

Barycentric coordinates or Barycentrics

A file of the [Geometrikon](#) gallery by [Paris Pamfilos](#)

The artist finds a greater pleasure in painting
than in having completed the picture.

Seneca, Letter to Lucilius

Contents

(Last update: 12-04-2026)

| | |
|---|----|
| 1 Preliminaries and definition | 2 |
| 2 Traces, ratios, harmonic conjugation | 4 |
| 3 Lines in barycentric coordinates | 4 |
| 4 Ceva's and Menelaus' theorems in barycentrics | 6 |
| 5 Tripolar and tripole in barycentrics | 7 |
| 6 Relation to Cartesian coordinates, inner product | 9 |
| 7 The circumcircle of ABC in barycentrics | 10 |
| 8 Displacement vectors, inner product, distance | 11 |
| 9 Orthogonality of lines, orthocentroidal circle | 13 |
| 10 Distance from line, bisectors, distance of parallels | 14 |
| 11 Meaning of line coefficients | 17 |
| 12 Power of a point, general circle, Euler circle | 18 |
| 13 Centroid, Incenter, Circumcenter, Symmedian point | 22 |
| 14 Euler line, Orthocenter, center of Euler's circle | 22 |
| 15 Triangle Area in Barycentrics | 23 |
| 16 Circumcevian triangle of a point | 25 |
| 17 Circle through three points, Brocard circle | 25 |
| 18 Barycentrics relative to two distinct triangles | 27 |
| 19 The associated affine transformation | 28 |
| 20 Relations between barycentrics and Cartesian coordinates | 30 |
| 21 Affine transformations represented in barycentrics | 32 |
| 22 Polar, self-conjugate, de Longchamps circles | 34 |
| 23 Remarks on working with barycentrics | 38 |
| 24 The Steiner point of the triangle | 40 |
| 25 Barycentric product | 40 |

1 Preliminaries and definition

“Barycentric coordinates” or “barycentrics” belong to the geometry of the “affine plane” ([6, p.191]), which deals mainly with coincidences of points and lines, ratios of lengths of collinear points, ratios of areas, cross ratios etc. Distances of points and more general metrical properties of shapes belong to Euclidean geometry, but can be handled also in the frame of this geometry, though some times the corresponding formulas appear more complicated than the corresponding ones of Euclidean geometry. Below we’ll have abundant examples of this.

In the subsequent discussion points $\{A, B, \dots\}$ of the plane are identified with two dimensional vectors $\{(a_1, a_2), (b_1, b_2) \dots\}$. Next properties are easily verified ([23, p.30]):

1. Three points of the plane $\{A, B, C\}$ are collinear, if and only, there are three numbers $\{x, y, z\}$ not all zero, such that

$$xA + yB + zC = 0 \quad \text{and} \quad x + y + z = 0.$$

2. Let the points of the plane $\{A, B, C\}$ be non-collinear. Then, for every other point D of the plane there are unique numbers $\{x, y, z\}$ with

$$x + y + z = 1 \quad \text{and} \quad D = xA + yB + zC.$$

3. For non-collinear points $\{A, B, C\}$ the point D , defined by the equation

$$OD = x \cdot OA + y \cdot OB + z \cdot OC, \quad \text{with} \quad x + y + z = 1, \quad (1)$$

is independent of the choice of origin O and depends only on $\{x, y, z\}$.

4. The numbers $\{x, y, z\}$ with $x + y + z = 1$ expressing D in the preceding equation are equal to the quotients of signed areas:

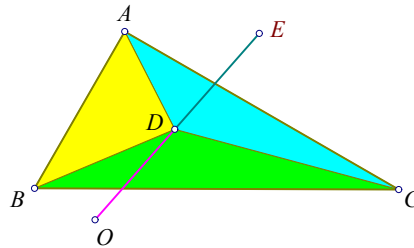


Figure 1: Barycentric coordinates $(x : y : z)$ of D

$$x = \frac{\text{Area}(DBC)}{\text{Area}(ABC)}, \quad y = \frac{\text{Area}(DCA)}{\text{Area}(ABC)}, \quad z = \frac{\text{Area}(DAB)}{\text{Area}(ABC)}. \quad (2)$$

Nr-1. If $x + y + z = 0$ and $xA + yB + zC = 0$, then

$$(-y - z)A + yB + zC = 0 \quad \Rightarrow \quad y(B - A) + z(C - A) = 0$$

i.e. $\{A, B, C\}$ are collinear. For the converse, simply reverse the preceding implications.

Nr-2. Extend the two-dimensional vectors $\{(x, y)\}$ to three dimensional $\{(x, y, 1)\}$ and denote the corresponding space-points by $\{A', B', C', D'\}$. Apply first *nr-1* to see that $\{A', B', C'\}$ are independent. Then express D' in the basis $\{A', B', C'\}$ to find the uniquely defined (x, y, z) as required.

Nr-3. Using another point O' for origin, the same equation and the analogous numbers (x', y', z') to express

$$D : O'D = x' \cdot O'A + y' \cdot O'B + z' \cdot O'C$$

and subtracting we get:

$$\begin{aligned} OD - O'D &= (x \cdot OA + y \cdot OB + z \cdot OC) - (x' \cdot O'A + y' \cdot O'B + z' \cdot O'C) \\ &= (x - x')A + (y - y')B + (z - z')C - (xO - x'O' + yO - y'O' + zO - z'O') \Rightarrow \\ OO' &= O' - O = (x - x')A + (y - y')B + (z - z')C - (O - O') \Rightarrow \\ &(x - x')A + (y - y')B + (z - z')C = 0. \end{aligned}$$

By nr-1 this implies $\{(x - x') = 0, (y - y') = 0, (z - z') = 0\}$, as desired.

Nr-4. Using the independence of (x, y, z) from the position of O we select $O = D$ and use the relation $x = 1 - y - z$ implying

$$\begin{aligned} (1 - y - z)DA + yDB + zDC &= 0 \Rightarrow \\ DA + y(DB - DA) + z(DC - DA) &= 0 \Rightarrow \\ AD &= y(AB) + z'(AC). \end{aligned}$$

Multiplying externally by AB we get

$$AD \times AB = z(AC \times AB) \Leftrightarrow z(AB \times AC) = AB \times AD.$$

Analogous equations result also for y and x . Taking the positive orientation in direction $AB \times AD$ we have the stated result.

The numbers $\{x, y, z\}$ in (2) are called “absolute barycentric homogeneous coordinates” of the point D , relative to the triangle ABC . The multiples $(k \cdot x, k \cdot y, k \cdot z)$ by a constant $k \neq 0$ are called “barycentric homogeneous coordinates” or “barycentrics” of the point D ([3, p.64], [28, p.25], [11], [2]). The signs of the numbers result from the orientations of the corresponding triangles e.g. if $\{D, A\}$ are on the same side of BC then x is positive. Otherwise it is negative and for D on BC it is zero $x = 0$. In fact $x = 0$ characterizes line BC . Analogous properties are valid also for y and z . In the sequel we'll use for barycentrics, besides the notation (x, y, z) , also the symbol $(x : y : z)$.

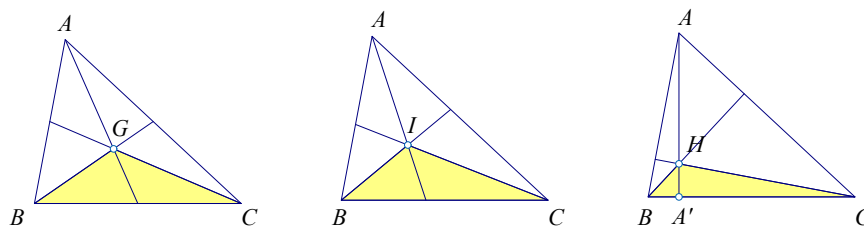
Remark 1. Figure 1 shows a point E defined by an equation as in nr-3, but without the restriction $\sigma = x + y + z = 1$. Point D is then obtained by dividing with σ . Thus, the points $\{D, E\}$ are collinear with O and defined by the equations

$$OE = xOA + yOB + zOC \quad \text{and} \quad OD = \frac{x}{\sigma}OA + \frac{y}{\sigma}OB + \frac{z}{\sigma}OC.$$

The barycentrics of the centroid (see figure 2) result from the fact that the areas of the triangles defined by G are $1/3$ of the area of ABC . This implies that $G = OA + OB + OC$ for any O of the plane. The triangles defined by the incenter I have areas respectively $\{ra/2, rb/2, rc/2\}$, where r the inradius and $\{a = |BC|, b = |CA|, c = |AB|\}$ the side-lengths of the triangle. This implies that $OI = (1/p)(aOA + bOB + cOC)$, with $p = a + b + c$ denoting the perimeter of the triangle.

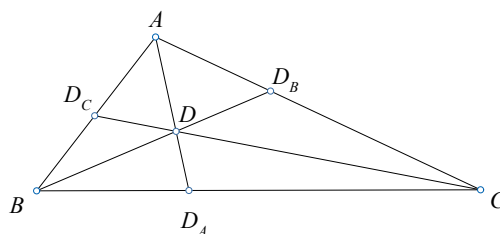
For the orthocenter we have $Area(HBC)/Area(ABC) = \cot(\widehat{B}) \cot(\widehat{C})$ and analogous relations for the triangles $\{HCA, HAB\}$ imply, for any point O of the plane

$$OH = \cot(\widehat{B}) \cot(\widehat{C})OA + \cot(\widehat{C}) \cot(\widehat{A})OB + \cot(\widehat{A}) \cot(\widehat{B})OC.$$

Figure 2: Barycentrics of Centroid G , Incenter I and orthocenter H

2 Traces, ratios, harmonic conjugation

Changing only the variable x , causes $D(x, y, z)$ to move along a fixed line through A . For $x = 0$, we get a point on the line BC : $D_A = yB + zC$, called “trace” of D on the side

Figure 3: Traces D_A, D_B, D_C of D on the sides of ABC

BC of $\triangle ABC$ (see figure 3). Analogously are defined the traces D_B and D_C . The oriented or “signed” ratio (used throughout this exposition [17, §5.14]) $r = D_{AB}/D_{AC}$, calculated using $y + z = 1$, gives the value $r = -(z/y)$.

Theorem 1. Given a point $P = yB + zC$ on the line BC the “harmonic conjugate” of P w.r. (B, C) is $P' = yB - zC$.

This follows from the expression of the ratio $PB/PC = -z/y$ and correspondingly for the conjugate P' : $P'B/P'C = z/y$, from which follows that the “cross ratio” has the value

$$(BC, PP') = (PB/PC) : (P'B/P'C) = -1,$$

proving that $\{P, P'\}$ are harmonic conjugate w.r.t. $\{B, C\}$.

Remark 2. Since any line can be described in “parametric form” $P = yB + zC$ and points $\{B, C\}$ can be complemented by a point A non-collinear with $\{B, C\}$ to a triangle of reference, the preceding property is generally valid for any two points on a line.

Remark 3. The “centroid” G of the triangle ABC , i.e. the intersection-point of its “medians,” defines triangles $\{GBC, GCA, GAB\}$ with equal areas. Thus it has “barycentric homogeneous coordinates” equal to $(1 : 1 : 1)$. This identifies the barycentric coordinates with the “projective homogeneous coordinates” [21] w.r.t. the “projective base” $\{A, B, C, G\}$ with G the “unit” point.

Exercise 1. Show that the point $P = (1 - t)A + tB$ is between the points $\{A, B\}$ when $0 < t < 1$.

3 Lines in barycentric coordinates

The following facts concerning the representation of lines with barycentric coordinates are easily verifiable:

1. A line ε is represented by an equation of the form

$$\varepsilon : ax + by + cz = 0. \quad (3)$$

2. The intersection point of line ε with the line $\varepsilon' : a'x + b'y + c'z = 0$ is given by the “vector product” of the vectors of the coefficients:

$$\varepsilon \cap \varepsilon' = (bc' - cb' : ca' - ac' : ab' - ba'). \quad (4)$$

3. The “line at infinity” is represented by equation (see equation 23)

$$\varepsilon_\infty : x + y + z = 0. \quad (5)$$

4. The point at infinity of the line $\varepsilon : ax + by + cz = 0$ i.e. its intersection with the line at infinity, identified with its “direction”, is given by

$$\varepsilon \cap \varepsilon_\infty = (b - c : c - a : a - b). \quad (6)$$

5. Two lines $\{\varepsilon, \varepsilon'\}$ are parallel if they meet at a point at infinity, hence their coefficients satisfy:

$$bc' - cb' + ca' - ac' + ab' - ba' = 0. \quad (7)$$

6. The line passing through two given points $\{P(a : b : c), P'(a' : b' : c')\}$ has coefficients the coordinates of the vector product and can be expressed by a determinant:

$$\text{line } PP' : (bc' - cb' : ca' - ac' : ab' - ba'), \quad \text{equation: } \begin{vmatrix} a & b & c \\ a' & b' & c' \\ x & y & z \end{vmatrix} = 0. \quad (8)$$

7. The direction of the line passing through the points $\{A = (a : b : c), B = (a' : b' : c')\}$, i.e. its point at infinity is given by the weighted difference of coordinates

$$\sigma_A \cdot (a' : b' : c') - \sigma_B \cdot (a : b : c) \quad \text{with} \quad \sigma_A = a + b + c, \quad \sigma_B = a' + b' + c'. \quad (9)$$

8. The middle M of two points $A(a : b : c)$ and $B(a' : b' : c')$ is found as the harmonic conjugate of the point at infinity of line AB . Applying the preceding calculations it is found to be (the equality being for the triples of coordinates)

$$M = \sigma_A B + \sigma_B A = (a + b + c)B + (a' + b' + c')A. \quad (10)$$

9. The line η parallel from $A(a : b : c)$ to the given line $\varepsilon : a'x + b'y + c'z = 0$ is the line joining A with the point at infinity of ε , which is $(b' - c' : c' - a' : a' - b')$. Thus the equation of this line is

$$\begin{vmatrix} b' - c' & c' - a' & a' - b' \\ a & b & c \\ x & y & z \end{vmatrix} = 0,$$

and the coefficients of this line, written in the form $px + qy + rz = 0$, are given by:

$$(p, q, r) = (a + b + c)(a', b', c') - (aa' + bb' + cc')(1, 1, 1). \quad (11)$$

Thus, the line η can be written as a linear combination of the given line ε and the line at infinity ε_∞ , in the form

$$\begin{aligned} \eta : (a + b + c)\varepsilon - (aa' + bb' + cc')\varepsilon_\infty &= 0 \quad \Leftrightarrow \\ \eta : (a + b + c)(a'x + b'y + c'z) - (aa' + bb' + cc')(x + y + z) &= 0. \end{aligned} \quad (12)$$

4 Ceva's and Menelaus' theorems in barycentrics

Three points $\{A', B', C'\}$ on the sides of the triangle of reference ABC are the "traces" of a point $P(x, y, z)$ if and only if they can be written in the form (see figure 4):

$$A'(0, y, z), \quad B'(x, 0, z), \quad C'(x, y, 0). \quad (13)$$

Their ratios on the triangle's sides satisfy then the equation:

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = -1. \quad (14)$$

As in the preceding section, this follows from the expressions of the ratios

$$\frac{A'B}{A'C} = -\frac{z}{y}, \quad \frac{B'C}{B'A} = -\frac{x}{z}, \quad \frac{C'A}{C'B} = -\frac{y}{x}.$$

Multiplying the three ratios gives the value -1 . The inverse is also obvious. If the product of the three ratios is -1 , then $\{x, y, z\}$ can be found such that $\{A', B', C'\}$ have the shown coordinates. This defines uniquely $P(x : y : z)$. For another more conventional view of Ceva's theorem look at the file [14].

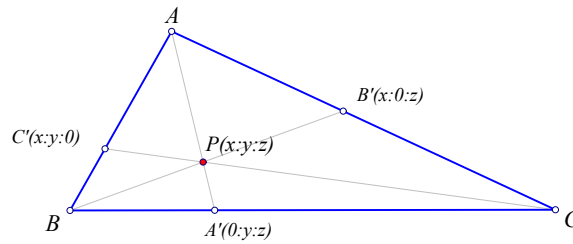


Figure 4: Ceva's theorem

It is obvious, that the points $\{A', B', C'\}$ on the sides of the triangle of reference ABC are on a line $px + qy + rz = 0$, if and only if they can be written in the form:

$$(0 : -r : q), \quad (r : 0 : -p), \quad (-q : p : 0).$$

Their ratios satisfy then the equation:

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1. \quad (15)$$

This follows from the expression of the ratio $DB/DC = -z/y$, for $D = yB + zC$, seen in section 2. By this $A' = -rB + qC \Rightarrow A'B/A'C = q/r$ etc. (see figure 5). For another more conventional view of Menelaus' theorem look at the file [18].

Exercise 2. Show that the points $\{X(0 : x_2 : x_3), Y(y_1 : 0 : y_3), Z(z_1 : z_2 : 0)\}$ on the sides of the triangle ABC define three lines $\{AX, BY, CZ\}$ passing through the same point, if and only if next determinant vanishes:

$$\begin{vmatrix} 0 & -x_3 & x_2 \\ y_3 & 0 & -y_1 \\ -z_2 & z_1 & 0 \end{vmatrix} = x_3y_1z_2 - x_2y_3z_1 = 0. \quad (16)$$

Hint: Determine the coefficients of the three lines $\{AX, BY, CZ\}$ and compute the determinant of their coefficients.

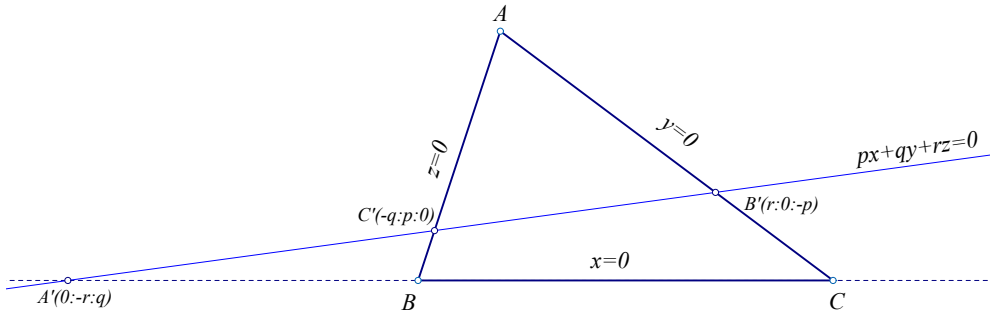


Figure 5: Menelaus' theorem

Exercise 3. Show that the points $\{X(0 : x_2 : x_3), Y(y_1 : 0 : y_3), Z(z_1 : z_2 : 0)\}$ on the sides of the triangle ABC are collinear if and only if next determinant vanishes:

$$\begin{vmatrix} 0 & x_2 & x_3 \\ y_1 & 0 & y_3 \\ z_1 & z_2 & 0 \end{vmatrix} = x_3y_1z_2 + x_2y_3z_1 = 0. \quad (17)$$

5 Tripolar and tripole in barycentrics

In this section the notion of “cross ratio” $(AB, CD) = \frac{CA}{CB} : \frac{DA}{DB}$ of four collinear points is needed and the associated notion of harmonic conjugate points $\{C, D\}$ w.r.t. $\{A, B\}$, defined by the condition $(AB, CD) = -1$. When four points satisfy the last condition we say D is “harmonic conjugate” to C w.r.t $\{A, B\}$ and write $D = C(AB)$. More on this can be found in the file [13].

The “trilinear polar” or “tripolar” ε_P of the point P w.r.t. the triangle ABC is the line containing the “harmonic conjugates” of the traces $\{A', B', C'\}$ of P on the sides of ABC :

$$A'' = A'(BC), \quad B'' = B'(AC), \quad C'' = C'(BA).$$

That these three points $\{A'', B'', C''\}$ are on a line follows by writing $P = uA + vB + wC$. Then the traces are given by $\{A' = vB + wC, P_B = wC + uA, P_C = uA + vB\}$. By section 2 the harmonic conjugates of the traces are then

$$A'' = vB - wC, \quad B'' = wC - uA, \quad C'' = uA - vB$$

and satisfy $A'' + B'' + C'' = 0$ showing that C'' is on the line of $A''B''$ (see figure 6).

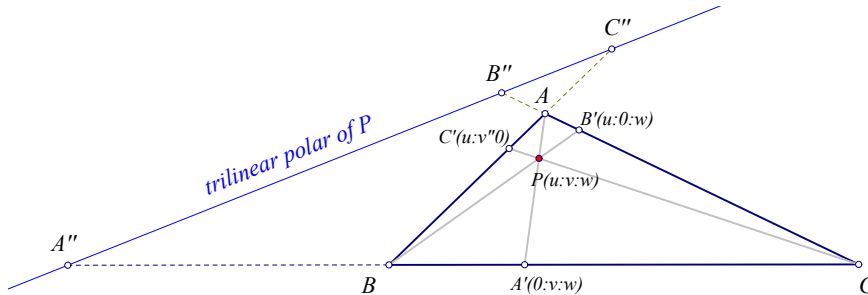


Figure 6: Trilinear polar of P

Alternatively, the barycentrics of $\{A'', B'', C''\}$ are respectively

$$A''(0 : v : -w), \quad B''(-u : 0 : w), \quad C''(u : -v : 0),$$

which up to a multiplicative factor can be written in the form

$$A''(0 : 1/w : -1/v), \quad B''(-1/w : 0 : 1/u), \quad C''(1/v : -1/u : 0),$$

implying that they are points of the line described by the next theorem:

Theorem 2. For every point $P(u : v : w)$ of the plane, different from the vertices of the triangle of reference ABC , the corresponding "tripolar" $tr(P)$ is the line described by the equation:

$$(vw)x + (wu)y + (uv)z = 0 \iff \frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0. \quad (18)$$

Conversely, every line $\varepsilon : pu + qv + rw = 0$ of the plane, not passing through a vertex of ABC is the tripolar $tr(P)$ of the point

$$P = (qr : rp : pq) \iff P = \left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r} \right), \quad (19)$$

which is called "tripole" of the line ε and denoted with the symbol $tr(\varepsilon)$

Remark 4. The map $P \mapsto tr(P)$ is one-to-one for all points of the plane except those lying on the side-lines of the triangle of reference ABC . For these points P , if they are different from a vertex and lie on a sideline ε , the corresponding line $tr(P)$ coincides with ε . For the vertices of ABC the trilinear polar cannot be uniquely defined.

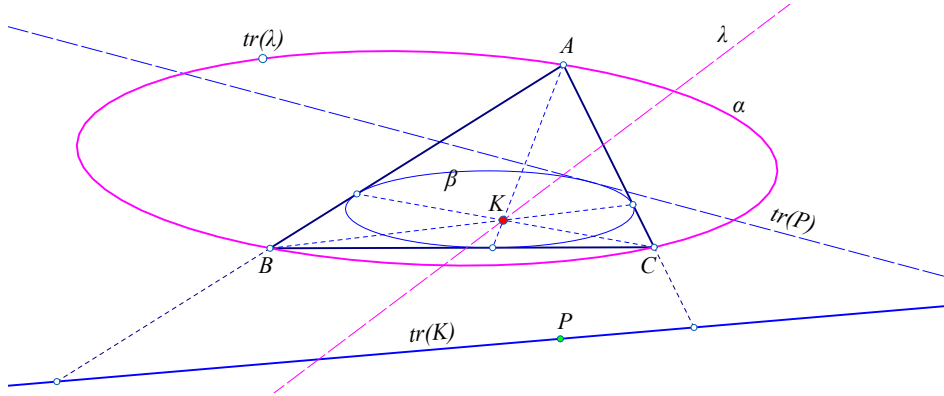


Figure 7: Conics created from a point K and its tripolar $tr(K)$

Remark 5. The map $P \mapsto tr(P)$ is also a prominent so-called "quadratic transformation" ([8, p.329], [7, II, pp.1-23]), useful in "triangle geometry", since it is at the basis of simple procedures to create conics inscribed and circumscribed in a triangle from lines and points of the plane. Figure 7 shows how this is done.

Taking an arbitrary point K , not lying on the sidelines of the triangle ABC , we consider all lines $\{\lambda\}$ through K . Their tripoles $tr(\lambda)$ describe then a conic circumscribing the triangle, a "circumconic" α .

Taking then the tripolar line $tr(K)$ we consider all points $\{P\}$ of $tr(K)$. Their tripolars $\{tr(P)\}$ are then lines enveloping a conic inscribed in the triangle, an "inconic" β .

For both conics point K is called "perspector" and the line $tr(P)$ is called "perspectrix" of the conic. $\{\alpha, \beta\}$ are considered as "dual" to each other, and it is easily seen that their equations in barycentrics can be expressed through the barycentric coordinates of $K(k_1 : k_2 : k_3)$:

$$\text{circumconic } \alpha : k_1 yz + k_2 zx + k_3 xy = 0, \quad (20)$$

$$\text{inconic } \beta : \frac{x^2}{k_1^2} + \frac{y^2}{k_2^2} + \frac{z^2}{k_3^2} - 2\frac{xy}{k_1 k_2} - 2\frac{yz}{k_2 k_3} - 2\frac{zx}{k_3 k_1} = 0. \quad (21)$$

Remark 6. Using the definition of the trilinear polar, it is easy to see, that for the points P of a fixed line ε passing through a vertex, C say, of the triangle of reference ABC , the corresponding trilinear polar $tr(P)$ passes through the “harmonic conjugate” point

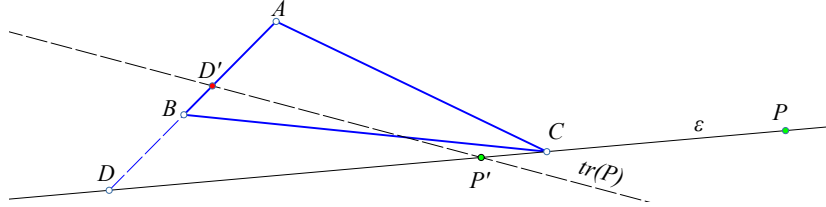


Figure 8: Trilinear polars for $P \in \varepsilon \ni C$ pass all through $D' = D(AB)$

$D' = D(AB)$, where $D = \varepsilon \cap AB$ (see figure 8). Further, for the point $P' = tr(P) \cap \varepsilon$ the cross ratio $(CD, PP') = -2$ is constant and, as the point P approaches C , the corresponding trilinear polar $tr(P)$ tends to coincide with the “harmonic conjugate” line CD' of $CD = \varepsilon$ w.r.t. the line-pair (CA, CB) .

Remark 7. From its proper definition follows that the trilinear polar of the “centroid” with equation

$$\varepsilon_\infty : u + v + w = 0,$$

is the “line at infinity” of the plane.

Exercise 4. Show that the polar ε_Q of a point Q w.r.t. to two lines $\{\gamma(X) = 0, \delta(X) = 0\}$ is the line

$$\varepsilon_Q(X) = \delta(Q)\gamma(X) + \gamma(Q)\delta(X) = 0.$$

Hint: Follows from the matrix representation of a conic $X^t M X = 0$ and the corresponding Q -polar line representation $\varepsilon_Q(X) = Q^t M X = 0$. The pair of lines represents a degenerate conic $\gamma(X)\delta(X) = 0$. The formula results by computing the corresponding matrix M .

Notice that this representation is valid in any projective system of coordinates, not only in Barycentrics.

6 Relation to Cartesian coordinates, inner product

Denoting by (x, y) the Cartesian coordinates and by $(u : v : w)$ the “absolute barycentrics” of the same point P , the relation between the two is expressed in matrix form by translating equation 1:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = M \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \quad (22)$$

This matrix and its inverse, studied further in sections 19 and 20, can be used to express other geometric objects through relations of the barycentrics. The equation of a line, which in Cartesian coordinates is

$$ax + by + c = (a, b, c) \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

translates into barycentrics to the equation

$$a'u + b'v + c'w = 0 \quad \text{with} \quad (a', b', c') = (a, b, c) \cdot M.$$

Thus, the equation of the line at infinity, described in Cartesians by $z = 0$, corresponds by the preceding rule, to the equation in barycentrics:

$$(0, 0, 1)M \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u + v + w = 0. \quad (23)$$

The “inner product” in Cartesian coordinates is also expressible through a “bilinear form”:

$$xx' + yy' = (x, y, 1) \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} - 1 = (u, v, w)M^t M \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} - 1. \quad (24)$$

The matrix $M^t M$ is easily seen to be

$$M^t M = \begin{pmatrix} A^2 & A \cdot B & A \cdot C \\ A \cdot B & B^2 & B \cdot C \\ A \cdot C & B \cdot C & C^2 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (25)$$

Taking into account that $u + v + w = u' + v' + w' = 1$, we find that

$$xx' + yy' = (u, v, w)N \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}, \quad \text{with } N = \begin{pmatrix} A^2 & A \cdot B & A \cdot C \\ A \cdot B & B^2 & B \cdot C \\ A \cdot C & B \cdot C & C^2 \end{pmatrix}. \quad (26)$$

7 The circumcircle of ABC in barycentrics

By the discussion in section 1 we can take the origin of Cartesian coordinates anywhere we like. Selecting it at the “circumcenter” O of the circumcircle of the triangle of reference ABC , we find that

$$A^2 = B^2 = C^2 = R^2, \quad A \cdot B = R^2 \cos(2\gamma), \quad A \cdot C = R^2 \cos(2\beta), \quad B \cdot C = R^2 \cos(2\alpha),$$

where $\{\alpha, \beta, \gamma\}$ the angles of the triangle opposite respectively to $\{BC, CA, AB\}$ and R is the circumradius of the triangle of reference ABC . Setting $\cos(2\gamma) = 1 - 2\sin^2(\gamma), \dots$ we find

$$\begin{aligned} N &= R^2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - 2R^2 \begin{pmatrix} 0 & \sin^2(\gamma) & \sin^2(\beta) \\ \sin^2(\gamma) & 0 & \sin^2(\alpha) \\ \sin^2(\beta) & \sin^2(\alpha) & 0 \end{pmatrix} \Leftrightarrow \\ N &= R^2 \cdot N_1 - \frac{1}{2}N_2 = R^2 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix}, \end{aligned} \quad (27)$$

where $\{a = |BC|, b = |CA|, c = |AB|\}$ the side lengths of the triangle. Taking into account the condition $\{u + v + w = 1, \dots\}$ for “absolute barycentric coordinates”, we find that the expression of the inner product $xx' + yy'$ in equation (26) in terms of the corresponding absolute barycentric coordinates is:

$$\begin{aligned} xx' + yy' &= (u, v, w)N \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} \\ &= R^2 - \frac{1}{2}(c^2(uv' + u'v) + b^2(wu' + w'u) + a^2(vw' + v'w)). \end{aligned} \quad (28)$$

Having the center of coordinates at the circumcenter of ABC , the equation of the “circumcircle” κ of ABC in barycentric coordinates can be found from the corresponding equation in Cartesian coordinates:

$$\begin{aligned} R^2 = x^2 + y^2 &= R^2 - (c^2 uv + b^2 wu + a^2 vw) \Leftrightarrow \\ a^2 vw + b^2 wu + c^2 uv &= 0. \end{aligned} \quad (29)$$

8 Displacement vectors, inner product, distance

Displacement vectors are differences of *absolute* barycentric coordinates of two points

$$(p : q : r) = (u : v : w) - (u' : v' : w').$$

Since $u + v + w = u' + v' + w' = 1$, they satisfy

$$p + q + r = 0$$

and represent points at infinity. Also through their corresponding Cartesian coordinate vectors, defined by equation (22), they represent the usual displacement from one point to the other. The inner product of the corresponding Cartesian coordinate vectors can be calculated using the same bilinear form (27). Thus, introducing the vectorial notation $\bar{x} = (x, y)$ and $\bar{P} = (p, q, r)$ for Cartesian, respectively barycentric coordinates, and also denoting by $\bar{x}_p = M\bar{P}$, we have:

$$\bar{x}_{pQ} = \bar{x}_Q - \bar{x}_p = M(\bar{Q} - \bar{P}) = M \cdot \bar{PQ}$$

and from this, the expression of the usual inner product:

$$\bar{x}_{pQ} \cdot \bar{x}_{p'Q'} = \bar{PQ}^t \cdot N \cdot \bar{P'Q'} = -\frac{1}{2}(p, q, r) \begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix} \begin{pmatrix} p' \\ q' \\ r' \end{pmatrix},$$

where we set $\bar{PQ} = Q - P = (p : q : r)$ and $\bar{P'Q'} = Q' - P' = (p' : q' : r')$. Expanding this, we get:

$$\bar{x}_{pQ} \cdot \bar{x}_{p'Q'} = -\frac{1}{2}[a^2(qr' + q'r) + b^2(rp' + r'p) + c^2(pq' + p'q)]. \quad (30)$$

This expresses the fundamental relation allowing the computation of Euclidean distances and angles in terms of barycentric coordinates. Thus, for example, the distance of two points represented in *absolute* barycentric coordinates $\{P, Q\}$ can be expressed through their “displacement vector” $\bar{PQ} = (p, q, r)$ and the formula:

$$|PQ|^2 = -a^2qr - b^2rp - c^2pq. \quad (31)$$

Introducing the “Conway symbols”

$$\begin{aligned} S_A &= bc \cos(\widehat{A}), & S_B &= ca \cos(\widehat{B}), & S_C &= ab \cos(\widehat{C}) \Leftrightarrow \\ S_A &= \frac{1}{2}(b^2 + c^2 - a^2), & S_B &= \frac{1}{2}(c^2 + a^2 - b^2), & S_C &= \frac{1}{2}(a^2 + b^2 - c^2), \end{aligned}$$

where $\{a, b, c\}$ are the side-lengths of the triangle of reference, and calculating the expression on the right of equation (30) we find, using conditions $\{p + q + r = p' + q' + r' = 0\}$, that it is equivalent to the following condition:

Theorem 3. The Euclidean inner product of two displacement vectors $\{(p : q : r), (p' : q' : r')\}$ is equal to

$$\bar{x}_{PQ} \cdot \bar{x}_{P'Q'} = S_A p p' + S_B q q' + S_C r r', \quad (32)$$

and the two vectors are orthogonal, if and only if they satisfy the equation:

$$S_A p p' + S_B q q' + S_C r r' = 0. \quad (33)$$

Corollary 1. Formula (31) is also equivalent to:

$$|PQ|^2 = S_A p^2 + S_B q^2 + S_C r^2, \quad (34)$$

where $\{p = u' - u : q = v' - v : r = w' - w\}$ and $\{P(u : v : w), Q(u : v : w)\}$ are in absolute barycentrics.

Remark 8. The three last formulas relate to the Euclidean inner product and the Pythagorean theorem, which is the basis of all metric relations of Euclidean geometry. V. Volenec, in his article [26], makes these the basis to deduce from them several other useful metric relations expressed in barycentrics.

Using the last formula we can easily find the equation of the “medial line” of two points $\{P, Q\}$, i.e. the points $X(X_u, X_v, X_w)$ whose distances $|XP| = |XQ|$. For this consider the quadratic form defined by the right side of the last formula

$$f(X, Y) = S_A X_u Y_u + S_B X_v Y_v + S_C X_w Y_w.$$

The equality $\{|XP|^2 = |XQ|^2\}$ by formula (34) is equivalent to

$$\begin{aligned} f(P - X, P - X) &= f(Q - X, Q - X) \quad \Leftrightarrow \\ f(P, P) - 2f(P, X) + f(X, X) &= f(Q, Q) - 2f(Q, X) + f(X, X) \quad \Leftrightarrow \\ f(2(P - Q), X) + f(Q, Q) - f(P, P) &= 0 \quad \Leftrightarrow \\ S_A(P_u - Q_u)X_u + S_B(P_v - Q_v)X_v + S_C(P_w - Q_w)X_w + k/2 &= 0 \end{aligned} \quad (35)$$

with

$$k = f(Q, Q) - f(P, P) = S_A(Q_u^2 - P_u^2) + S_B(Q_v^2 - P_v^2) + S_C(Q_w^2 - P_w^2).$$

Theorem 4. Equation (35) represents the medial line of two points $\{P, Q\}$ in absolute barycentrics.

Remark 9. It is easy verifiable that the “Conway symbols” satisfy the conditions:

$$S_A = S \cot(\widehat{A}), \quad S_B = S \cot(\widehat{B}), \quad S_C = S \cot(\widehat{C}), \quad (36)$$

$$S_B + S_C = a^2, \quad S_C + S_A = b^2, \quad S_A + S_B = c^2, \quad (37)$$

$$S_A S_B + S_B S_C + S_C S_A = S^2, \quad (38)$$

$$S_A S_B + c^2 S_C = S_B S_C + a^2 S_A = S_C S_A + b^2 S_B = S^2. \quad (39)$$

$$a^2 S_A + b^2 S_B + c^2 S_C = 2S^2. \quad (40)$$

where S denotes twice the area of the triangle of reference ABC ([28, p.33]).

More general, for an angle of measure ϕ we can define the compatible to the preceding symbol

$$S_\phi = S \cdot \cot(\phi), \quad (41)$$

used in several formulas involving calculations with barycentrics ([28]). More on this can be found in the file [15].

9 Orthogonality of lines, orthocentroidal circle

The parametric form of a line represented in barycentrics by the equation

$$pu + qv + rw = 0,$$

can be given in the form

$$(u : v : w) = (q : -p : 0) + t(r - q : p - r : q - p).$$

The last vector on the right is a formal “displacement vector”, satisfying $u + v + w = 0$, i.e. representing in barycentrics the “direction” of the line. Notice that it is also the “point at infinity” of that line, i.e. its intersection with the “line at infinity”, represented in barycentrics by the equation $u + v + w = 0$.

Thus, the lines $\{pu + qv + rw = 0, p'u + q'v + r'w = 0\}$ are orthogonal, if and only if their “directions”

$$(p_d : q_d : r_d) = (r - q : p - r : q - p), \quad (p'_d : q'_d : r'_d) = (r' - q' : p' - r' : q' - p')$$

are orthogonal displacements, i.e. they satisfy equation (33). Latter equation, together with the valid for points at infinity $u + v + w = 0$, imply the explicit form of the “direction” of the orthogonal line:

$$(p'_d : q'_d : r'_d) = ((S_B \cdot q_d - S_C \cdot r_d) : (S_C \cdot r_d - S_A \cdot p_d) : (S_A \cdot p_d - S_B \cdot q_d)). \quad (42)$$

Alternatively to this, i.e. using the line-coefficients themselves and not their differences, we arrive after a short calculation at the theorem ([10, II,p.57]):

Theorem 5. *The lines $\varepsilon : px + qy + rz = 0$ and $\varepsilon' : p'x + q'y + r'z = 0$ are orthogonal if and only if their coefficients satisfy the equation*

$$a^2pp' + b^2qq' + c^2rr' - S_A(qr' + q'r) - S_B(rp' + r'p) - S_C(pq' + p'q) = 0. \quad (43)$$

Remark 10. The bilinear form involved in equation (43) is degenerate, since the corresponding matrix H satisfies:

$$(1, 1, 1)H = (1, 1, 1) \begin{pmatrix} -a^2 & S_C & S_B \\ S_C & -b^2 & S_A \\ S_B & S_A & -c^2 \end{pmatrix} = 0. \quad (44)$$

Remark 11. Last equation conforms to the fact that if lines $\{\varepsilon = 0, \varepsilon' = 0\}$ are orthogonal the same is true for $\{\varepsilon + k \cdot \varepsilon_\infty, \varepsilon'\}$, where ε_∞ is the line at infinity. By section 3-nr-5 the lines $\{\varepsilon + k \cdot \varepsilon_\infty = 0\}$ represent, for variable k , all the parallels to line $\varepsilon = 0$.

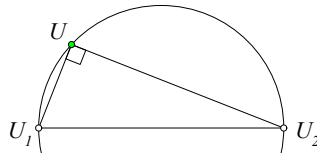


Figure 9: Circle on diameter U_1U_2 in absolute barycentrics

As an application of the orthogonality of displacement vectors we obtain the equation of the circle having for diameter the segment defined by the points

$$U_1 = (u_1 : v_1 : w_1) \quad \text{and} \quad U_2 = (u_2 : v_2 : w_2).$$

The points $U(u : v : w)$ of this circle are characterized by the orthogonality of the displacement vectors $\{UU_1, UU_2\}$ implying the theorem:

Theorem 6. The points $U(u : v : w)$ of the circle on diameter U_1U_2 satisfy the equation in absolute barycentrics:

$$S_A(u - u_1)(u - u_2) + S_B(v - v_1)(v - v_2) + S_C(w - w_1)(w - w_2) = 0. \quad (45)$$

An application of this leads to a compact expression for the “orthocentroidal” circle of a triangle ABC , defined to be the circle with diametral points the centroid G and the orthocenter H of the triangle. Taking into account formula (79) for H and doing some calculation we find its expression in absolute barycentrics:

$$3(S_Au^2 + S_Bv^2 + S_Cw^2) - (S_Au + S_Bv + S_Cw) = 0, \quad (46)$$

where (u, v, w) are assumed to be absolute coordinates.

10 Distance from line, bisectors, distance of parallels

Representing the coordinates by $\bar{U} = (u : v : w)$, the “direction” of the line

$$\varepsilon : f(\bar{U}) = pu + qv + rw = 0 \quad (47)$$

is its intersection

$$(r - q : p - r : q - p) \quad (48)$$

with the line at infinity as explained in section 9. The orthogonal direction to this can be calculated using theorem 3 and leads to a multiple of the displacement vector \bar{V} :

$$(v_1 = S_B(p - r) - S_C(q - p), \quad v_2 = S_C(q - p) - S_A(r - q), \quad v_3 = S_A(r - q) - S_B(p - r)).$$

Which in matrix notation is represented by the equation:

$$\bar{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 & S_B & -S_C \\ -S_A & 0 & S_C \\ S_A & -S_B & 0 \end{pmatrix} \cdot \begin{pmatrix} r - q \\ p - r \\ q - p \end{pmatrix}. \quad (49)$$

The line ε' , passing through the point $\bar{U}_0 = (u_0, v_0, w_0)$, and orthogonal to ε has the



Figure 10: Calculate the distance $|U_0U_1|$ in barycentrics

parametric representation, the point U_0 in absolute barycentrics:

$$\varepsilon' : \bar{U}_t = \bar{U}_0 + t\bar{V}.$$

Its intersection point \bar{U}_t with ε satisfies the equation of line ε :

$$f(\bar{U}_0 + t\bar{V}) = 0 \quad \Leftrightarrow \quad f(\bar{U}_0) + tf(\bar{V}) = 0 \quad \Leftrightarrow \quad t = -\frac{f(\bar{U}_0)}{f(\bar{V})}.$$

Taking into account equation (49), we find that

$$\begin{aligned} f(\bar{V}) &= (p, q, r) \begin{pmatrix} 0 & S_B & -S_C \\ -S_A & 0 & S_C \\ S_A & -S_B & 0 \end{pmatrix} \cdot \begin{pmatrix} r - q \\ p - r \\ q - p \end{pmatrix} \Leftrightarrow \\ f(\bar{V}) &= S_A(r - q)^2 + S_B(p - r)^2 + S_C(q - p)^2. \end{aligned} \quad (50)$$

Thus, the displacement vector $\bar{U}_t - \bar{U}_0$ and its length is, according to equation (31):

$$\bar{U}_t - \bar{U}_0 = -\frac{f(\bar{U}_0)}{f(\bar{V})} \cdot V \quad \Rightarrow \quad |\bar{U}_t - \bar{U}_0|^2 = \frac{f(\bar{U}_0)^2}{f(\bar{V})^2} (-a^2 v_2 v_3 - b^2 v_3 v_1 - c^2 v_1 v_2). \quad (51)$$

The parenthesis on the right is

$$V^2 = -a^2 v_2 v_3 - b^2 v_3 v_1 - c^2 v_1 v_2 = -\frac{1}{2} (v_1, v_2, v_3) \begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

which, taking into account equation (49), leads to

$$V^2 = -\frac{1}{2} (r - q, p - r, q - p) \cdot K \cdot \begin{pmatrix} r - q \\ p - r \\ q - p \end{pmatrix}, \quad (52)$$

where K is the matrix

$$K = \begin{pmatrix} 0 & -S_A & S_A \\ S_B & 0 & -S_B \\ -S_C & S_C & 0 \end{pmatrix} \begin{pmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & S_B & -S_C \\ -S_A & 0 & S_C \\ S_A & -S_B & 0 \end{pmatrix}.$$

Carrying out the computation we find that

$$K = 2 \begin{pmatrix} -a^2 S_A^2 & S_A S_B S_C & S_A S_B S_C \\ S_A S_B S_C & -b^2 S_B^2 & S_A S_B S_C \\ S_A S_B S_C & S_A S_B S_C & -c^2 S_C^2 \end{pmatrix}. \quad (53)$$

This, using the relations in section 9 of the Conway symbols becomes

$$\begin{aligned} -a^2 S_A^2 + S_A S_B S_C &= S_A (S_B S_C - a^2 S_A) \\ &= S_A (S_B S_C - (S_B + S_C) S_A) \\ &= S_A (2S_B S_C - (S_B S_C + S_C S_A + S_A S_B)) \\ &= 2S_A S_B S_C - S_A S^2 \quad \Rightarrow \\ -a^2 S_A^2 &= S_A S_B S_C - S_A S^2. \end{aligned}$$

Thus, matrix K can be written in the form

$$\frac{K}{2} = S_A S_B S_C \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - S^2 \begin{pmatrix} S_A & 0 & 0 \\ 0 & S_B & 0 \\ 0 & 0 & S_C \end{pmatrix}. \quad (54)$$

Introducing this into equation (52) we find that

$$V^2 = S^2 (S_A (r - q)^2 + S_B (p - r)^2 + S_C (q - p)^2). \quad (55)$$

This introduced to equation (51) leads finally to the expression for the distance of the point \bar{U}_0 from the line $\varepsilon : f(\bar{U}) = pu + qv + rw = 0$:

Theorem 7. The distance $\text{dist}(U_0, \varepsilon)$ of the point $\bar{U}_0(u_0 : v_0 : w_0)$ given in absolute barycentrics from the line $\varepsilon : f(\bar{U}) = pu + qv + rw = 0$ is given by the formula:

$$\text{dist}(U_0, \varepsilon)^2 = |\bar{U}_t \bar{U}_0|^2 = \frac{S^2(pu_0 + qv_0 + rw_0)^2}{S_A(r - q)^2 + S_B(p - r)^2 + S_C(q - p)^2}. \quad (56)$$

As an application of this formula we can consider the bisectors of two distinct intersecting lines $\{\varepsilon, \varepsilon'\}$. Denoting as above ε and $\varepsilon' : f'(\bar{U}) = p'u + q'v + r'w = 0$, the equations of the bisectors of the angles of the two lines are given by

$$\begin{aligned} \text{dist}(\bar{U}, \varepsilon)^2 - \text{dist}(\bar{U}, \varepsilon')^2 &= 0 \quad \Leftrightarrow \\ f(\bar{U})^2 \cdot S(f') - f'(\bar{U})^2 \cdot S(f) &= 0 \quad (57) \\ \text{with } S(f) &= S_A(r - q)^2 + S_B(p - r)^2 + S_C(q - p)^2. \end{aligned}$$

Measuring the distance $\text{dist}(A, \varepsilon)$ of the vertex $A(1 : 0 : 0)$ from the line ε and doing this also for the other vertices, we find the following property:

Theorem 8. The coefficients of the line $\varepsilon : pu + qv + rw = 0$ are proportional to the distances of the vertices of the triangle of reference ABC from the line:

$$\frac{\text{dist}(A, \varepsilon)^2}{p^2} = \frac{\text{dist}(B, \varepsilon)^2}{q^2} = \frac{\text{dist}(C, \varepsilon)^2}{r^2} = \frac{S^2}{S_A(r - q)^2 + S_B(p - r)^2 + S_C(q - p)^2}. \quad (58)$$

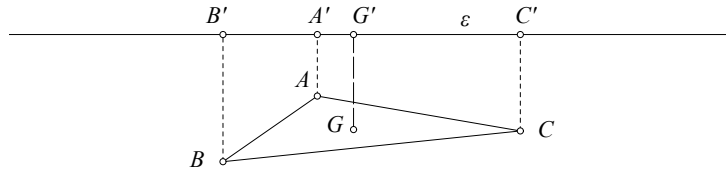


Figure 11: Distance of centroid from the line ε

Remark 12. Another aspect of the ratio on the right side of the last formula is found by considering a line not passing through the the centroid $G(1/3 : 1/3 : 1/3)$ of $\triangle ABC$, i.e. satisfying $p + q + r \neq 0$. Equation (56) leads in this case to:

$$\begin{aligned} \text{dist}(G, \varepsilon)^2 &= \frac{S^2(\sigma/3)^2}{S_A(r - q)^2 + S_B(p - r)^2 + S_C(q - p)^2}, \quad \text{with } \sigma = p + q + r \quad \Leftrightarrow \\ \frac{S^2}{S_A(r - q)^2 + S_B(p - r)^2 + S_C(q - p)^2} &= \left(\frac{3\text{dist}(G, \varepsilon)}{\sigma} \right)^2 \quad (59) \end{aligned}$$

From theorem 7 and the preceding remark follows immediately the form of the equation of the bisector lines of the angle of two lines and the form of the distance of two parallel lines:

Theorem 9. The bisector lines of the angle of two intersecting lines, which do not pass through the centroid G :

$$\varepsilon : f(u, v, w) = pu + qv + rw = 0, \quad \varepsilon' : f'(u, v, w) = p'u + q'v + r'w = 0,$$

with normalized coefficients satisfying $\{p + q + r = p' + q' + r' = 1\}$ are given by the equations

$$\text{dist}(G, \varepsilon)f(u, v, w) \pm \text{dist}(G, \varepsilon')f'(u, v, w) = 0. \quad (60)$$

Theorem 10. The distance of two parallel lines $\{\varepsilon, \varepsilon'\}$:

$$f(u, v, w) = pu + qv + rw, \quad f'(u, v, w) = f(u, v, w) + k \quad \text{with } \sigma = p + q + r \neq 0,$$

is given by

$$\text{dist}(\varepsilon, \varepsilon') = \frac{3|k| \cdot \text{dist}(G, \varepsilon)}{|\sigma|}. \quad (61)$$

Exercise 5. Show that the distance $d(P, t_P)$ of a point $P(p : q : r)$ expressed in absolute barycentrics from its tripolar $t_P : \frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 0$ is

$$d(P, t_P)^2 = \frac{(3Spqr)^2}{a^2qr(p-q)(p-r) + b^2rp(q-r)(q-p) + c^2pq(r-p)(r-q)}. \quad (62)$$

11 Meaning of line coefficients

The meaning of line coefficients, expressed through equation (58), can be explained geometrically and more directly, than it was done in section 10, whose purpose was the general formula of distance of a point from a line, as described by equation (56). For this it suffices to use the results of section 4. In fact, the intersection points $\{A', B', C'\}$ of the line $\varepsilon : pu + qv + rw = 0$ with the sides of the triangle of reference, have corresponding coordinates (see figure 12)

$$A'(0 : -r : q), \quad B'(-r : 0 : p), \quad C'(-q : p : 0).$$

And the ratio of the distances $\{AA'', BB'', CC''\}$ is

$$\frac{BB''}{CC''} = \frac{A'B}{A'C} = \frac{q}{r} \quad \Rightarrow \quad \frac{BB''}{q} = \frac{CC''}{r}.$$

Analogously is seen that $BB''/q = AA''/p$. Notice that in order to have valid equations

$$\frac{AA''}{p} = \frac{BB''}{q} = \frac{CC''}{r},$$

the distances must be signed and their signs must be properly chosen.

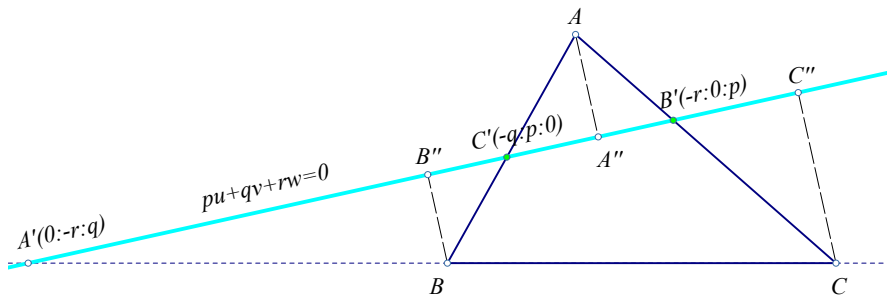


Figure 12: Coefficients proportional to distances $\frac{AA''}{p} = \frac{BB''}{q} = \frac{CC''}{r}$

12 Power of a point, general circle, Euler circle

The expression in barycentrics of the “power” of a point $P(u : v : w)$ relative to a circle, can be found from the corresponding expression in Cartesian coordinates. In fact, in Cartesian coordinates the power of the point P w.r.t. the circle $\kappa(O, r)$ is

$$PO^2 - r^2,$$

which, using equation (31) for points $\{O(u_0 : v_0 : w_0), P(u : v : w)\}$ in absolute barycentrics becomes:

$$-(PO^2 - r^2) = a^2(v_0 - v)(w_0 - w) + b^2(u_0 - u)(w_0 - w) + c^2(u_0 - u)(v_0 - v) + r^2.$$

The equation of the circle $\kappa(O, r)$ results by equating this to zero:

$$\begin{aligned} & a^2vw + b^2wu + c^2uv \\ & + a^2v_0w_0 + b^2u_0w_0 + c^2u_0v_0 + r^2 \\ & - a^2(v_0w + vw_0) - b^2(w_0u + wu_0) - c^2(u_0v + uv_0) = 0. \end{aligned} \quad (63)$$

Taking into account that $u + v + w = 1$, the two last rows are seen to sum into the linear term:

$$(r^2 - OA^2)u + (r^2 - OB^2)v + (r^2 - OC^2)w.$$

Thus the power of $P(u : v : w)$ w.r.t. the circle $\kappa(O, r)$ in absolute barycentrics becomes

$$PO^2 - r^2 = -a^2vw - b^2wu - c^2uv + (OA^2 - r^2)u + (OB^2 - r^2)v + (OC^2 - r^2)w, \quad (64)$$

and the equation $PO^2 - r^2 = 0$ of the circle $\kappa(O, r)$ in absolute barycentrics becomes

$$a^2vw + b^2wu + c^2uv - (OA^2 - r^2)u - (OB^2 - r^2)v - (OC^2 - r^2)w = 0. \quad (65)$$

Absolute barycentrics satisfy $x + y + z = 1$ and last equation can be homogenized to

$$a^2vw + b^2wu + c^2uv - [(OA^2 - r^2)u + (OB^2 - r^2)v + (OC^2 - r^2)w][x + y + z] = 0. \quad (66)$$

Equation (65) represents the equation of the circle κ as a sum of the corresponding expression of the circumcircle κ_0 of the triangle of reference ABC and a certain line λ . Thus, the difference of the expressions of the two circles κ and κ_0 is the expression of their “radical axis” λ and the following theorem is valid.

Theorem 11. *Equation*

$$(OA^2 - r^2)u + (OB^2 - r^2)v + (OC^2 - r^2)w = 0 \quad (67)$$

represents the radical axis of the circle $\kappa(O, r)$ and the circumcircle κ_0 of the triangle of reference ABC . Equation (64) represents the power of the point $P(u : v : w)$ w.r.t. κ , equation (65) represents the circle κ in absolute barycentrics, and equation (66) represents the homogeneous equation of κ in barycentrics (not necessarily absolute).

Corollary 2. *The equation of the general circle $\kappa(O, r)$ in barycentrics can be brought to the form*

$$a^2vw + b^2wu + c^2uv - (ku + lv + mw)(u + v + w) = 0, \quad (68)$$

where the line $\varepsilon : kx + ly + mz = 0$ represents the radical axis of this circle with the circumcircle κ_0 of the triangle of reference ABC , and the left side of the equation represents the negative of the power $PO^2 - r^2$ of the point $P(u : v : w)$ w.r.t. κ .

Remark 13. Equation (68) must be applied with caution. You can somehow find the equation of the radical axis $k'u + l'v + m'w = 0$ of κ and κ_0 and use it with this formula to find the equation of κ . This will not work if (k', l', m') do not coincide with the coefficients (k, l, m) in equation (67) but are instead a multiple of them. The circle κ' represented with such a multiple of the coefficients of equation (67) will be a member of the pencil of circles generated by κ and κ_0 (see figure 13).

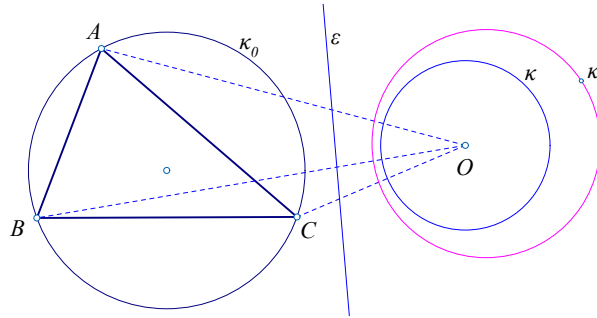


Figure 13: κ' member of pencil generated by κ and κ_0

As an example application of this equation we consider the “Euler circle” λ of the triangle of reference ABC . The powers of the vertices $\{A, B, C\}$ w.r.t. $\lambda(N, r)$ are easily seen (see figure 14) to be

$$OA^2 - r^2 = AB' \cdot AB'' = \frac{1}{2}bc \cos(\widehat{A}) = \frac{1}{2}S_A$$

and the analogous expressions obtained by the cyclic permutations of $\{A, B, C\}$. Here $\{B', B''\}$ are respectively the traces on AC of the median and altitude from B . Thus, equation (66) in the case of the Euler circle becomes:

$$(a^2vw + b^2wu + c^2uv) - \frac{1}{2}(S_Au + S_Bv + S_Cw)(u + v + w) = 0 \iff (69)$$

$$a^2vw + b^2wu + c^2uv - (S_Au^2 + S_Bv^2 + S_Cw^2) = 0. (70)$$

Figure 14 illustrates the case showing also the radical axis of the Euler circle and the

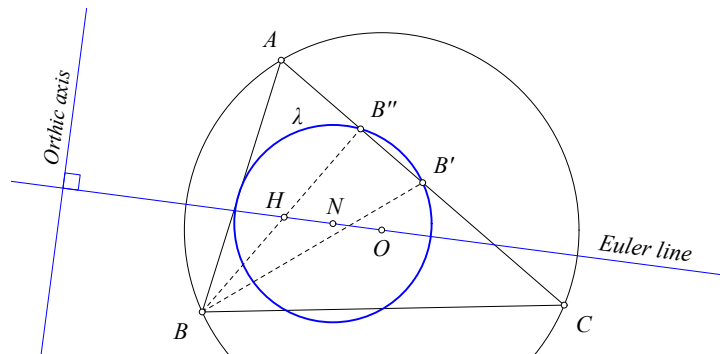


Figure 14: The Euler circle of the triangle ABC

circumcircle. This is the “orthic axis” of the triangle, identified by theorem 11 with the line

$$S_Au + S_Bv + S_Cw = 0, (71)$$

and coinciding with the *trilinear polar* of the orthocenter H of ABC .

Exercise 6. Given the equation of a circle $\kappa(O, r)$ in barycentrics in the general form

$$a^2vw + b^2wu + c^2uv - (ku + lv + mw)(u + v + w) = 0, \quad (72)$$

determine its center O and its radius r in dependence of (k, l, m) .

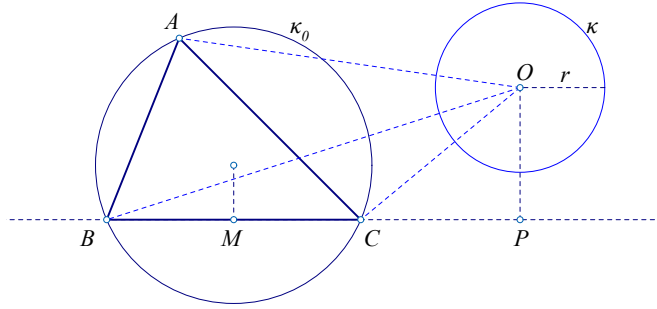


Figure 15: Finding center O and radius r from the equation

Hint: It is $\{k = OA^2 - r^2, l = OB^2 - r^2, m = OC^2 - r^2\}$, hence $OB^2 - OC^2 = l - m$. But $OB^2 - OC^2 = 2BC \cdot MP$, with M the middle of BC and P the projection of O on BC . Thus, $l - m = 2BC \cdot MP$ and the location of P and consequently the line $\lambda_A = OP$ orthogonal to the sideline BC at P can be found. Analogously the line λ_B through O orthogonal to CA can be found, etc.

Exercise 7. Show that the conditions for the general equation of the conic

$$Au^2 + Bv^2 + Cw^2 + 2Huv + 2Guw + 2Fvw = 0 \Leftrightarrow (u : v : w) \begin{pmatrix} A & H & G \\ H & B & F \\ G & F & C \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0,$$

to represent a circle are ([27, p.244])

$$\frac{B + C - 2F}{a^2} = \frac{C + A - 2G}{b^2} = \frac{A + B - 2H}{c^2}. \quad (73)$$

Hint: Expand equation (72) and determine the matrix M' of the (circle) conic:

$$X^t M' X = (u : v : w) \begin{pmatrix} -k & (-l - k + c^2)/2 & (-m - k + b^2)/2 \\ (-l - k + c^2)/2 & -l & (-m - l + a^2)/2 \\ (-m - k + b^2)/2 & (-m - l + a^2)/2 & -m \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0.$$

The matrix M corresponding to the given conic equation should have the form of a non-zero multiple $M = \lambda M'$.

Remark 14. Alternatively to the solution suggested in exercise 6, one can use the preceding matrix M' to find the center O of the circle coinciding with the pol relative to κ of the line at infinity $\varepsilon_\infty : u + v + w = 0$ with coefficients $(1, 1, 1)$. This is found by applying the inverse $M'' = M'^{-1}$ of the matrix to this vector. Doing some computation we find:

$$\begin{pmatrix} o_1 \\ o_2 \\ o_3 \end{pmatrix} = \begin{pmatrix} -2a^2k + (c^2 - a^2 - b^2)(-l) + (b^2 - a^2 - c^2)(-m) + a^2(b^2 + c^2 - a^2) \\ (c^2 - a^2 - b^2)(-k) - 2b^2l + (a^2 - b^2 - c^2)(-m) + b^2(c^2 + a^2 - b^2) \\ (b^2 - a^2 - c^2)(-k) + (a^2 - b^2 - c^2)(-l) - 2c^2m + c^2(a^2 + b^2 - c^2) \end{pmatrix}.$$

Using the Conway triangle symbols, this can be written

$$\begin{pmatrix} o_1 \\ o_2 \\ o_3 \end{pmatrix} = \begin{pmatrix} -a^2 & S_C & S_B \\ S_C & -b^2 & S_A \\ S_B & S_A & -c^2 \end{pmatrix} \begin{pmatrix} k \\ l \\ m \end{pmatrix} - \begin{pmatrix} a^2 S_A \\ b^2 S_B \\ c^2 S_C \end{pmatrix}. \quad (74)$$

We notice that the matrix product on the right side in equation (74) represents the direction of the line which is orthogonal to the radical axis $ku + lv + mw = 0$.

The radius r of the circle can be computed from the power of a point P w.r.t. to the given circle. This is equal to the negative of the left side of equation (72), i.e. it is equal to $r^2 - PC^2$. Taking $P = G = \frac{1}{3}(1 : 1 : 1)$, the centroid of the triangle of reference, we have:

$$\begin{aligned} r^2 - GC^2 &= \frac{1}{9}(a^2 + b^2 + c^2) - \frac{1}{3}(k + l + m) \quad \Leftrightarrow \\ r^2 &= GC^2 + \frac{1}{9}(a^2 + b^2 + c^2) - \frac{1}{3}(k + l + m). \end{aligned} \quad (75)$$

Exercise 8. Prove Feuerbach's theorem saying, that the incircle κ and the Euler circle κ' of a triangle ABC touch at a point P (see figure 16).

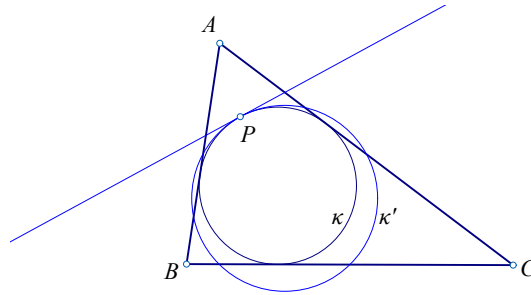


Figure 16: Feuerbach's theorem

Hint: This can be done in three steps:

(1) Use equation (69) of the Euler circle, and the equation of the incircle

$$a^2vw + b^2wu + c^2uv - (\tau_a^2u + \tau_b^2v + \tau_c^2w)(u + v + w) = 0, \quad (76)$$

with $\{\tau_a = (b + c - a)/2, \tau_b = (c + a - b)/2, \tau_c = (a + b - c)/2\}$.

(2) Taking the difference of the left members of their equations we see that the coefficients of the radical axis of the circles $\{\kappa, \kappa'\}$ are $\{\tau_a^2 - S_A/2, \tau_b^2 - S_B/2, \tau_c^2 - S_C/2\}$.

(3) Show that this radical axis is tangent to the incircle κ . This means that these coefficients satisfy the tangential equation of κ , which is: $\tau_a v w + \tau_b w u + \tau_c u v = 0$. The matrix of the "tangential equation" of a conic is a multiple of the inverse of the matrix representing that conic ([24, p.152], [10, II, p.62], [3, p.161]).

Remark 15. Point P of exercise 8 is called "Feuerbach center" of the triangle ABC . Its notation in Kimberling's list [9] and its barycentrics are

$$X(11) = (\tau_a(b - c)^2 : \tau_b(c - a)^2 : \tau_c(a - b)^2). \quad (77)$$

Also the radical axis $\tau_a^2u + \tau_b^2v + \tau_c^2w = 0$ of the incircle κ and the circumcircle of ABC appearing in equation (76) is the trilinear polar of the triangle center $X(279)$.

13 Centroid, Incenter, Circumcenter, Symmedian point

These remarkable points are the simplest examples of “triangle centers” of the triangle of reference ABC , whose barycentric coordinates can be easily calculated. The first, the “centroid” $X(2)$ in Kimberling’s notation ([9]), is the unit point of the projective base $\{A, B, C, G\}$ defining the system of “barycentrics” and has coordinates $(1 : 1 : 1)$. The incenter $X(1)$ of the triangle, using the definition of barycentrics through areas, is easily seen to be $(a : b : c)$. The circumcenter $X(3)$ of the triangle ABC is also calculated using the area definition of barycentrics. In fact, the area of the triangle $X(3)BC$ is

$$(X(3)BC) = \frac{1}{4}BC^2 \cot(\alpha) = \dots = \frac{R}{4abc}a^2S_A \Rightarrow X(3) = (a^2S_A : b^2S_B : c^2S_C),$$

with corresponding absolute barycentrics:

$$X(3) = \frac{1}{2S^2}(a^2S_A : b^2S_B : c^2S_C). \tag{78}$$

The symmedian point $X(6)$ ([16]) is characterized by its property to have distances from the sides analogous to these sides. Thus, the ratio of the areas

$$\frac{(X(6)AB)}{(X(6)AC)} = \frac{c^2}{b^2} \Rightarrow X(6) = (a^2 : b^2 : c^2).$$

The equalities $X(k) = (u : v : w)$ for barycentrics must be understood in a wider sense and often is used the notation $(u : v : w)$, instead of (x, y, z) , in order to stress the fact that the barycentrics are defined up to multiplicative constants, so that only their relative ratios are uniquely defined. Alternatively to the equality symbol, we use also the symbol $X \cong (u : v : w)$.

Exercise 9. Show that the symmedian center K of the triangle ABC is the unique point of the plane for which the projecting to the sides segments, translated parallel and made successive, close to a triangle τ similar to ABC (see figure 17-(I)).

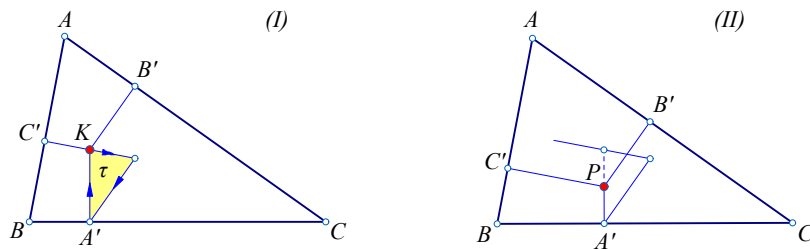


Figure 17: Triangle τ formed by the projecting segments $\{KA', KB', KC'\}$

Hint: For any point P the lines extending the projecting segments on the sides form a triangle similar to ABC (see figure 17-(II)). The characteristic property of the symmedian point K , by which the distances from the sides are proportional to those sides, causes the projecting segments, parallel translated and made successive, to close.

14 Euler line, Orthocenter, center of Euler’s circle

Having the barycentrics $\{(1 : 1 : 1), (a^2S_A : b^2S_B : c^2S_C)\}$ of the centroid and the circumcenter, the Euler line, which passes through them, is easily seen (section 3) to have coefficients

$$\text{Euler line : } (b^2S_B - c^2S_C)u + (c^2S_C - a^2S_A)v + (a^2S_A - b^2S_B)w = 0.$$

Its *point at infinity* $X(30)$ i.e. its intersection with the line at infinity $u + v + w = 0$ is then calculated to be

$$X(30) = (2a^2S_A - b^2S_B - c^2S_C : \dots) = ((2a^4 - (b^2 + c^2)a^2 - (b^2 - c^2)^2 : \dots),$$

where the dots indicate the remaining coordinates resulting by cyclic permutations of the letters $\{a, b, c\}$ and $\{A, B, C\}$. On the Euler line are located some other *triangle centers*, like the orthocenter H or $X(4)$ and the “center of the Euler circle” E or $X(5)$. The barycentrics

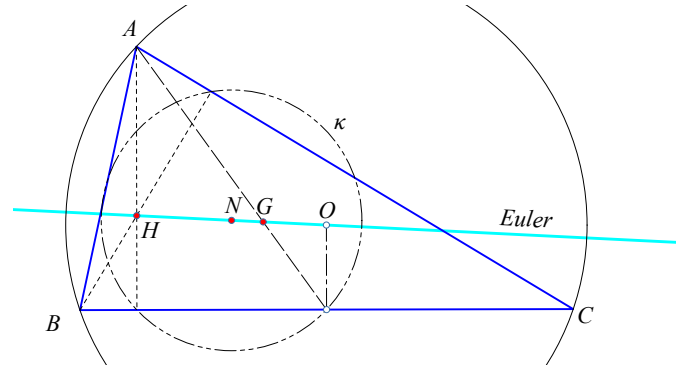


Figure 18: Points $\{H = X_4, N = X_5\}$ on the Euler line

of the *orthocenter* are easily found using the well known ratio $HO/HG = 3/2$. Taking into account the ratio rules of section 19, by which the condition $P = (1 - t)X + tY$ for points in Euclidean coordinates transfers to the same relation for these points expressed in *absolute* barycentrics, we have:

$$\begin{aligned} H &= 3 \frac{G}{\sigma_G} - 2 \frac{O}{\sigma_O} = (1 : 1 : 1) - 2 \frac{(a^2S_A, \dots)}{\sigma_O} \\ &= \left(1 - 2 \frac{a^2S_A}{2S^2}, \dots \right) = \frac{1}{S^2} (S_B S_C : \dots) \cong (S_B S_C : \dots) \Rightarrow \\ H &= (S_B S_C : S_C S_A : S_A S_B). \end{aligned} \tag{79}$$

where σ_X is the sum of barycentrics of X . Analogously are computed the barycentrics of the center $N = X(5)$ of the Euler circle, which is the middle of the segment HO :

$$N \cong \sigma_O \cdot H + \sigma_H \cdot O = 2S^2 \cdot H + S^2 \cdot O \cong (S^2 + S_B S_C : S^2 + S_C S_A : S^2 + S_A S_B). \tag{80}$$

Remark 16. The expression of the center $N = X_5 \cong (S^2 + S_B S_C, \dots)$ as a linear combination of the centers $\{G, H\}$ defining the *Euler line* generalizes for all notable triangle centers *lying on that line*, which can be also written as linear combinations

$$X = (m \cdot S^2 + n \cdot S_B S_C : \dots).$$

The coefficients $\{m, n\}$ are called “*Shinagawa coefficients*” of X ([9]).

15 Triangle Area in Barycentrics

Consider the triangle of reference ABC and a second $\triangle DEF$, whose vertices have absolute barycentric coordinates w.r.t. $ABC : D(d_1 : d_2 : d_3), E(e_1 : e_2 : e_3), F(f_1 : f_2 : f_3)$. The basic relations between barycentric and Cartesian coordinates have been discussed

in section 6. Denoting by (X_1, X_2) the Cartesian coordinates of the points X of the plane and using equation (22), we obtain:

$$\begin{pmatrix} D_1 & E_1 & F_1 \\ D_2 & E_2 & F_2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{pmatrix}. \quad (81)$$

Taking the determinants, we obtain the relation of the areas

$$(DEF) = (ABC) \cdot \begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix}, \quad (82)$$

expressing the area (DEF) in terms of the absolute barycentrics of the vertices of the triangle.

As an application, we can easily compute the area of the cevian triangle $A'B'C'$ of a point $P(x : y : z)$ (see figure 4). The traces of P on the sides of the triangle of reference ABC are $A'(0 : y : z)$, $B'(x : 0 : z)$, $C'(x : y : 0)$. With a short calculation we find then

$$(A'B'C') = \frac{(ABC)}{(y+z)(z+x)(x+y)} \begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{vmatrix} = \frac{2(ABC)xyz}{(y+z)(z+x)(x+y)}. \quad (83)$$

Some more effort in calculation is required for the "pedal" triangle (see figure 19) of a point $P(u : v : w)$ w.r.t. the triangle of reference ABC . From the discussion in section 10 we can compute the coordinates of the projections $\{A', B', C'\}$ on the sides:

$$\begin{aligned} A' &= (0 : a^2v + uS_C : a^2w + uS_B), \\ B' &= (b^2u + vS_C : 0 : b^2w + vS_A), \\ C' &= (c^2u + wS_B : c^2v + wS_A : 0). \end{aligned}$$

Then, assuming absolute barycentrics, satisfying $u + v + w = 1$, setting $S = 2(ABC)$ and

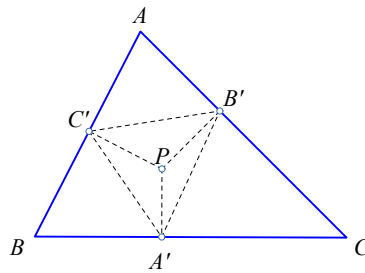


Figure 19: Pedal triangle $A'B'C'$ of ABC w.r.t. P

using equation (39), we find the crucial determinant

$$\begin{vmatrix} 0 & b^2u + vS_C & c^2u + wS_B \\ a^2v + uS_C & 0 & c^2v + wS_A \\ a^2w + uS_B & b^2w + vS_A & 0 \end{vmatrix} = S^2(a^2vw + b^2wu + c^2uv). \quad (84)$$

This, turning to general, not necessarily *absolute* barycentrics, implies the formula for the area $(A'B'C')$ of the pedal of the point $P(u : v : w)$ w.r.t. the triangle of reference:

$$(A'B'C') = \frac{S^3}{a^2b^2c^2} \cdot \frac{a^2vw + b^2wu + c^2uv}{2(u + v + w)^2}. \quad (85)$$

16 Circumcevian triangle of a point

Given a triangle ABC and a point P the “circumcevian” triangle of P w.r.t. ABC is the triangle $A'B'C'$ formed by the second intercepts $\{A', B', C'\}$ of the cevians $\{AP, BP, CP\}$ of P with the circum-circle κ of ABC (see figure 20). The main property of the *circumcevian triangle* is:

Theorem 12. *The circumcevian triangle of P is similar to the corresponding pedal triangle of P .*

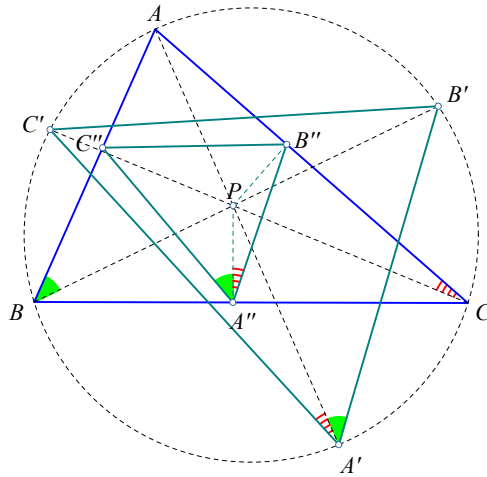


Figure 20: The circumcevian triangle $A'B'C'$ of P w.r.t. ABC

Figure 20 shows the way to prove that the two triangles have the same angles. Returning to computations, the equations of the cevians are:

$$AA' : -zv + yw = 0, \quad BB' : -xw + zu = 0, \quad CC' : -yu + xv = 0. \quad (86)$$

The points $\{A', B', C'\}$ are calculated and seen to be:

$$A' = \begin{pmatrix} -a^2yz \\ b^2yz + c^2y^2 \\ c^2yz + b^2z^2 \end{pmatrix}, \quad B' = \begin{pmatrix} a^2xz + c^2x^2 \\ -b^2xz \\ c^2xz + a^2z^2 \end{pmatrix}, \quad C' = \begin{pmatrix} a^2xy + b^2x^2 \\ b^2xy + a^2y^2 \\ -c^2xy \end{pmatrix}. \quad (87)$$

Their determinant is found:

$$\det(A', B', C') = (a^2yz + b^2zx + c^2xy)^3. \quad (88)$$

The determination of the area ($A'B'C'$) requires the division with the product $\sigma_{A'}\sigma_{B'}\sigma_{C'}$ of the sums of the coordinates of these points:

$$\begin{aligned} \sigma_{A'} &= (b^2 + c^2 - a^2)yz + b^2z^2 + c^2y^2 = S_A(y + z)^2 + S_By^2 + S_Cz^2, \\ \sigma_{B'} &= (c^2 + a^2 - b^2)zx + c^2x^2 + a^2z^2 = S_B(z + x)^2 + S_Cz^2 + S_Ax^2, \\ \sigma_{C'} &= (a^2 + b^2 - c^2)xy + a^2y^2 + b^2x^2 = S_C(x + y)^2 + S_Ax^2 + S_By^2. \end{aligned}$$

17 Circle through three points, Brocard circle

The expression in barycentrics of the circle passing through three points $\{P_i(u_i : v_i : w_i)\}$, can be found from the corresponding expression in Cartesian coordinates, using the transformation $X_i = MP_i$ of section 6. In fact, using Cartesian coordinates $\{X_i = (x_i, y_i)\}$, the

circle through these points is represented by the equation:

$$\begin{aligned}
0 &= \begin{vmatrix} X_1^2 & X_2^2 & X_3^2 & X^2 \\ x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ 1 & 1 & 1 & 1 \end{vmatrix} \\
&= X_1^2 \begin{vmatrix} x_2 & x_3 & x \\ y_2 & y_3 & y \\ 1 & 1 & 1 \end{vmatrix} - X_2^2 \begin{vmatrix} x_1 & x_3 & x \\ y_1 & y_3 & y \\ 1 & 1 & 1 \end{vmatrix} + X_3^2 \begin{vmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \\ 1 & 1 & 1 \end{vmatrix} - X^2 \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} \\
&= X_1^2 |MP_2, MP_3, MP| - X_2^2 |MP_1, MP_3, MP| + X_3^2 |MP_1, MP_2, MP| - X^2 |MP_1, MP_2, MP_3| \\
&= (X_1^2 |P_2, P_3, P| - X_2^2 |P_1, P_3, P| + X_3^2 |P_1, P_2, P| - X^2 |P_1, P_2, P_3|) |M| \Leftrightarrow \\
&\quad X_1^2 |P_2, P_3, P| + X_2^2 |P_3, P_1, P| + X_3^2 |P_1, P_2, P| - X^2 |P_1, P_2, P_3| = 0. \tag{89}
\end{aligned}$$

Here $|M|$ denotes the determinant of the matrix M and $|P, Q, R|$ denotes the determinant of the matrix of the columns of *absolute* barycentric coordinates vectors $\{P, Q, R\}$. By equation (28) the inner products are:

$$X_i^2 = R^2 - (a^2 v_i w_i + b^2 w_i u_i + c^2 u_i v_i).$$

Replacing with this in equation (89), we see that the terms involving R^2 factor into

$$|P_2, P_3, P| + |P_3, P_1, P| + |P_1, P_2, P| - |P_1, P_2, P_3| = 0,$$

because this is the determinant of the matrix having two equal rows:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

From this follows that the circle through the three points is described by the equation:

$$\begin{aligned}
&(a^2 v_1 w_1 + b^2 w_1 u_1 + c^2 u_1 v_1) |P_2, P_3, P| \\
&+ (a^2 v_2 w_2 + b^2 w_2 u_2 + c^2 u_2 v_2) |P_3, P_1, P| \\
&+ (a^2 v_3 w_3 + b^2 w_3 u_3 + c^2 u_3 v_3) |P_1, P_2, P| \\
&- (a^2 v w + b^2 w u + c^2 u v) |P_1, P_2, P_3| = 0. \tag{90}
\end{aligned}$$

Notice that the three first rows are the expressions of line equations, and the determinant-equations $\{|P_2, P_3, P| = 0, \dots\}$ represent respectively the side-lines of the triangle $P_1 P_2 P_3$. Since the expression in the last row is the one of the equation of the circumcircle of ABC , the linear part of the first three rows represents the equation of the radical axis of the circle κ through the three points and the circumcircle κ_0 of the triangle of reference ABC . Thus, we have the theorem

Theorem 13. *The equation*

$$\begin{aligned}
&(a^2 v_1 w_1 + b^2 w_1 u_1 + c^2 u_1 v_1) |P_2, P_3, P| \\
&+ (a^2 v_2 w_2 + b^2 w_2 u_2 + c^2 u_2 v_2) |P_3, P_1, P| \\
&+ (a^2 v_3 w_3 + b^2 w_3 u_3 + c^2 u_3 v_3) |P_1, P_2, P| = 0, \tag{91}
\end{aligned}$$

represents the radical axis of the circle κ through the three points $\{P_1, P_2, P_3\}$ expressed in absolute barycentrics and the circumcircle κ_0 of the triangle of reference ABC .

Equation (90) leads to a condition on four points to belong to the same circle by setting $P = P_4$. The equation can be expressed also through a 4×4 determinant. Doing the calculation and some simplifications the formula takes the form:

$$\begin{vmatrix} s(P_1) & s(P_2) & s(P_3) & s(P_4) \\ P_{11} & P_{21} & P_{31} & P_{41} \\ P_{12} & P_{22} & P_{32} & P_{42} \\ P_{13} & P_{23} & P_{33} & P_{43} \end{vmatrix} = 0, \quad (92)$$

where in each column appears the expression

$$s(X) = \frac{a^2 x_2 x_3 + b^2 x_3 x_1 + c^2 x_1 x_2}{(x_1 + x_2 + x_3)}$$

and the not necessarily absolute barycentrics of the involved points.

Equation (92) applied to the two Brocard points $\Omega \left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2} \right)$, $\Omega' \left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2} \right)$ and the symmedian point $K(a^2 : b^2 : c^2)$, leads to the "Brocard circle" passing through these three points:

$$b^2 c^2 u^2 + c^2 a^2 v^2 + a^2 b^2 w^2 - a^4 v w - b^4 w u - c^4 u v = 0. \quad (93)$$

18 Barycentrics relative to two distinct triangles

Here we discuss the relation of barycentrics $(x : y : z)$ w.r.t. the triangle ABC and the barycentrics $(x' : y' : z')$ of the same point X w.r.t. the triangle $A'B'C'$ (see figure 21). Assuming everything expressed w.r.t. ABC in absolute barycentrics and representing

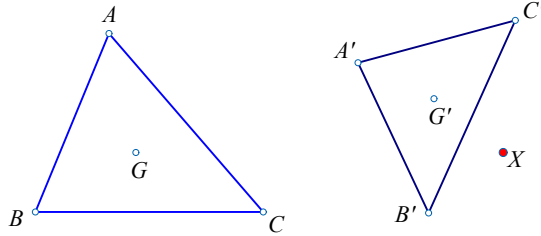


Figure 21: Relating barycentrics relative to ABC and $A'B'C'$

the coordinates with column vectors, we have, for the matrices (X, Y, Z) with columns $\{X, Y, Z\}$, the matrix equations, up to non-zero factors:

$$X = (A, B, C) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (A', B', C') \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (A', B', C') \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix},$$

latter equality being valid because (A, B, C) is the identity matrix. This, if

$$A' = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, B' = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, C' = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = (A', B', C')^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} b_2 c_3 - b_3 c_2 & b_3 c_1 - b_1 c_3 & b_1 c_2 - b_2 c_1 \\ c_2 a_3 - c_3 a_2 & c_3 a_1 - c_1 a_3 & c_1 a_2 - c_2 a_1 \\ a_2 b_3 - a_3 b_2 & a_3 b_1 - a_1 b_3 & a_1 b_2 - a_2 b_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (B \times C)X \\ (C \times A)X \\ (A \times B)X \end{pmatrix}.$$

As an example, consider the centroid G' of the orthic triangle $A'B'C'$ of the triangle of reference ABC . The barycentrics of G' w.r.t. $A'B'C'$ are $(1/3)(1 : 1 : 1)$ and the (absolute) barycentrics of its vertices are

$$A' = \frac{1}{a^2} \begin{pmatrix} 0 \\ S_C \\ S_B \end{pmatrix}, \quad B' = \frac{1}{b^2} \begin{pmatrix} S_C \\ 0 \\ S_A \end{pmatrix}, \quad C' = \frac{1}{c^2} \begin{pmatrix} S_B \\ S_C \\ 0 \end{pmatrix}.$$

Performing the matrix multiplication and simplifying we find that the barycentrics of G' w.r.t. ABC are

$$(a^2((b^2 - c^2)^2 - a^2(b^2 + c^2)) : \dots : \dots),$$

the dots indicating cyclic permutations of $\{a, b, c\}$. This identifies G' with the triangle center $X(51)$ relative to ABC in Kimberling's notation [9].

19 The associated affine transformation

Here we re-examine the matrix M of section 6, defining the transformation from absolute barycentrics to Cartesian coordinates.

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = M \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \quad (94)$$

The matrix M is invertible and its inverse, denoting by S twice the signed area of $\triangle ABC$, is

$$\begin{aligned} M^{-1} &= \frac{1}{A_1(B_2 - C_2) + B_1(C_2 - A_2) + C_1(A_2 - B_2)} \begin{pmatrix} B_2 - C_2 & C_1 - B_1 & B_1C_2 - B_2C_1 \\ C_2 - A_2 & A_1 - C_1 & C_1A_2 - C_2A_1 \\ A_2 - B_2 & B_1 - A_1 & A_1B_2 - A_2B_1 \end{pmatrix} \\ &= \frac{1}{S} \cdot \begin{pmatrix} B_2 - C_2 & C_1 - B_1 & B_1C_2 - B_2C_1 \\ C_2 - A_2 & A_1 - C_1 & C_1A_2 - C_2A_1 \\ A_2 - B_2 & B_1 - A_1 & A_1B_2 - A_2B_1 \end{pmatrix}. \end{aligned} \quad (95)$$

The matrix M defines an invertible linear transformation L_M (an isomorphism) of \mathbb{R}^3 onto itself and the set of all *absolute barycentrics* satisfying $u + v + w = 1$ represents the plane ε of \mathbb{R}^3 , which is orthogonal to the vector $(1, 1, 1) \in \mathbb{R}^3$ and passes through the point $\frac{1}{3}(1, 1, 1) \in \mathbb{R}^3$. The image $\varepsilon' = L_M(\varepsilon)$ is the plane of \mathbb{R}^3 parallel to the coordinate $\{z = 0\}$ -plane through the point $z = 1$. The linear transformation L_M introduces by its restriction on plane ε : $\mathcal{M} = L_M|_{\varepsilon}$ an "affine" transformation ([6, p.199]) between the planes $\{\varepsilon, \varepsilon'\}$:

$$\mathcal{M} : \varepsilon \longrightarrow \varepsilon', \quad \text{with} \quad \mathcal{M}(u, v, w) = (x, y, 1).$$

It is interesting to see some consequences of this interpretation as, for example, the transformation of lines to lines, the preservation of ratios on lines and the preservation of quotients of areas. These are general properties of the affine transformations but can be also deduced here directly for this special case. In fact, a line in ε is the intersection of ε with a *plane through the origin* η of \mathbb{R}^3 : represented by an equation of the form $\eta : pu + qv + rw = 0$. The image-plane $\eta' = \mathcal{M}(\eta)$ is found by writing the equation using matrix notation:

$$pu + qv + rw = 0 \quad \Leftrightarrow \quad 0 = (p, q, r) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = (p, q, r) M^{-1} M \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Hence the image of the plane η is the plane represented by the equation

$$\eta' : p'x + q'y + r'z = 0, \quad \text{where } (p', q', r') = (p, q, r)M^{-1}, \quad (96)$$

and the line of the plane ε' is the intersection $\varepsilon' \cap \eta'$. In particular, the line at infinity of ε , represented through its intersection with the plane $\eta : u + v + w = 0$, maps to the line at infinity, which is the intersection of the plane ε' with the plane with coefficients

$$\eta' : (1, 1, 1)M^{-1} = (0, 0, 1) \quad \text{i.e the plane } z = 0.$$

Another consequence of the *affine* property of the transformation M and its inverse is the pre-servation of ratios along lines. Thus, for two points $\{P, Q\}$ of the plane and their line

$$S(t) = (1 - t)P + tQ, \quad \text{with } r = \frac{t}{t - 1} \quad \text{equal to the signed ratio: } r = \frac{SP}{SQ}, \quad (97)$$

the corresponding barycentrics satisfy the same relation:

$$M^{-1}S(t) = M^{-1} \begin{pmatrix} S_1(t) \\ S_2(t) \\ 1 \end{pmatrix} = M^{-1} \begin{pmatrix} (1 - t)P_1 + tQ_1 \\ (1 - t)P_2 + tQ_2 \\ 1 - t + t \end{pmatrix} = (1 - t)(M^{-1}P) + t(M^{-1}Q).$$

Notice that the function $r = f(t) = t/(t - 1)$ has inverse $f^{-1} = f$, so that given the ratio $r = SP/SQ$ and taking $t = r/(r - 1)$, and setting $\{P', Q', \dots\}$ for the corresponding barycentric vectors of the points $\{P, Q, \dots\}$, we have that

$$S'(t) = (1 - t)P' + tQ' \quad \Leftrightarrow \quad S'(r) = \frac{1}{1 - r}(P' - rQ'), \quad (98)$$

satisfies precisely the relation $SP/SQ = r$. If the ratio r is expressed in the form $r = m/n$, then for the point S_r satisfying this condition and the corresponding barycentrics vectors we obtain ([12]):

$$\frac{S_r P}{S_r Q} = r = \frac{m}{n} \quad \Rightarrow \quad S'_r = \frac{1}{n - m}(nP' - mQ'). \quad (99)$$

As application, we prove the collinearity of the *incenter* $I(a : b : c)$, centroid $G(1 : 1 : 1)$

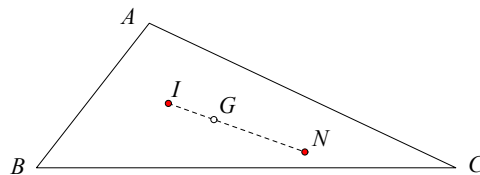


Figure 22: The collinearity of $\{I, G, N\}$

and *Nagel point* $N(b + c - a : \dots)$ ([19]). The relation to verify is (see figure 22):

$$GN/GI = r = -2 \quad \Rightarrow \quad t = r/(r - 1) = 2/3.$$

Thus, turning to absolute barycentrics by dividing the previous barycentrics vectors with the sum $\sigma_N = \sigma_I = a + b + c = 2\sigma$, we obtain:

$$(1 - t)\frac{N}{2\sigma} + t\frac{I}{2\sigma} = \frac{1}{6\sigma}(N + 2I) = \frac{1}{6\sigma}((b + c - a) + 2a : \dots) = \frac{1}{3}(1 : 1 : 1) = G.$$

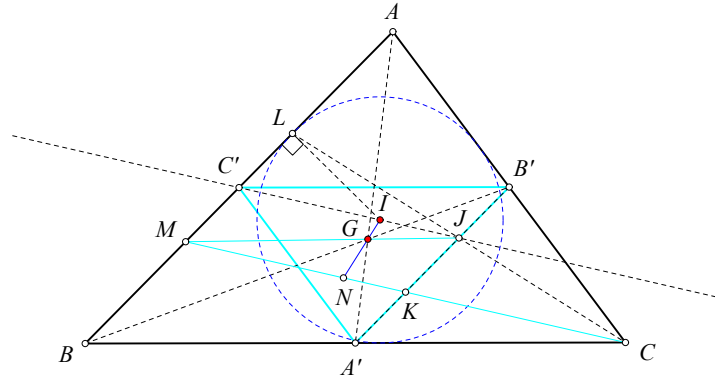


Figure 23: The Nagel point of the anticomplementary

As an application of the preceding result we obtain the property (see figure 23) of “the incenter I to be the Nagel point of the anticomplementary triangle $A'B'C'$ of the middles of sides of ABC ”. This follows from the fact that the homothety with center G and ratio $1/2$, as an affine map, preserves the barycentrics and maps ABC to $A'B'C'$ and also maps the Nagel point N of ABC to I . By the way, we notice also that the triples like the points $\{C', I, J\}$ are collinear (see exercise 11). The triple consisting of the middle of AB , the incenter and the intersection $J = A'B' \cap CL$, where L , the touch point of the incircle with AB . A key fact to prove this synthetically is the equality $AL = MB = \tau - a$, where τ the half-perimeter and $a = |BC|$.

A third property of the affine transformation M , already seen in section 15, is the multiplication of areas by a constant, in this case expressed through equation (82).

20 Relations between barycentrics and Cartesian coordinates

Here we give a second look at the relations between Cartesian coordinates and barycentrics initiated in section 6 using a particular Cartesian coordinate system. From now on the symbols $\{X, Y, \dots\}$ will denote Euclidean two-dimensional position vectors with Cartesian coordinates $X(X_x, X_y)$ w.r.t. to a system with origin at the centroid G of the triangle of reference ABC , assumed to have the positive orientation. The absolute barycentric coordinates of the same point will be denoted by (X_u, X_v, X_w) . The two sets of coordinates are related by the matrix M representing the affine transformation L_M of the preceding section:

$$\begin{pmatrix} X_x \\ X_y \\ 1 \end{pmatrix} = M \cdot \begin{pmatrix} X_u \\ X_v \\ X_w \end{pmatrix}.$$

Under the preceding assumption of the particular Cartesian coordinate system, the two matrices $\{M, M^{-1}\}$ take the form:

$$M = \begin{pmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ 1 & 1 & 1 \end{pmatrix}, \quad M^{-1} = \frac{1}{S} \begin{pmatrix} J(BC)_x & J(BC)_y & S/3 \\ J(CA)_x & J(CA)_y & S/3 \\ J(AB)_x & J(AB)_y & S/3 \end{pmatrix}. \quad (100)$$

Here S is twice the area of $\triangle ABC$ and J denotes the positive rotation by $\pi/2$ acting on vectors $X = (X_x, X_y)$ and transforming them to $J(X) = (-X_y, X_x)$, so that the inner product of two vectors equals their determinant:

$$\langle J(U), V \rangle = -U_y V_x + U_x V_y = |U, V|.$$

This implies, that M^{-1} applied to a point has the form:

$$M^{-1} \cdot \begin{pmatrix} X_x \\ X_y \\ 1 \end{pmatrix} = \frac{1}{S} \begin{pmatrix} |BC, X| \\ |CA, X| \\ |AB, X| \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (101)$$

It is interesting to notice the behavior of the absolute coordinates in the case of the sum of two Euclidean vectors and the multiplication of a vector by a number.

$$\begin{aligned} \begin{pmatrix} (X+Y)_u \\ (X+Y)_v \\ (X+Y)_w \end{pmatrix} &= M^{-1} \begin{pmatrix} (X+Y)_x \\ (X+Y)_y \\ 1 \end{pmatrix} = M^{-1} \left(\begin{pmatrix} X_x \\ X_y \\ 1 \end{pmatrix} + \begin{pmatrix} Y_x \\ Y_y \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \Rightarrow \\ &= \begin{pmatrix} X_u \\ X_v \\ X_w \end{pmatrix} + \begin{pmatrix} Y_u \\ Y_v \\ Y_w \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} (X+Y)_u \\ (X+Y)_v \\ (X+Y)_w \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \end{aligned} \quad (102)$$

Showing that the sum of the barycentrics does not represent the point resulting by summing the corresponding Cartesian coordinates. Analogously for the multiplication by a number λ we have:

$$\begin{aligned} \begin{pmatrix} (\lambda X)_u \\ (\lambda X)_v \\ (\lambda X)_w \end{pmatrix} &= M^{-1} \begin{pmatrix} (\lambda X)_x \\ (\lambda X)_y \\ 1 \end{pmatrix} = M^{-1} \left(\lambda \begin{pmatrix} X_x \\ X_y \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1-\lambda \end{pmatrix} \right) \Rightarrow \\ &= \lambda \begin{pmatrix} X_u \\ X_v \\ X_w \end{pmatrix} + \frac{1-\lambda}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \end{aligned} \quad (103)$$

In particular, we see that taking the negative barycentrics corresponds to considering the symmetric point w.r.t. the origin G (see also formula (109)):

$$\begin{pmatrix} (-X)_u \\ (-X)_v \\ (-X)_w \end{pmatrix} = - \begin{pmatrix} X_u \\ X_v \\ X_w \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (104)$$

Remark 17. Equation (102) written in the form

$$\begin{pmatrix} (X+Y)_u \\ (X+Y)_v \\ (X+Y)_w \end{pmatrix} = \begin{pmatrix} X_u \\ X_v \\ X_w \end{pmatrix} + \begin{pmatrix} Y_u \\ Y_v \\ Y_w \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (105)$$

can be considered as the result in barycentrics of an Euclidean translation by the vector Y . We see that this is not described by the addition of corresponding coordinates as is the case of the expression of the translation in Cartesian coordinates. Also the expression of the *displacement vectors* (section 8), which are differences of barycentrics vectors, do not correspond exactly to the difference of the Cartesian coordinates:

$$\begin{pmatrix} Y_u \\ Y_v \\ Y_w \end{pmatrix} - \begin{pmatrix} X_u \\ X_v \\ X_w \end{pmatrix} = \begin{pmatrix} (Y-X)_u \\ (Y-X)_v \\ (Y-X)_w \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

21 Affine transformations represented in barycentrics

Using the embedding of the plane $(x, y) \in \mathbb{R}^2 \mapsto (x, y, 1)$ into \mathbb{R}^3 , the affine transformation L_M and its inverse, represented by the matrices $\{M, M^{-1}\}$ in (100), we can easily arrive at the matrix representation by barycentrics of the affine transformations of the plane, which comprise the isometries and the similarities as special cases. The commutative diagram below leads to a method to express this matrix in terms of the Cartesian coordinates of the involved points and transformations.

$$\begin{array}{ccc} \mathbb{R}^3(\text{Cartesian coordinates}) & \xrightarrow{M_f} & \mathbb{R}^3(\text{Cartesian coordinates}) \\ \uparrow M & & \downarrow M^{-1} \\ \mathbb{R}^3(\text{absolute barycentrics}) & \xrightarrow{N_f} & \mathbb{R}^3(\text{absolute barycentrics}) \end{array}$$

The representation of a general affine transformation f of the plane by a matrix M_f in Cartesian coordinates has the form

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \xrightarrow{f} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} P_1 & Q_1 & V_1 \\ P_2 & Q_2 & V_2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \Leftrightarrow f(X) = M_f \cdot X,$$

with the determinant $P_1Q_2 - P_2Q_1 \neq 0$. The matrix representing the same transformation in barycentrics results as a product of matrices:

$$N_f = M^{-1} \cdot M_f \cdot M.$$

The matrices $\{M, M^{-1}\}$, introduced in section 6 and having the form (100), explained in section 20, lead to the following expression for the product:

$$M^{-1} \cdot M_f = \frac{1}{S} \begin{pmatrix} |BC, P| & |BC, Q| & |BC, V| + S/3 \\ |CA, P| & |CA, Q| & |CA, V| + S/3 \\ |AB, P| & |AB, Q| & |AB, V| + S/3 \end{pmatrix}.$$

This, multiplying with M , leads finally to the matrix $N_f = M^{-1} \cdot M_f \cdot M$ expressed in terms of the Euclidean coordinates of the vectors $\{A, B, C\}$ w.r.t. a system having its origin at the centroid G of the triangle of reference ABC . The symbol $f(X)$ in the formula below denotes the application of the transformation f to $X(x, y)$ i.e. $f(X) = xP + yQ + V$ all this expressed as a two dimensional vector (omitting the last 1 from $(x, y, 1)$).

$$N_f = \frac{1}{S} \begin{pmatrix} |BC, f(A)| & |BC, f(B)| & |BC, f(C)| \\ |CA, f(A)| & |CA, f(B)| & |CA, f(C)| \\ |AB, f(A)| & |AB, f(B)| & |AB, f(C)| \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (106)$$

Remark 18. It is easily verified that the determinants $\{|BC, V|, \dots\}$ are related to the absolute coordinates $\{u, v, w\}$ of V by the simple formula:

$$\begin{pmatrix} |BC, V| \\ |CA, V| \\ |AB, C| \end{pmatrix} = \frac{S}{3} \begin{pmatrix} 3u - 1 \\ 3v - 1 \\ 3w - 1 \end{pmatrix}. \quad (107)$$

Applying this to formula (106) we find that the matrix N_f can be expressed also in the form

$$N_f = \begin{pmatrix} f(A)_u & f(B)_u & f(C)_u \\ f(A)_v & f(B)_v & f(C)_v \\ f(A)_w & f(B)_w & f(C)_w \end{pmatrix}, \quad (108)$$

where the symbols $(f(X)_u, f(X)_v, f(X)_w)$ denote the absolute barycentrics of the point $f(X)$. Note that this representation of the affinity by N_f is not valid for a general projectivity f , since this would imply that all projectivities fixing the vertices of $\triangle ABC$ have the same matrix representation, which is not true. Affinities, though, are completely determined by their action on the vertices of $\triangle ABC$. Formula (108) conforms to this fact and shows, that affinities, expressed in absolute barycentrics, act as linear transformations and their values are completely determined from those on the “base” consisting of the vertices $\{A, B, C\}$.

In equation (105) we saw the expression of a translation in barycentrics. Another simple example of affine transformation is the *point-symmetry*. This, using formula (106) or its equivalent (108) and the rules of section 20, is easily seen to be expressible in barycentrics in the same typical form as in Cartesian coordinates:

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = -\begin{pmatrix} u \\ v \\ w \end{pmatrix} + 2 \begin{pmatrix} W_u \\ W_v \\ W_w \end{pmatrix}, \quad (109)$$

where (W_u, W_v, W_w) are the barycentrics of the center W of the symmetry. This conforms to the remarks on the ratio of segments along a fixed line of section 19 and could be deduced trivially from the arguments used there. This remark applies also to *homotheties*, which generalize the point-symmetry and in Cartesian coordinates are described by the formula:

$$f(X) = Y = W + k(X - W), \quad (110)$$

where $\{W, k\}$ are respectively the *center* and the *homothety ratio*. The corresponding formula in absolute barycentrics has the same form.

Exercise 10. Show, using formula (108), that the reflection f_{BC} in the side BC of $\triangle ABC$ is represented in barycentrics by the matrix

$$f_{BC}(X) = \begin{pmatrix} -1 & 0 & 0 \\ 2S_C/a^2 & 1 & 0 \\ 2S_B/a^2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Exercise 11. Using absolute barycentrics, the formula (51) for the projection of a point on a line and formula (109), show that the reflection U' of a point $U(u : v : w)$ in the line, expressed by the equation $f(u, v, w) = pu + qv + rw = 0$, is given by the formula

$$U' = U - 2 \frac{f(U)}{f(V)} V, \quad \text{with } V = \begin{pmatrix} S_B(p - r) - S_C(q - p) \\ S_C(q - p) - S_A(r - q) \\ S_A(r - q) - S_B(p - r) \end{pmatrix}. \quad (111)$$

Remark 19. Notice that V represents in barycentrics the orthogonal direction to that of the line $f(u, v, w) = pu + qv + rw = 0$. Also, if you interpret this equation as one defining a “plane” in \mathbb{R}^3 through its origin and V as an orthogonal vector to this plane, which in this case is a multiple of (p, q, r) , then the reflection in this plane expressed in Cartesian coordinates of \mathbb{R}^3 is formally the same with equation (111).

Exercise 12. Referring to figure 23 and using formula (110), show that the barycentrics of the intersection point $J = A'B' \cap CL$ are a non-zero multiple of $(\tau - b : \tau - a : 2\tau - a - b)$. Use this to show the collinearity of points $\{C', I, J\}$.

22 Polar, self-conjugate, de Longchamps circles

These are two circles, real only for obtuse triangles. The polar circle $\lambda(H, r)$ centered at the orthocenter H of the triangle ABC is defined by the property of this point

$$HA \cdot HA' = HB \cdot HB' = HC \cdot HC' = r^2,$$

where $\{A', B', C'\}$ denote the *traces* of the orthocenter on the sides of the triangle. Starting from some simple geometric properties of the corresponding figure 24 we compute its radius. For this we use material from the preceding discussion and in particular identities for the Conway triangle symbols of section 8 and also from the file [15]. From the figure results easily the well known property $HA' = A'E$ and denoting the side-lengths by $\{a = |BC|, b = |CA|, c = |AB|\}$, we have

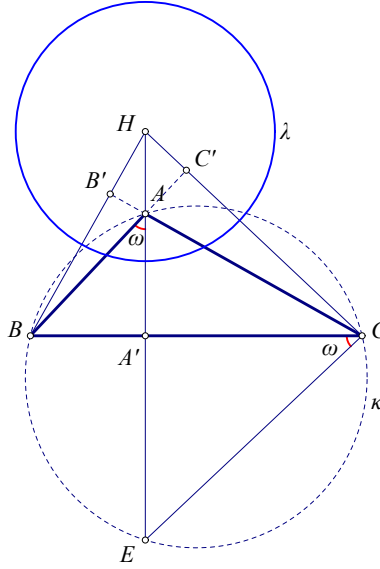


Figure 24: The polar circle $\lambda(H, r)$ of the obtuse triangle ABC

$$\begin{aligned} r^2 &= HA \cdot HA' = HA \cdot A'E = (A'E - A'A) \cdot A'E = \\ &= (CA' \tan(\omega) - c \sin(\widehat{B})) \cdot (CA' \tan(\omega)) = \\ &= (CA' \cot(\widehat{B}) - c \sin(\widehat{B})) \cdot (CA' \cot(\widehat{B})) = \\ &= (b \cos(\widehat{C}) \cot(\widehat{B}) - c \sin(\widehat{B})) \cdot (b \cos(\widehat{C}) \cot(\widehat{B})) = \\ &= \left(\left(b \frac{S_C}{ab} \right) \left(\frac{S_B}{S} \right) - c \sin(\widehat{B}) \right) \cdot \left(b \frac{S_C}{ab} \right) \left(\frac{S_B}{S} \right) = \\ &= \left(\frac{S_C S_B}{aS} - c \sin(\widehat{B}) \right) \left(\frac{S_C S_B}{aS} \right) = \\ &= \left(\frac{S^2 - a^2 S_A}{aS} - c \sin(\widehat{B}) \right) \left(\frac{S^2 - a^2 S_A}{aS} \right) = \\ &= (S^2 - a^2 S_A - Sac \sin(\widehat{B})) \left(\frac{S^2 - a^2 S_A}{a^2 S^2} \right) = \\ &= (S^2 - a^2 S_A - S^2) \left(\frac{S^2 - a^2 S_A}{a^2 S^2} \right) = -S_A \left(\frac{S^2 - a^2 S_A}{S^2} \right) = \end{aligned}$$

$$\frac{a^2 S_A^2 - S^2 S_A}{S^2} = \frac{a^2(b^2 c^2 - S^2) - S^2 S_A}{S^2} = \frac{a^2 b^2 c^2 - S^2(a^2 + S_A)}{S^2} = \quad (112)$$

$$\frac{a^2 b^2 c^2 - S^2 \frac{1}{2}(a^2 + b^2 + c^2)}{S^2} = \frac{4R^2 S^2 - S^2 \frac{1}{2}(a^2 + b^2 + c^2)}{S^2} \Rightarrow$$

$$r^2 = 4R^2 - \frac{1}{2}(a^2 + b^2 + c^2). \quad (113)$$

For the equation of the polar circle we use the formula 65 for the general circle with center H and radius r , taking into account the absolute coordinates of the orthocenter resulting from equation (79)

$$H = \frac{1}{S^2} (S_B S_C : S_C S_A : S_A S_B). \quad (114)$$

Applying formula (34) we have

$$\begin{aligned} HA^2 &= S_A \left(\frac{S_B S_C}{S^2} - 1 \right)^2 + S_B \left(\frac{S_C S_A}{S^2} \right)^2 + S_C \left(\frac{S_A S_B}{S^2} \right)^2 = \\ &= \frac{1}{S^4} (S_A S_B^2 S_C^2 + S_B S_C^2 S_A^2 + S_C S_A^2 S_B^2) - \frac{2}{S^2} S_A S_B S_C + S_A = \\ &= \frac{1}{S^4} S_A S_B S_C (S_B S_C + S_C S_A + S_A S_B) - \frac{2}{S^2} S_A S_B S_C + S_A = \\ &= \frac{1}{S^4} S_A S_B S_C (S^2) - \frac{2}{S^2} S_A S_B S_C + S_A = \\ &= S_A - \frac{1}{S^2} S_A S_B S_C. \end{aligned}$$

Using this and equation (112) for the radius of the polar circle, we get

$$\begin{aligned} r^2 - HA^2 &= \frac{a^2 S_A^2 - S^2 S_A}{S^2} - \left(S_A - \frac{1}{S^2} S_A S_B S_C \right) = \\ &= \frac{a^2 S_A^2 - 2S^2 S_A + S_A S_B S_C}{S^2} = \frac{S_A}{S^2} (a^2 S_A - 2S^2 + S_B S_C) = \\ &= \frac{S_A}{S^2} (-S^2) = -S_A, \end{aligned}$$

and analogously $r^2 - HB^2 = -S_B$ and $r^2 - HC^2 = -S_C$. Replacing these in equation (65) leads then to the equation for the polar circle

$$\begin{aligned} a^2 vw + b^2 wu + c^2 uv - (S_A u + S_B v + S_C w)(u + v + w) &= 0 \Leftrightarrow \\ S_A u^2 + S_B v^2 + S_C w^2 &= 0. \end{aligned} \quad (115)$$

Remark 20. Last equation implies that the triangle of reference ABC is “self-polar” or “self-conjugate” w.r.t. to its polar circle κ , i.e. each sideline of ABC is the polar relative to κ of the opposite vertex. In fact, it is easily seen, that every conic κ having this property, i.e. making the triangle of reference ABC self-conjugate relative to κ , can be expressed in barycentrics through an equation in the form:

$$pu^2 + qv^2 + rw^2 = 0. \quad (116)$$

The de Longchamps circle of the triangle ABC (see figure 25) can be defined as the circle μ homothetic to the polar circle λ in ratio $-2 : 1$ w.r.t. to the centroid G of the triangle [22]. Applying formula (110) for this homothety f we find, working with absolute

barycentrics

$$\begin{aligned}
 Y(u' : v' : w') &= f(X(u : v : w)) = G - 2(X - G) \quad \Leftrightarrow \\
 X &= G + \frac{1}{2}(G - Y) \quad \Leftrightarrow \\
 (u : v : w) &= \frac{1}{3}(1 : 1 : 1) + \frac{1}{2} \left(\frac{1}{3}(1 : 1 : 1) - (u' : v' : w') \right) \quad \Rightarrow \\
 (u : v : w) &= \frac{1}{2}(1 - u' : 1 - v' : 1 - w').
 \end{aligned}$$

Replacing this into the polar equation and dropping the primes, we find the equation of de Longchamps' circle

$$a^2u^2 + b^2v^2 + c^2w^2 + 2S_Cuv + 2S_Avw + 2S_Bwu = 0. \quad (117)$$

The triangle center $X(20)$, which is the center of the de Longchamps' circle, is found by

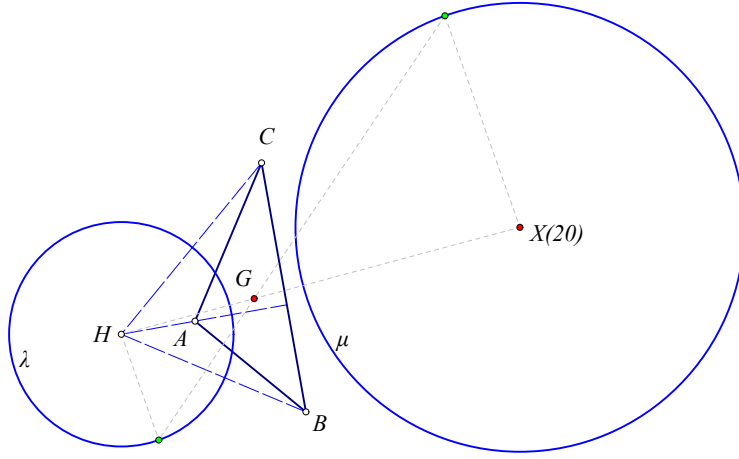


Figure 25: The polar λ and de Longchamps' μ circles of $\triangle ABC$

applying the homothety f to the center H of the polar circle. In absolute barycentrics:

$$X(20) = G - 2(H - G) = \quad (118)$$

$$\begin{aligned}
 &\frac{1}{3}(1 : 1 : 1) - 2 \left(\frac{1}{S^2}(S_B S_C : S_C S_A : S_A S_B) - \frac{1}{3}(1 : 1 : 1) \right) \quad \Rightarrow \\
 X(20) &= \frac{1}{S^2} (S^2 - 2S_B S_C : S^2 - 2S_C S_A : S^2 - 2S_A S_B). \quad (119)
 \end{aligned}$$

Similarly, the radius ρ of this circle is twice the radius (113) of the polar circle:

$$\rho^2 = 16R^2 - 2(a^2 + b^2 + c^2). \quad (120)$$

Exercise 13. Given a triangle ABC and a pair (D, δ) of a point and a line, with $\{A, B, C, D, \delta\}$ in general position, there is precisely one conic κ , relative to which ABC is self-conjugate and δ is the polar of D (see figure 26).

Hint: Take ABC as triangle of reference for barycentric coordinates. Then the conic κ will be represented through an equation of the form $k_1u^2 + k_2v^2 + k_3w^2 = 0$ and the point $D(d_1 : d_2 : d_3)$ and the coefficients $(\delta_1 : \delta_2 : \delta_3)$ of the line δ should satisfy the corresponding pol-polar relation expressed through the matrix equation

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \lambda \begin{pmatrix} \delta_1/d_1 \\ \delta_2/d_2 \\ \delta_3/d_3 \end{pmatrix}, \text{ with some } \lambda \neq 0.$$

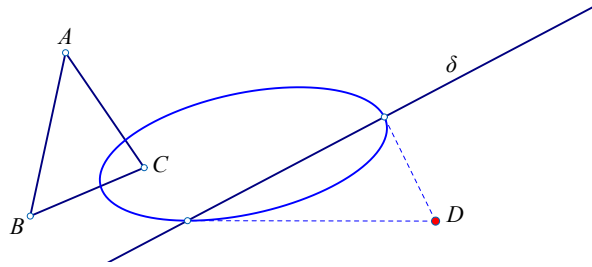


Figure 26: Conic κ with self-conjugate ABC and polar δ w.r.t. D

Theorem 14. *Two triangles $\{ABC, DEF\}$ which are self-polar w.r.t. to some conic κ have their vertices on a conic κ_1 and their sidelines tangent to a conic κ_2 (see figure 27).*

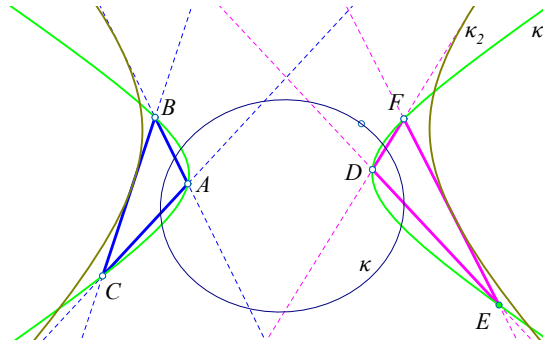


Figure 27: ABC and DEF are self-polar w.r.t. κ

We show the first claim, that the vertices $\{A, B, C, D, E, F\}$ are on a conic κ_1 . To simplify calculations we can use a projectivity J fixing the points $\{A, B, C\}$ and mapping D to the centroid $G(1 : 1 : 1)$ of the triangle ABC . Since projectivities preserve all the properties involved in the theorem, we have to show the first claim for the points $\{A, B, C, G, X_1, X_2\}$, with $\{X_1 = J(E), X_2 = J(F)\}$.

The six points are considered to be in general position. Thus, there is a conic κ_1 passing through the first five $\{A, B, C, G, X_1\}$ and we have to show, that under theorem's hypothesis also the sixth point $X_2 \in \kappa_1$. Considering barycentrics w.r.t. ABC , the matrix N of the conic κ_1 and its equation is of the form

$$X^t N X = X^t \begin{pmatrix} 0 & h & g \\ h & 0 & f \\ g & f & 0 \end{pmatrix} X = 0 \quad \text{with} \quad 0 = X_1^t N X_1 = 2(f y_1 z_1 + g z_1 x_1 + h x_1 y_1). \quad (*)$$

$$\text{Since also } G \in \kappa_1 \Leftrightarrow 0 = G^t N G = 2(f, g, h) \quad (**)$$

$$\xrightarrow{(*), (**)} (f, g, h) = (x_1(y_1 - z_1), y_1(z_1 - x_1), z_1(x_1 - y_1)). \quad (+)$$

Denoting by M the matrix of κ , the self-conjugate hypothesis for the triangle $G X_1 X_2$ implies that the matrix M is diagonal and the pol-polar relation for the point-line pairs

$\{(X_1, GX_2), (X_2, GX_1)\}$ implies

$$M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \quad GX_2 = MX_1 \Leftrightarrow (y_2 - z_2, z_2 - x_2, x_2 - y_2) = (m_1x_1, m_2y_1, m_3z_1),$$

$$GX_1 = MX_2 \Leftrightarrow (y_1 - z_1, z_1 - x_1, x_1 - y_1) = (m_1x_2, m_2y_2, m_3z_2),$$

$$X_1X_2 = MG \Leftrightarrow (y_1z_2 - y_2z_1, z_1x_2 - z_2x_1, x_1y_2 - x_2y_1) = (m_1, m_2, m_3).$$

Using these relations we must show $X_2^tNX_2 = 2(fy_2z_2 + gz_2x_2 + hx_2y_2) = 0$. In fact, it is

$$\begin{aligned} fy_2z_2 + gz_2x_2 + hx_2y_2 &= f \frac{z_1 - x_1}{m_2} \frac{x_1 - y_1}{m_3} + g \frac{x_1 - y_1}{m_3} \frac{y_1 - z_1}{m_1} + h \frac{y_1 - z_1}{m_1} \frac{z_1 - x_1}{m_2} \\ \stackrel{(+)}{=} x_1(y_1 - z_1) \frac{z_1 - x_1}{m_2} \frac{x_1 - y_1}{m_3} + y_1(z_1 - x_1) \frac{x_1 - y_1}{m_3} \frac{y_1 - z_1}{m_1} + z_1(x_1 - y_1) \frac{y_1 - z_1}{m_1} \frac{z_1 - x_1}{m_2} \\ &= \frac{(y_1 - z_1)(z_1 - x_1)(x_1 - y_1)}{m_1m_2m_3} (m_1x_1 + m_2y_1 + m_3z_3) \\ &= \frac{(y_1 - z_1)(z_1 - x_1)(x_1 - y_1)}{m_1m_2m_3} ((y_2 - z_2) + (z_2 - x_2) + (x_2 - y_2)) = 0. \end{aligned}$$

This shows that $X_2 \in \kappa_1$ and completes the first part of the theorem. For the second part, we show, using the preceding relations, that the coefficients of the sidelines of the two triangles $\{ABC, GX_1X_2\}$ satisfy the well known determinantal condition for the existence of a conic passing through six points $\{A, B, C, D, E, F\}$:

$$\begin{vmatrix} A_1^2 & A_2^2 & A_3^2 & A_1A_2 & A_2A_3 & A_3A_1 \\ B_1^2 & B_2^2 & B_3^2 & B_1B_2 & B_2B_3 & B_3B_1 \\ C_1^2 & C_2^2 & C_3^2 & C_1C_2 & C_2C_3 & C_3C_1 \\ D_1^2 & D_2^2 & D_3^2 & D_1D_2 & D_2D_3 & D_3D_1 \\ E_1^2 & E_2^2 & E_3^2 & E_1E_2 & E_2E_3 & E_3E_1 \\ F_1^2 & F_2^2 & F_3^2 & F_1F_2 & F_2F_3 & F_3F_1 \end{vmatrix} = 0.$$

For the coefficients of the lines $\{BC, CA, AB, X_1X_2, GX_2, GX_1\}$ of our configuration this equation becomes:

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ m_1^2 & m_2^2 & m_3^2 & m_1m_2 & m_2m_3 & m_3m_1 \\ (m_1x_1)^2 & (m_2y_1)^2 & (m_3z_1)^2 & (m_1x_1)(m_2y_1) & (m_2y_1)(m_3z_1) & (m_3z_1)(m_1x_1) \\ (m_1x_2)^2 & (m_2y_2)^2 & (m_3z_2)^2 & (m_1x_2)(m_2y_2) & (m_2y_2)(m_3z_2) & (m_3z_2)(m_1x_2) \end{vmatrix} =$$

$$m_1^2m_2^2m_3^2 ((y_2z_1 - y_1z_2)x_1x_2 + (x_1z_2 - x_2z_1)y_1y_2 + (x_2y_1 - x_1y_2)z_1z_2) =$$

$$-m_1^2m_2^2m_3^2 (m_1x_1x_2 + m_2y_1y_2 + m_3z_1z_2) = 0.$$

Last equation being valid because $\{X_1, X_2\}$ are conjugate points w.r.t. the conic κ .

23 Remarks on working with barycentrics

The vectors of barycentric coordinates $\{U_P, U_Q, U_R, \dots \in \mathbb{R}^3\}$ representing the points of the plane $\{P, Q, R, \dots \in \mathbb{R}^2\}$ are not the points we see. What we see are the points of \mathbb{R}^2 .

This is clearly understood in the case of the triangle of reference ABC . The barycentric-vectors representing it are $\{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$ and define the vertices of an equilateral triangle lying on the plane $\varepsilon : u + v + w = 1$ of \mathbb{R}^3 . The map L_M of section 19 is the affine transformation mapping the equilateral $A'B'C'$ onto ABC . By means of it, all properties of the triangle correspond to properties of the equilateral and vice versa.

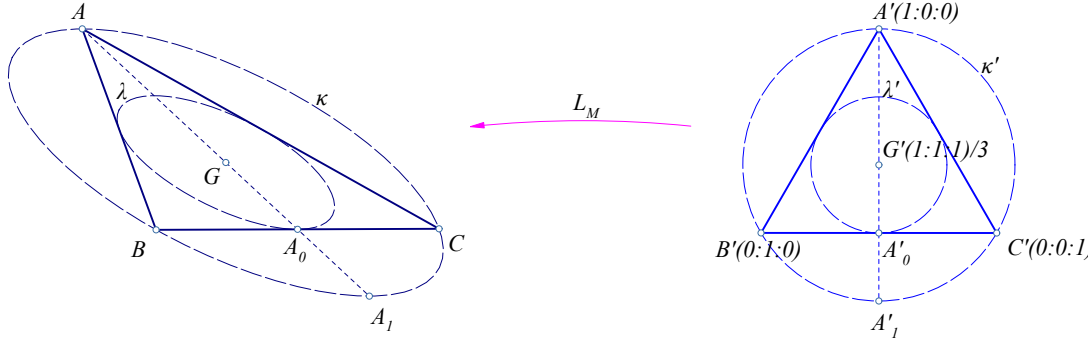


Figure 28: We see ABC and work with $A'B'C'$

In particular, properties of the equilateral which are preserved by affine transformations map to similar properties of ABC . A characteristic example is the circumcircle κ' of the equilateral $A'B'C'$, which carries also the symmetric $\{A'_1, B'_1, C'_1\}$ of the vertices w.r.t. the center G' of the equilateral (see figure 28). The map L_M transforms G' to the centroid G of ABC and the circumcircle of $A'B'C'$ to the “Steiner (outer) ellipse” κ of ABC , characterized by the fact to pass through the vertices of ABC and their symmetric A_1, B_1, C_1 w.r.t. G , point G being its center. Similarly, L_M maps the incircle λ' of $A'B'C'$ to the “Steiner in-ellipse” λ of the triangle ABC , characterized by its tangency at the middles A_0, B_0, C_0 of the sides of ABC . The homothety of $\{\kappa', \lambda'\}$ with center G' and ratio $2 : 1$ translates to the homothety of $\{\kappa, \lambda\}$ with center G and the same ratio.

Invertible affine transformations or *affinities* like L_M , besides the preservation of ratios along lines, map also areas σ to multiples $k\sigma$, with k a constant factor. This implies, that points P' with barycentrics $(u' : v' : w')$ w.r.t. $A'B'C'$ map to corresponding points $P = L_M(P')$ with the same barycentrics $(u : v : w) = (u' : v' : w')$ w.r.t. ABC . Thus, for example, the circumcircle of the equilateral $A'B'C'$ with side $a = |B'C'|$, whose equation, according to section 7 is

$$a^2v'w' + a^2w'u' + a^2u'v' = 0 \quad \Leftrightarrow \quad v'w' + w'u' + u'v' = 0,$$

maps to the Steiner outer ellipse, which in barycentrics w.r.t. ABC must satisfy the same equation:

$$vw + wu + uv = 0. \tag{121}$$

On the other hand, the incircle of the equilateral $A'B'C'$, which, according to section 12, is represented in barycentrics w.r.t. $A'B'C'$ by the equation

$$a^2(vw + wu + uv) - \frac{a^2}{4}(u + v + w)^2 = 0 \quad \Leftrightarrow \quad u^2 + v^2 + w^2 - 2(vw + wu + uv) = 0,$$

maps to the Steiner inner ellipse, which in barycentrics w.r.t. ABC is represented by the same equation:

$$u^2 + v^2 + w^2 - 2(vw + wu + uv) = 0. \tag{122}$$

24 The Steiner point of the triangle

Two non degenerate circumconics κ, κ' of the triangle ABC , i.e. genuine conics passing through its vertices, have a fourth intersection point L (see figure 29), which can be easily determined from the equations of the two conics in barycentrics. These are of the form

$$\begin{aligned} \kappa : p\sigma w + q\omega u + ruv = 0 & \Leftrightarrow \frac{p}{u} + \frac{q}{v} + \frac{r}{w} = 0, \\ \kappa' : p'\sigma w + q'\omega u + r'uv = 0 & \Leftrightarrow \frac{p'}{u} + \frac{q'}{v} + \frac{r'}{w} = 0, \end{aligned}$$

their second equivalent expression having a meaning at points different from the vertices of the triangle, where two of the coordinates vanish. Point L cannot have a vanishing coordinate, since this would imply that it lies on a side of the triangle and the conic would contain a whole line, i.e. would be degenerate, which is something we exclude. Thus, for a point $L(u : v : w)$ with $uvw \neq 0$ the two equations represent a linear homogeneous system for the vector $(1/u : 1/v : 1/w)$ with solution expressed (up to non-zero multiplicative factor) through the vector product of the coefficient vectors

$$\begin{aligned} \left(\frac{1}{u} : \frac{1}{v} : \frac{1}{w}\right) &= (qr' - rq' : rp' - pr' : pq' - qp') \Leftrightarrow \\ (u : v : w) &= \left(\frac{1}{qr' - rq'} : \frac{1}{rp' - pr'} : \frac{1}{pq' - qp'}\right). \end{aligned} \quad (123)$$

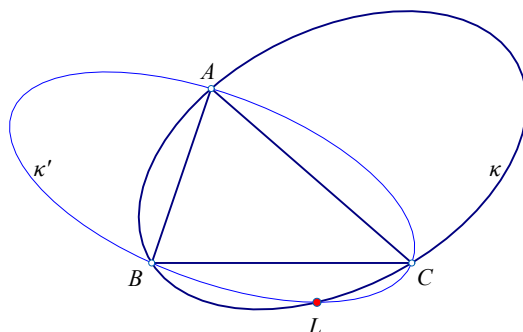


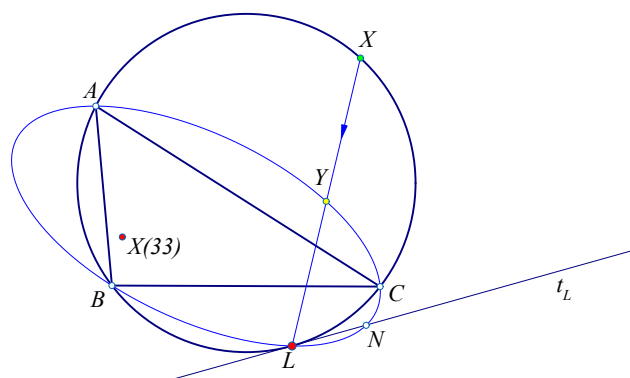
Figure 29: Fourth intersection point L of two circumconics $\{\kappa, \kappa'\}$

The Steiner point, $L = X(99)$ in Kimberling's notation ([9]) (see figure 30), is per definition the fourth intersection point of the circumcircle $\kappa : a^2yx + b^2zx + c^2xy = 0$, with the Steiner ellipse of the triangle $\kappa' : yx + zx + xy = 0$. Applying formula 123 in this special case we obtain its expression in barycentrics

$$L(u : v : w) = \left(\frac{1}{b^2 - c^2} : \frac{1}{c^2 - a^2} : \frac{1}{a^2 - b^2}\right). \quad (124)$$

25 Barycentric product

Figure 30 shows another characteristic of the Steiner ellipse, related to the second intersection N of the ellipse with the tangent t_L of the circumcircle at L and the "third power center" $X(33) = (a^{-2} : b^{-2} : c^{-2})$ of ABC .

Figure 30: The Steiner point $L = X(99)$ of the triangle ABC

Using the “barycentric product” ([28, p.99]) of two points:

$$X(x_1 : x_2 : x_3), Y(y_1 : y_2 : y_3), \quad X \cdot Y = (x_1 y_1 : x_2 y_2 : x_3 y_3),$$

we can easily verify that the “radial projection” $X \mapsto Y$ with center at L , of the points $\{X\}$ of the circle to corresponding points $\{Y\}$ on the Steiner ellipse, is given by the simple rule $Y = X(33) \cdot X$. Using also the corresponding to the homonymous product, barycentric division: $X/Y = (x_1/y_1 : x_2/y_2 : x_3/y_3)$, we can easily verify that

$$N = \left(\frac{b^2 c^2}{b^2 - c^2} : \frac{c^2 a^2}{c^2 - a^2} : \frac{a^2 b^2}{a^2 - b^2} \right) \quad \text{and} \quad N/L = X(33).$$

We should notice that the transformation $g_M(X) = M \cdot X$ for a fixed point M of the plane, like the preceding $g_{X(33)}(X) = X(33) \cdot X$, represents a “homography” (projective transformation or projectivity) of the plane fixing the vertices of the triangle of reference ABC .

In fact, every homography f of the plane, having precisely three fixed points coinciding with the vertices of the triangle of reference ABC , is expressible through the barycentric product with an appropriate point

$$X_f : Y = f(X) \quad \Leftrightarrow \quad Y = X_f \cdot X.$$

This is an immediate consequence of the “fundamental theorem of projective geometry”, according to which ([25, I, p. 96], [1, I, p.97]):

“A homography is uniquely determined by prescribing its values at four points in general position”.

If the homography f has precisely three fixed points coinciding with the vertices of ABC , consider a point X_0 in general position w.r.t. ABC and its image $Y_0 = f(X_0)$ (see figure 31). Define then the barycentric quotient, point $X_f = Y_0/X_0$. The transformation of the plane $g(X) = X_f \cdot X$ is a homography which fixes the vertices of ABC and also maps X_0 to Y_0 . Thus, on the quadruple $\{A, B, C, X_0\}$ it coincides with the homography f , consequently, according to the aforementioned fundamental theorem, g coincides with f everywhere.

Figure 31 suggests also another aspect of the homographies fixing the vertices of ABC . This is related to another “fundamental theorem”, according to which ([25, I, p.213], [4, p.167, 169], [5, p. 91], [1, II, p.179]):

“Given a triple $\{A, B, C\}$ of distinct points on a conic κ_1 and a triple of distinct points $\{A', B', C'\}$ on the conic κ_2 , there is a unique homography f mapping κ_1 onto κ_2 and also mapping the three points $\{A, B, C\}$ correspondingly to $\{A', B', C'\}$ ”.

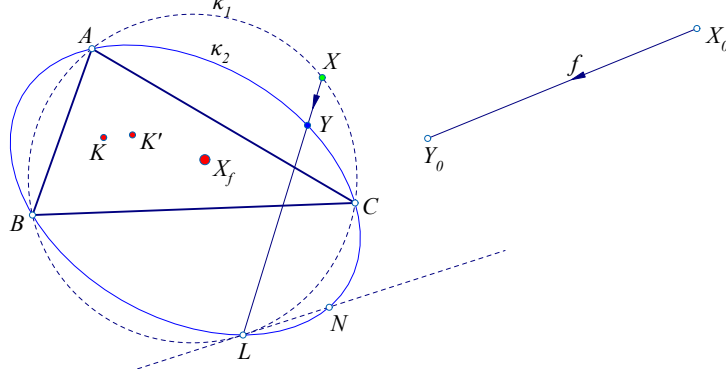


Figure 31: Homography f fixing the vertices of the triangle of reference ABC

This theorem can be applied in the case of the configuration involving two distinct circumconics $\{\kappa_1, \kappa_2\}$, as in figure 31, with $\{A' = A, B' = B, C' = C\}$. By that theorem, there is a homography f fixing $\{A, B, C\}$ and mapping κ_1 onto κ_2 . By the preceding discussion, this homography is expressible through the barycentric product and a fixed point $X_f : f(X) = X_f \cdot X$. In addition, for this configuration the following theorem is valid.

Theorem 15. *Two circumconics $\{\kappa_1, \kappa_2\}$ of the triangle of reference define a unique homography f fixing the vertices of the triangle of reference ABC and mapping κ_1 onto κ_2 . In addition, each point $X \in \kappa_1$ maps to $Y = f(X) \in \kappa_2$ and the line XY passes through the fourth intersection point L of the two conics (see figure 31).*

Proof. In view of the preceding discussion, we need to prove only the second claim. A crucial point is that the perspector $K(f_1 : g_1 : h_1)$ of κ_1 maps via f to the perspector $K'(f_2 : g_2 : h_2)$ of κ_2 . This means that in the description of the two conics

$$\begin{aligned} \kappa_1 : \frac{f_1}{x} + \frac{g_1}{y} + \frac{h_1}{z} = 0 \quad \text{and} \quad \kappa_2 : \frac{f_2}{x} + \frac{g_2}{y} + \frac{h_2}{z} = 0 &\Rightarrow \\ K' = f(K) = X_f \cdot K &\Leftrightarrow (f_2 : g_2 : h_2) = (z_1 f_1 : z_2 g_1 : z_3 h_1) \quad \text{for} \\ (z_1 : z_2 : z_3) &= X_f = \left(\frac{f_2}{f_1} : \frac{g_2}{g_1} : \frac{h_2}{h_1} \right). \end{aligned}$$

Having that, and using the fourth point formula (123) we compute

$$\begin{aligned} L &= \left(\frac{1}{g_1 h_1 (z_3 - z_2)} : \frac{1}{h_1 f_1 (z_1 - z_3)} : \frac{1}{f_1 g_1 (z_2 - z_1)} \right) \Rightarrow \\ \det(X, Y, L) &= \frac{1}{f_1 g_1 h_1} (f_1 x_2 x_3 + g_1 x_3 x_1 + h_1 x_2 x_1) = 0, \end{aligned}$$

showing that the three points $\{X, Y, L\}$ are collinear for every point $X \in \kappa_1$. \square

Remark 21. We notice that this radial projection of $f : \kappa_1 \rightarrow \kappa_2$, with $Y = f(X)$ and $\{X, Y, L\}$ collinear, is valid only for its restriction on κ_1 . For example, in figure 31 f maps the perspector K of κ_1 to the perspector K' of κ_2 but the line KK' does not pass through L .

Remark 22. Letting X converge towards $L \in \kappa_1$, we see that the line XY tends to coincide with the tangent t_L to κ_1 at L and $Y = X_f \cdot L$ coincides with the second intersection N of t_L with the conic κ_2 . Thus $N = X_f L$ and homography f and its corresponding X_f could be also defined through the barycentric quotient $X_f = N/L$.

Remark 23. The preceding discussion concerning the fourth intersection point of two circumconics of the triangle of reference ABC can be generalized to an arbitrary pair of conics intersecting at four points in general position. We can select three out of the four points and name them $\{A, B, C\}$ and the remaining fourth denote by L . Then, the preceding discussion can be applied and analogous results can be proved.

Exercise 14. Given a triangle ABC and two pairs $\{(D, \delta), (E, \varepsilon)\}$, of a point and a line, the seven elements $\{A, B, C, D, E, \delta, \varepsilon\}$ in general position, there is a conic κ for which ABC is self-conjugate and for each pair the line is the polar of the corresponding point, if and only if the barycentric products with the tripoles of the lines satisfy:

$$D \cdot \text{tr}(\delta) = E \cdot \text{tr}(\varepsilon).$$

Hint: For the definition of self-polar see remark 20. The proof results immediately from exercise 13.

Remark 24. Let $\mathcal{M} = \{P(x : y : z) : xyz \neq 0\}$ denote the complement in the plane of the sidelines of the triangle of reference ABC . It can be easily checked that the points of \mathcal{M} form a group w.r.t. to the barycentric product with unit the centroid G of the triangle.

It is likely that there is no irreducible algebraic curve of degree $n > 1$ whose points in \mathcal{M} form a subgroup w.r.t. to the barycentric product. Some argument used below can be applied to show this for odd degrees $n = 2k + 1$ with $k > 0$. Next exercise deals with this question for the case $n = 2$ of conics.

Exercise 15. Show that the only degenerate conics (their points in \mathcal{M}) which are subgroups w.r.t. the barycentric product are the following:

1. The union of two lines $\{\mu, \mu'\}$ with μ a median line, like AM , and the corresponding parallel μ' to BC (see figure 32-(I)).
2. A median line μ , like line AM in the figure.
3. The segment through a vertex and the middle of the opposite side, like AM in the figure.

Show also that there are no genuine (non-degenerate) conics κ whose points lying in \mathcal{M} form a subgroup w.r.t. the barycentric product.

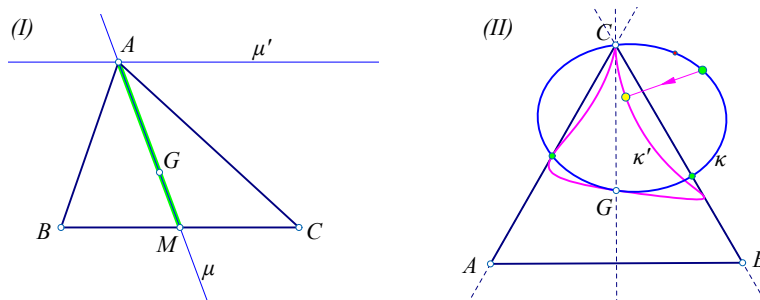


Figure 32: Barycentric product of points of a conic through $\{G, C\}$

Hint: Every subgroup must contain G and every point X of the plane has X^2 lying in the inner domain of ABC . If a line through G is invariant under the barycentric product,

then it must go through a vertex of ABC . Also the barycentric square X^2 of the points X on a line through A map on a segment running from A to a point A' on the side BC , etc. In [20] there is an account of the conics generated from lines by applying to their points the transformation of the barycentric square $X \mapsto X^2$.

Regarding the genuine conics, notice that every such conic κ must go through some vertex of the triangle, C say. To see this consider a point $X(x : y : z) \in \kappa$ and take the powers $\{X^n\}$ after dividing $\{x, y, z\}$ with the greater of them. The conic κ must have points lying on some external domain of the triangle whose. To see this look at the tangent at C . The barycentric squares X^2 are again points of the conic lying in the interior of the triangle. This denies the existence of such conics. Figure 32 shows the image κ' of a conic κ through $\{G, A\}$ under the map $X \mapsto X^2$.

References

- [1] M. Berger. *Geometry vols I, II*. Springer Verlag, Heidelberg, 1987.
- [2] F. Capitan. Coordenadas baricéntricas. <https://garciacapitan.epizy.com/baricentricas/?i=1>, 2025.
- [3] J. Casey. *A treatise on the analytical geometry of the point, line, circle and conic sections*. Hodges Figgis and Co., Dublin, 1893.
- [4] M. Chasles. *Traite de Sections Coniques*. Gauthier-Villars, Paris, 1865.
- [5] H. Coxeter. *The Real Projective Plane*. McGraw-Hill, New York, 1949.
- [6] H. Coxeter. *Introduction to Geometry*. John Wiley and Sons Inc., New York, 1961.
- [7] K. Doehlemann. *Geometrische Transformationen I, II*. Göschen'sche Verlagshandlung, Leipzig, 1908.
- [8] G. Glaeser, H. Stachel, and B. Odehnal. *The universe of conics*. Springer, Berlin, 2016.
- [9] K. Kimberling. Encyclopedia of triangle centers. <https://faculty.evansville.edu/ck6/encyclopedia/ETC.html>, 2025.
- [10] S. Loney. *The elements of coordinate geometry I, II*. AITBS publishers, Delhi, 1991.
- [11] A. Montesdeoca. Geometria metrica y proyectiva en el plano con coordenadas baricéntricas. algunos topicos. <https://amontes.webs.ull.es/pdf/geoba.pdf>, 2017.
- [12] P. Moses and C. Kimberling. Intersections of lines and circles. *Missouri J. Math. Sci*, 19:176–187, 2007.
- [13] P. Pamfilos. Cross Ratio. <http://users.math.uoc.gr/~pamfilos/eGallery/problems/CrossRatio.pdf>, 2019.
- [14] P. Pamfilos. Ceva's theorem. <http://users.math.uoc.gr/~pamfilos/eGallery/problems/Ceva.pdf>, 2019.
- [15] P. Pamfilos. Conway triangle symbols. <http://users.math.uoc.gr/~pamfilos/eGallery/problems/Conway.pdf>, 2019.

- [16] P. Pamfilos. Symmedian. <http://users.math.uoc.gr/~pamfilos/eGallery/problems/Symmedian.pdf>, 2019.
- [17] P. Pamfilos. *Lectures on Euclidean Geometry vols. I, II*. Springer, Heidelberg, 2024.
- [18] P. Pamfilos. Menelaus' theorem. <http://users.math.uoc.gr/~pamfilos/eGallery/problems/Menelaus.pdf>, 2024.
- [19] P. Pamfilos. Nagel center. <http://users.math.uoc.gr/~pamfilos/eGallery/problems/Nagel.pdf>, 2024.
- [20] P. Pamfilos. A note on the barycentric square. *Global Journal of Advanced Research on Classical and Modern Geometries*, 13:119–130, 2024.
- [21] P. Pamfilos. The projective plane. <http://users.math.uoc.gr/~pamfilos/eGallery/problems/ProjectivePlane.pdf>, 2024.
- [22] I. Patrascu and F. Smarandache. De Longchamps' point, line and circle. <https://www.researchgate.net/publication/262036841>, 1998.
- [23] D. Pedoe. *A course of Geometry*. Dover, New York, 1990.
- [24] G. Salmon. *A treatise on Conic Sections*. Longmans, Green and Co., London, 1917.
- [25] O. Veblen and J. Young. *Projective Geometry vol. I, II*. Ginn and Company, New York, 1910.
- [26] V. Volenec. Metrical relations in barycentric coordinates. *Mathematical Communications*, 8:55–68, 2003.
- [27] A. Whitworth. *Trilinear Coordinates*. Cambridge University Press, 1866.
- [28] P. Yiu. Introduction to the geometry of the triangle. <http://users.math.uoc.gr/~pamfilos/Yiu.pdf>, 2013.