$ pass all through $D' = D(AB)$

Remark 5. Using the definition of the trilinear polar, it is easy to see, that for the points P of a fixed line ε passing through a vertex, C say, of the triangle of reference ABC, the corresponding trilinear polar $tr(P)$ passes through the *"harmonic conjugate"* point $D' = D(AB)$, where $D = \varepsilon \cap AB$ (see figure 6). Further, for the point $P' = tr(P) \cap \varepsilon$ the cross ratio $(CD, PP') = -2$ is constant and, as the point P approaches C, the corresponding trilinear polar $tr(P)$ tends to coincide with the *"harmonic conjugate"* line CD' of $CD = \varepsilon$ w.r.t. the line-pair (CA, CB).

Remark 6. From its proper definition follows that the trilinear polar of the *"centroid"* with equation

$$\varepsilon_\infty \; : \; u \; + \; v \; + \; w \; = \; 0,$$

is the *"line at infinity"* of the plane.

Exercise 1. *Show that the polar* ε_Q *of a point* Q *w.r.t. to two lines* $\{\gamma(X) = 0\,,\ \delta(X) = 0\}$ *is the line*

$$\varepsilon_Q(X) \;=\; \delta(Q)\gamma(X) \;+\; \gamma(Q)\delta(X) \;=\; 0\,.$$

Hint: Follows from the matrix representation of a conic $X^t M X = 0$ and the corresponding $Q-$polar line representation $\varepsilon_Q(X) = Q^t M X = 0$. The pair of lines represents a degenerate conic $\gamma(X)\delta(X) = 0$. The formula results by computing the corresponding matrix M.

Notice that this representation is valid in any projective system of coordinates, not only in Barycentrics.

6 Relation to Cartesian coordinates, inner product

Denoting by (x, y) the Cartesian coordinates and by (u, v, w) the *"absolute barycentrics"* of the same point P, the relation between the two is expressed in matrix form by translating equation 1:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = M \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{18}$$

This matrix and its inverse, studied further in sections 18 and 19, can be used to express other geometric objects through relations of the barycentrics. The equation of a line, which in Cartesian coordinates is

$$ax + by + c = (a, b, c) \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

translates into barycentrics to the equation

$$a'u + b'v + c'w = 0 \quad \text{with} \quad (a', b', c') = (a, b, c) \cdot M.$$

Thus, the equation of the line at infinity, described in Cartesians by $z = 0$, corresponds by the preceding rule, to the equation in barycentrics:

$$(0, 0, 1) M \begin{pmatrix} u \\ v \\ w \end{pmatrix} \;=\; u + v + w = 0. \tag{19}$$

The *"inner product"* in Cartesian coordinates is also expressible through a *"bilinear form"*:

$$xx' + yy' = (x, y, 1)\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} - 1 = (u, v, w)M^tM\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} - 1. \tag{20}$$

The matrix M^tM is easily seen to be

$$M^tM = \begin{pmatrix} A^2 & A\cdot B & A\cdot C \\ A\cdot B & B^2 & B\cdot C \\ A\cdot C & B\cdot C & C^2 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{21}$$

Taking into account that $u + v + w = u' + v' + w' = 1,$ we find that

$$xx' + yy' = (u, v, w)N\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}, \quad \text{with} \quad N = \begin{pmatrix} A^2 & A\cdot B & A\cdot C \\ A\cdot B & B^2 & B\cdot C \\ A\cdot C & B\cdot C & C^2 \end{pmatrix}. \tag{22}$$

7 The circumcircle of ABC in barycentrics

By the discussion in section 1 we can take the origin of Cartesian coordinates anywhere we like. Selecting it at the *"circumcenter"* O of the circumcircle of the triangle of reference $ABC,$ we find that

$$A^2 = B^2 = C^2 = R^2, \quad A\cdot B = R^2\cos(2\gamma), \;\; A\cdot C = R^2\cos(2\beta), \;\; B\cdot C = R^2\cos(2\alpha),$$

where $\{\alpha, \beta, \gamma\}$ the angles of the triangle opposite respectively to $\{BC, CA, AB\}$ and R is the circumradius of the triangle of reference ABC. Setting $\cos(2\gamma) = 1 - 2\sin^2(\gamma), \dots$ we find

$$N = R^2\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - 2R^2\begin{pmatrix} 0 & \sin^2(\gamma) & \sin^2(\beta) \\ \sin^2(\gamma) & 0 & \sin^2(\alpha) \\ \sin^2(\beta) & \sin^2(\alpha) & 0 \end{pmatrix} \quad \Leftrightarrow$$

$$N = R^2\cdot N_1 - \frac{1}{2}N_2 = R^2\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix}, \tag{23}$$

where $\{a = |BC|, b = |CA|, c = |AB|\}$ the side lengths of the triangle. Taking into account the condi-tion $\{u + v + w = 1, \dots\}$ for *"absolute barycentric coordinates"*, we find that the expression of the inner product $xx' + yy'$ in terms of the corresponding *absolute* barycentric coordinates is:

$$\begin{aligned} xx' + yy' &= (u, v, w)N\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} \\ &= R^2 - \frac{1}{2}(c^2(uv' + u'v) + b^2(wu' + w'u) + a^2(vw' + v'w)). \end{aligned} \tag{24}$$

Having the center of coordinates at the circumcenter of $ABC,$ the equation of the *"circumcircle"* κ of ABC in barycentric coordinates can be found from the corresponding equation in Cartesian coordinates:

$$\begin{aligned} R^2 = x^2 + y^2 \;&=\; R^2 - (c^2uv + b^2wu + a^2vw) \quad \Leftrightarrow \\ & a^2vw + b^2wu + c^2uv = 0. \end{aligned} \tag{25}$$

8 Displacement vectors, inner product, distance

Displacement vectors are differences of *absolute* barycentric coordinates of two points

$$(p,q,r) = (u,v,w) - (u',v',w').$$

Since $u+v+w = u'+v'+w' = 1,$ they satisfy

$$p+q+r=0$$

and represent points at infinity. Also through their corresponding Cartesian coordinate vectors, defined by equation (18), they represent the usual displacement from one point to the other. The inner product of the corresponding Cartesian coordinate vectors can be calculated using the same bilinear form (23). Thus, introducing the vectorial notation $\overline{x} = (x,y)$ and $\overline{P} = (p,q,r)$ for Cartesian, respectively barycentric cordinates, and also denoting by $\overline{x}_P = M\overline{P},$ we have:

$$\overline{x}_{PQ} = \overline{x}_Q - \overline{x}_P = M(\overline{Q}-\overline{P}) = M\cdot\overline{PQ}$$

and from this, the expression of the usual inner product:

$$\overline{x}_{PQ}\cdot\overline{x}_{P'Q'} = \overline{PQ}^t\cdot N\cdot\overline{P'Q'} = -\frac{1}{2}(p,q,r)\begin{pmatrix}0 & c^2 & b^2\\ c^2 & 0 & a^2\\ b^2 & a^2 & 0\end{pmatrix}\begin{pmatrix}p'\\ q'\\ r'\end{pmatrix},$$

where we set $\overline{PQ} = Q-P = (p,q,r)$ and $\overline{P'Q'} = Q'-P' = (p',q',r')$. Expanding this, we get:

$$\overline{x}_{PQ}\cdot\overline{x}_{P'Q'} = -\frac{1}{2}[a^2(qr'+q'r)+b^2(rp'+r'p)+c^2(pq'+p'q)]. \tag{26}$$

This expresses the fundamental relation allowing the computation of euclidean distances and angles in terms of barycentric coordinates. Thus, for example, the distance of two points represented in *absolute* barycentric coordinates $\{P,Q\}$ can be expressed through their *"displacement vector"* $\overline{PQ} = (p,q,r)$ and the formula:

$$|PQ|^2 = -a^2qr - b^2rp - c^2pq. \tag{27}$$

Introducing the *"Conway symbols"*

$$\begin{array}{llll} S_A = bc\cos(\widehat{A}), & S_B = ca\cos(\hat{B}), & S_C = ab\cos(\widehat{C}) & \Leftrightarrow\\ S_A = \frac{1}{2}(b^2+c^2-a^2), & S_B = \frac{1}{2}(c^2+a^2-b^2), & S_C = \frac{1}{2}(a^2+b^2-c^2), & \end{array}$$

where $\{a,b,c\}$ are the side-lengths of the triangle of reference, and calculating the expression on the right of equation (26) we find, using conditions $\{p+q+r = p'+q'+r' = 0\}$, that it is equivalent to

$$\overline{x}_{PQ}\cdot\overline{x}_{P'Q'} \;=\; S_App' \;+\; S_Bqq' \;+\; S_Crr'. \tag{28}$$

Consequently, formula (27) is also equivalent to:

$$|PQ|^2 \;=\; S_Ap^2 \;+\; S_Bq^2 \;+\; S_Cr^2, \tag{29}$$

where $\{p = u'-u\ ,\ q = v'-v\ ,\ r = w'-w\}$ and $\{P(u,v,w), Q(u,v,w)\}$ are in *absolute barycentrics.*

Using the last formula we can easily find the equation of the *"medial line"* of two points $\{P, Q\}$, i.e. the points $X(X_u, X_v, X_w)$ whose distances $|XP| = |XQ|$. For this consider the quadratic form defined by the right side of the last formula

$$f(X, Y) \;=\; S_A X_u Y_u + S_B X_v Y_v + S_C X_w Y_w \,.$$

The equality $\{|XP|^2 = |XQ|^2\}$ by formula (29) is equivalent to

$$\begin{aligned} f(P - X, P - X) = f(Q - X, Q - X) \quad &\Leftrightarrow \\ f(P, P) - 2f(P, X) + f(X, X) = f(Q, Q) - 2f(Q, X) + f(X, X) \quad &\Leftrightarrow \\ f(2(P - Q), X) + f(Q, Q) - f(P, P) = 0 \quad &\Leftrightarrow \\ S_A(P_u - Q_u)X_u + S_B(P_v - Q_v)X_v + S_C(P_w - Q_w)X_w + k/2 = 0 & \end{aligned} \tag{30}$$

with

$$k \;= f(Q, Q) - f(P, P) \;=\; S_A(Q_u^2 - P_u^2) + S_B(Q_v^2 - P_v^2) + S_C(Q_w^2 - P_w^2)\,.$$

Theorem 3. *Equation (30) represents the medial line of two points $\{P, Q\}$ in absolute barycentrics.*

Remark 7. It is easy verifiable that the *"Conway symbols"* satisfy the conditions:

$$S_B + S_C = a^2, \qquad S_C + S_A = b^2, \qquad S_A + S_B = c^2, \tag{31}$$
$$S_A S_B + S_B S_C + S_C S_A = S^2, \tag{32}$$
$$S_A S_B + c^2 S_C \;=\; S_B S_C + a^2 S_A \;=\; S_C S_A + b^2 S_B \;=\; S^2. \tag{33}$$
$$a^2 S_A + b^2 S_B + c^2 S_C = 2S^2. \tag{34}$$

where S denotes twice the area of the triangle of reference ABC ([9, p.33]).

More general, for an angle of measure ϕ we can define the compatible to the preceding symbol

$$S_\phi = S \cdot \cot(\phi), \tag{35}$$

used in several formulas involving calculations with barycentrics ([9]). More on this can be found in the file **Conway triangle symbols**.

Theorem 4. *Two displacement vectors $\{(p, q, r), (p', q', r')\}$ are orthogonal, if and only if they satisfy the equation:*

$$S_A pp' \;+\; S_B qq' \;+\; S_C rr' \;=\; 0. \tag{36}$$

9 Orthogonality of lines, orthocentroidal circle

The parametric form of a line represented in barycentrics by the equation

$$pu \;+\; qv \;+\; rw \;=\; 0,$$

can be given in the form

$$(u,\ v,\ w) = (q,\ -p,\ 0) + t(r - q,\ p - r,\ q - p).$$

The last vector on the right is a formal *"displacement vector"*, satisfying $u + v + w = 0$, i.e. representing in barycentrics the *"direction"* of the line. Notice that it is also the *"point at infinity"* of that line, i.e. its intersection with the *"line at infinity"*, represented in barycentrics by the equation $u + v + w = 0$.

Thus, the lines $\{pu + qv + rw = 0,\ p'u + q'v + r'w = 0\}$ are orthogonal, if and only if their *"directions"*

$$(p_d,\ q_d,\ r_d) = (r - q,\ p - r,\ q - p), \quad (p'_d,\ q'_d,\ r'_d) = (r' - q',\ p' - r',\ q' - p')$$

are orthogonal displacements, i.e. they satisfy equation (36). Latter equation, together with the valid for points at infinity $u + v + w = 0$, imply the explicit form of the *"direction"* of the orthogonal line:

$$(p'_d, q'_d, r'_d) = (\ (S_B \cdot q_d - S_C \cdot r_d)\ ,\ (S_C \cdot r_d - S_A \cdot p_d)\ ,\ (S_A \cdot p_d - S_B \cdot q_d)\ )\ . \tag{37}$$

Alternatively to this, i.e. using the line-coefficients themselves and not their differences, we arive after a short calculation at the theorem ([4, II,p.57]):

Theorem 5. *The lines* $\varepsilon:\ px + qy + rz = 0$ *and* $\varepsilon'\ p'x + q'y + r'z = 0$ *are orthogonal if and only if their coefficients satisfy the equation*

$$a^2pp' + b^2qq' + c^2rr' - S_A(qr' + q'r) - S_B(rp' + r'p) - S_C(pq' + p'q) = 0. \tag{38}$$

Remark 8. The bilinear form involved in equation (38) is degenerate, since the corresponding matrix H satisfies:

$$(1,1,1)H = (1,1,1)\begin{pmatrix} -a^2 & S_C & S_B \\ S_C & -b^2 & S_A \\ S_B & S_A & -c^2 \end{pmatrix} = 0. \tag{39}$$

Remark 9. Last equation conforms to the fact that if lines $\{\varepsilon = 0,\ \varepsilon' = 0\}$ are orthogonal the same is true for $\{\varepsilon + k \cdot \varepsilon_\infty,\ \ \varepsilon'\}$, where ε_∞ is the line at infinity. By section 3-*nr-5* the lines $\{\varepsilon + k \cdot \varepsilon_\infty = 0\}$ represent, for variable k, all the parallels to line $\varepsilon = 0$.

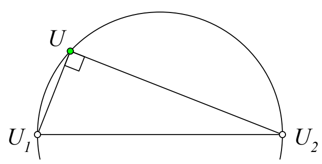

Figure 7: Circle on diameter U_1U_2 in absolute barycentrics

As an application of the orthogonality of displacement vectors we obtain the equation of the circle having for diameter the segment defined by the points

$$U_1 \ = \ (u_1, v_1, w_1) \quad \text{and} \quad U_2 \ = \ (u_2, v_2, w_2).$$

The points $U(u, v, w)$ of this circle are characterized by the orthogonality of the displacement vectors $\{UU_1, UU_2\}$ implying the theorem:

Theorem 6. *The points* $U(u, v, w)$ *of the circle on diameter* U_1U_2 *satisfy the equation in absolute barycentrics:*

$$S_A(u - u_1)(u - u_2) \ + \ S_B(v - v_1)(v - v_2) \ + \ S_C(w - w_1)(w - w_2) \ = \ 0. \tag{40}$$

An application of this leads to a compact expression for the *"orthocentroidal"* circle of a triangle ABC, defined to be the circle with diametral points the centroid G and the orthocenter H of the triangle. Taking into account formula (66) for H and doing some calculation we find its expression in barycentrics:

$$3(S_Au^2 + S_Bv^2 + S_Cw^2) - (S_Au + S_Bv + S_Cw) \ = \ 0\,, \tag{41}$$

where (u, v, w) are assumed to be absolute coordinates.

10 Distance of a point from a line, distance of two parallels

Representing the coordinates by $\overline{U} = (u, v, w)$, the *"direction"* of the line

$$\varepsilon : \quad f(\overline{U}) = pu + qv + rw = 0 \tag{42}$$

is its intersection

$$(r - q, \ \ p - r, \ \ q - p) \tag{43}$$

with the line at infinity as explained in section 9. The orthogonal direction to this can be calculated using theorem 4 and leads to a multiple of the displacement vector $\overline{V}$:

$$(v_1 = \ S_B(p-r) - S_C(q-p), \ \ v_2 = S_C(q-p) - S_A(r-q), \ \ v_3 = S_A(r-q) - S_B(p-r)).$$

Which in matrix notation is represented by the equation:

$$\overline{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 & S_B & -S_C \\ -S_A & 0 & S_C \\ S_A & -S_B & 0 \end{pmatrix} \cdot \begin{pmatrix} r-q \\ p-r \\ q-p \end{pmatrix}. \tag{44}$$

The line ε', passing through the point $\overline{U}_0 = (u_0, v_0, w_0)$, and orthogonal to ε has the

Figure 8: Calculate the distance $|U_0U_1|$ in barycentrics

parametric rerpesentation, the point U_0 in absolute barycentrics:

$$\varepsilon' : \quad \overline{U}_t = \overline{U}_0 + t\overline{V}.$$

Its intersection point $\overline{U}_t$ with ε satisfies the equation of line ε :

$$f(\overline{U}_0 + t\overline{V}) = 0 \quad \Leftrightarrow \quad f(\overline{U}_0) + tf(\overline{V}) = 0 \quad \Leftrightarrow \quad t = -\frac{f(\overline{U}_0)}{f(\overline{V})}.$$

Taking into account equation (44), we find that

$$f(\overline{V}) = (p, q, r) \begin{pmatrix} 0 & S_B & -S_C \\ -S_A & 0 & S_C \\ S_A & -S_B & 0 \end{pmatrix} \cdot \begin{pmatrix} r-q \\ p-r \\ q-p \end{pmatrix} \quad \Leftrightarrow$$

$$f(\overline{V}) = S_A(r-q)^2 + S_B(p-r)^2 + S_C(q-p)^2. \tag{45}$$

Thus, the displacement vector $\overline{U}_t - \overline{U}_0$ and its length is, according to equation (27):

$$\overline{U}_t - \overline{U}_0 = -\frac{f(\overline{U}_0)}{f(V)} \cdot V \quad \Rightarrow \quad |\overline{U}_t - \overline{U}_0|^2 = \frac{f(\overline{U}_0)^2}{f(V)^2}(-a^2v_2v_3 - b^2v_3v_1 - c^2v_1v_2). \tag{46}$$

The parenthesis on the right is

$$V^2 = -a^2v_2v_3 - b^2v_3v_1 - c^2v_1v_2 = -\frac{1}{2}(v_1, v_2, v_3) \begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

which, taking into account equation (44), leads to

$$V^2 = -\frac{1}{2}(r-q, p-r, q-p) \cdot K \cdot \begin{pmatrix} r-q \\ p-r \\ q-p \end{pmatrix}, \tag{47}$$

where K is the matrix

$$K = \begin{pmatrix} 0 & -S_A & S_A \\ S_B & 0 & -S_B \\ -S_C & S_C & 0 \end{pmatrix} \begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & S_B & -S_C \\ -S_A & 0 & S_C \\ S_A & -S_B & 0 \end{pmatrix}.$$

Carrying out the computation we find that

$$K = 2 \begin{pmatrix} -a^2 S_A^2 & S_A S_B S_C & S_A S_B S_C \\ S_A S_B S_C & -b^2 S_B^2 & S_A S_B S_C \\ S_A S_B S_C & S_A S_B S_C & -c^2 S_C^2 \end{pmatrix}. \tag{48}$$

This, using the relations in section 9 of the Conway symbols becomes

$$\begin{aligned} -a^2 S_A^2 + S_A S_B S_C &= S_A(S_B S_C - a^2 S_A) \\ &= S_A(S_B S_C - (S_B + S_C) S_A) \\ &= S_A(2 S_B S_C - (S_B S_C + S_C S_A + S_A S_B)) \\ &= 2 S_A S_B S_C - S_A S^2 \quad \Rightarrow \\ -a^2 S_A^2 &= S_A S_B S_C - S_A S^2. \end{aligned}$$

Thus, matrix K can be written in the form

$$\frac{K}{2} = S_A S_B S_C \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - S^2 \begin{pmatrix} S_A & 0 & 0 \\ 0 & S_B & 0 \\ 0 & 0 & S_C \end{pmatrix}. \tag{49}$$

Introducing this into equation (47) we find that

$$V^2 = S^2 (S_A (r-q)^2 + S_B (p-r)^2 + S_C (q-p)^2). \tag{50}$$

This introduced to equation (46) leads finally to the expression for the distance of the point $\overline{U}_0$ from the line $\varepsilon : f(\overline{U}) = pu + qv + rw = 0$:

Theorem 7. *The distance $dist(U_0, \varepsilon)$ of the point $\overline{U}_0(u_0, v_0, w_0)$ given in absolute barycentrics from the line $\varepsilon : f(\overline{U}) = pu + qv + rw = 0$ is given by the formula:*

$$dist(U_0, \varepsilon)^2 = |\overline{U}_t \overline{U}_0|^2 = \frac{S^2 (pu_0 + qv_0 + rw_0)^2}{S_A (r-q)^2 + S_B (p-r)^2 + S_C (q-p)^2}. \tag{51}$$

Measuring the distance $dist(A, \varepsilon)$ of the vertex $A(1,0,0)$ from the line ε and doing this also for the other vertices, we find the following property:

Theorem 8. *The coefficients of the line $\varepsilon : pu + qv + rw = 0$ are proportional to the distances of the vertices of the triangle of reference ABC from the line:*

$$\frac{dist(A, \varepsilon)^2}{p^2} = \frac{dist(B, \varepsilon)^2}{q^2} = \frac{dist(C, \varepsilon)^2}{r^2} = \frac{S^2}{S_A (r-q)^2 + S_B (p-r)^2 + S_C (q-p)^2}. \tag{52}$$

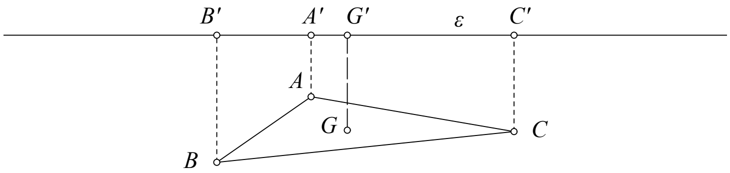

Figure 9: Distance of centroid from the line ε

Remark 10. Another aspect of the ratio on the right side of the last formula is found by considering a line not passing through the the centroid $G(1/3, 1/3, 1/3)$ of $\triangle ABC$, i.e. satisfying $p + q + r \neq 0$. Equation (51) leads in this case to:

$$dist(G, \varepsilon)^2 = \frac{S^2(\sigma/3)^2}{S_A(r-q)^2 + S_B(p-r)^2 + S_C(q-p)^2}, \quad \text{with } \sigma = p+q+r \quad \Leftrightarrow$$
$$\frac{S^2}{S_A(r-q)^2 + S_B(p-r)^2 + S_C(q-p)^2} = \left(\frac{3dist(G,\varepsilon)}{\sigma}\right)^2 \tag{53}$$

From theorem 7 and the preceding remark follows immediately the form of the equation of the bisector lines of the angle of two lines and the form of the distance of two parallel lines:

Theorem 9. *The bisector lines of the angle of two intersecting lines, which do not pass through the centroid G:*

$$\varepsilon \ : \ f(u,v,w) = pu + qv + rw = 0\,, \quad \varepsilon \ : \ f'(u,v,w) = p'u + q'v + r'w = 0\,,$$

with normalized coefficients satisfying $\{p + q + r \ = \ p' + q' + r' \ = \ 1\}$ are given by the equations

$$dist(G,\varepsilon)f(u,v,w) \pm dist(G,\varepsilon')f'(u,v,w) \ = \ 0\,. \tag{54}$$

Theorem 10. *The distance of two parallel lines $\{\varepsilon, \varepsilon'\}$:*

$$f(u,v,w) = pu + qv + rw, \quad f'(u,v,w) = f(u,v,w) + k \quad \textit{with} \quad \sigma = p+q+r \neq 0,$$

is given by

$$dist(\varepsilon,\varepsilon') \ = \ \frac{3|k| \cdot dist(G,\varepsilon)}{|\sigma|}\,. \tag{55}$$

11 Meaning of line coefficients

The meaning of line coefficients, expressed through equation (52), can be explained geometrically and more directly, than it was done in section 10, whose purpose was the general formula of distance of a point from a line, as described by equation (51). For this it suffices to use the results of section 4. In fact, the intersection points $\{A', B', C'\}$ of the line $\varepsilon: \ pu + qv + rw = 0$ with the sides of the triangle of reference (See Figure 10), have corresponding coordinates

$$A'(0, -r, q), \quad B'(-r, 0, p), \quad C'(-q, p, 0).$$

And the ratio of the distances $\{AA'', BB'', CC''\}$ is

$$\frac{BB''}{CC''} = \frac{A'B}{A'C} = \frac{q}{r} \quad \Rightarrow \quad \frac{BB''}{q} = \frac{CC''}{r}.$$

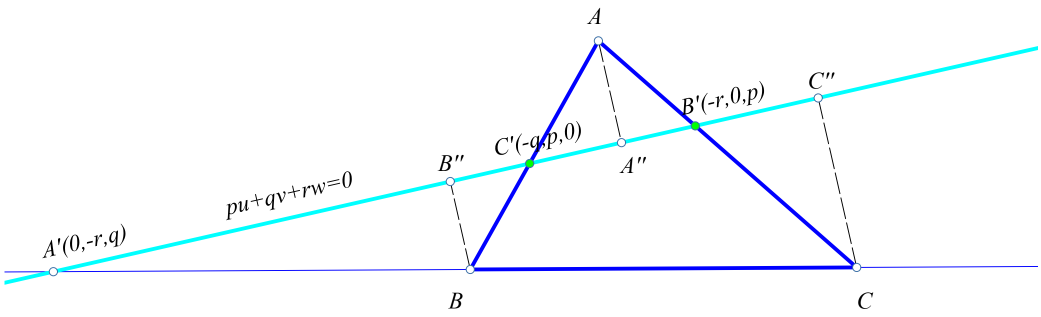

Figure 10: Coefficients proportional to distances $\frac{AA''}{p} = \frac{BB''}{q} = \frac{CC''}{r}$

Analogously is seen that $BB''/q = AA''/p$. Notice that in order to have valid equations

$$\frac{AA''}{p} = \frac{BB''}{q} = \frac{CC''}{r},$$

the distances must be signed and their signs must be properly chosen.

12 Power of a point, general circle, Euler circle

The expression in barycentrics of the *"power"* of a point $P(u, v, w)$ relative to a circle, can be found from the corresponding expression in Cartesian coordinates. In fact, in Cartesian coordinates the power of the point P w.r.t. the circle $\kappa(O, r)$ is

$$PO^2 - r^2,$$

which, using equation (27) for points $\{O(u_0, v_0, w_0), P(u, v, w)\}$ in absolute barycentrics becomes:

$$-(PO^2 - r^2) = a^2(v_0 - v)(w_0 - w) + b^2(u_0 - u)(w_0 - w) + c^2(u_0 - u)(v_0 - v) + r^2.$$

The equation of the circle $\kappa(O, r)$ results by equating this to zero:

$$\begin{aligned} & a^2vw + b^2wu + c^2uv \\ & +a^2v_0w_0 + b^2u_0w_0 + c^2u_0v_0 + r^2 \\ & -a^2(v_0w + vw_0) - b^2(w_0u + wu_0) - c^2(u_0v + uv_0) = 0. \end{aligned} \tag{56}$$

Taking into account that $u + v + w = 1$, the two last rows are seen to sum into the linear term:

$$(r^2 - OA^2)u + (r^2 - OB^2)v + (r^2 - OC^2)w. \tag{57}$$

Thus, the equation of the circle $\kappa(O, r)$ becomes

$$a^2vw + b^2wu + c^2uv + (r^2 - OA^2)u + (r^2 - OB^2)v + (r^2 - OC^2)w = 0, \tag{58}$$

which represents the equation of the circle κ as a sum of the corresponding expression of the circumcircle of the triangle of reference and a certain line. Thus, the difference of the expressions of the two circles is the expression of the *"radical line"* of the two circles and the following theorem is valid.

Theorem 11. *The equation*

$$(r^2 - OA^2)u + (r^2 - OB^2)v + (r^2 - OC^2)w = 0 \tag{59}$$

represents the radical axis of the circle $\kappa(O, r)$ *and the circumcircle* κ_0 *of the triangle of reference* ABC. *The expression of the equation of the general circle in absolute barycentric coordinates is the sum the corresponding expressions of* κ_0 *and this line:*

$$\kappa(O, r) \; : \; a^2vw + b^2wu + c^2uv + (r^2 - OA^2)u + (r^2 - OB^2)v + (r^2 - OC^2)w = 0. \tag{60}$$

Remark 11. By multiplying with $1 = x + y + z$, we can *"homogenize"* the last equation, thus leading to the homogeneous one:

$$\begin{aligned} a^2vw \; + \; b^2wu \; + \; c^2uv \; + \\ ((r^2 - OA^2)u + (r^2 - OB^2)v + (r^2 - OC^2)w)(u + v + w) \; = \; 0 \, . \end{aligned} \tag{61}$$

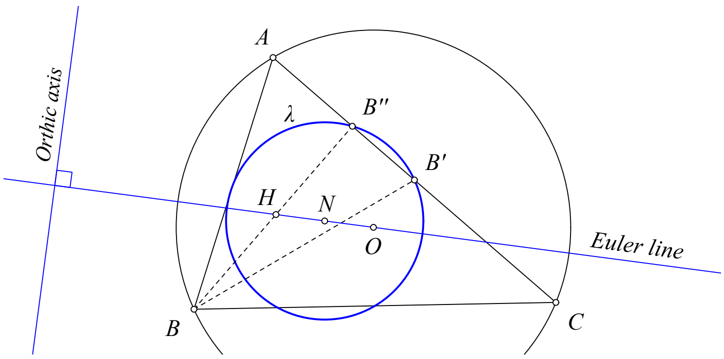

Figure 11: The Euler circle of the triangle ABC

As an example application of this equation we consider the *"Euler circle"* λ of the triangle of reference ABC. The powers of the vertices $\{A, B, C\}$ w.r.t. $\lambda(N, r)$ are easily seen (see figure 11) to be

$$OA^2 - r^2 \; = \; AB' \cdot AB'' \; = \; \frac{1}{2} bc \cos(\widehat{A}) \; = \; \frac{1}{2} S_A$$

and the analogous expressions obtained by the cyclic permutations of $\{A, B, C\}$. Here $\{B', B''\}$ are respectively the traces on AC of the median and altitude from B. Thus, equations (60) resp. (61) become in this case

$$(a^2vw + b^2wu + c^2uv) - \frac{1}{2}(S_Au + S_Bv + S_Cw)(u + v + w) \; = \; 0 \quad \Leftrightarrow \tag{62}$$

$$a^2vw + b^2wu + c^2uv - (S_Au^2 + S_Bv^2 + S_Cw^2) \; = \; 0 \, . \tag{63}$$

Figure 11 illustrates the case showing also the radical axis of the Euler circle and the circumcircle. This is the *"orthic axis"* of the triangle, identified by theorem 11 with the line

$$S_Au + S_Bv + S_Cw \; = \; 0, \tag{64}$$

and coinciding with the *trilinear polar* of the orthocenter H of ABC.

13 Centroid, Incenter, Circumcenter, Symmedian point

These remarkable points are the simplest examples of *"triangle centers"* of the triangle of reference ABC, whose barycentric coordinates can be easily calculated. The first, the *"centroid"* $X(2)$ in Kimberling's notation ([3]), is the unit point of the projective base $\{A, B, C, G\}$ defining the system of *"barycentrics"* and has coordinates $(1, 1, 1)$. The *incenter* $X(1)$, using the definition of barycentrics through areas, is easily seen to be (a, b, c). The *circumcenter* $X(3)$ of the triangle ABC is also calculated using the area definition of barycentrics. In fact, the area of the triangle $X(3)BC$ is

$$(X(3)BC) = \frac{1}{4}BC^2\cot(\alpha) = \ldots = \frac{R}{4abc}a^2S_A \quad \Rightarrow \quad X(3) = (a^2S_A,\ b^2S_B,\ c^2S_C),$$

with corresponding absolute barycentrics:

$$X(3) = \frac{1}{2S^2}(a^2S_A,\ b^2S_B,\ c^2S_C). \tag{65}$$

The *symmedian point* $X(6)$ (see file **Symmedian point of the triangle**) is characterized by its property to have distances from the sides analogous to these sides. Thus, the ratio of the areas

$$\frac{(X(6)AB)}{(X(6)AC)} = \frac{c^2}{b^2} \quad \Rightarrow \quad X(6) = (a^2, b^2, c^2).$$

The equalities $X(k) = (u, v, w)$ for barycentrics must be understood in a wider sense and often is used the symbol $X(k) = (u : v : w)$ to stress the fact that the barycentrics are defined up to multiplicative constants, so that only their relative ratios are uniquely defined. Alternatively, we use also the symbol $X \cong (u, v, w)$.

14 Euler line, Orthocenter, center of Euler's circle

Having the barycentrics $\{(1, 1, 1), (a^2S_A, b^2S_B, c^2S_C)\}$ of the centroid and the circumcenter, the *Euler line*, which passes through them, is easily seen (section 3) to have coefficients

$$\text{Euler line}\ :\quad (b^2S_B - c^2S_C)u \ +\ (c^2S_C - a^2S_A)v \ +\ (a^2S_A - b^2S_B)w \ =\ 0.$$

Its *point at infinity* $X(30)$ i.e. its intersection with the line at infinity $u + v + w = 0$ is then calculated to be

$$X(30) = (2a^2S_A - b^2S_B - c^2S_C, \ldots) = ((2a^4 - (b^2 + c^2)a^2 - (b^2 - c^2)^2, \ldots),$$

where the dots indicate the remaining coordinates resulting by cyclic permutations of the letters $\{a, b, c\}$ and $\{A, B, C\}$. On the Euler line are located some other *triangle centers*, like the orthocenter H or $X(4)$ and the *"center of the Euler circle"* E or $X(5)$. The barycentrics of the *orthocenter* are easily found using the well known ratio $HO/HG = 3/2$. Taking into account the ratio rules of section 18, by which the condition $P = (1-t)X + tY$ for points in euclidean coordinates transfers to the same relation for these points expressed in *absolute* barycentrics, we have:

$$\begin{aligned}
H &= 3\frac{G}{\sigma_G} - 2\frac{O}{\sigma_O} = (1, 1, 1) - 2\frac{(a^2S_A, \ldots)}{\sigma_O} \\
&= \left(1 - 2\frac{a^2S_A}{2S^2}, \ldots\right) = \frac{1}{S^2}(S_BS_C, \ldots) \sim (S_BS_C, \ldots) \qquad \Rightarrow \\
H &= (\ S_BS_C\ :\ S_CS_A\ :\ S_AS_B\ ).
\end{aligned} \tag{66}$$

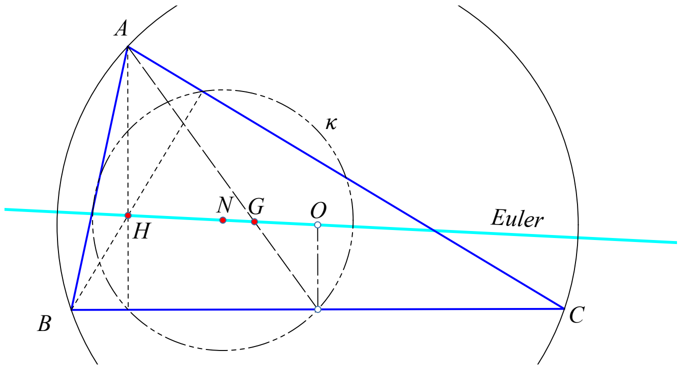

Figure 12: Points $\{H = X_4,\ \ N = X_5\}$ on the Euler line

where σ_X is the sum of barycentrics of X. Analogously are computed the barycentrics of the center $N = X_5$ of the Euler circle, which is the middle of the segment HO :

$$N \sim \sigma_O \cdot H + \sigma_H \cdot O = 2S^2 \cdot H + S^2 \cdot O \sim (\ S^2 + S_B S_C\ ,\ S^2 + S_C S_A\ ,\ S^2 + S_A S_B\ ). \quad (67)$$

Remark 12. The expression of the center $N = X(5) \cong (S^2 + S_B S_C, \dots)$ as a linear combination of the centers $\{G, H\}$ defining the *Euler line* generalizes for all notable triangle centers *lying on that line*, which can be also written as linear combinations

$$X = (\ m \cdot S^2\ +\ n \cdot S_B S_C,\ \dots).$$

The coefficients $\{m, n\}$ are called *"Shinagawa coefficients"* of X ([3]).

15 Triangle Area in Barycentrics

Consider the triangle of reference ABC and a second $\triangle DEF$, whose vertices have absolute bary-centric coordinates w.r.t. ABC : $D(d_1, d_2, d_3)$, $E(e_1, e_2, e_3)$, $F(f_1, f_2, f_3)$. The basic relations be-tween barycentric and Cartesian coordinates have been discussed in section 6. Denoting by (X_1, X_2) the Cartesian coordinates of the points X of the plane and using equation (18), we obtain:

$$\begin{pmatrix} D_1 & E_1 & F_1 \\ D_2 & E_2 & F_2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix}. \quad (68)$$

Taking the determinants, we obtain the relation of the areas

$$(DEF) = (ABC) \cdot \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix}, \quad (69)$$

expressing the area (DEF) in terms of the absolute barycentrics of the vertices of the triangle.

As an application, we can easily compute the area of the cevian triangle $A'B'C'$ of a point $P(x, y, z)$ (see figure 3). The *traces* of P on the sides of the triangle of reference ABC are $A'(0, y, z)$, $B'(x, 0, z)$, $C'(x, y, 0)$. With a short calculation we find then

$$(A'B'C') = \frac{(ABC)}{(y+z)(z+x)(x+y)} \begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{vmatrix} = \frac{2(ABC)xyz}{(y+z)(z+x)(x+y)}. \quad (70)$$

Some more effort in calculation is required for the *"pedal"* triangle (see figure 13) of a point $P(u,v,w)$ w.r.t. the triangle of reference ABC. From the discussion in section 10 we can compute the coordinates of the projections $\{A',B',C'\}$ on the sides:

$$\begin{aligned}A' &= (0\ ,\ a^2v+uS_C\ ,\ a^2w+uS_B),\\ B' &= (b^2u+vS_C\ ,\ 0\ ,\ b^2w+vS_A),\\ C' &= (c^2u+wS_B\ ,\ c^2v+wS_A\ ,\ 0).\end{aligned}$$

Then, assuming absolute barycentrics, satisfying $u+v+w=1$, setting $S=2(ABC)$ and

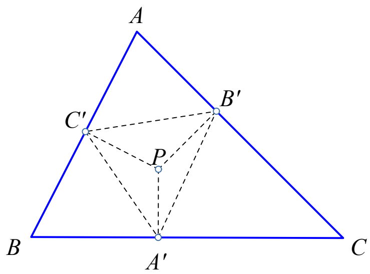

Figure 13: Pedal triangle $A'B'C'$ of ABC w.r.t. P

using equation (33), we find the crucial determinant

$$\begin{vmatrix} 0 & b^2u+vS_C & c^2u+wS_B\\ a^2v+uS_C & 0 & c^2v+wS_A\\ a^2w+uS_B & b^2w+vS_A & 0\end{vmatrix} = S^2(a^2vw+b^2wu+c^2uv). \tag{71}$$

This, turning to general, not necessarily *absolute* barycentrics, implies the formula for the area $(A'B'C')$ of the pedal of the point $P(u,v,w)$ w.r.t. the triangle of reference:

$$(A'B'C') = \frac{S^3}{a^2b^2c^2}\cdot\frac{a^2vw+b^2wu+c^2uv}{2(u+v+w)^2}. \tag{72}$$

16 Circumcevian triangle of a point

Given a triangle ABC and a point P the *"circumcevian"* triangle of P w.r.t. ABC is the triangle $A'B'C'$ formed by the second intercepts $\{A',B',C'\}$ of the cevians $\{AP,BP,CP\}$ of P with the circum-circle κ of ABC (see figure 14). The main property of the *circumcevian triangle* is:

Theorem 12. *The circumcevian triangle of P is similar to the corresponding pedal triangle of P.*

Figure 14 shows the way to prove that the two triangles have the same angles. Returning to compu-tations, the equations of the cevians are:

$$AA':\ -zv+yw=0,\quad BB':\ -xw+zu=0,\quad CC':\ -yu+xv=0. \tag{73}$$

The points $\{A',B',C'\}$ are calculated and seen to be:

$$A' = \begin{pmatrix}-a^2yz\\ b^2yz+c^2y^2\\ c^2yz+b^2z^2\end{pmatrix},\quad B' = \begin{pmatrix}a^2xz+c^2x^2\\ -b^2xz\\ c^2xz+a^2z^2\end{pmatrix},\quad C' = \begin{pmatrix}a^2xy+b^2x^2\\ b^2xy+a^2y^2\\ -c^2xy\end{pmatrix}. \tag{74}$$

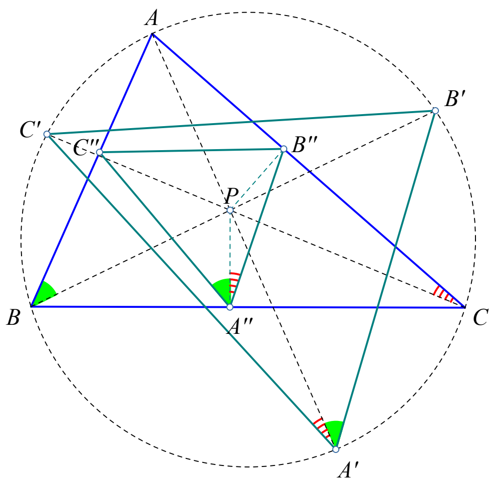

Figure 14: The circumcevian triangle $A'B'C'$ of P w.r.t. ABC

Their determinant is found:

$$det(A', B', C') = (a^2yz + b^2zx + c^2xy)^3. \tag{75}$$

The determination of the area $(A'B'C')$ requires the division with the product $\sigma_{A'}\sigma_{B'}\sigma_{C'}$ of the sums of the coordinates of these points:

$$\begin{aligned}
\sigma_{A'} &= (b^2 + c^2 - a^2)yz + b^2z^2 + c^2y^2 &&= S_A(y+z)^2 + S_By^2 + S_Cz^2,\\
\sigma_{B'} &= (c^2 + a^2 - b^2)zx + c^2x^2 + a^2z^2 &&= S_B(z+x)^2 + S_Cz^2 + S_Ax^2,\\
\sigma_{C'} &= (a^2 + b^2 - c^2)xy + a^2y^2 + b^2x^2 &&= S_C(x+y)^2 + S_Ax^2 + S_By^2.
\end{aligned}$$

17 Circle through three points, Brocard circle

The expression in barycentrics of the circle passing through three points $\{P_i(u_i, v_i, w_i)\}$, can be found from the corresponding expression in Cartesian coordinates, using the transformation $X_i = MP_i$ of section 6. In fact, using Cartesian coordinates $\{X_i = (x_i, y_i)\}$, the circle through these points is represented by the equation:

$$\begin{aligned}
0 &= \begin{vmatrix} X_1^2 & X_2^2 & X_3^2 & X^2 \\ x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ 1 & 1 & 1 & 1 \end{vmatrix}\\
&= X_1^2 \begin{vmatrix} x_2 & x_3 & x \\ y_2 & y_3 & y \\ 1 & 1 & 1 \end{vmatrix} - X_2^2 \begin{vmatrix} x_1 & x_3 & x \\ y_1 & y_3 & y \\ 1 & 1 & 1 \end{vmatrix} + X_3^2 \begin{vmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \\ 1 & 1 & 1 \end{vmatrix} - X^2 \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}\\
&= X_1^2|MP_2, MP_3, MP| - X_2^2|MP_1, MP_3, MP| + X_3^2|MP_1, MP_2, MP| - X^2|MP_1, MP_2, MP_3|\\
&= \left(X_1^2|P_2, P_3, P| - X_2^2|P_1, P_3, P| + X_3^2|P_1, P_2, P| - X^2|P_1, P_2, P_3| \right)|M| \quad \Leftrightarrow
\end{aligned}$$

$$X_1^2|P_2, P_3, P| + X_2^2|P_3, P_1, P| + X_3^2|P_1, P_2, P| - X^2|P_1, P_2, P_3| = 0. \tag{76}$$

Here $|M|$ denotes the determinant of the matrix M and $|P, Q, R|$ denotes the determinant of the matrix of the columns of *absolute* barycentric coordinates vectors $\{P, Q, R\}$. By equation (24) the inner products are:

$$X_i^2 = R^2 - (a^2v_iw_i + b^2w_iu_i + c^2u_iv_i).$$

Replacing with this in equation (76), we see that the terms involving R^2 factor into

$$|P_2, P_3, P| + |P_3, P_1, P| + |P_1, P_2, P| - |P_1, P_2, P_3| = 0,$$

because this is the determinant of the matrix having two equal rows:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

From this follows that the circle through the three points is described by the equation:

$$\begin{aligned} &(a^2v_1w_1 + b^2w_1u_1 + c^2u_1v_1)|P_2, P_3, P| \\ &+(a^2v_2w_2 + b^2w_2u_2 + c^2u_2v_2)|P_3, P_1, P| \\ &+(a^2v_3w_3 + b^2w_3u_3 + c^2u_3v_3)|P_1, P_2, P| \\ &-(a^2vw + b^2wu + c^2uv)|P_1, P_2, P_3| = 0. \end{aligned} \tag{77}$$

Notice that the three first rows are the expressions of line equations, and the determinant-equations $\{|P_2, P_3, P| = 0, \ldots\}$ represent respectively the side-lines of the triangle $P_1P_2P_3$. Since the expression in the last row is the one of the equation of the circumcircle of ABC, the linear part of the first three rows represents the equation of the radical axis of the circle κ through the three points and the circumcircle κ_0 of the triangle of reference ABC. Thus, we have the theorem

Theorem 13. *The equation*

$$\begin{aligned} &(a^2v_1w_1 + b^2w_1u_1 + c^2u_1v_1)|P_2, P_3, P| \\ &+(a^2v_2w_2 + b^2w_2u_2 + c^2u_2v_2)|P_3, P_1, P| \\ &+(a^2v_3w_3 + b^2w_3u_3 + c^2u_3v_3)|P_1, P_2, P| = 0, \end{aligned} \tag{78}$$

represents the radical axis of the circle κ through the three points $\{P_1, P_2, P_3\}$ expressed in absolute barycentrics and the circumcircle κ_0 of the triangle of reference ABC.

Equation (77) leads to a condition on four points to belong to the same circle by setting $P = P_4$. The equation can be expressed also through a 4×4 determinant. Doing the calculation and some simplifications the formula takes the form:

$$\begin{vmatrix} s(P_1) & s(P_2) & s(P_3) & s(P_4) \\ P_{11} & P_{21} & P_{31} & P_{41} \\ P_{12} & P_{22} & P_{32} & P_{42} \\ P_{13} & P_{23} & P_{33} & P_{43} \end{vmatrix} = 0\,, \tag{79}$$

where in each column appears the expression

$$s(X) = \frac{a^2x_2x_3 + b^2x_3x_1 + c^2x_1x_2}{(x_1 + x_2 + x_3)}$$

and the not necessarily absolute barycentrics of the involved points.

Equation (79) applied to the two Brocard points $\Omega\left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right), \Omega'\left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}\right)$ and the symmedian point $K(a^2 : b^2 : c^2)$, leads to the *"Brocard circle"* passing through these three points:

$$b^2c^2u^2 + c^2a^2v^2 + a^2b^2w^2 - a^4vw - b^4wu - c^4uv = 0\,. \tag{80}$$

18 The associated affine transformation

Here we re-examine the matrix M of section 6, defining the transformation from absolute barycentrics to Cartesian coordinates.

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = M \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{81}$$

The matrix M is invertible and its inverse, denoting by S twice the signed area of $\triangle ABC$, is

$$\begin{aligned} M^{-1} &= \frac{1}{A_1(B_2-C_2)+B_1(C_2-A_2)+C_1(A_2-B_2)} \begin{pmatrix} B_2-C_2 & C_1-B_1 & B_1C_2-B_2C_1 \\ C_2-A_2 & A_1-C_1 & C_1A_2-C_2A_1 \\ A_2-B_2 & B_1-A_1 & A_1B_2-A_2B_1 \end{pmatrix} \\ &= \frac{1}{S} \cdot \begin{pmatrix} B_2-C_2 & C_1-B_1 & B_1C_2-B_2C_1 \\ C_2-A_2 & A_1-C_1 & C_1A_2-C_2A_1 \\ A_2-B_2 & B_1-A_1 & A_1B_2-A_2B_1 \end{pmatrix}. \end{aligned} \tag{82}$$

The matrix M defines an invertible linear transformation L_M (an isomorphism) of $\mathbb{R}^3$ onto itself and the set of all *absolute barycentrics* satisfying $u+v+w=1$ represents the plane ε of $\mathbb{R}^3$, which is orthogonal to the vector $(1,1,1) \in \mathbb{R}^3$ and passes through the point $\frac{1}{3}(1,1,1) \in \mathbb{R}^3$. The image $\varepsilon' = L_M(\varepsilon)$ is the plane of $\mathbb{R}^3$ parallel to the coordinate $\{z=0\}$-plane through the point $z=1$. The linear transformation L_M introduces by its restriction on plane $\varepsilon :$ $\mathcal{M} = L_M|\varepsilon$ an *"affine"* transformation ([2, p.199]) between the planes $\{\varepsilon, \varepsilon'\}$:

$$\mathcal{M} \ : \ \varepsilon \longrightarrow \varepsilon', \quad \text{with} \quad \mathcal{M}(u,v,w) = (x,y,1).$$

It is interesting to see some consequences of this interpretation as, for example, the transformation of lines to lines, the preservation of ratios on lines and the preservation of quotiens of areas. These are general properties of the affine transformations but can be also deduced here directly for this special case. In fact, a line in ε is the intersection of ε with a *plane through the origin* η of $\mathbb{R}^3$: represented by an equation of the form $\eta : pu+qv+rw=0$. The image-plane $\eta' = \mathcal{M}(\eta)$ is found by writing the equation using matrix notation:

$$pu+qv+rw=0 \quad \Leftrightarrow \quad 0 = (p,q,v)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = (p,q,v)M^{-1}M\begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Hence the image of the plane η is the plane represented by the equation

$$\eta' \ : \ p'x+q'y+r'z=0, \quad \text{where} \quad (p',q',r') = (p,q,r)M^{-1}, \tag{83}$$

and the line of the plane ε' is the intersection $\varepsilon' \cap \eta'$. In particular, the line at infinity of ε, represented through its intersection with the plane $\eta \ : \ u+v+w=0$, maps to the line at infinity, which is the intersection of the plane ε' with the plane with coefficients

$$\eta' \ : \ (1,1,1)M^{-1} = (0,0,1) \quad \text{i.e the plane} \quad z=0.$$

Another consequence of the *affine* property of the transformation $\mathcal{M}$ and its inverse is the pre-servation of ratios along lines. Thus, for two points $\{P,Q\}$ of the plane and their line

$$S(t) = (1-t)P+tQ, \quad \text{with} \quad r = \frac{t}{t-1} \quad \text{equal to the signed ratio:} \quad r = \frac{SP}{SQ}, \tag{84}$$

the corresponding barycentrics satisfy the same relation:

$$M^{-1}S(t) = M^{-1}\begin{pmatrix} S_1(t) \\ S_2(t) \\ 1 \end{pmatrix} = M^{-1}\begin{pmatrix} (1-t)P_1 + tQ_1 \\ (1-t)P_2 + tQ_2 \\ 1-t+t \end{pmatrix} = (1-t)(M^{-1}P) + t(M^{-1}Q).$$

Notice that the function $r = f(t) = t/(t-1)$ has inverse $f^{-1} = f$, so that given the ratio $r = SP/SQ$ and taking $t = r/(r-1)$, and seting $\{P', Q', ...\}$ for the corresponding barycentric vectors of the points $\{P, Q, ...\}$, we have that

$$S'(t) = (1-t)P' + tQ' \quad \Leftrightarrow \quad S'(r) = \frac{1}{1-r}(P' - rQ'), \tag{85}$$

satisfies precisely the relation $SP/SQ = r$. If the ratio r is expressed in the form $r = m/n$, then for the point S_r satisfying this condition and the corresponding barycentrics vectors we obtain ([6]):

$$\frac{S_rP}{S_rQ} = r = \frac{m}{n} \quad \Rightarrow \quad S'_r = \frac{1}{n-m}(nP' - mQ'). \tag{86}$$

As application, we prove the collinearity of the *incenter* $I(a:b:c)$, centroid $G(1:1:1)$

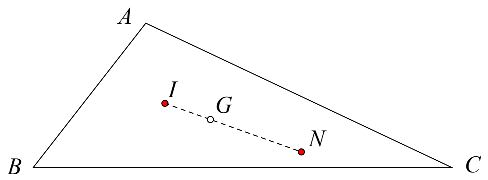

Figure 15: The collinearity of $\{I, G, N\}$

and *Nagel point* $N(b+c-a, ...)$ (see file **Nagel point of the triangle**). The relation to verify is (see figure 15):

$$GN/GI = r = -2 \quad \Rightarrow \quad t = r/(r-1) = 2/3.$$

Thus, turning to absolute barycentrics by dividing the previous barycentrics vectors with the sum $\sigma_N = \sigma_I = a+b+c = 2\sigma$, we obtain:

$$(1-t)\frac{N}{2\sigma} + t\frac{I}{2\sigma} = \frac{1}{6\sigma}(N+2I) = \frac{1}{6\sigma}((b+c-a)+2a, ...) = \frac{1}{3}(1,1,1) = G.$$

As an application of the preceding result we obtain the property (see figure 16) of *"the incenter I to be the Nagel point of the anticomplementary triangle $A'B'C'$ of the middles of sides of ABC"*. This follows from the fact that the homothety with center G and ratio $1/2$, as an affine map, preserves the barycentrics and maps ABC to $A'B'C'$ and also maps the Nagel point N of ABC to I. By the way, we notice also that the triples like the points $\{C', I, J\}$ are collinear (see exercise 3). The triple consisting of the middle of AB, the incenter and the intersection $J = A'B' \cap CL$, where L, the touch point of the incircle with AB. A key fact to prove this synthetically is the equality $AL = MB = \tau - a$, where τ the half-perimeter and $a = |BC|$.

A third property of the affine transformation $\mathcal{M}$, already seen in section 15, is the multiplication of areas by a constant, in this case expressed through equation (69).

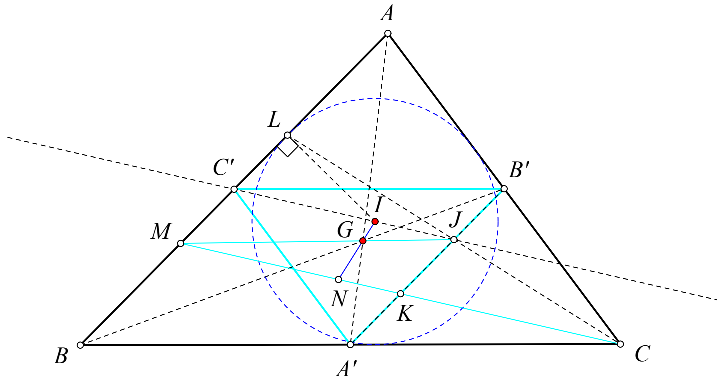

Figure 16: The Nagel point of the anticomplementary

19 Relations between barycentrics and Cartesian coordinates

Here we give a second look at the relations between Cartesian coordinates and barycentrics initiated in section 6 using a particular Cartesian coordinate system. From now on the symbols $\{X, Y, ...\}$ will denote euclidean two-dimensional position vectors with *Cartesian coordinates* $X(X_x, X_y)$ w.r.t. to *a system with origin at the centroid* G *of the triangle of reference* ABC, assumed to have the positive orientation. The *absolute barycentric* coordinates of the same point will be denoted by $\{X_u, X_v, X_w\}$. The two sets of coordinates are related by the matrix M representing the affine transformation L_M of the preceding section:

$$\begin{pmatrix} X_x \\ X_y \\ 1 \end{pmatrix} = M \cdot \begin{pmatrix} X_u \\ X_v \\ X_w \end{pmatrix} .$$

Under the preceding assumption of the particular Cartesian coordinate system, the two matrices $\{M, M^{-1}\}$ take the form:

$$M = \begin{pmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{pmatrix}, \quad M^{-1} = \frac{1}{S} \begin{pmatrix} J(BC)_x & J(BC)_y & S/3 \\ J(CA)_x & J(CA)_y & S/3 \\ J(AB)_x & J(AB)_y & S/3 \end{pmatrix}. \tag{87}$$

Here S is twice the area of $\triangle ABC$ and J denotes the positive rotation by $\pi/2$ acting on vectors $X = (X_x, X_y)$ and transforming them to $J(X) = (-X_y, X_x)$, so that the inner product of two vectors equals their determinant:

$$\langle J(U), V \rangle = -U_y V_x + U_x V_y = |U, V| .$$

This implies, that M^{-1} applied to a point has the form:

$$M^{-1} \cdot \begin{pmatrix} X_x \\ X_y \\ 1 \end{pmatrix} = \frac{1}{S} \begin{pmatrix} |BC, X| \\ |CA, X| \\ |AB, X| \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} . \tag{88}$$

It is interesting to notice the behavior of the absolute coordinates in the case of the sum of two euclidean vectors and the multiplication of a vector by a number.

$$\begin{pmatrix}(X+Y)_u\\(X+Y)_v\\(X+Y)_w\end{pmatrix} = M^{-1}\begin{pmatrix}(X+Y)_x\\(X+Y)_y\\1\end{pmatrix} = M^{-1}\left(\begin{pmatrix}X_x\\X_y\\1\end{pmatrix}+\begin{pmatrix}Y_x\\Y_y\\1\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}\right) \quad \Rightarrow$$

$$\begin{pmatrix}X_u\\X_v\\X_w\end{pmatrix}+\begin{pmatrix}Y_u\\Y_v\\Y_w\end{pmatrix} = \begin{pmatrix}(X+Y)_u\\(X+Y)_v\\(X+Y)_w\end{pmatrix}+\frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}, \tag{89}$$

Showing that the sum of the barycentrics does not represent the point resulting by summing the corresponding Cartesian coordinates. Analogously for the multiplication by a number λ we have:

$$\begin{pmatrix}(\lambda X)_u\\(\lambda X)_v\\(\lambda X)_w\end{pmatrix} = M^{-1}\begin{pmatrix}(\lambda X)_x\\(\lambda X)_y\\1\end{pmatrix} = M^{-1}\left(\lambda\begin{pmatrix}X_x\\X_y\\1\end{pmatrix}+\begin{pmatrix}0\\0\\1-\lambda\end{pmatrix}\right) \quad \Rightarrow$$

$$\begin{pmatrix}(\lambda X)_u\\(\lambda X)_v\\(\lambda X)_w\end{pmatrix} = \lambda\begin{pmatrix}X_u\\X_v\\X_w\end{pmatrix}+\frac{1-\lambda}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}. \tag{90}$$

In particular, we see that taking the negative barycentrics corresponds to considering the symmetric point w.r.t. the origin G (see also formula (96)):

$$\begin{pmatrix}(-X)_u\\(-X)_v\\(-X)_w\end{pmatrix} = -\begin{pmatrix}X_u\\X_v\\X_w\end{pmatrix}+\frac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}. \tag{91}$$

Remark 13. Equation (89) written in the form

$$\begin{pmatrix}(X+Y)_u\\(X+Y)_v\\(X+Y)_w\end{pmatrix} = \begin{pmatrix}X_u\\X_v\\X_w\end{pmatrix}+\begin{pmatrix}Y_u\\Y_v\\Y_w\end{pmatrix}-\frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}, \tag{92}$$

can be considered as the result in barycentrics of an euclidean translation by the vector Y. We see that this is not described by the addition of corresponding coordinates as is the case of the expression of the translation in Cartesian coordinates. Also the expression of the *displacement vectors* (section 8), which are differences of barycentrics vectors, do not correspond exactly to the difference of the Cartesian coordinates:

$$\begin{pmatrix}Y_u\\Y_v\\Y_w\end{pmatrix}-\begin{pmatrix}X_u\\X_v\\X_w\end{pmatrix} = \begin{pmatrix}(Y-X)_u\\(Y-X)_v\\(Y-X)_w\end{pmatrix}-\frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

20 Affine transformations represented in barycentrics

Using the embedding of the plane $(x,y)\in\mathbb{R}^2\mapsto(x,y,1)$ into $\mathbb{R}^3$, the affine transformation L_M and its inverse, represented by the matrices $\{M,M^{-1}\}$ in (87), we can easily arrive at the matrix representation by barycentrics of the affine transformations of the plane, which comprise the isometries and the similarities as special cases. The commutative diagram below leads to a method to express this matrix in terms of the Cartesian coordinates of the involved points and transformations.

$$\begin{array}{ccc}
\mathbb{R}^3\text{(Cartesian coordinates)} & \xrightarrow{M_f} & \mathbb{R}^3\text{(Cartesian coordinates)} \\
{\scriptstyle M}\uparrow & & \downarrow{\scriptstyle M^{-1}} \\
\mathbb{R}^3\text{(absolute barycentrics)} & \xrightarrow{N_f} & \mathbb{R}^3\text{(absolute barycentrics)}
\end{array}$$

The representation of a general affine transformation f of the plane by a matrix M_f in Cartesian coordinates has the form

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \overset{f}{\mapsto} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} P_1 & Q_1 & V_1 \\ P_2 & Q_2 & V_2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad \Leftrightarrow \quad f(X) = M_f \cdot X \, ,$$

with the determinant $P_1Q_2 - P_2Q_1 \neq 0$. The matrix representing the same transformation in barycentrics results as a product of matrices:

$$N_f = M^{-1} \cdot M_f \cdot M \, .$$

The matrices $\{M, M^{-1}\}$, introduced in section 6 and having the form (87), explained in section 19, lead to the following expression for the product:

$$M^{-1} \cdot M_f = \frac{1}{S} \begin{pmatrix} |BC,P| & |BC,Q| & |BC,V| + S/3 \\ |CA,P| & |CA,Q| & |CA,V| + S/3 \\ |AB,P| & |AB,Q| & |AB,V| + S/3 \end{pmatrix} .$$

This, multiplying with M, leads finally to the matrix $N_f = M^{-1} \cdot M_f \cdot M$ expressed in terms of the euclidean coordinates of the vectors $\{A, B, C\}$ w.r.t. a system having its origin at the centroid G of the triangle of reference ABC. The symbol $f(X)$ in the formula below denotes the application of the transformation f to $X(x,y)$ i.e. $f(X) = xP + yQ + V$ all this expressed as a two dimensional vector (omitting the last 1 from $(x, y, 1)$).

$$N_f = \frac{1}{S} \begin{pmatrix} |BC,f(A)| & |BC,f(B)| & |BC,f(C)| \\ |CA,f(A)| & |CA,f(B)| & |CA,f(C)| \\ |AB,f(A)| & |AB,f(B)| & |AB,f(C)| \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} . \tag{93}$$

Remark 14. It is easily verified that the determinants $\{|BC, V|, ...\}$ are related to the absolute coordi-nates $\{u, v, w\}$ of V by the simple formula:

$$\begin{pmatrix} |BC, V| \\ |CA, V| \\ |AB, C| \end{pmatrix} = \frac{S}{3} \begin{pmatrix} 3u - 1 \\ 3v - 1 \\ 3w - 1 \end{pmatrix} . \tag{94}$$

Applying this to formula (93) we find that the matrix N_f can be expressed also in the form

$$N_f = \begin{pmatrix} f(A)_u & f(B)_u & f(C)_u \\ f(A)_v & f(B)_v & f(C)_v \\ f(A)_w & f(B)_w & f(C)_w \end{pmatrix} , \tag{95}$$

where the symbols $\{f(X)_u, f(X)_v, f(X)_w\}$ denote the absolute barycentrics of the point $f(X)$. Note that this representation of the affinity by N_f is not valid for a general projectivity f, since this would imply that all projectivities fixing the vertices of $\triangle ABC$ have the same matrix representation, which is not true. Affinities, though, are completely determined by their action on the vertices of $\triangle ABC$. Formula (95) conforms to this fact and shows, that affinities, expressed in absolute barycentrics, act as linear transformations and their values are completely determined from those on the *"base"* consisting of the vertices $\{A, B, C\}$.

In equation (92) we saw the expression of a translation in barycentrics. Another simple example of affine transformation is the *point-symmetry*. This, using formula (93) or its equivalent (95) and the rules of section 19, is easily seen to be expressible in barycentrics in the same typical form as in Cartesian coordinates:

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = -\begin{pmatrix} u \\ v \\ w \end{pmatrix} + 2\begin{pmatrix} W_u \\ W_v \\ W_w \end{pmatrix}, \tag{96}$$

where $\{W_u, W_v, W_w\}$ are the barycentrics of the center W of the symmetry. This conforms to the remarks on the ratio of segments along a fixed line of section 18 and could be deduced trivially from the arguments used there. This remark applies also to *homotheties*, which generalize the point-symmetry and in Cartesian coordinates are described by the formula:

$$f(X) = Y = W + k(X - W), \tag{97}$$

where $\{W, k\}$ are respectively the *center* and the *homothety ratio*. The corresponding formula in absolute barycentrics has the same form.

Exercise 2. *Show, using formula (95), that the reflection f_{BC} in the side BC of $\triangle ABC$ is represented in barycentrics by the matrix*

$$f_{BC}(X) = \begin{pmatrix} -1 & 0 & 0 \\ 2S_C/a^2 & 1 & 0 \\ 2S_B/a^2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Exercise 3. *Using absolute barycentrics, the formula (46) for the projection of a point on a line and formula (96), show that the reflection U' of a point $U(u,v,w)$ in the line, expressed by the equation $f(u,v,w) = pu + qv + rw = 0$, is given by the formula*

$$U' = U - 2\frac{f(U)}{f(V)}V, \quad \textit{with} \quad V = \begin{pmatrix} S_B(p-r) - S_C(q-p) \\ S_C(q-p) - S_A(r-q) \\ S_A(r-q) - S_B(p-r) \end{pmatrix}. \tag{98}$$

Remark 15. Notice that V represents in barycentrics the orthogonal direction to that of the line $f(u,v,w) = pu + qv + rw = 0$. Also, if you interpret this equation as one defining a *"plane"* in $\mathbb{R}^3$ through its origin and V as an orthogonal vector to this plane, which in this case is a multiple of (p,q,r), then the reflection in this plane expressed in Cartesian coordinates of $\mathbb{R}^3$ is formaly the same with equation (98).

Exercise 4. *Referring to figure 16 and using formula (97), show that the barycentrics of the intersection point $J = A'B' \cap CL$ are a non-zero multiple of $(\tau - b, \tau - a, 2\tau - a - b)$. Use this to show the collinearity of points $\{C', I, J\}$.*

21 Polar and de Longchamps circles

These are two circles, real only for obtuse triangles. The polar circle $\lambda(H, r)$ centered at the orthocenter H of the triangle ABC is defined by the property of this point

$$HA \cdot HA' = HB \cdot HB' = HC \cdot HC' = r^2,$$

where $\{A', B', C'\}$ denote the *traces* of the orthocenter on the sides of the triangle. Starting from some simple geometric properties of the corresponding figure 17 we compute its radius. For this we use material from the preceding discussion and in particular identities

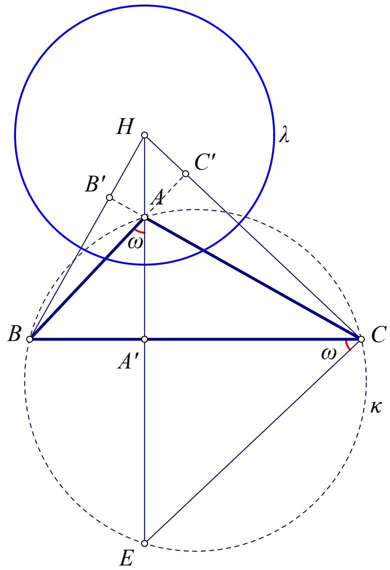

Figure 17: The polar circle $\lambda(H,r)$ of the obtuse triangle ABC

for the Conway triangle symbols of section 8 and also from the file **Conway triangle symbols**. From the figure results easily the well known property $HA' = A'E$ and denoting by $\{a = |BC|, b = |CA|, c = |AB|\}$ the side-lengths, we have

$$\begin{aligned}
&r^2 = HA \cdot HA' = HA \cdot A'E = (A'E - A'A) \cdot A'E = \\
&(CA' \tan(\omega) - c\sin(\hat{B})) \cdot (CA' \tan(\omega)) = \\
&(CA' \cot(\hat{B}) - c\sin(\hat{B})) \cdot (CA' \cot(\hat{B}) = \\
&(b\cos(\widehat{C}) \cot(\hat{B}) - c\sin(\hat{B})) \cdot (b\cos(\widehat{C}) \cot(\hat{B})) = \\
&\left(\left(b\frac{S_C}{ab}\right)\left(\frac{S_B}{S}\right) - c\sin(\hat{B})\right) \cdot \left(b\frac{S_C}{ab}\right)\left(\frac{S_B}{S}\right) = \\
&\left(\frac{S_C S_B}{aS} - c\sin(\hat{B})\right)\left(\frac{S_C S_B}{aS}\right) = \\
&\left(\frac{S^2 - a^2 S_A}{aS} - c\sin(\hat{B})\right)\left(\frac{S^2 - a^2 S_A}{aS}\right) = \\
&(S^2 - a^2 S_A - Sac\sin(\hat{B}))\left(\frac{S^2 - a^2 S_A}{a^2 S^2}\right) = \\
&(S^2 - a^2 S_A - S^2))\left(\frac{S^2 - a^2 S_A}{a^2 S^2}\right) = -S_A\left(\frac{S^2 - a^2 S_A}{S^2}\right) = \\
&\frac{a^2 S_A^2 - S^2 S_A}{S^2} = \frac{a^2(b^2c^2 - S^2) - S^2 S_A}{S^2} = \frac{a^2b^2c^2 - S^2(a^2 + S_A)}{S^2} = \qquad (99) \\
&\frac{a^2b^2c^2 - S^2\frac{1}{2}(a^2+b^2+c^2)}{S^2} = \frac{4R^2S^2 - S^2\frac{1}{2}(a^2+b^2+c^2)}{S^2} \quad \Rightarrow
\end{aligned}$$

$$r^2 = 4R^2 - \frac{1}{2}(a^2+b^2+c^2)\,. \qquad (100)$$

For the equation of the polar circle we use the formula 58 for the general circle with center H and and radius r, taking into account the absolute coordinates of the orthocenter resulting from equation (66)

$$H = \frac{1}{S^2}(\, S_B S_C \; : \; S_C S_A \; : \; S_A S_B \,)\,. \tag{101}$$

Applying formula (29) we have

$$\begin{gathered}
HA^2 = S_A\left(\frac{S_B S_C}{S^2}-1\right)^2 + S_B\left(\frac{S_C S_A}{S^2}\right)^2 + S_C\left(\frac{S_A S_B}{S^2}\right)^2 = \\
\frac{1}{S^4}\left(S_A S_B^2 S_C^2 + S_B S_C^2 S_A^2 + S_C S_A^2 S_B^2\right) - \frac{2}{S^2} S_A S_B S_C + S_A = \\
\frac{1}{S^4} S_A S_B S_C \left(S_B S_C + S_C S_A + S_A S_B\right) - \frac{2}{S^2} S_A S_B S_C + S_A = \\
\frac{1}{S^4} S_A S_B S_C (S^2) - \frac{2}{S^2} S_A S_B S_C + S_A = \\
S_A - \frac{1}{S^2} S_A S_B S_C\,.
\end{gathered}$$

Using this and equation (99) for the radius of the polar circle, we get

$$\begin{gathered}
r^2 - HA^2 \;=\; \frac{a^2 S_A^2 - S^2 S_A}{S^2} - \left(S_A - \frac{1}{S^2} S_A S_B S_C\right) = \\
\frac{a^2 S_A^2 - 2S^2 S_A + S_A S_B S_C}{S^2} = \frac{S_A}{S^2}\left(a^2 S_A - 2S^2 + S_B S_C\right) = \\
\frac{S_A}{S^2}(-S^2) = -S_A\,,
\end{gathered}$$

and analogously $r^2 - HB^2 = -S_B$ and $r^2 - HC^2 = -S_C$. Replacing these in equation (58) leads then to the equation for the polar circle

$$\begin{aligned}
a^2 vw + b^2 wu + c^2 uv - (S_A u + S_B v + S_C w)(u+v+w) \;&=\; 0 \qquad \Leftrightarrow \\
S_A u^2 + S_B v^2 + S_C w^2 \;&=\; 0\,.
\end{aligned} \tag{102}$$

The de Longchamps circle of the triangle ABC (see figure 18) can be defined as the circle μ homothetic to the polar circle λ in ratio $-2:1$ w.r.t. to the centroid G of the triangle [7]. Applying formula (97) for this homothety f we find, working with absolute barycentrics

$$\begin{gathered}
Y(u':v':w') = f(X(u:v:w)) = G - 2(X-G) \qquad \Leftrightarrow \\
X = G + \frac{1}{2}(G - Y) \qquad \Leftrightarrow \\
(u:v:w) = \frac{1}{3}(1:1:1) + \frac{1}{2}\left(\frac{1}{3}(1:1:1) - (u':v':w')\right) \qquad \Rightarrow \\
(u:v:w) = \frac{1}{2}(1-u' : 1-v' : 1-w')\,.
\end{gathered}$$

Replacing this into the polar equation and dropping the primes, we find the equation of de Longchamps' circle

$$a^2u^2 + b^2v^2 + c^2w^2 + 2S_C uv + 2S_A vw + 2S_B wu \;=\; 0\,. \tag{103}$$

The triangle center $X(20)$, which is the center of the de Longchamps' circle, is found by

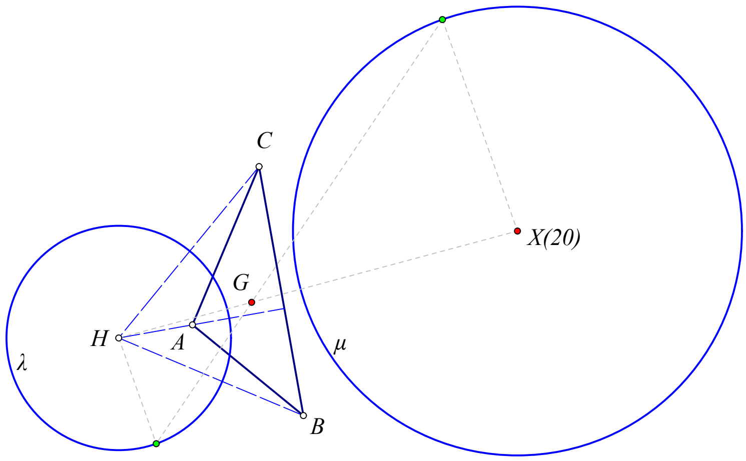

Figure 18: The polar λ and de Longchamps' μ circles of $\triangle ABC$

applying the homothety f to the center H of the polar circle. In absolute barycentrics:

$$X(20) = G - 2(H - G) = \tag{104}$$

$$\frac{1}{3}(1:1:1) - 2\left(\frac{1}{S^2}(S_BS_C : S_CS_A : S_AS_B) - \frac{1}{3}(1:1:1)\right) \quad \Rightarrow$$

$$X(20) = \frac{1}{S^2}\left(S^2 - 2S_BS_C : S^2 - 2S_CS_A : S^2 - 2S_AS_B\right) . \tag{105}$$

Similarly, the radius ρ of this circle is twice the radius (100) of the polar circle:

$$\rho^2 = 16R^2 - 2(a^2 + b^2 + c^2) . \tag{106}$$

22 Remarks on working with barycentrics

The vectors of barycentric coordinates $\{U_P, U_Q, U_R \cdots \in \mathbb{R}^3\}$ representing the points of the plane $\{P, Q, R, \cdots \in \mathbb{R}^2\}$ are not the points we see. What we see are the points of $\mathbb{R}^2$. This is clearly understood in the case of the triangle of reference ABC. The barycentrics-vectors representing it are $\{(1,0,0), (0,1,0), (0,0,1)\}$ and define the vertices of an equilateral triangle lying on the plane $\varepsilon : u + v + w = 1$ of $\mathbb{R}^3$. The map L_M of section 18 is the affine transformation mapping the equilateral $A'B'C'$ onto ABC. By means of it, all properties of the triangle correspond to properties of the equilateral and vice versa. In particular, properties of the equilateral which are preserved by affine transformations map to similar properties of ABC. A characteristic example is the circum-circle κ' of the equilateral $A'B'C'$, which carries also the symmetrics $\{A'_1, B'_1, C'_1\}$ of the vertices w.r.t. the center G' of the equilateral (see figure 19). The map L_M transforms G' to the centroid G of ABC and the circumcircle of $A'B'C'$ to the *"Steiner (outer) ellipse"* κ of ABC, characterized by the fact to pass through the vertices of ABC and their symmetrics A_1, B_1, C_1 w.r.t. G, point G being its center. Similarly, L_M maps the incircle λ' of $A'B'C'$ to the *"Steiner in-ellipse"* λ of the triangle ABC, characterized by its tangency at the middles A_0, B_0, C_0 of the sides of ABC. The homothety of $\{\kappa', \lambda'\}$ with center G' and ratio $2:1$ translates to the homothety of $\{\kappa, \lambda\}$ with center G and the same ratio.

Invertible affine transformations or *affinities* like L_M, besides the preservation of ratios along lines, map also areas σ to multiples $k\sigma$, with k a constant factor. This implies,

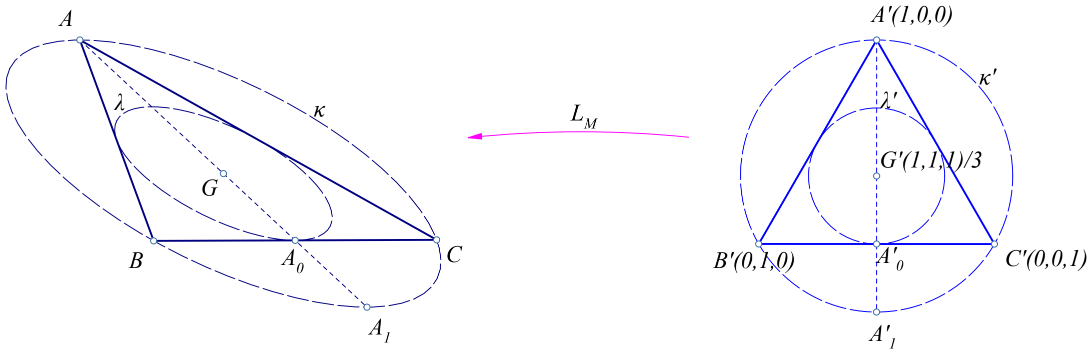

Figure 19: We see ABC and work with $A'B'C'$

that points P' with barycentrics $(u' : v' : w')$ w.r.t. $A'B'C'$ map to corresponding points $P = L_M(P')$ with the same barycentrics $(u : v : w) = (u' : v' : w')$ w.r.t. ABC. Thus, for example, the circumcircle of the equilateral $A'B'C'$ with side $a = |B'C'|$, whose equation, according to section 7 is

$$a^2v'w' + a^2w'u' + a^2u'v' = 0 \quad \Leftrightarrow \quad v'w' + w'u' + u'v' = 0,$$

maps to the Steiner outer ellipse, which in barycentrics w.r.t. ABC must satisfy the same equation:

$$vw + wu + uv = 0. \tag{107}$$

On the other hand, the incircle of the equilateral $A'B'C'$, which, according to section 12, is represented in barycentrics w.r.t. $A'B'C'$ by the equation

$$a^2(vw + wu + vw) - \frac{a^2}{4}(u + v + w)^2 = 0 \quad \Leftrightarrow \quad u^2 + v^2 + w^2 - 2(vw + wu + uv) = 0,$$

maps to the Steiner inner ellipse, which in barycentrics w.r.t. ABC is represented by the same equation:

$$u^2 + v^2 + w^2 - 2(vw + wu + uv) = 0. \tag{108}$$

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Related topics

1. **Cross Ratio**
2. **Ceva's theorem**
3. **Conway triangle symbols**
4. **Menelaus' theorem**
5. **Nagel point of the triangle**
6. **Projective line**
7. **Projective plane**
8. **Symmedian point of the triangle**

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