1 The two main types of pencils

The first main type of a “pencil of circles” or “coaxal system of circles” ([Ped90, p.106]) is the set of all circles passing through two points \(A, B\) called “base points” of the pencil (See Figure 1). All pairs of circles of this pencil share the same “radical axis”, which is the line \(AB\) and is called the “radical axis of the pencil”. The orthogonal bisector line of the segment \(AB\) carries the centers of all the circles of the pencil and is called the “line of
1 The two main types of pencils

This type of pencil often called “hyperbolic” is also characteristically called “intersecting pencil of circles.” The pencil has a smallest circle $\kappa$, having as diameter the segment $AB$. There is no biggest circle in this pencil. With growing radius the circle tends to the common radical axis, the line $AB$, which is considered also as a degenerate member of the pencil (a circle of infinite radius).

The second main type is the set of “Apollonian circles of a segment $AB$” (See Figure 2). By their definition the points $\{X\}$ of such a circle $\kappa$ satisfy the condition

$$X \in \kappa \iff \frac{XA}{XB} = k, \text{ with a constant } k.$$

The circles have now no common points and the pencil, often called “elliptic”, is also characteristically called a “non-intersecting pencil of circles”. The circles of the pencil have in this case also a common radical axis which coincides with the orthogonal bisector line of the segment $AB$, called the “radical axis of the pencil”. The “line of centers” of the pencil is now line $AB$. The points $\{A, B\}$ are called “limit points” of the pencil. They are considered as members of the pencil with radius zero (see file Apollonian circles). The circles $\kappa$ of this pencil can be identified with those of the following theorem.

**Theorem 1.** The limit points $\{A, B\}$ of a non intersecting pencil are inverse relative to every circle $\kappa$ of the pencil. They are also harmonic conjugate relative to the diametrically opposite points $\{U, V\}$ of $\kappa$, which are defined from its intersection with the center line of the pencil. Thus, every pair $(U, V)$ of “harmonic conjugates” to $(A, B)$ is the diameter of a circle of the pencil and vice-versa.
2 The tangential type of pencils

The two previous are the main types of pencils. There are also some other pencils, which can be considered as limiting cases of these two. Next pencil can be considered as limiting case of a pencil of intersecting type, resulting when one of the common points, \( B \) say, tends to coincide with the other common point \( A \). Figure 3 displays the resulting pencil of circles often called “parabolic”. Its members are all tangent to a line \( \varepsilon \) at a fixed point \( A \) of this line, therefore it is characteristically called a “tangential pencil of circles”. The line \( \varepsilon \) is the common radical axis of all pairs of circles of the pencil, called the “radical axis of the pencil” and the orthogonal to it at \( A \) is the “line of centers of the pencil”.

3 Pencils described by equations

The equation of a circle in cartesian coordinates results by expanding the simple expression deriving directly from the definition of a circle \( \kappa(X_0, r) \) with center \( X_0 = (x_0, y_0) \) and radius \( r \):

\[
\kappa : \quad (x - x_0)^2 + (y - y_0)^2 = r^2 \quad \Leftrightarrow \quad x^2 + y^2 - 2x_0x - 2y_0y + (x_0^2 + y_0^2 - r^2) = 0.
\]

The slightly more general equation involving a function of two variables

\[
f(x, y) = a(x^2 + y^2) + 2bx + 2cy + d = 0 \quad (1)
\]

comprises the case of lines \((a = 0)\), considered as circles of infinite radius. Notice that the equation comprises, for appropriate coefficients, the case of single points

\[
(x - x_0)^2 + (y - y_0)^2 = 0,
\]

considered as circles with radius \( r = 0 \). Comparing equation (1) with the one preceding it, we see that the center of the circle is given by

\[
x_0 = -\frac{b}{a}, \quad y_0 = -\frac{c}{a} \quad \text{and the radius}
\]

\[
x_0^2 + y_0^2 - r^2 = \frac{d}{a} \quad \Leftrightarrow \quad r^2 = \frac{b^2 + c^2 - ad}{a^2}. \quad (2)
\]

The pencils of circles are “lines” of circles defined as linear combinations of two “points” in the space of all circles expressed through equations similar to equation (1). This means
that they are described by equations of the form

\[ \lambda f_1(x, y) + \mu f_2(x, y) = 0 \Leftrightarrow \lambda(a_1(x^2 + y^2) + 2b_1x + 2c_1y + d_1) + \mu(a_2(x^2 + y^2) + 2b_2x + 2c_2y + d_2) = 0. \]

(3)

The various members result by varying the pair of parameters \((\lambda, \mu)\). Pairs of parameters satisfying \((\lambda', \mu') = k(\lambda, \mu)\) with \(k \neq 0\), define the same circle-member, so that essentially only the quotient \((\lambda : \mu)\) is relevant for the definition of a member. The unique member of such a pencil, which degenerates to a line, is obtained by setting

\[ \lambda a_1 + \mu a_2 = 0 \Leftrightarrow (\lambda : \mu) = (-a_2 : a_1) \]

and the resulting line-equation for these values of \((\lambda, \mu)\)

\[ 2(\lambda b_1 + \mu b_2)x + 2(\lambda c_1 + \mu c_2)y + (\lambda d_1 + \mu d_2) = 0 \]

expresses then the “radical axis” of the pencil. The kind of the pencil can be recognized by the number of circles of radius \(r = 0\) i.e. points it comprises. This in turn, by means of equation (2), is controlled by the number of solutions of equation

\[ b^2 + c^2 - ad = 0 \Leftrightarrow (\lambda b_1 + \mu b_2)^2 + (\lambda c_1 + \mu c_2)^2 - (\lambda a_1 + \mu a_2)(\lambda d_1 + \mu d_2) = 0, \]

which is homogeneous w.r. to \((\lambda, \mu)\) and setting \(t = \lambda/\mu\) leads to the quadratic

\[ (tb_1 + b_2)^2 + (tc_1 + c_2)^2 - (ta_1 + a_2)(td_1 + d_2) = 0. \]

(5)

If this quadratic has two real solutions, then the pencil has two point-members, which is the case of the “elliptic” pencil of “non-intersecting” type, the points being its “limit points”. If there is only one solution then the pencil is “parabolic” of the “tangential” type, and if there are no real solutions, then the pencil is “hyperbolic” of the “intersecting” type and all circles pass through two fixed points \(\{A, B\}\), the “base points” of the pencil.

### 4 Common properties to all types of pencils

Next properties, formulated as separate propositions, are easy consequences of the definitions and can be proved either geometrically or using the analytic description of the previous section 3 (see also [Mcl91]).

**Proposition 1.** Every pair consisting of a circle \(\mu(N, r)\) and a line \(\varepsilon\), not passing through the center \(N\) of the circle, defines exactly one pencil of circles which contains the circle \(\mu\) and has corresponding radical axis the line \(\varepsilon\).

**Proposition 2.** Every pair of different points \((A, B)\) of the plane defines a unique pencil of non-intersecting type having them as limit points.

**Proposition 3.** Every pair \((A, \varepsilon)\) of a point \(A\) and a line or circle \(\varepsilon\) of the plane defines a unique pencil of non-intersecting type or tangential type containing them as members.

**Proposition 4.** Every pair of non concentric circles defines exactly one circle pencil which contains them as members.
The previous propositions show that given a pair \((\alpha, \beta)\) whose members are points, lines or circles, with the exclusion of two lines, defines a pencil of circles containing them, for which we say that it is "generated" by the pair.

**Proposition 5.** Given a pencil of circles, for every point \(X\) of the plane there exists exactly one pencil-member \(\kappa\) which passes through it.

**Proposition 6.** Given a circle pencil and a circle \(\kappa\), which does not belong to the pencil and does not have its center at the center line of the pencil, the radical axes of the pairs of circles \((\kappa, \mu)\), where \(\mu\) is a circle that belongs to the pencil, pass all through a fixed point \(C\) of the radical axis of the pencil (See Figure 4).

**Proposition 7.** Given the circles \(\{\kappa(A, \alpha), \lambda(B, \beta)\}\), the geometric locus of points \(X\) for which the ratio of powers relative to these two circles is constant is a circle \(\mu\), which belongs to the pencil defined by \(\{\kappa, \lambda\}\).

**Proposition 8.** The circle \(\mu\) belongs to the pencil \(\mathcal{P}\), which is generated by the circles \(\{\kappa, \lambda\}\) if and only if, for every point \(X\) of \(\mu\) the ratio of powers of \(X\) relative to the circles \(\{\kappa, \lambda\}\) is constant.

**Proposition 9.** Given the circles \(\kappa(A, \alpha)\) and \(\lambda(B, \beta)\), the geometric locus of points \(X\) for which the ratio of the lengths of the tangents to the two circles \(\kappa\) and \(\lambda\) is fixed, is a circle \(\mu\) which belongs to the pencil defined by \(\kappa\) and \(\lambda\).

**Proposition 10.** Given the circle \(\kappa(O, R)\) and a point \(T\), different from the center \(O\) of the circle, the geometric locus of the points \(A\) for which the ratio of the lengths \(\frac{|AT|}{|AX|} = k\) is constant, where \(AX\) is the tangent to \(\kappa\) from \(A\), is a circle \(\lambda\) of the pencil generated by the circle \(\kappa\) and the point \(T\) (See Figure 6).
Proposition 11. Given the circle \( \kappa(O, r) \) and a point \( T \), three other points \( \{A, B, C\} \) have the same ratio of lengths of tangents to their distance from \( T \)

\[
\frac{|AX|}{|AT|} = \frac{|BY|}{|BT|} = \frac{|CZ|}{|CT|}
\]

if and only if, the circumscribed circle \( \lambda \) of triangle \( ABC \) belongs to the pencil generated by the circle \( \kappa \) and the point \( T \) (See Figure 6).

Exercise 1. Given two circles \( \{\kappa_1(O_1, r_1), \kappa_2(O_2, r_2)\} \) show that the distance \( d_\ell \) of the limit points of the pencil they generate is

\[
d_\ell = \frac{2r_1r_2}{|O_1O_2|} \sqrt{\frac{r_1^2 + r_2^2 - |O_1O_2|^2}{4r_1^2r_2^2}} - 1.
\]

which has a value: real for elliptic, zero for parabolic and imaginary for hyperbolic pencils.

5 Orthogonal circles

Orthogonal circles are formed with the help of tangents to circles from a given point. The tangents \( \{PA, PB\} \) to the circle \( \kappa(O, r) \) from a point \( P \) are equal, therefore they define a circle \( \lambda(P, r') \), which has these tangents as radii (See Figure 7-I). At the intersection point

Figure 7: Right circles

A of the two circles the angle between their radii is a right one. This is a mutual relation. The circle \( \kappa \) can be considered that it results the same way, from the tangents to \( \lambda \) from \( O \). Two intersecting circles, whose radii at the intersection points are orthogonal are called “orthogonal circles”. By definition, therefore, this is equivalent to:

“At one of their intersection points, the radius of one is tangent to the other”.

From this characteristic property follow also the two next corollaries.
Corollary 1. Two circles \( \{ \kappa(O,r), \lambda(P,s) \} \) are orthogonal, if and only if
\[
|OP|^2 = r^2 + s^2.
\]

Corollary 2. Two circles \( \{ \kappa, \lambda \} \) are orthogonal, if and only if the diametrically opposite points \( \{ A, B \} \) of one and \( \{ C, D \} \) of the other, on their center line, build a harmonic quadruple (See Figure 7-II).

This follows directly from the characteristic property of a harmonic quadruple of points.

Exercise 2. Construct a circle \( \lambda \), orthogonal to a given circle \( \kappa(O,r) \) and having its center at a given point \( P \) external to \( \kappa \).

Exercise 3. Show that the circles \( \{ \kappa, \lambda \} \), intersecting at points \( \{ A, B \} \) are orthogonal, if and only if one of the following conditions holds:

1. Every line through point \( A \) defines points \( \{ M, C \} \) on circles \( \{ \kappa, \lambda \} \) such that the angle \( \angle CBM \) is a right one.
2. For every point \( C \) of \( \lambda \), the lines \( \{ CA, CB \} \) intersect again the circle \( \kappa \) at diametrically opposite points \( \{ M, N \} \).
3. The tangents to the circles at \( \{ M, C \} \) intersect orthogonally.

Hint: The triangles \( EAZ \) and \( BMC \) are similar even when the circles are not orthogonal (See Figure 8-I). (2) draw \( BM \) and use (a) (See Figure 8-II).

Theorem 2. The circles \( \mu \), which are simultaneously orthogonal to the non concentric circles \( \{ \kappa, \lambda \} \), have their centers on the radical axis of \( \{ \kappa, \lambda \} \).

If the circles \( \mu(S,\rho) \) and \( \kappa(O,\rho') \) are orthogonal, then their radii at one of their intersection points \( A \) will be orthogonal (See Figure 9-I), therefore line \( SA \) will be tangent to \( \kappa \).

Consequently the tangents from point \( S \) towards the given circles will be equal and point \( S \) will lie on the radical axis of \( \kappa \) and \( \lambda \).

Corollary 3. A circle \( \mu \) orthogonal to two other circles \( \{ \kappa, \lambda \} \) is simultaneously orthogonal also to every circle of the pencil generated by \( \{ \kappa, \lambda \} \).

If \( \mu \) is orthogonal to \( \{ \kappa, \lambda \} \) its center will be contained in the radical axis of \( \{ \kappa, \lambda \} \), which is also the radical axis of the pencil produced by \( \{ \kappa, \lambda \} \). Consequently \( \mu \) will have the same power relative to all the circles of this pencil. Thus, if \( \mu(S,r) \) intersects a third circle \( v \) of the pencil at \( A \), then the radius \( SA \) of \( \mu \) will also be tangent to \( v \), therefore the two circles will be orthogonal.
6 Orthogonal pencils

By means of corollary 3 pencils “go in pairs”. Each pencil defines a second, consisting of all circles which are orthogonal to all circles of the first pencil. In the case of an intersecting pencil, the orthogonal one is a non-intersecting (See Figure 10-I) and in the case of non-intersecting the orthogonal is an intersecting pencil. Thus, orthogonality interchanges these two types of pencils. The radical axis of one becomes line of centers of the other.

In the case of a tangential pencil, its orthogonal consists of all circles passing through the same contact point $A$ of the first pencil, but this time tangent there at the line which is orthogonal to the common tangent of the first pencil. Thus, in this case the orthogonal pencil of a tangential pencil is also a tangential pencil.

7 Three exeptional types of pencils

Next figure displays the three last types of pencils consisting, in the first case, of lines passing from a fixed point (See Figure 12-I). These lines are considered as very large circles of an intersecting pencil, whose one of the common points $B$ is at infinity. The pencil in this case coincides with a “pencil of lines”. The next type results from a tangential pencil, whose contact point $A$ is at infinity, hence the circles become parallel lines (See Figure 12-II). The third pencil consists of concentric circles (See Figure 12-III) and can be considered as limiting case of a non-intersecting pencil, in which the point $B$ tends to
Figure 11: Pencil orthogonal to a tangential one

Figure 12: Pencils resulting from the previous as limiting cases

Figure 13: Pencils orthogonal to exceptional pencils

Three exceptional types of pencils

ε
(I) (II) (III)

Theorem 3. An inversion w.r. to a circle \( \kappa \) maps a pencil of circles to another pencil of circles.

The theorem follows from the property of inversions to map the set \( \mathcal{K} \) of all circles and all lines of the plane into \( \mathcal{K} \) itself ([Joh60, p.44]). As will be seen below, using an appropriate inversion, it is possible to invert a usual pencil of circles to an exceptional one and vice-versa.
8 Orthogonals of the exceptional types of pencils

In the case of "exceptional" pencils, the orthogonals are seen in figure 13. In accordance with the main types from which they result as limiting cases, here we have again an interchange of types $(I) \leftrightarrow (III)$ and an orthogonal of the same type in figure 13-II.

9 Reduction to exceptional cases

Using an appropriate "inversion transformation" we can transform the main types of pencils and the tangential one to corresponding exceptional types. For the intersecting type we can apply an inversion with center at one of the common points $\{A, B\}$ of the pencil, and an arbitrary radius. Figure 14 shows the inversion on circle $\lambda(B, |AB|)$ of a pencil of intersecting type. The inverted of the circle members are the lines through $A$. Thus, by this inversion the pencil transforms to the pencil of lines through $A$. Each circle transforms to the line passing through $A$ and its second intersection with $\lambda$. The smallest circle of the pencil $\kappa$ transforms to the line $\kappa'$ tangent to $\kappa$ at $A$.

Figure 14: Intersecting pencil inverted to pencil of lines

Figure 15 shows the inversion on circle $\lambda(B, |AB|)$ of a pencil of non-intersecting type. The inverted of the circle members are the concentric circles with center at $A$. Each circle member of the pencil $\kappa$ transforms to the circle with center at $A$ and passing through its center.
intersection points with $\lambda$. The radical axis $\varepsilon$ of the pencil transforms to such a circle $\varepsilon'$ coinciding with the symmetric of $\lambda$ w.r. to $\varepsilon$. The reason for this behaviour is the fact that *inversions preserve angles*. In fact, the orthogonal to the original pencil is of intersecting type, containing members such as the circle $\kappa$ in figure 15. Such circles transform by the inversion to lines through $A$, like line $\kappa'$ in figure 15. Consequently the original pencil transforms to the orthogonal of the pencil of lines through $A$, which is the pencil of concentric circles.

![Figure 16: Tangential pencil inverted to a pencil of parallel lines](image)

Finally figure 16 shows the inversion on a circle $\lambda(B)$ of a pencil of tangential type. The inverted of the circle members are lines parallel to the common tangent $\varepsilon$ of the original pencil.

A typical application of this reduction to the exceptional cases is the following theorem known as "Haruki's lemma"

**Theorem 4.** The intersection points of two pairs of circles $\{(\alpha, \beta), (\gamma, \delta)\}$, the first belonging to a circle pencil and the second pair belonging to its orthogonal pencil, lie by four on eight circles.

![Figure 17: Intersections of pairs of orthogonal circles](image)

**Proof.** Figure 17 shows two such pairs and two, out of the eight, circles passing through intersection points of the circle pairs. The theorem becomes trivial if we perform an inversion w.r. to a circle of arbitrary radius but center at the base point $O$ of the intersecting pencil. The configuration transforms by this inversion to the one shown in figure 18, for which the proof is trivial.

$\square$
Figure 18: Inverted intersections of pairs of orthogonal circles

Figure 19 shows the form obtained by the configuration of the previous theorem in the case of orthogonal pencils of tangential type. The proof for this case, in which there is one only circle, uses a similar argument to the previous one.

10 Two prominent pencils

There are two much studied pencils of circles connected, the first with a triangle and the second with a quadrangle. The first is the pencil generated by the three "Apollonian circles" \( \{ \kappa_A, \kappa_B, \kappa_C \} \) of the triangle \( ABC \). The circles are defined by the property of their points \( \{ X \} \) to have constant ratio of distances from the vertices (See Figure 20):

\[
\kappa_A = \left\{ X : \frac{XB}{XC} = \frac{AB}{AC} \right\}, \quad \kappa_B = \left\{ X : \frac{XC}{XA} = \frac{BC}{BA} \right\}, \quad \kappa_C = \left\{ X : \frac{XA}{XB} = \frac{CA}{CB} \right\}.
\]

The three circles intersect at two points, the "isodynamic points" of the triangle and the line they define is called "Brocard axis" of the triangle. It passes through the "symmedian point" \( K \) and the "circumcenter" \( O \) of the triangle. The centers of the circles lie on the "Lemoine axis" of the triangle, which is orthogonal to the Brocard axis. The Lemoine axis is the "trilinear polar" of the symmedian point \( K \) and also the "polar" of \( K \) w.r. to the circumcircle \( \kappa \). The three circles are also orthogonal to the circumcircle \( \kappa \). More on this can be found in the file Apollonian circles.
The second prominent pencil (see figure 21) is connected with the “diagonals” of a complete quadrangle. It is generated by the three circles having as diameters the three “diagonals” \{AC, BD, EF\} of the quadrangle. The centers of these circles lie on the “Newton line” of the quadrangle joining the middles of the diagonals of the quadrangle.

11 Polars and poles w.r. to members of a pencil

**Theorem 5.** The polars \( e_P \) of a point \( P \) w.r. to the member-circles \( \kappa \) of a pencil \( \mathcal{P} \) pass through the point \( P' \), which is the diametral of \( P \) w.r. to the circle \( \Lambda \) of the orthogonal pencil of the pencil \( \mathcal{P} \).

The proof follows by considering the projection \( Q \) of \( P \) on the polar \( e_P \) of \( P \) w.r. to the circle \( \kappa \). This is the inverse of \( P \) w.r. to \( \kappa \) and every circle passing through \( \{P, Q\} \) is
orthogonal to \( \kappa \). In particular the circle \( \lambda \) passing through \( \{P, Q\} \) and with center on the radical axis \( \epsilon \) of the pencil \( \mathcal{P} \) defines the unique circle of the orthogonal pencil passing through \( P \). The polar \( \epsilon_P \) is orthogonal to \( PQ \) at \( Q \) and passes through the diametral point \( P' \) of \( \lambda \), which is the same for all members \( \kappa \in \mathcal{P} \).

Considering the poles of a fixed line \( \epsilon \) w.r. to the circle-members \( \{\kappa\} \) of a pencil we find them defining certain hyperbolas and in some cases parabolas and degenerate conics.

**Theorem 6.** The poles of a fixed line w.r. to the circle-members of a pencil lie on a conic, which is a hyperbola or parabola or a degenerate conic.

Here we discuss the case of an intersecting pencil, leaving the other cases as exercises. The pencil we consider has base points \( \{A, B\} \) lying symmetric w.r. to the origin on the y-axis (See Figure 23). All circles \( \kappa(K, r) \) of this pencil pass through \( \{A(0, a), B(0, -a)\} \). The circles are easily seen to have centers \( K(k, 0) \) and be defined by equations of the form:

\[
x^2 + y^2 - 2kx - a^2 = 0 \quad \text{with} \quad k = \frac{u^2 - a^2}{2u}, \quad r = |KU| = \frac{u^2 + a^2}{2|u|},
\]  

(6)
where we denote by \( U(u,0) \) a running point on the x-axis, defining the circle \( \kappa(K,r) \).
Assuming also the line to be given by an equation of the form
\[
\frac{x}{s} + \frac{y}{t} = 1, \tag{7}
\]
we find, using the relation \( KQ \cdot KP = r^2 \) and some simple calculations, the expression satisfied by the poles \( P \) of \( \varepsilon \) w.r. to \( \kappa \):
\[
s \cdot x^2 - t \cdot xy - st \cdot y + sa^2 = 0, \tag{8}
\]
which represents in general \((st \neq 0)\) a hyperbola with an asymptote orthogonal to the line \( \varepsilon \) at \( S \) and the other asymptote parallel to the y-axis and passing through \(-S\). The exclusion of ellipses was expected, since, as the center \( K \) of \( \kappa \) approaches \( S \), the corresponding pole \( P \) tends to infinity, hence the geometric locus of points \( \{P\} \) is not bounded. Next sections show some other instances of the intimate relation of circle pencils with hyperbolas.

12 Quadratic transformation defined by a pencil

Returning to theorem 5 and the corresponding figure 22, we consider a pencil \( \mathcal{P} \) and the associated to it transformation \( f : \mathcal{P} \mapsto \mathcal{P}' \), which to every point \( P \) associates the common point \( P' \) of all polars of \( P \) w.r. to the circle members of the pencil. The recipe of construction of \( P' \) involves the orthogonal pencil \( Q \) of \( \mathcal{P} \). For each point \( P \) we select the unique circle member \( \lambda \in Q \), and define \( P' \) as the diametral point of \( P \) w.r. to \( \lambda \). It is obvious that this is an “involutorial” transformation, i.e. it satisfies \( f^2 = e \), the symbol \( e \) denoting the identity transformation. In other words, the transformation \( f \) behaves like a reflection, being identical with its inverse transformation i.e. satisfying \( f(P') = P \).

Next theorem gives a more precise description of it ([Pon65, I,p.43]).

**Theorem 7.** The transformation \( f \) is a “quadratic” one, transforming lines to conics of all kinds except ellipses.

![Figure 24: The quadratic transformation \( f : P \mapsto P' \)](image)

We handle here the case of a pencil \( \mathcal{P} \) of non-intersecting type, leaving the other cases as exercises. In this case the circle members \( \{\lambda\} \) of the orthogonal pencil \( Q \) pass through
two fixed points \( \{A, B\} \), which can be taken on the x-axis symmetric w.r.t. to the origin (See Figure 24). A short calculation shows then that the transformation is given by

\[
P(p_1, p_2) \mapsto P'(p'_1, p'_2) = f(P) \quad \text{with} \quad \begin{pmatrix} p'_1 = -p_1, \\ p'_2 = \frac{p_1^2 - a^2}{p_2} \end{pmatrix},
\]

(9)

where \( A(a, 0) \). It is easily seen from the formula and also from the figure, that the coordinate vectors have a constant inner product \( P \cdot P' = -a^2 \).

Assuming the line \( \varepsilon \) represented in parametric form

\[
\varepsilon : x = \alpha t + \beta, \quad y = \gamma t + \delta,
\]

taking \( f(x, y) \) and eliminating the parameter \( t \), we find the equation satisfied by

\[
f(\varepsilon) : \quad \alpha \cdot x^2 + \gamma \cdot xy + (\beta \gamma - \alpha \delta) \cdot y - a^2 \alpha = 0,
\]

(10)

representing, for \( \gamma = 0 \) a parabola and for \( \gamma \neq 0 \) a hyperbola.

**Remark-1** The name “quadratic” stems from the expression (9), which in homogeneous coordinates is described by the equations

\[
x' = -xy, 
\]

(11)

\[
y' = x^2 - a^2 z^2, 
\]

(12)

\[
z' = yz, 
\]

(13)

which on the right have quadratic polynomials. The transformation belongs to the more general group of “Cremona transformations”, studied in the context of “algebraic geometry” ([Sa70, p.19], [Ode16, p.329]). The transformation is well defined and 1-1 everywhere except for the points of three lines, which are the x-axis and the two parallels to the y-axis at \( x = \pm a \). On these two lines the transformation is not 1-1, but sends each line to a point. The three points in homogeneous coordinates

\[
(a, 0, 1), \quad (-a, 0, 1), \quad (0, 1, 0),
\]

which are the solutions of the system of equations \( x' = y' = z' = 0 \), and represent the intersections of the three exceptional lines are called “fundamental points” of the transformation.

**Remark-2** An analogous quadratic transformation can be defined more generally for a “pencil of conics” through four fixed points (See Figure 25). The fundamental points in this case are the three diagonal points \( \{A, B, C\} \) of the complete quadrangle of the four fixed points. The corresponding quadratic transformation \( f \) is defined as before: the polars of \( P \) w.r.t. to the conics of the pencil pass through the same point \( P' = f(P) \). The transformation, referred to the “projective coordinate system” of the triangle of these three points, takes then the form of the usual “isogonal transformation”

\[
x' = yz, \quad y' = zx, \quad z' = xy.
\]

The images \( f(\varepsilon) \) of lines \( \{\varepsilon\} \) are again conics passing through the fundamental points \( \{A, B, C\} \). These are the “triangle conics” or “circumconics” of the triangle \( ABC \) ([Yiu13, p. 109]), forming the so called “homaloidal net” of conics in the context of Cremona quadratic
Remark-3 From equation (9) follows that lines \( \varepsilon \) parallel to the y-axis map under \( f \) to degenerate conics consisting of two lines: the x-axis and the symmetric to \( \varepsilon \) w.r. to the y-axis. This refines the type of the quadratic transformation \( f \) to the type of "de Jonquieres transformations", which are Cremona transformations of the plane preserving a pencil of lines ([Des09, p.51]).

Remark-4 For a member circle \( \kappa \in \mathcal{P} \) the quadratic transformation maps it to a curve \( \kappa' = f(\kappa) \) of degree four. Figure 26 displays such an example for a circle \( \kappa \) of the pencil of non-intersecting type with limit points \( \{A(a,0), B(-a,0)\} \) lying on the x-axis symmetric.
w.r. to the origin. The circle can be parameterized by its diametral points \( \{U(u,0), V(v,0)\} \) satisfying \( uv = a^2 \). A short computation shows that the curve \( \kappa' \) satisfies the equation
\[
y^2(ux + a^2)(x + u) + u(x^2 - a^2)^2 = 0.
\]

13 Hyperbola from a pencil of circles

Next properties supply an alternative way to define a hyperbola using a pencil of circles passing through two fixed base points. The appropriate configuration consists of two lines \( \{OA, OB\} \) and two points \( \{F, F'\} \), taken symmetrically relative to \( O \) and also lying on a bisector of the angle of the two lines. We then consider all circles \( \{\alpha\} \) passing through \( \{F, F'\} \) and intersecting the lines along the chords \( \{AB, CD\} \) (See Figure 27).

**Theorem 8.** Under the previous conventions, the following are valid properties.

1. The triangles \( \{AF'O, BOF', ABF\} \) are similar.
2. The triangles \( \{F'BA, OFA\} \) are similar.
3. The middle \( P \) of \( AB \) moves on a hyperbola with focal points \( \{F, F'\} \).
4. The line \( AB \) is tangent at \( P \) to the previous hyperbola.

Nrs 1-2 follow by observing the angles inscribed in the circle \( \alpha \).

For nr-3 define first \( \{x = AF, y = AF', d = PF, e = PF', m = AO\} \). Then from the similarity of the triangles we have
\[
\frac{d}{x} = \frac{ON}{m}, \quad \frac{e}{y} = \frac{OM}{m} \quad \Rightarrow \quad e - d = \frac{1}{m}(yOM - xON) = \frac{1}{2m}(y^2 - x^2).
\]

But the difference \( y^2 - x^2 = F'S^2 - FS^2 = 2FF' \cdot OS \). Hence
\[
\frac{1}{2m}(y^2 - x^2) = (FF' \cdot OS)/m = FF' \cos(\phi).
\]

This shows that \( P \) is on the hyperbola with focal points \( \{F, F'\} \) and major axis \( 2a = FF' \cos(\phi) \). Since \( 2c = FF' \), we have also that \( b = c \sin(\phi) \) and the two lines \( \{OA, OB\} \) are asymptotes of the hyperbola.
Nr-4 follows by showing that $PB$ is a bisector of the angle $F'PF$. In fact, by the equality of the angles $AOB = 2\phi = \overline{AEB}$, where $E$ is the circumcenter of $AFF'$, follows that points $\{A,E,O,B,T\}$ are concyclic on a circle $\beta$ and $TE$ is a diameter of it, which passes through the middle $P$ of $AB$. Then, point $T$ is the pole of $AB$ and $P(FF'BT)$ is a harmonic pencil with two orthogonal rays $\{PB,PT\}$. Hence these rays are bisectors of the angle of the other two.

Next figure summarizes again the same properties stressing the role of the angles involved.

![Figure 28: Property of the general asymptotic triangle](image)

**Theorem 9.** The asymptotic triangle $OAB$ defines three triangles $\{FAB,OBF',OF'A\}$, which are similar and the angle $AFB$ is constant and equal to $F'OB = (360° - \omega)/2$, where $\omega$ is the angle between the asymptotes.

**Theorem 10.** There is a unique hyperbola having for asymptotes the diagonals $\{AB',A'B\}$ of an isosceles trapezium $AA'B'B$ and passing through the middles of the non-parallel sides.

From the similarity of triangles $\{AOF',F'OB\}$, follows that the distance of $O$ from the sides $\{F'A,F'B\}$ of the triangle is proportional to the lengths of these sides, something that characterizes the “symmedian line” of a triangle, hence the property.

**Theorem 11.** All triangles $ABF'$, created from the asymptotic triangle $OAB$, have the angle $\widehat{F}F'$ constant and equal to half the angle of the asymptotes. In addition $FF'$ is the symmedian line from $F'$, hence the symmedian point $K$ of this triangle lies on line $FF'$, which is also a bisector of the angle $AOB$.

**Theorem 12.** A hyperbola, whose asymptotes $\{OA,OB\}$ make an angle $\omega$, is generated by rotating an angle $AF'B = \omega/2$ or $AFB = 180° - \omega/2$, rotating about their fixed vertices $\{F,F'\}$, which are are points of a bisector of the angle $\omega$. The rotating angle intersects the fixed angle $\omega$ at points $\{A,B\}$ and the hyperbola is the locus of the middle $D$ of $AB$. Lines $AB$ are tangents to this hyperbola.

14 Rectangular hyperbolas related to pencils of circles

**Theorem 13.** The circles having for diameter chords of a rectangular hyperbola parallel to the direction $\alpha$, define a “pencil” $\mathcal{D}$. Their centers lie on the conjugate direction $\varepsilon$ of $\alpha$ and their radical axis $\zeta$ is the orthogonal to $\varepsilon$ through $O$ (See Figure 29-II). The orthogonal to this pencil $\mathcal{D}'$ is created analogously by the orthogonal to $\alpha$ direction $\beta$ of parallel chords.
The circle equation \((x-x_0)^2 + (y-y_0)^2 - r^2 \iff x^2 + y^2 - 2(x_0x + y_0y) + x_0^2 + y_0^2 - r^2 = 0\), in the case in which the diametral points are on the same branch, takes a simple form. Using it we find that the radical axis of two such circles \(\{\kappa, \kappa'\}\), characterized by the constants \(\{k, k'\}\) is given by the equation \(x_1x + y_1y = 0\), where \(E(x_1, y_1)\) is an intersection of \(\varepsilon\) with the hyperbola. Point \(E\) is in this case a limit point of the pencil and \(|OE|\) is the length of the tangent from \(O\) to every member-circle of the pencil.

In figure 29 the radical axis \(\zeta\) does not intersect the hyperbola and the pencil \(D\) is of non-intersecting type. The orthogonal pencil \(D'\) consists of member-circles \(\lambda\) on diameters with endpoints on different branches.

**Theorem 14.** Given a pencil of circles of intersecting or non-intersecting type and a direction \(\eta\), the diametral points of the diameters of member-circles, which are parallel to \(\eta\) generate rectangular hyperbolas.

In figure-29 notice the line \(\tau\), which is the radical axis of the pair of orthogonal circles \(\{\kappa, \lambda\}\) and coincides with the altitude from \(B'\) of the right triangle \(A'B'C'\).

**Theorem 15.** To the circles of a pencil with “base points” \(\{A, B\}\), i.e. all circles passing through \(\{A, B\}\), tangents at a fixed given direction are drawn. The geometric locus of contact points \(P\) is a rectangular hyperbola.
Consider such a circle with center $K$ and adopt for coordinate axes the lines $\{Ox, Oy\}$ inclined to the medial line $OK$ of $AB$ by half the fixed angle $\phi = \frac{TKP}{2}$, defined by the orthogonal $KP$ to the fixed direction of the tangents. Setting $\{x = YP, y = OY\}$, we have

$$\frac{x}{y} = \frac{TY}{y} \Rightarrow xy = TY \cdot SY = TY \cdot YP \cot(\phi) = TY(TP - TY) \cot(\phi)$$

$$= TY(TP - TY) \cot(\phi) = (OT \sin(\phi))(2KT \sin(\phi) - OT \sin(\phi)) \cot(\phi)$$

$$= OT(2KT - OT) \cos(\phi) = OA^2 \cos(\phi).$$

Figure-31 results from figure-30 by considering the other intersection point $P'$ of the locus-hyperbola for the fixed direction of the tangent line $\eta$ of the circle $\alpha$ of the intersecting pencil $D$ with base points $\{A, B\}$. The circle $\beta$ with diameter $PP'$ belongs to the orthogonal pencil $D'$ and its tangent $\varepsilon$ at $P'$ has also fixed direction, namely the orthogonal to that of $\eta$. Thus the same hyperbola is the locus of tangent points of tangents in the fixed direction $\varepsilon$ to the circles of the non-intersecting pencil $D'$ with “limit points” $\{A, B\}$.

It is trivial to see that the analogous problem for a “tangential pencil” of circles leads to a pair of orthogonal lines, which are identical with the asymptotes $\{Ox, Oy\}$ of theorem 15 if the common tangent to the pencil is $AB$ and the common contact point is $O$ (See Figure-30).

**Bibliography**


