The artist finds a greater pleasure in painting than in having completed the picture.

_Seneca, Letter to Lucilius_
1 Preliminaries and definition

“Barycentric coordinates” do not use distances of points, but only ratios of segments. Thus, they belong to the geometry of the “affine plane”, which deals with parallels and ratios of collinear segments without to use a notion of distance between two points, as it does euclidean geometry.

In the subsequent discussion points \{A, B, \ldots\} of the plane are identified with two dimensional vectors \{(a_1, a_2), (b_1, b_2) \ldots\}. Next properties are easily verified ([Ped90, p.30]):

1. Three points of the plane \{A, B, C\} are collinear, if and only, there are three numbers \{x, y, z\} not all zero, such that
   \[ xA + yB + cZ = 0 \quad \text{and} \quad x + y + z = 0. \]

2. Let the points of the plane \{A, B, C\} be non-collinear. Then, for every other point \(D\) of the plane there are unique numbers \{x, y, z\} with
   \[ x + y + z = 1 \quad \text{and} \quad D = xA + yB + zC. \]

3. For non-collinear points \{A, B, C\} the point \(D\), defined by the equation
   \[ OD = x \cdot OA + y \cdot OB + z \cdot OC, \] with arbitrary origin \(O\) and \(x + y + z = 1\), (1)
   is independent of the choice of origin and depends only on \(x, y, z\).

4. The numbers \(x, y, z\) with \(x + y + z = 1\) expressing \(D\) in the previous equation are equal to the quotients of signed areas:

\[
x = \frac{\text{Area}(DBC)}{\text{Area}(ABC)}, \quad y = \frac{\text{Area}(DCA)}{\text{Area}(ABC)}, \quad z = \frac{\text{Area}(DAB)}{\text{Area}(ABC)}. \tag{2}
\]

Nr-1. If \(x + y + z = 0\) and \(xA + yB + zC = 0\), then
\[ (-y - z)A + yB + zC = 0 \quad \Rightarrow \quad y(B - A) + z(C - A) = 0 \]
i.e. \(\{A, B, C\}\) are collinear. For the converse, simply reverse the previous implications.

Nr-2. Extend the two-dimensional vectors \((x, y)\) to three dimensional \((x, y, 1)\) and denote the corresponding space-points by \(\{A', B', C', D'\}\). Apply first nr-1 to see that \(\{A', B', C'\}\) are independent. Then express \(D'\) in the basis \(\{A', B', C'\}\) to find the uniquely defined \((x, y, z)\) as required.

Nr-3. Using another point \(O'\) for origin, the same equation and the analogous numbers \((x', y', z')\) to express
\[ D : O'D = x' \cdot O'A + y' \cdot O'B + z' \cdot O'C \]
and substracting we get:

\[
OD - O'D = (x \cdot OA + y \cdot OB + z \cdot OC) - (x' \cdot O'A + y' \cdot O'B + z' \cdot O'C)
\]

\[
= (x - x')A + (y - y')B + (z - z')C - (xO - x'O' + yO - y'O' + zO - z'O') \quad \Rightarrow
\]

\[
OO' = O' - O = (x - x')A + (y - y')B + (z - z')C - (O - O') \quad \Rightarrow
\]

\[
(x - x')A + (y - y')B + (z - z')C = 0.
\]

By \(nr-1\) this implies \((x - x') = 0, (y - y') = 0, (z - z') = 0\), as desired.

\(Nr-4\). Using the independence of \((x, y, z)\) from the position of \(O\) we select \(O = D\) and use the relation \(x = 1 - y - z\) implying

\[
(1 - y - z)DA + yDB + zDC = 0 \quad \Rightarrow
\]

\[
DA + y(DB - DA) + z(DC - DA) = 0 \quad \Rightarrow
\]

\[
AD = y(AB) + z'(AC).
\]

Multiplying externally by \(AB\) we get i

\[
AD \times AB = z(AC \times AB) \quad \Leftrightarrow \quad z(AB \times AC) = AB \times AD.
\]

Analogous equations result also for \(y\) and \(x\). Taking the positive orientation in the direction \(AB \times AD\) we have the stated result.

The numbers \((x, y, z)\) in (2) are called “absolute barycentric homogeneous coordinates” of the point \(D\), relative to the triangle \(ABC\). The multiples by a constant \(k \neq 0 \colon (k \cdot x, k \cdot y, k \cdot z)\) are called “barycentric homogeneous coordinates” or “barycentrics” of the point \(D\) ([Cas93, p.64], [Yiu13, p.25], [Yiu00], [Mon17]). The signs of the numbers result from the orientations of the corresponding triangles e.g. if \((D, A)\) are on the same side of \(BC\) then \(x\) is positive. Otherwise it is negative and for \(D\) on \(BC\) it is zero \(x = 0\). In fact \(x = 0\) characterizes line \(BC\). Analogous properties are valid also for \(y\) and \(z\).

**Remark-1** Figure 1 shows a point \(E\) defined by an equation as in \(nr-3\), but without the restriction \(s = x + y + z = 1\). Point \(D\) is then obtained by dividing with \(s\). Thus, the points are collinear with \(O\) and defined by the equations

\[
OE = xOA + yOB + zOC \quad \text{and} \quad OD = \frac{x}{s}OA + \frac{y}{s}OB + \frac{z}{s}OC.
\]

## 2 Traces, ratios, harmonic conjugation

Varying \(x\) alone, moves \(D\) along a fixed line through \(A\). For \(x = 0\), we get a point \(D_A = yB + zC\) on the line \(BC\). Point \(D_A\) is called the “trace” of \(D\) on the side \(BC\) of the triangle \(ABC\). Analogously are defined the traces \(D_B\) and \(D_C\). The oriented ratio

\[
r = D_AB/D_A C,\]

calculated using \(y + z = 1\), gives the value \(r = -(z/y)\).

**Theorem 1.** Given a point \(P = yB + zC\) on the line \(BC\) the “harmonic conjugate” of \(P\) w.r. \((B, C)\) is \(P' = yB - zC\).

This follows from the expression of the ratio \(PB/PC = -z/y\) and corresponding for the conjugate \(P' : P'B/P'C = z/y\), from which follows that \((PB/PC) : (P'B/P'C) = -1\).

**Remark-1** Since any line can be described in “parametric form” \(P = yB + zC\) and points
{B, C} can be complemented by a point A non-collinear with {B, C} to a triangle of reference, the previous property is generally valid for any two points on a line.

Remark-2 The “centroid” G of the triangle ABC, i.e. the intersection-point of its “medians,” defines triangles {GBC, GCA, GAB} with equal areas. Thus it has “barycentric homogeneous coordinates” equal to (1, 1, 1). This identifies these coordinates with the projective homogeneous coordinates w.r. to the “projective base” {A, B, C, G} with G the “unit” point (see file Projective plane).

3 Lines in barycentric coordinates

The following facts concerning the representation of lines with barycentric coordinates are easily verifiable:

1. A line \( \varepsilon \) is represented by an equation of the form

\[
\varepsilon : \ ax + by + cz = 0.\tag{3}
\]

2. The intersection point of line \( \varepsilon \) with the line \( \varepsilon' : \ a'x + b'y + c'z = 0 \) is given by the “vector product” of the vectors of the coefficients:

\[
\varepsilon \cap \varepsilon' = (bc' - cb', \ ca' - ac', \ ab' - ba'). \tag{4}
\]

3. The “line at infinity” is represented by the equation (see equation 20)

\[
\varepsilon_\infty : \ x + y + z = 0. \tag{5}
\]

4. The point at infinity of the line \( \varepsilon : \ ax + by + cz = 0 \) i.e. its intersection with the line at infinity, identified with its “direction”, is given by

\[
\varepsilon \cap \varepsilon_\infty = (b - c, \ c - a, \ a - b). \tag{6}
\]

5. Two lines \( \{ \varepsilon, \varepsilon' \} \) are parallel if they meet at a point at infinity, hence their coefficients satisfy:

\[
bc' - cb' + ca' - ac' + ab' - ba' = 0. \tag{7}
\]

6. The line passing through two given points \( \{ P(a, b, c), P'(a', b', c') \} \) has coefficients the coordinates of the vector product and can be expressed by a determinant:

\[
\text{line } PP' : \ (bc' - cb', \ ca' - ac', \ ab' - ba'), \quad \text{equation: } \begin{vmatrix} a & b & c \\ a' & b' & c' \\ x & y & z \end{vmatrix} = 0. \tag{8}
\]
7. The direction of the line passing through the points \( A = (a, b, c) \), \( B = (a', b', c') \), i.e. its point at infinity is given by the weighted difference of coordinates

\[
    s_A \cdot (a', b', c') - s_B \cdot (a, b, c) \quad \text{with} \quad s_A = a + b + c, \quad s_B = a' + b' + c'.
\]

(9)

8. The middle \( M \) of two points \( A(a, b, c) \) and \( B(a', b', c') \) is found as the harmonic conjugate of the point at infinity of line \( AB \). Applying the previous calculations it is found to be (the equality being for the triplets of coordinates)

\[
    M = \frac{a + b + c}{B} + \frac{a' + b' + c'}{A}.
\]

(10)

9. The line \( \eta \) parallel from \( A(a, b, c) \) to the given line \( \varepsilon : a'x + b'y + c'z = 0 \) is the line joining \( A \) with the point at infinity of the line \( \varepsilon \), which is \( (b' - c', c' - a', a' - b') \). Thus the equation of this line is

\[
    \begin{vmatrix}
        b' - c' & c' - a' & a' - b' \\
        a & b & c \\
        x & y & z
    \end{vmatrix} = 0,
\]

and the coefficients of this line, written in the form \( px + qy + rz = 0 \), are given by:

\[
    (p, q, r) = (a + b + c)(a', b', c') - (aa' + bb' + cc')(1, 1, 1).
\]

(11)

Thus, the line \( \eta \) can be written as a linear combination of the given line \( \varepsilon \) and the line at infinity \( \varepsilon_{\infty} \), in the form

\[
    \eta : \quad (a + b + c)e - (aa' + bb' + cc')e_{\infty} = 0 \quad \iff \quad (a + b + c)(a'x + b'y + c'z) - (aa' + bb' + cc')(x + y + z) = 0.
\]

(12) (13)

4 Ceva’s and Menelaus’ theorems in barycentrics

Three points \( \{A', B', C'\} \) on the sides of the triangle of reference \( ABC \) are the “traces” of a point \( P(x, y, z) \) if and only if they can be written in the form:

\[
    A'(0, y, z), \quad B'(x, 0, z), \quad C'(x, y, 0).
\]

(14)

Their ratios on the triangle’s sides satisfy then the equation:

\[
    \frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = -1.
\]

(15)
As in the previous section, this follows from the expressions of the ratios

\[
\frac{A'B}{A'C} = -\frac{z}{y}, \quad \frac{B'C}{B'A} = -\frac{x}{z}, \quad \frac{C'A}{C'B} = -\frac{y}{x}.
\]

Multiplying the three ratios gives the value −1. The inverse is also obvious. If the product of the three ratios is −1, then \((x, y, z)\) can be found such that \(\{A', B', C'\}\) have the shown coordinates. This defines uniquely \(P(x, y, z)\). For another more conventional view of Ceva’s theorem look at the file Ceva’s theorem.

It is obvious, that the points \(\{A', B', C'\}\) on the sides of the triangle of reference \(ABC\) are on a line \(px + qy + rz = 0\), if and only if they can be written in the form:

\((0, -r, q), \quad (r, 0, -p), \quad (-q, p, 0)\).

Their ratios satisfy then the equation:

\[
\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1.
\]

This follows from the expression of the ratio \(DB/DC = -z/y\), for \(D = yB + zC\), seen in section 2. By this \(A' = -rB + qC \Rightarrow A'B/A'C = q/r\) etc. (See Figure 4). For another more conventional view of Menelaus’ theorem look at the file Menelaus’ theorem.

5 Trilinear polar in barycentrics

The “trilinear polar” or “tripolar” \(\epsilon_P\) of the point \(P\) w.r. to the triangle \(ABC\) is the line containing the “harmonic conjugates” of the traces \(\{A', B', C'\}\) of \(P\) on the sides of \(ABC\):

\[
A'' = A'(BC), \quad B'' = B'(AC), \quad C'' = C'(BA).
\]

That these three points \(\{A'', B'', C''\}\) are on a line follows by writing \(P = uA + vB + wC\). Then the traces are given by \(\{A' = vB + wC, P_B = wC + uA, P_C = uA + vB\}\). By section 2 the harmonic conjugates of the traces are then \(\{A'' = vB - wC, B'' = wC - uA, C'' = uA - vB\}\) and satisfy \(A'' + B'' + C'' = 0\) showing that \(C''\) is on the line of \(A''B''\).

Alternatively, the barycentrics of \(\{A'', B'', C''\}\) are respectively

\[
A''(0, v, -w), \quad B''(-u, 0, w), \quad C''(u, -v, 0),
\]

which up to a multiplicative factor can be written in the form

\[
A''(0, 1/w, -1/v), \quad B''(-1/w, 0, 1/u), \quad C''(1/v, -1/u, 0),
\]

implying that they are points of the line described by the next theorem:
Theorem 2. For every point \( P(u,v,w) \) of the plane, different from the vertices of the triangle of reference \( ABC \), the corresponding "trilinear polar" \( \text{tr}(P) \) is described by the equation:

\[
\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.
\]

Conversely, every line \( \varepsilon : pu + qv + rw = 0 \) of the plane, not passing through a vertex of \( ABC \) is the trilinear polar \( \text{tr}(P) \) of the point

\[
P\left(\frac{1}{p}, \frac{1}{q}, \frac{1}{r}\right),
\]

which is called "tripole" of the line \( \varepsilon \).

Remark-1 The map \( P \mapsto \text{tr}(P) \) is one-to-one for all points of the plane except those lying on the side-lines of the triangle of reference \( ABC \). For these points \( P \), if they are different from a vertex and lie on a side-line \( \varepsilon \), the corresponding line \( \text{tr}(P) \) coincides with \( \varepsilon \). For the vertices of \( ABC \) the trilinear polar cannot be uniquely defined.

Remark-2 Using the definition of the trilinear polar, it is easy to see, that for the points \( P \) of a line \( \varepsilon \) passing through a vertex, \( C \) say, of the triangle of reference \( ABC \), the corresponding trilinear polar \( \text{tr}(P) \) passes through the "harmonic conjugate" point \( D' = D(AB) \), where \( D = \varepsilon \cap AB \). As the point \( P \) approaches \( C \), the corresponding trilinear polar \( \text{tr}(P) \) tends to coincide with the "harmonic conjugate" line \( CD' \) of \( CD \) w.r. to the line-pair (\( CA, CB \)).

Remark-3 From its proper definition follows that the trilinear polar of the "centroid" with equation

\[
\varepsilon_{\infty} : u + v + w = 0,
\]
is the “line at infinity” of the plane.

6 Relation to cartesian coordinates, inner product

Denoting by \((x, y)\) the cartesian coordinates and by \((u, v, w)\) the “absolute barycentrics” of the same point \(P\), the relation between the two is expressed in matrix form by translating equation 1:

\[
\begin{pmatrix}
    x \\
    y \\
    1
\end{pmatrix} = M \cdot \begin{pmatrix}
    u \\
    v \\
    w
\end{pmatrix} = \begin{pmatrix}
    A_1 & B_1 & C_1 \\
    A_2 & B_2 & C_2 \\
    1 & 1 & 1
\end{pmatrix} \cdot \begin{pmatrix}
    u \\
    v \\
    w
\end{pmatrix}.
\]

(19)

This can be used to express other geometric objects through relations of the barycentrics. The equation of a line, which in cartesian coordinates is

\[
a x + b y + c = (a, b, c) \cdot \begin{pmatrix}
    x \\
    y \\
    1
\end{pmatrix} = 0
\]

translates into barycentrics to the equation

\[
a' u + b' v + c' w = 0 \quad \text{with} \quad (a', b', c') = (a, b, c) \cdot M.
\]

Thus, the equation of the line at infinity, described in cartesians by \(z = 0\), corresponds by the preceding rule, to the equation in barycentrics:

\[
(0, 0, 1)M \begin{pmatrix}
    u \\
    v \\
    w
\end{pmatrix} = u + v + w = 0.
\]

(20)

The “inner product” in cartesian coordinates is also expressible through a “bilinear form”:

\[
xx' + yy' = (x, y, 1) \begin{pmatrix}
    x' \\
    y' \\
    1
\end{pmatrix} - 1 = (u, v, w)M' M \begin{pmatrix}
    u' \\
    v' \\
    w'
\end{pmatrix} - 1.
\]

(21)

The matrix \(M' M\) is easily seen to be

\[
M' M = \begin{pmatrix}
    A^2 & A \cdot B & A \cdot C \\
    A \cdot B & B^2 & B \cdot C \\
    A \cdot C & B \cdot C & C^2
\end{pmatrix} + \begin{pmatrix}
    1 & 1 & 1 \\
    1 & 1 & 1 \\
    1 & 1 & 1
\end{pmatrix}.
\]

(22)

Taking into account that \(u + v + w = u' + v' + w' = 1\), we find that

\[
x x' + y y' = (u, v, w)N \begin{pmatrix}
    u' \\
    v' \\
    w'
\end{pmatrix}, \quad \text{with} \quad N = \begin{pmatrix}
    A^2 & A \cdot B & A \cdot C \\
    A \cdot B & B^2 & B \cdot C \\
    A \cdot C & B \cdot C & C^2
\end{pmatrix}.
\]

(23)

7 The circumcircle of ABC in barycentrics

By the discussion in section 1 we can take the origin of cartesian coordinates anywhere we like. Selecting it at the “circumcenter” \(O\) of the circumcircle of the triangle of reference \(ABC\), we find that

\[
A^2 = B^2 = C^2 = R^2, \quad A \cdot B = R^2 \cos(2\gamma), \quad A \cdot C = R^2 \cos(2\beta), \quad B \cdot C = R^2 \cos(2\alpha),
\]
where \( \{a, \beta, \gamma\} \) the angles of the triangle opposite respectively to \( \{BC, CA, AB\} \) and \( R \) is the circumradius of the triangle of reference \( ABC \). Setting \( \cos(2\gamma) = 1 - 2\sin(\gamma)^2 \), we find

\[
N = R^2 \left( \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array} \right) - 2R^2 \left( \begin{array}{ccc}
0 & \sin(\gamma)^2 & \sin(\beta)^2 \\
\sin(\gamma)^2 & 0 & \sin(\alpha)^2 \\
\sin(\beta)^2 & \sin(\alpha)^2 & 0 \\
\end{array} \right) \implies
\]

\[
N = R^2 \cdot N_1 - \frac{1}{2} N_2 = R^2 \left( \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array} \right) - \frac{1}{2} \left( \begin{array}{ccc}
c^2 & b^2 \\
c^2 & 0 & a^2 \\
0 & b^2 & a^2 \\
\end{array} \right),
\]

(24)

where \( \{a = |BC|, b = |CA|, c = |AB|\} \) the side lengths of the triangle. Taking into account the condition \( \{u + v + w = 1, \ldots\} \) for “absolute barycentric coordinates”, we find that the expression of the inner product \( xx' + yy' \) in terms of the corresponding absolute barycentric coordinates:

\[
xx' + yy' = (u, v, w)N \left( \begin{array}{c}
u' \\
v' \\
w' \\
\end{array} \right) = R^2 - \frac{1}{2}(c^2(uv' + u'v) + b^2(wu' + w'u) + a^2(vw' + v'w)).
\]

(25)

Having selected the center of coordinates at the circumcenter of \( ABC \), the equation of the “circumcircle” \( \kappa \) of \( ABC \) in barycentric coordinates can be found from the corresponding equation in cartesian coordinates:

\[
R^2 = x^2 + y^2 = R^2 - (c^2uv + b^2wu + a^2vw) \implies
\]

\[
a^2vw + b^2wu + c^2uv = 0.
\]

(26) (27)

8 Displacement vectors, inner product, distance

Displacement vectors are differences of absolute barycentric coordinates of two points

\[
(p, q, r) = (u, v, w) - (u', v', w').
\]

Since \( u + v + w = u' + v' + w' = 1 \), they satisfy

\[
p + q + r = 0
\]

and represent points at infinity. Also through their corresponding cartesian coordinate vectors, defined by equation (19), they represent the usual displacement from one point to the other. The inner product of the corresponding cartesian coordinate vectors can be calculated using the same bilinear form (24). Thus, introducing the vectorial notation \( \overline{\xi} = (x, y) \) and \( \overline{P} = (p, q, r) \) for cartesian, respectively barycentric coordinates, and also denoting by \( \overline{\xi_P} = M\overline{P} \), we have:

\[
\overline{\xi_{PQ}} = \overline{\xi_Q} - \overline{\xi_P} = M(\overline{Q} - \overline{P}) = M \cdot \overline{PQ}
\]

and from this, the expression of the usual inner product:

\[
\overline{\xi_{PQ}} \cdot \overline{\xi_{P'Q'}} = \overline{PQ} \cdot N \cdot \overline{P'Q'} = -\frac{1}{2} (p, q, r) \left( \begin{array}{ccc}
c^2 & b^2 \\
c^2 & 0 & a^2 \\
b^2 & a^2 & 0 \\
\end{array} \right) \left( \begin{array}{c}
p' \\
q' \\
r' \\
\end{array} \right),
\]

where we set \( \overline{PQ} = Q - P = (p, q, r) \) and \( \overline{P'Q'} = Q' - P' = (p', q', r') \). Expanding this, we get:

\[
\overline{\xi_{PQ}} \cdot \overline{\xi_{P'Q'}} = -\frac{1}{2} [a^2(qr' + q'r) + b^2(rp' + r'p) + c^2(pq' + p'q)].
\]

(28)
This expresses the fundamental relation allowing the computation of euclidean distances and angles in terms of barycentric coordinates. Thus, for example, the distance of two points represented in \textit{absolute} barycentric coordinates \( \{P, Q\} \) can be expressed through their “displacement vector” \( \overrightarrow{PQ} = (p, q, r) \) and the formula:

\[
|PQ|^2 = -a^2 qr - b^2 rp - c^2 pq .
\] (29)

Introducing the “Conway symbols”

\[
S_A = (b^2 + c^2 - a^2)/2, \quad S_B = (c^2 + a^2 - b^2)/2, \quad S_C = (a^2 + b^2 - c^2)/2,
\]

where \( \{a, b, c\} \) are the side-lengths of the triangle of reference, and calculating the expression on the right of equation \( \text{28} \) we find, using the conditions \( p + q + r = p' + q' + r' = 0 \), that it is equivalent to

\[
\overline{PQ} \cdot \overline{P'Q'} = S_A p p' + S_B q q' + S_C r r'.
\] (30)

Consequently, formula \( \text{29} \) is also equivalent to:

\[
|PQ|^2 = S_A p^2 + S_B q^2 + S_C r^2 .
\] (31)

\textbf{Remark-1} \hspace{1em} It is easy verifiable that the “Conway symbols” satisfy the conditions:

\[
S_B + S_C = a^2, \quad S_C + S_A = b^2, \quad S_A + S_B = c^2, \quad S_A S_B + S_B S_C + S_C S_A = S^2, \quad S_A S_B + c^2 S_C = S_B S_C + a^2 S_A = S_C S_A + b^2 S_B = S^2. \quad S^2
\]

where \( S \) denotes twice the area of the triangle of reference \( ABC \) ([Yiu13, p.33]).

More general, for an angle of measure \( \phi \) we can define the compatible to the previous symbol

\[
S_\phi = S \cdot \cot(\phi),
\] (36)

used in several formulas involving calculations with barycentrics ([Yiu13]). More on this can be found in the file \textit{Conway triangle symbols}.

\textbf{Theorem 3.} Two displacement vectors \( \{(p, q, r), (p', q', r')\} \) are orthogonal, if and only if they satisfy the equation:

\[
S_A p p' + S_B q q' + S_C r r' = 0.
\] (37)

\section{Orthogonality of lines}

The parametric form of a line represented in barycentrics by the equation

\[
pu + qv + rw = 0,
\]

can be given in the form

\[
(u, v, w) = (q, -p, 0) + t(r - q, p - r, q - p).
\]

The last vector on the right is a formal “displacement vector”, i.e. it satisfies the equation \( u + v + w = 0 \), and represents in barycentrics the direction of the line. Notice that it is also
10 Distance of a point from a line

the “point at infinity” of that line, i.e. its intersection with the “line at infinity”, represented in barycentrics by the equation \( u + v + w = 0 \).

Thus, two lines \( \{pu + qv + rw = 0, p'u + q'v + r'w = 0\} \) are orthogonal, precisely when their “directions”

\[
\begin{align*}
(p_d, q_d, r_d) &= (r - q, p - r, q - p), \\
(p'_d, q'_d, r'_d) &= (r' - q', p' - r', q' - p')
\end{align*}
\]

are orthogonal displacements, i.e. they satisfy equation (37). Latter equation, together with the valid for points at infinity \( u + v + w = 0 \), imply the explicit form of the “direction” of the orthogonal line:

\[
(p'_d, q'_d, r'_d) = \begin{pmatrix}
S_B \cdot q_d - S_C \cdot r_d \\
S_C \cdot r_d - S_A \cdot p_d \\
S_A \cdot p_d - S_B \cdot q_d
\end{pmatrix}
\]

Alternatively to this, i.e. using the line-coefficients themselves and not their differences, we arrive after a short calculation at the theorem ([Lon91, II,p.57]):

**Theorem 4.** The lines \( \varepsilon : px + qy + rz = 0 \) and \( \varepsilon' : p'x + q'y + r'z = 0 \) are orthogonal if and only if their coefficients satisfy the equation

\[
a^2p' + b^2q' + c^2r' - S_A(qr' + q'r) - S_B(rp' + r'p) - S_C(pq' + p'q) = 0.
\]

**Remark-1** The bilinear form involved in equation (39) is degenerate, since the corresponding matrix \( H \) satisfies:

\[
(1,1,1)H = (1,1,1) \begin{pmatrix}
-a^2 & S_C & S_B \\
S_C & -b^2 & S_A \\
S_B & S_A & -c^2
\end{pmatrix} = 0.
\]

**Remark-2** Last equation conforms to the fact that if lines \( \{\varepsilon = 0, \varepsilon' = 0\} \) are orthogonal the same is true for \( \{\varepsilon + k \cdot \varepsilon_{\infty}, \varepsilon'\} \), where \( \varepsilon_{\infty} \) is the line at infinity. By section 3-nr-5 the lines \( \{\varepsilon + k \cdot \varepsilon_{\infty} = 0\} \) represent, for variable \( k \), all the parallels to line \( \varepsilon = 0 \).

As an application of the orthogonality of displacement vectors we obtain the equation of the circle having for diameter the segment defined by the points

\[
U_1 = (u_1, v_1, w_1) \text{ and } U_2 = (u_2, v_2, w_2).
\]

Its points \( U(u,v,w) \) are characterized by the orthogonality of the displacement vectors \( \{UU_1, UU_2\} \) implying the theorem:

**Theorem 5.** The points \( U(u,v,w) \) of the circle on diameter \( U_1U_2 \), all points expressed in absolute barycentrics, satisfy the equation:

\[
S_A(u - u_1)(u - u_2) + S_B(v - v_1)(v - v_2) + S_C(w - w_1)(w - w_2) = 0.
\]

10 Distance of a point from a line

Representing the coordinates by \( \bar{U} = (u,v,w) \), the “direction” of the line

\[
\varepsilon : f(\bar{U}) = pu + qv + rw = 0
\]

is its intersection

\[
(r - q, p - r, q - p)
\]
with the line at infinity as explained in section 9. The orthogonal direction to this can be calculated using theorem 3 and leads to a multiple of the displacement vector $\overline{V}$:

$$(v_1 = S_B(p - r) - S_C(q - p), \ v_2 = S_C(q - p) - S_A(r - q), \ v_3 = S_A(r - q) - S_B(p - r)).$$

Which in matrix form is represented by the equation:

$$\overline{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 & S_B & -S_C \\ -S_A & 0 & S_C \\ S_A & -S_B & 0 \end{pmatrix} \begin{pmatrix} r - q \\ p - r \\ q - p \end{pmatrix}. \tag{44}$$

The line $\varepsilon'$, passing through the point $\overline{U}_0 = (u_0, v_0, w_0)$, and orthogonal to $\varepsilon$ has the parametric representation

$$\varepsilon': \quad \overline{U}_t = \overline{U}_0 + t \overline{V}.$$

Its intersection point $\overline{U}_t$ with $\varepsilon$ satisfies the equation of line $\varepsilon$:

$$f(\overline{U}_0 + t \overline{V}) = 0 \quad \Leftrightarrow \quad f(\overline{U}_0) + tf(\overline{V}) = 0 \quad \Leftrightarrow \quad t = -\frac{f(\overline{U}_0)}{f(\overline{V})}.$$

Taking into account equation (44), we find that

$$f(\overline{V}) = (p, q, r) \begin{pmatrix} 0 & S_B & -S_C \\ -S_A & 0 & S_C \\ S_A & -S_B & 0 \end{pmatrix} \begin{pmatrix} r - q \\ p - r \\ q - p \end{pmatrix} = S_A(r - q)^2 + S_B(p - r)^2 + S_C(q - p)^2. \tag{45}$$

Thus, the displacement vector $\overline{U}_t - \overline{U}_0$ and its length is, according to equation (29):

$$\overline{U}_t - \overline{U}_0 = -\frac{f(\overline{U}_0)}{f(\overline{V})} \cdot \overline{V} \quad \Rightarrow \quad |\overline{U}_t - \overline{U}_0|^2 = \frac{f(\overline{U}_0)^2}{f(\overline{V})^2} (-a^2 v_2 v_3 - b^2 v_3 v_1 - c^2 v_1 v_2). \tag{46}$$

The parenthesis on the right is

$$V^2 = -a^2 v_2 v_3 - b^2 v_3 v_1 - c^2 v_1 v_2 = -\frac{1}{2} (v_1, v_2, v_3) \begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

which, taking into account equation (44), leads to

$$V^2 = -\frac{1}{2} (r - q, p - r, q - p) \cdot K \cdot \begin{pmatrix} r - q \\ p - r \\ q - p \end{pmatrix}, \tag{47}$$

where $K$ is the matrix

$$K = \begin{pmatrix} 0 & -S_A & S_A \\ S_B & 0 & -S_B \\ -S_C & S_C & 0 \end{pmatrix} \begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & S_B & -S_C \\ -S_A & 0 & S_C \\ S_A & -S_B & 0 \end{pmatrix}.$$
Carrying out the computation we find that

\[ K = 2 \begin{pmatrix} -a^2 S_A^2 & S_A S_B S_C & S_A S_B S_C \\ S_A S_B S_C & -b^2 S_B^2 & S_A S_B S_C \\ S_A S_B S_C & S_A S_B S_C & -c^2 S_C^2 \end{pmatrix}. \tag{48} \]

This, using the relations in section 9 of the Conway symbols becomes

\[-a^2 S_A^2 + S_A S_B S_C = S_A(S_B S_C - a^2 S_A) = S_A(S_B S_C - (S_B + S_C)S_A) = S_A(2S_B S_C - (S_B S_C + S_C S_A + S_A S_B)) = 2S_A S_B S_C - S_A S^2 \Rightarrow -a^2 S_A^2 = S_A S_B S_C - S_A S^2.\]

Thus, matrix \( K \) can be written in the form

\[ \frac{K}{2} = S_A S_B S_C \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - S^2 \begin{pmatrix} S_A & 0 & 0 \\ 0 & S_B & 0 \\ 0 & 0 & S_C \end{pmatrix}. \tag{49} \]

Introducing this into equation (47) we find that

\[ V^2 = S^2 (S_A(r-q)^2 + S_B(p-r)^2 + S_C(q-p)^2). \tag{50} \]

This introduced to equation (46) leads finally to the expression for the distance of the point \( \overline{U}_0 \) from the line \( e : f(\overline{U}) = pu + qv + rw = 0 \):

**Theorem 6.** The distance \( \text{dist}(U_0, e) \) of the point \( \overline{U}_0 \) from the line \( e : f(\overline{U}) = pu + qv + rw = 0 \) is given by the formula:

\[ \text{dist}(U_0, e)^2 = |\overline{U}_0 \overline{U}|^2 = \frac{S^2(pu_0 + qv_0 + rw_0)^2}{S_A(r-q)^2 + S_B(p-r)^2 + S_C(q-p)^2}. \tag{51} \]

Measuring the distance \( \text{dist}(A, e) \) of the vertex \( A(1,0,0) \) from the line \( e \) and doing this also for the other vertices, we find the following property:

**Theorem 7.** The coefficients of the line \( e : pu + qv + rw = 0 \) are proportional to the distances of the vertices of the triangle of reference \( ABC \) from the line:

\[ \frac{\text{dist}(A, e)^2}{p^2} = \frac{\text{dist}(B, e)^2}{q^2} = \frac{\text{dist}(C, e)^2}{r^2} = \frac{S^2}{S_A(r-q)^2 + S_B(p-r)^2 + S_C(q-p)^2}. \tag{52} \]

Another aspect of the ratio on the right is found by considering the “normalized” coefficients of the line, satisfying \( p + q + r = 1 \), and the centroid of the triangle \( G(1,1,1) \). These, replaced into equation (51), lead to:

\[ S_A(r-q)^2 + S_B(p-r)^2 + S_C(q-p)^2 = \frac{S^2}{\text{dist}(G, e)^2}. \tag{53} \]

### 11 Meaning of line coefficients

The meaning of line coefficients, expressed through equation (52), can be explained geometrically and more directly, than it was done in section 10, whose purpose was the
general formula of distance of a point from a line, as described by equation (51). For this it suffices to use the results of section 4. In fact, the intersection points \( \{A', B', C'\} \) of the line \( \varepsilon : \pu + \qv + \rw = 0 \) with the sides triangle of reference (See Figure 8), have corresponding coordinates

\[
A'(0, -r, q), \quad B'(-r, 0, p), \quad C'(-q, p, 0).
\]

And the ratio of the distances \( \{AA'', BB'', CC''\} \) is

\[
\frac{BB''}{CC''} = \frac{A'B}{A'C} = \frac{q}{r} \quad \Rightarrow \quad \frac{BB''}{q} = \frac{CC''}{r}.
\]

Analogously is seen that \( BB''/q = AA''/p \). Notice that in order to have valid equations

\[
\frac{AA''}{p} = \frac{BB''}{q} = \frac{CC''}{r},
\]

the distances must be signed and their signs must be properly chosen.

12 Medial line in barycentrics

The middle of a segment \( AB \) determined by two points \( A(u, v, w) \) and \( B(u', v', w') \) is found by 3-nr-7 to be \( N = (u + v + w)B + (u' + v' + w')A \). The line \( \varepsilon = AB \) is found by 3-nr-6 to have coefficients

\[ \varepsilon : (p, q, r) = A \times B = (vw' - v'w, w'u' - w'u, uv' - u'v). \]

By 3-nr-9 a line with coefficients \( \varepsilon' \) with coefficients \( \varepsilon : (p', q', r') \) orthogonal to \( \varepsilon \) satisfies the equation involving the symmetric matrix \( H \):

\[
(p', q', r')H \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0,
\]

Thus \( \varepsilon' \), seen as a triple satisfying simultaneously \( \varepsilon'(M\varepsilon') = 0 \) and \( \varepsilon' \cdot N = 0 \), is (its coefficients) a multiple of the vector product

\[
\varepsilon' = (\varepsilon M) \times N.
\]
A short computation shows that this is equal to the vector consisting of the minors of the matrix:
\[
\begin{pmatrix}
  a^2(3p - s_x) & b^2(3q - s_x) & c^2(3r - s_x) \\
  s_A u' + s_B u & s_A v' + s_B v & s_A w' + s_B w
\end{pmatrix}
\]
where \( s_X \) represents the sum of the coordinates of \( X \) and \( \{a, b, c\} \) are, as usual, the side-lengths of the triangle of reference. This vector product can be represented in the form
\[
\varepsilon' = [\text{diag}(a^2, b^2, c^2)(3\varepsilon - s_x\varepsilon_{\infty})] \times [s_A B + s_B A],
\]
where \( \text{diag}(a^2, b^2, c^2) \) is the diagonal matrix with these elements, and \( \varepsilon_0 \) is identified with the vector \((1, 1, 1)\) representing the coefficients of the line at infinity. In this formula we can replace line \( \varepsilon \) with a parallel to it \( \varepsilon_0 \) for which \( s_{\varepsilon_0} = 0 \), line \( \varepsilon_0 \) being defined by
\[
\varepsilon_0 = \varepsilon - \frac{s_x}{3}\varepsilon_{\infty}.
\]
Making this modification the coefficients of the medial line of \( AB \) are given by
\[
\varepsilon' = [\text{diag}(a^2, b^2, c^2)\varepsilon_0] \times [s_A B + s_B A].
\]

### 13 Power of a point, general circle, Euler circle

The expression in barycentrics of the “power” of a point \( P(u,v,w) \) relative to a circle, can be found from the corresponding expression in cartesian coordinates. In fact, in cartesian coordinates the power of the point \( P \) w.r. to the circle \( \kappa(O,r) \) is
\[
PO^2 - r^2,
\]
which, using equation (29) for the respective absolute barycentrics \( \{O(u_0,v_0,w_0), P(u,v,w)\} \) becomes:
\[
-(PO^2 - r^2) = a^2(v_0 - v)(w_0 - w) + b^2(u_0 - u)(w_0 - w) + c^2(u_0 - u)(v_0 - v) + r^2.
\]
The equation of the circle \( \kappa(O,r) \) results by equating this to zero:
\[
\begin{align*}
  a^2 v w + b^2 w u + c^2 u v + a^2 v_0 w_0 + b^2 u_0 w_0 + c^2 u_0 v_0 + r^2 \\
  -a^2 (v_0 w + v w_0) - b^2 (w_0 u + w u_0) - c^2 (u_0 v + u v_0) = 0.
\end{align*}
\]
Taking into account that \( u + v + w = 1 \), the two last rows are seen to sum into the linear term:
\[
(r^2 - OA^2)u + (r^2 - OB^2)v + (r^2 - OC^2)w.
\]
Thus the equation of the circle \( \kappa(O,r) \) becomes
\[
a^2 v w + b^2 w u + c^2 u v + (r^2 - OA^2)u + (r^2 - OB^2)v + (r^2 - OC^2)w = 0,
\]
which represents the equation of the circle \( \kappa \) as a sum of the corresponding expression of the circumcircle of the triangle of reference and a certain line. Thus, the difference of the expressions of the two circles is the expression of the “radical line” of the two circles and the following theorem is valid.
Theorem 8. The equation
\[(r^2 - OA^2)u + (r^2 - OB^2)v + (r^2 - OC^2)w = 0 \tag{61}\]
represents the radical axis of the circle \(\kappa(O, r)\) and the circumcircle \(\kappa_0\) of the triangle of reference \(ABC\). The expression of the equation of the general circle in absolute barycentric coordinates is the sum the corresponding expressions of \(\kappa_0\) and this line:
\[a^2vw + b^2wu + c^2uv + ((r^2 - OA^2)u + (r^2 - OB^2)v + (r^2 - OC^2)w)(u + v + w) = 0. \tag{62}\]

Remark-1 The previous equation can be “homogenized” by multiplying with \(1 = x + y + z\), thus leading to the homogeneous equation:
\[a^2vw + b^2wu + c^2uv + ((r^2 - OA^2)u + (r^2 - OB^2)v + (r^2 - OC^2)w)(u + v + w) = 0. \tag{63}\]

As an example application of this equation we consider the “Euler circle” \(\lambda\) of the triangle of reference \(ABC\). The powers of the vertices \(\{A, B, C\}\) w.r. to \(\lambda(N, r)\) are easily seen to be
\[OA^2 - r^2 = AB' \cdot AB'' = bc \cos(\hat{A}) = S_A/2\]
and the analogous expressions resulting by cyclically permuting the letters \(\{A, B, C\}\). Here \(\{B', B''\}\) are respectively the traces on \(AC\) of the median and altitude from \(B\). Thus, equations (62) resp. (63) become in this case
\[2(a^2vw + b^2wu + c^2uv) - (S_Au + S_Bv + S_Cw) = 0 \quad \Leftrightarrow \quad \tag{64}\]
\[a^2vw + b^2wu + c^2uv - (S_Au^2 + S_Bv^2 + S_Cw^2) = 0. \tag{65}\]

Figure 9 illustrates the case showing also the radical axis of the Euler circle and the circumcircle. This is the “orthic axis” of the triangle, identified by theorem 8 with the line
\[S_Au + S_Bv + S_Cw = 0, \tag{66}\]
and coinciding with the trilinear polar of the orthocenter \(H\) of \(ABC\).

14 Centroid, Incenter, Circumcenter, Symmedian point

These remarkable points are the simplest examples of “triangle centers” of the triangle of reference \(ABC\), whose barycentric coordinates can be easily calculated. The first, the “centroid” \(X_2\) in the notation of Kimberling ([Kim18]), is the unit point of the projective
base \( \{A, B, C, G\} \) defining the "barycentrics" and has coordinates \((1, 1, 1)\). The incenter \(X_1\), using the definition of barycentrics through areas, is easily seen to be \((a, b, c)\). The circumcenter \(X_3\) of the triangle \(ABC\) is also calculated using the area definition of barycentrics. In fact, the area of the triangle \(X_3BC\) is
\[
(X_3BC) = \frac{1}{4}BC^2 \cot(\alpha) = \ldots = \frac{R}{4abc} a^2S_A \quad \Rightarrow \quad X_3 = (a^2S_A, b^2S_B, c^2S_C),
\]
with corresponding absolute barycentrics:
\[
X_3 = \frac{1}{2S^2}(a^2S_A, b^2S_B, c^2S_C).
\]
(67)

The symmedian point \(X_6\) (see file Symmedian point of the triangle) is characterized by its property to have distances from the sides analogous to these sides. Thus, the ratio of the areas
\[
\frac{(X_6AB)}{(X_6AC)} = \frac{c^2}{b^2} \quad \Rightarrow \quad X_6 = (a^2, b^2, c^2).
\]

The equalities \(X_k = (u, v, w)\) for barycentrics must be understood in a wider sense and often is used the symbol \(X_k = (u : v : w)\) to stress the fact that the barycentrics are defined up to multiplicative constants, so that only their relative ratios are uniquely defined. Alternatively, we use also the symbol \(X \sim (u, v, w)\).

15 Euler line, Orthocenter, center of Euler’s circle

Having the barycentrics \(\{(1, 1, 1), (a^2S_A, b^2S_B, c^2S_C)\}\) of the centroid and the circumcenter, the Euler line, which passes through them, is easily seen (section 3) to have coefficients
\[
\text{Euler line} : \quad (b^2S_B - c^2S_C)u + (c^2S_C - a^2S_A)v + (a^2S_A - b^2S_B)w = 0.
\]

Its point at infinity \(X_{30}\) i.e. its intersection with the line at infinity \(u + v + w = 0\) is then calculated to be
\[
X_{30} = (2a^2S_A - b^2S_B - c^2S_C, \ldots) = ((2a^4 - (b^2 + c^2)a^2 - (b^2 - c^2)^2, \ldots),
\]

where the dots indicate the remaining coordinates resulting by cyclic permutations of the letters \(\{a, b, c\}\) and \(\{A, B, C\}\). On the Euler line are located some other triangle centers, like the orthocenter \(H\) or \(X_4\) and the "center of the Euler circle" \(E\) or \(X_5\). The barycentrics of

![Euler line diagram](image_url)
the orthocenter are easily found using the ratio rules of section 2 and the well known ratio
$\text{HO}/\text{HG} = 3/2$:

$$
H = 3 \frac{G}{s_G} - 2 \frac{O}{s_O} = (1, 1, 1) - 2 \frac{(a^2 S_A, \ldots)}{s_O}
$$

$$
= \left( 1 - 2 \frac{a^2 S_A}{2S^2}, \ldots \right) = \frac{1}{S^2} (S_BS_C, \ldots) \sim (S_BS_C, \ldots) \Rightarrow
$$

$$
H = (S_BS_C : S_CS_A : S_AS_B).
$$

where $s_X$ is the sum of coordinates of $X$. Analogously are computed the barycentrics of
the center $N = X_5$ of the Euler circle, which is the middle of the segment $HO$:

$$
N \sim s_O \cdot H + s_H \cdot O = 2S^2 \cdot H + S^2 \cdot O \sim (S^2 + S_BS_C, S^2 + S_CS_A, S^2 + S_AS_B). \quad (68)
$$

Remark-1 The expression of the center $N = X_5 \sim (S^2 + S_BS_C, \ldots)$ as a linear combination
of the centers $(G, H)$ defining the Euler line generalizes for all notable triangle centers lying
on that line, which can be also written as linear combinations

$$
X = (m \cdot S^2 + n \cdot S_BS_C, \ldots).
$$

The coefficients $(m, n)$ are called “Shinagawa coefficients” of the triangle center $X$ ([Kim18]).

## 16 Triangle Area in Barycentrics

Consider the triangle of reference $ABC$ and a second one $DEF$, whose vertices have
absolute barycentric coordinates w.r. to $ABC : D(d_1, d_2, d_3), E(e_1, e_2, e_3), F(f_1, f_2, f_3)$. The
basic relations between barycentric and cartesian coordinates have been discussed in section 6. Denoting by $(X_1, X_2)$ the cartesian coordinates of the points $X$ of the plane and
using equation (19), we obtain:

$$
\begin{pmatrix}
D_1 & E_1 & F_1 \\
D_2 & E_2 & F_2 \\
1 & 1 & 1
\end{pmatrix}
= \begin{pmatrix}
A_1 & B_1 & C_1 \\
A_2 & B_2 & C_2 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
d_1 & e_1 & f_1 \\
d_2 & e_2 & f_2 \\
d_3 & e_3 & f_3
\end{pmatrix}. \quad (69)
$$

Taking the determinants, we obtain the relation of the areas

$$
(DEF) = (ABC) \cdot \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix}, \quad (70)
$$

expressing the area $(DEF)$ in terms of the absolute barycentrics of the vertices of the triangle.

As an application, we can easily compute the area of the cevian triangle $A'B'C'$ of a
point $P(x, y, z)$ (see figure 3). The traces of $P$ on the sides of the triangle of reference $ABC$
are $\{A'(0, y, z), B'(x, 0, z), C'(x, y, 0)\}$. With a short calculation we find then

$$
(A'B'C') = \frac{(ABC)}{(y + z)(z + x)(x + y)} \begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & 0 & 0 \end{vmatrix} = \frac{xyzS}{(y + z)(z + x)(x + y)}. \quad (71)
$$

Some more effort in calculation is required for the “pedal” triangle of a point $P(u, v, w)$
Their determinant is seen to be

\[ \det(A',B',C') = (a^2yz + b^2zx + c^2xy)^3. \]  

Figure 11: Pedal triangle \( A'B'C' \) of \( ABC \) w.r. to \( P \)

w.r. to the triangle of reference \( ABC \) (See Figure 11). From the discussion in section 10 we can compute the coordinates of the points \( \{A',B',C'\} \) found to be:

\[
A'(0 , a^2v + uS_C , a^2w + uS_B), \quad B'(b^2u + vS_C , 0 , b^2w + vS_A), \quad (c^2u + wS_B , c^2v + wS_A , 0).
\]

Then, assuming the coordinates in normal form i.e. satisfying \( u + v + w = 1 \), and using equation (34), we find the crucial determinant

\[
\begin{vmatrix}
0 & b^2u + vS_C & c^2u + wS_B \\
a^2v + uS_C & 0 & c^2v + wS_A \\
a^2w + uS_B & b^2w + vS_A & 0
\end{vmatrix} = S^2(a^2vw + b^2wu + c^2uv).
\]  

(72)

This, for general, not necessarily normal barycentric coordinates, implies the formula for the area \( (A'B'C') \) of the pedal of the point \( P(u,v,w) \) w.r. to the triangle of reference:

\[
(A'B'C') = \frac{S^3}{2(u + v + w)^2} \cdot \frac{a^2vw + b^2wu + c^2uv}{a^2b^2c^2}.
\]  

(73)

17 Circumcevian triangle of a point

Given a triangle \( ABC \) and a point \( P \) the “circumcevian” triangle of \( P \) w.r. to \( ABC \) is the triangle \( A'B'C' \) formed by the second intercepts \( \{A',B',C'\} \) of the cevians \( \{AP, BP, CP\} \) of \( P \) with the circumcircle \( \kappa \) of \( ABC \) (See Figure 12). The main property of the circumcevian triangle is:

**Theorem 9.** The circumcevian triangle of \( P \) is similar to the corresponding pedal triangle of \( P \).

Figure 12 shows the way to prove that the two triangles have the same angles.

Returning to computations, the equations of the cevians are:

\[
AA' : -zy + wy = 0, \quad BB' : -xz + zu = 0, \quad CC' : -yu + xv = 0.
\]  

(74)

The points \( \{A',B',C'\} \) are calculated to be:

\[
A' = \left( \frac{-a^2yz}{b^2yz + c^2y^2}, \frac{a^2xz + c^2x^2}{c^2yz + b^2z^2} \right), \quad B' = \left( \frac{a^2xy + b^2x^2}{b^2yz + c^2y^2}, \frac{-b^2xz}{c^2xz + a^2z^2} \right), \quad C' = \left( \frac{b^2xy + a^2y^2}{c^2yz + b^2z^2}, \frac{-c^2xy}{c^2xz + a^2z^2} \right).
\]  

(75)
The determination of the area \((A'B'C')\) requires the division with the product \(s_{A'}s_{B'}s_{C'}\) of the sums of the coordinates of these points:

\[
\begin{align*}
s_{A'} &= (b^2 + a^2 - c^2)yz + b^2z^2 + c^2y^2 = S_A(y + z)^2 + S_By^2 + S_Cz^2, \\
s_{B'} &= (c^2 + a^2 - b^2)zx + c^2x^2 + a^2z^2 = S_B(z + x)^2 + S_Cz^2 + S_Ax^2, \\
s_{C'} &= (a^2 + b^2 - c^2)xy + a^2y^2 + b^2x^2 = S_C(x + y)^2 + S_Ax^2 + S_By^2.
\end{align*}
\]

18 Circle through three points

The expression in barycentrics of the circle passing through three points \(\{P_i(u_i,v_i,w_i)\}\), can be found from the corresponding expression in cartesian coordinates, using the transformation \(X_i = MP_i\) of section 6. In fact, using cartesian coordinates \(\{X_i = (x_i,y_i)\}\), the circle through these points is represented by the equation:

\[
0 = \begin{vmatrix}
X_1^2 & X_2^2 & X_3^2 & X^2 \\
x_1 & x_2 & x_3 & x \\
y_1 & y_2 & y_3 & y \\
1 & 1 & 1 & 1 \\
\end{vmatrix}
\]

\[
= X_1^2\left|MP_2,MP_3,MP\right| - X_2^2\left|MP_1,MP_3,MP\right| + X_3^2\left|MP_1,MP_2,MP\right| - X^2\left|MP_1,MP_2,MP_3\right|
\]

\[
= \left( X_1^2|P_2,P_3,P| - X_2^2|P_1,P_3,P| + X_3^2|P_1,P_2,P| - X^2|P_1,P_2,P_3| \right) |M| \iff X_1^2|P_2,P_3,P| + X_2^2|P_3,P_1,P| + X_3^2|P_1,P_2,P| - X^2|P_1,P_2,P_3| = 0. \quad (77)
\]

Here \(|M|\) denotes the determinant of the matrix \(M\) and \(|P,Q,R|\) denotes the determinant of the matrix of the columns of absolute barycentric coordinates vectors \(\{P,Q,R\}\). By equation (25) the inner products are:

\[X_i^2 = R^2 - (a^2v_i w_j + b^2w_i u_j + c^2u_i v_j).\]
Replacing with this in equation (77), we see that the terms involving $R^2$ factor into

$$|P_2, P_3, P| + |P_3, P_1, P| + |P_1, P_2, P| - |P_1, P_2, P_3| = 0,$$

because this is the determinant of the matrix having two equal rows:

$$
\begin{vmatrix}
1 & 1 & 1 & 1 \\
x_1 & x_2 & x_3 & x \\
y_1 & y_2 & y_3 & y \\
1 & 1 & 1 & 1
\end{vmatrix} = 0.
$$

From this follows that the circle through the three points is described by the equation:

$$
(a^2v_1w_1 + b^2w_1u_1 + c^2u_1v_1)|P_2, P_3, P| \\
+ (a^2v_2w_2 + b^2w_2u_2 + c^2u_2v_2)|P_3, P_1, P| \\
+ (a^2v_3w_3 + b^2w_3u_3 + c^2u_3v_3)|P_1, P_2, P| \\
- (a^2vw + b^2wu + c^2uv)|P_1, P_2, P_3| = 0.
$$

(78)

Notice that the three first rows are the expressions of line equations, and the determinants $\{|P_2, P_3, P| = 0, \ldots \}$ represent respectively the side-lines of the triangle $P_1P_2P_3$. Since the expression in the last row is the one of the equation of the circumcircle of $ABC$, the linear part of the first three rows represents the equation of the radical axis of the circle $\kappa$ through the three points and the circumcircle $\kappa_0$ of the triangle of reference $ABC$. Thus, we have the theorem

**Theorem 10.** The equation

$$
(a^2v_1w_1 + b^2w_1u_1 + c^2u_1v_1)|P_2, P_3, P| \\
+ (a^2v_2w_2 + b^2w_2u_2 + c^2u_2v_2)|P_3, P_1, P| \\
+ (a^2v_3w_3 + b^2w_3u_3 + c^2u_3v_3)|P_1, P_2, P| = 0,
$$

(79)

represents the radical axis of the circle $\kappa$ through the three points $\{P_1, P_2, P_3\}$ expressed in absolute barycentrics and the circumcircle $\kappa_0$ of the triangle of reference $ABC$.

Equation (78) leads to a condition on four points to belong to the same circle by setting $P = P_4$. The equation can be expressed also through a $4 \times 4$ determinant using the non-normalized barycentrics of the points. Doing the calculation and some simplifications the formula takes the form:

$$
\begin{vmatrix}
p_{11} & p_{12} & p_{13} & s(P_1) \\
p_{21} & p_{22} & p_{23} & s(P_2) \\
p_{31} & p_{32} & p_{33} & s(P_3) \\
p_{41} & p_{42} & p_{43} & s(P_4)
\end{vmatrix} = 0,
$$

(80)

where in each row appear the barycentrics, not necessarily normalized, of the corresponding point and the expression

$$
s(X) = \frac{a^2x_2x_3 + b^2x_3x_1 + c^2x_1x_2}{x_1 + x_2 + x_3}.
$$
19  The associated affine transformation

Here we return in section 6 and the matrix $M$ defining the transformation from barycentrics to cartesian coordinates.

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = M \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$  

The matrix $M$ is invertible and its inverse is

$$M^{-1} = \frac{1}{A_1(B_2 - C_2) + B_1(C_2 - A_2) + C_1(A_2 - B_2)} \begin{pmatrix} B_2 - C_2 & C_1 - B_1 & B_1C_2 - B_2C_1 \\ C_2 - A_2 & A_1 - C_1 & C_1A_2 - C_2A_1 \\ A_2 - B_2 & B_1 - A_1 & A_1B_2 - A_2B_1 \end{pmatrix}.$$  

The matrix $M$ defines an invertible linear transformation $L_M$ (an isomorphism) of $\mathbb{R}^3$ onto itself and the set of all absolute barycentrics satisfying $u + v + w = 1$ represents the plane $\varepsilon$ of $\mathbb{R}^3$, which is orthogonal to the vector $(1,1,1) \in \mathbb{R}^3$ and passes through the point $\frac{1}{3}(1,1,1) \in \mathbb{R}^3$. The image $\varepsilon' = L_M(\varepsilon)$ is the plane of $\mathbb{R}^3$ parallel to the $(x,y) – plane$ through the point $z = 1$. The linear transformation $L_M$ introduces by its restriction on plane $\varepsilon : M = L_M|\varepsilon$ an “affine” transformation between the planes $\{\varepsilon, \varepsilon'\}$:

$$M : \varepsilon \rightarrow \varepsilon', \text{ with } M(u,v,w) = (x,y,1).$$

It is interesting to see some consequences of this interpretation as, for example, the transformation of lines to lines, the preservation of ratios on lines and the preservation of quotients of areas. These are general properties of the affine transformations but can be also deduced here directly for this special case. In fact, a line in $\varepsilon$ is the intersection of $\varepsilon$ with a plane through the origin $\eta$ of $\mathbb{R}^3$: represented by an equation of the form $\eta : pu + qv + rw = 0$. The image-plane $\eta' = M(\eta)$ is found by writing the equation using matrix notation:

$$pu + qv + rw = 0 \iff 0 = (p,q,v)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = (p,q,v)M^{-1}M\begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$  

Hence the image of the plane $\eta$ is the plane represented by the equation

$$\eta' : p'x + q'y + r'z = 0, \text{ where } (p',q',r') = (p,q,r)M^{-1},$$  

and the line of the plane $\varepsilon'$ is the intersection $\varepsilon' \cap \eta'$. In particular, the line at infinity of $\varepsilon$, represented through its intersection with the plane $\eta : u + v + w = 0$, maps to the line at infinity, which is the intersection of the plane $\varepsilon'$ with the plane with coefficients

$$\eta' : (1,1,1)M^{-1} = (0,0,1) \quad \text{i.e the plane} \quad z = 0.$$

Another consequence of the affine property of the transformation $M$ and its inverse is the preservation of ratios along lines. Thus, for two points $\{P,Q\}$ of the plane and their line

$$S(t) = (1-t)P + tQ, \text{ with } r = \frac{t}{t-1} \text{ equal to the signed ratio: } r = \frac{SP}{SQ},$$  

where
the corresponding barycentrics satisfy the same relation:

\[ M^{-1}S(t) = M^{-1} \begin{pmatrix} S_1(t) \\ S_2(t) \\ 1 \end{pmatrix} = M^{-1} \begin{pmatrix} (1-t)P_1 + tQ_1 \\ (1-t)P_2 + tQ_2 \\ 1-t+t \end{pmatrix} = (1-t)(M^{-1}P) + t(M^{-1}Q). \]

Notice that the function \( r = f(t) = t/(t-1) \) has inverse \( f^{-1} = f \), so that given the ratio \( r = SP/SQ \) and taking \( t = r/(r-1) \), and setting \( \{P',Q',\ldots\} \) for the corresponding barycentric vectors of the points \( \{P,Q,\ldots\} \), we have that

\[ S'(t) = (1-t)P' + tQ' \iff S'(r) = \frac{1}{1-r}(P' - rQ'), \]

satisfies precisely the relation \( SP/SQ = r \). If the ratio \( r \) is expressed in the form \( r = m/n \), then for the point \( S_0 \) satisfying this condition and the corresponding barycentrics vectors we obtain ([MP07]):

\[ \frac{S_1P}{S_1Q} = \frac{m}{n} \Rightarrow S'_1 = \frac{1}{n-m}(nP' - mQ'). \]

As an application, we prove the collinearity of the *incenter* \( I(a:b:c) \), centroid \( G(1:1:1) \) and *Nagel point* \( N(b+c-a,\ldots) \) (see file [Nagel point of the triangle](#)). The relation to verify is

\[ GN/GI = r = -2 \Rightarrow t = r/(r-1) = 2/3. \]

Thus, turning to absolute barycentrics by dividing the previous barycentrics vectors with \( s_N = s_I = a + b + c = 2s \), we obtain:

\[
(1-t)\frac{N}{2s} + t\frac{I}{2s} = \frac{1}{6s}(N + 2I) = \frac{1}{6s}((b + c - a) + 2a, \ldots) = \frac{1}{3}(1,1,1) = G.
\]

A third property of the affine transformation \( \mathcal{M} \), already seen in section 16, is the multiplication of areas by a constant, in this case expressed through equation (70).

### 20 Remarks on working with barycentrics

The vectors of barycentric coordinates \( \{U_P,U_Q,U_R,\ldots \in \mathbb{R}^3 \} \) representing the points of the plane \( \{P,Q,R,\ldots \in \mathbb{R}^2 \} \) are not the points we see. What we see are the points of \( \mathbb{R}^2 \). This is clearly understood in the case of the triangle of reference \( ABC \). The barycentrics-vectors representing it are \( \{(1,0,0),(0,1,0),(0,0,1)\} \) and define the vertices of an equilateral triangle lying on the plane \( \varepsilon : u + v + w = 1 \) of \( \mathbb{R}^3 \). The map \( L_M \) of section 19 is the affine transformation mapping the equilateral \( A'B'C' \) onto \( ABC \). By means of it, all properties of the triangle correspond to properties of the equilateral and vice versa. In particular,
Figure 14: We see \( ABC \) and work with \( A'B'C' \)

properties of the equilateral which are preserved by affine transformations map to similar properties of \( ABC \). A characteristic example is the circumcircle \( \kappa' \) of the equilateral \( A'B'C' \), which carries also the symmetrics \( \{A'_1, B'_1, C'_1\} \) of the vertices w.r. to the center \( G' \) of the equilateral (See Figure 14). The map \( L_M \) transforms \( G' \) to the centroid \( G \) of \( ABC \) and the circumcircle of \( A'B'C' \) to the “Steiner outer ellipse” \( \kappa \) of \( ABC \), characterized by the fact to pass through the vertices of \( ABC \) and their symmetrics \( A_1, B_1, C_1 \) w.r. to \( G \), point \( G \) being its center. Similarly, \( L_M \) maps the incircle \( \lambda' \) of \( A'B'C' \) to the “Steiner inner ellipse” \( \lambda \) of the triangle \( ABC \), characterized by its tangency at the middles \( A_0, B_0, C_0 \) of the sides of \( ABC \). The homothety of \( \{\kappa', \lambda'\} \) with center \( G' \) and ratio \( 2 : 1 \) translates to the homothety of \( \{\kappa, \lambda\} \) with center \( G \) and the same ratio.

Invertible affine transformations or “affinities” like \( L_M \), besides the preservation of ratios along lines, map also areas \( \sigma \) to multiples \( k\sigma \), with \( k \) a constant factor. This implies, that points \( P' \) with barycentrics \( (u': v': w') \) w.r. to \( A'B'C' \) map to corresponding points \( P = L_M(P') \) with the same barycentrics \( (u : v : w) = (u' : v' : w') \) w.r. to \( ABC \). Thus, for example, the circumcircle of the equilateral \( A'B'C' \) with side \( a = |B'C'| \), whose equation, according to section 7 is

\[
a^2v'w' + a^2w'u' + a^2u'v' = 0 \quad \Leftrightarrow \quad v'w' + w'u' + u'v' = 0,
\]

maps to the Steiner outer ellipse, which in barycentrics w.r. to \( ABC \) must satisfy the same equation:

\[
vw + wu + uv = 0. \tag{86}
\]

On the other hand, the incircle of the equilateral \( A'B'C' \), which, according to section 13, is represented in barycentrics w.r. to \( A'B'C' \) by the equation

\[
a^2(vw + wu + vw) - \frac{a^2}{4}(u + v + w)^2 = 0 \quad \Leftrightarrow \quad u^2 + v^2 + w^2 - 2(vw + wu + uv) = 0,
\]

maps to the Steiner inner ellipse, which in barycentrics w.r. to \( ABC \) is represented by the same equation:

\[
u^2 + v^2 + w^2 - 2(vw + wu + uv) = 0. \tag{87}
\]

Bibliography


**Related topics**

1. Cross Ratio
2. Ceva’s theorem
3. Conway triangle symbols
4. Menelaus’ theorem
5. Nagel point of the triangle
6. Projective line
7. Projective plane
8. Symmedian point of the triangle