Pencils of circles or coaxal circles

A file of the Geometrikon gallery by Paris Pamfilos

The reason why we are on a higher imaginative level is not because we have finer imagination, but because we have better instruments.

A.N. Whitehead, Science and the Modern World VII, p.166

Contents

(Last update: 12-12-2021)

1 The two main types of pencils

The first main type of a *"pencil of circles"* or *"coaxal system of circles"* ([Ped90, p.106]) is the set of all circles passing through two points $\{A, B\}$ called *"base points"* of the pencil (see

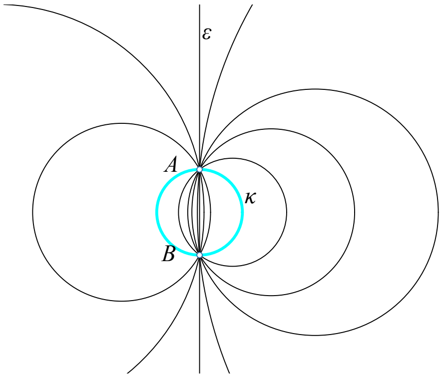

Figure 1: Pencil of *"intersecting type"*

figure 1). All pairs of circles of this pencil share the same *"radical axis"*, which is the line AB and is called the *"radical axis of the pencil"*. The orthogonal bisector line of the segment AB carries the centers of all the circles of the pencil and is called the *"line of centers of the pencil"* . This type of pencil often called *"elliptic"* ([Cox61, p.85]) is also characteristically called *"intersecting pencil of circles."* The pencil has a smallest circle κ, having as diameter the segment AB. There is no biggest circle in this pencil. With growing radius the circle tends to the common radical axis, the line AB, which is considered also as a degenerate member of the pencil (a circle of infinite radius).

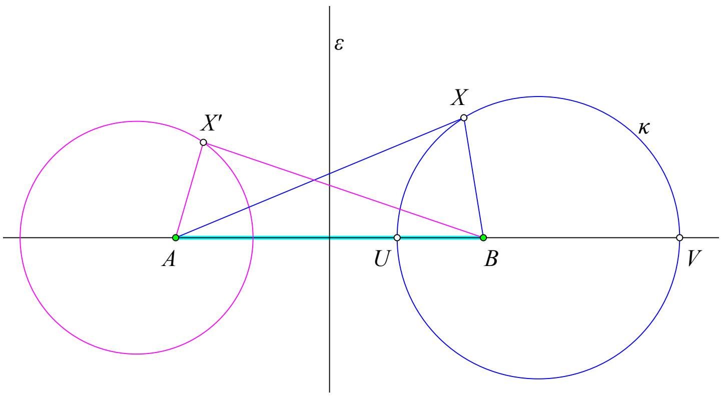

Figure 2: Pencil of *"non-intersecting type"*

The second main type is the set of *"Apollonian circles of a segment AB"* (see figure 2). By their definition the points $\{X\}$ of such a circle κ satisfy the condition

$$X \in \kappa \quad \Leftrightarrow \quad \frac{XA}{XB} = k, \quad \text{with a constant} \quad k.$$

The circles have now no common points and the pencil, often called *"hyperbolic"*, is also characteristically called a *"non-intersecting pencil of circles"*. The circles of the pencil have in this case also a common radical axis which coincides with the orthogonal bisector line of the segment AB, called the *"radical axis of the pencil"*. The *"line of centers"* of the pencil is now line AB. The points $\{A, B\}$ are called *"limit points"* of the pencil. They are considered as members of the pencil with radius zero (see file **Apollonian circles**). The circles κ of this pencil can be identified with those of the following theorem.

Theorem 1. *The limit points $\{A, B\}$ of a non intersecting pencil are inverse relative to every circle κ of the pencil. They are also harmonic conjugate relative to the diametrically opposite points $\{U, V\}$ of κ, which are defined from its intersection with the center line of the pencil. Thus, every pair (U, V) of "harmonic conjugates" to (A, B) is the diameter of a circle of the pencil and vice-versa.*

2 The tangential type of pencils

The two previous are the main types of pencils. There are also some other pencils, which can be considered as limiting cases of these two. Next pencil can be considered as limiting case of a pencil of intersecting type, resulting when one of the common points, B say, tends to coincide with the other common point A. Figure 3 displays the resulting pencil

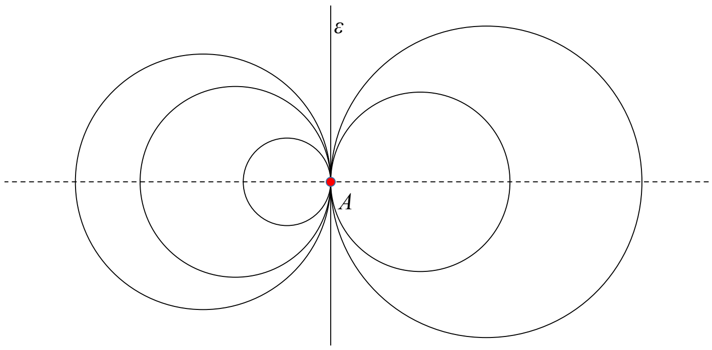

Figure 3: Tangential pencil of circles

of circles often called *"parabolic"*. Its members are all tangent to a line ε at a fixed point A of this line, therefore it is characteristically called a *"tangential pencil of circles"*. The line ε is the common radical axis of all pairs of circles of the pencil, called the *"radical axis of the pencil"* and the orthogonal to it at A is the *"line of centers of the pencil"*.

3 Pencils described by equations

The equation of a circle in cartesian coordinates results by expanding the simple expression deriving directly from the definition of a circle $\kappa(X_0, r)$ with center $X_0 = (x_0, y_0)$ and radius r :

$$\kappa \ : \quad (x - x_0)^2 + (y - y_0)^2 = r^2 \quad \Leftrightarrow \quad x^2 + y^2 - 2x_0x - 2y_0y + (x_0^2 + y_0^2 - r^2) = 0.$$

The slightly more general equation involving a function of two variables

$$f(x, y) \ = \ a(x^2 + y^2) + 2bx + 2cy + d \ = \ 0 \tag{1}$$

comprises the case of lines $(a = 0)$, considered as circles of infinite radius. Notice that the equation comprises, for appropriate coefficients, the case of single points

$$(x - x_0)^2 + (y - y_0)^2 = 0,$$

considered as circles with radius $r = 0$. Comparing equation (1) with the one preceding it, we see that the center of the circle is given by

$$x_0 = -\frac{b}{a}, \quad y_0 = -\frac{c}{a} \quad \text{and the radius}$$
$$x_0^2 + y_0^2 - r^2 = \frac{d}{a} \quad \Leftrightarrow \quad r^2 = \frac{b^2 + c^2 - ad}{a^2}. \tag{2}$$

The pencils of circles are *"lines"* of circles defined as linear combinations of two *"points"* in the space of all circles expressed through equations similar to equation (1). This means that they are described by equations of the form

$$\lambda f_1(x,y) + \mu f_2(x,y) = 0 \quad \Leftrightarrow$$
$$\lambda(a_1(x^2+y^2) + 2b_1x + 2c_1y + d_1) + \mu(a_2(x^2+y^2) + 2b_2x + 2c_2y + d_2) = 0. \quad \Leftrightarrow$$
$$(\lambda a_1 + \mu a_2)(x^2+y^2) + 2(\lambda b_1 + \mu b_2)x + 2(\lambda c_1 + \mu c_2)y + (\lambda d_1 + \mu d_2) = 0. \tag{3}$$

The various members result by varying the pair of parameters (λ, μ). Pairs of parameters satisfying $(\lambda', \mu') = k(\lambda, \mu)$ with $k \neq 0$, define the same circle-member, so that essentially only the quotient $(\lambda : \mu)$ is relevant for the definition of a member. The unique member of such a pencil, which degenerates to a line, is obtained by setting

$$\lambda a_1 + \mu a_2 = 0 \quad \Leftrightarrow \quad (\lambda : \mu) = (-a_2 : a_1)$$

and the resulting line-equation for these values of (λ, μ)

$$2(\lambda b_1 + \mu b_2)x + 2(\lambda c_1 + \mu c_2)y + (\lambda d_1 + \mu d_2) = 0 \tag{4}$$

expresses then the *"radical axis"* of the pencil. The kind of the pencil can be recognized by the number of circles of radius $r = 0$ i.e. *points* it comprises. This in turn, by means of equation (2), is controlled by the number of solutions of equation

$$b^2 + c^2 - ad = 0 \quad \Leftrightarrow \quad (\lambda b_1 + \mu b_2)^2 + (\lambda c_1 + \mu c_2)^2 - (\lambda a_1 + \mu a_2)(\lambda d_1 + \mu d_2) = 0,$$

which is homogeneous w.r. to (λ, μ) and setting $t = \lambda/\mu$ leads to the quadratic

$$(tb_1 + b_2)^2 + (tc_1 + c_2)^2 - (ta_1 + a_2)(td_1 + d_2) = 0, \quad \Leftrightarrow$$
$$(b_1^2 + c_1^2 - a_1d_1)t^2 - 2(b_1b_2 + c_1c_2 - a_1d_2 - a_2d_1)t + (b_2^2 + c_2^2 - a_2d_2) = 0. \tag{5}$$

If this quadratic has two real solutions, then the pencil has two point-members, which is the case of the *"hyperbolic"* pencil of *"non-intersecting"* type, the points being its *"limit points"*. If there is only one solution then the pencil is *"parabolic"* of the *"tangential"* type, and if there are no real solutions, then the pencil is of *"elliptic"* or *"intersecting"* type and all circles pass through two fixed points $\{A, B\}$, the *"base points"* of the pencil.

4 Common properties to all types of pencils

Next properties, formulated as separate propositions, are easy consequences of the definitions and can be proved either geometrically or using the analytic description of the previous section 3 (see also [Mcl91]).

Proposition 1. *Every pair consisting of a circle $\mu(N, r)$ and a line ε defines exactly one pencil of circles which contains the circle μ and has corresponding radical axis the line ε.*

Proposition 2. *Every pair of different points (A, B) of the plane defines a unique pencil of non-intersecting type having them as limit points.*

Proposition 3. *Every pair (A, ε) of a point A and a line or circle ε of the plane defines a unique pencil of non-intersecting type or tangential type containing them as members.*

Proposition 4. *Every pair of non concentric circles defines exactly one circle pencil which contains them as members.*

The previous propositions show that given a pair (α, β) whose members are points, lines or circles, with the exclusion of two lines, defines a *pencil of circles* containing them, for which we say that it is *"generated"* by the pair.

Proposition 5. *Given a pencil of circles, for every point X of the plane there exists exactly one pencil-member κ which passes through it.*

Proposition 6. *Given a circle pencil and a circle κ, which does not belong to the pencil and does not have its center at the center line of the pencil, the radical axes of the pairs of circles (κ, μ), where μ is a circle that belongs to the pencil, pass all through a fixed point C of the radical axis of the pencil (see figure 4).*

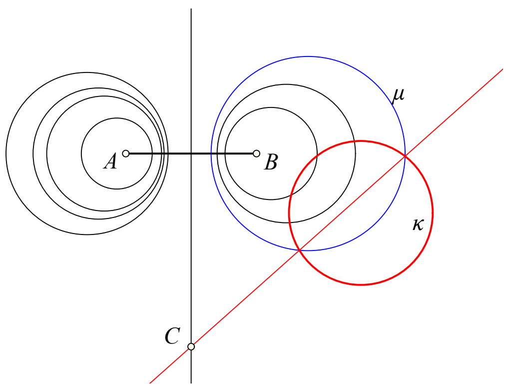

Figure 4: All radical axes of pairs $\{(\mu, \kappa)\}$ pass through C

Since the function representing the circle $f(x, y) = (x - x_0)^2 + (y - y_0)^2 - r^2$ expresses also the *"power"* of the point $X(x, y)$ w.r.t. the circle, we have the following property:

Proposition 7. *The circle μ belongs to the pencil $\mathcal{P}$, which is generated by the circles $\{\kappa, \lambda\}$ if and only if, for every point X of μ the ratio of powers of X relative to the circles $\{\kappa, \lambda\}$ is constant.*

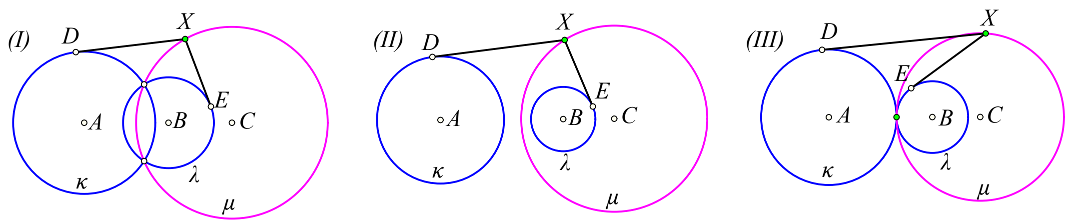

Figure 5: Locus of points with constant ratio of tangents to two circles

Proposition 8. *Given the circles $\kappa(A, \alpha)$ and $\lambda(B, \beta)$, the geometric locus of points X for which the ratio of the lengths of the tangents to the two circles κ and λ is fixed, is a circle μ which belongs to the pencil defined by κ and λ.*

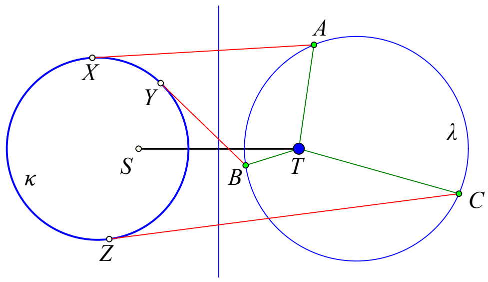

Figure 6: $\frac{|AX|}{|AT|} = \frac{|BY|}{|BT|} = \frac{|CZ|}{|CT|}$

Proposition 9. *Given the circle $\kappa(O,R)$ and a point T, different from the center O of the circle, the geometric locus of the points A for which the ratio of the lengths $\frac{|AT|}{|AX|} = k$ is constant, where AX is the tangent to κ from A, is a circle λ of the pencil generated by the circle κ and the point T (see figure 6).*

Proposition 10. *Given the circle $\kappa(O,r)$ and a point T, three other points $\{A,B,C\}$ have the same ratio of lengths of tangents to their distance from T*

$$\frac{|AX|}{|AT|} = \frac{|BY|}{|BT|} = \frac{|CZ|}{|CT|}$$

if and only if, the circumscribed circle λ of triangle ABC belongs to the pencil generated by the circle κ and the point T (see figure 6).

Exercise 1. *Given two circles $\{\kappa_1(O_1,r_1),\kappa_2(O_2,r_2)\}$ show that the distance d_ℓ of the limit points of the pencil they generate is*

$$d_\ell = \frac{2r_1r_2}{|O_1O_2|}\sqrt{\frac{(r_1^2+r_2^2-|O_1O_2|^2)^2}{4r_1^2r_2^2}-1}\,.$$

which has a value: real for hyperbolic, zero for parabolic and imaginary for elliptic pencils.

5 Orthogonal circles

Orthogonal circles are formed with the help of tangents to circles from a given point. The tangents $\{PA,PB\}$ to the circle $\kappa(O,r)$ from a point P are equal, therefore they define a circle $\lambda(P,r')$, which has these tangents as radii (see figure 7-(I)). At the intersection point

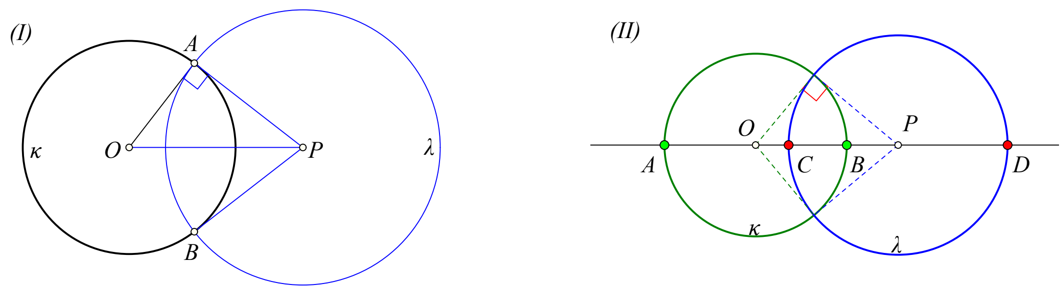

Figure 7: Right circles Harmonic quadruple $(ABCD) = -1$

A of the two circles the angle between their radii is a right one. This is a mutual relation.

The circle κ can be considered that it results the same way, from the tangents to λ from O. Two intersecting circles, whose radii at the intersection points are orthogonal are called *"orhogonal circles"*. By definition, therefore, this is equivalent to:

"At one of their intersection points, the radius of one is tangent to the other".

From this characteristic property follow also the two next corollaries.

Corollary 1. *Two circles $\{\kappa(O,r), \lambda(P,s)\}$ are orthogonal, if and only if*

$$|OP|^2 = r^2 + s^2.$$

Corollary 2. *Two circles $\{\kappa, \lambda\}$ are orthogonal, if and only if the diametrically opposite points $\{A, B\}$ of one and $\{C, D\}$ of the other, on their center line, build a harmonic quadruple (see figure 7-(II)).*

This follows directly from the characteristic property of a harmonic quadruple of points.

Exercise 2. *Construct a circle λ, orthogonal to a given circle $\kappa(O,r)$ and having its center at a given point P external to κ.*

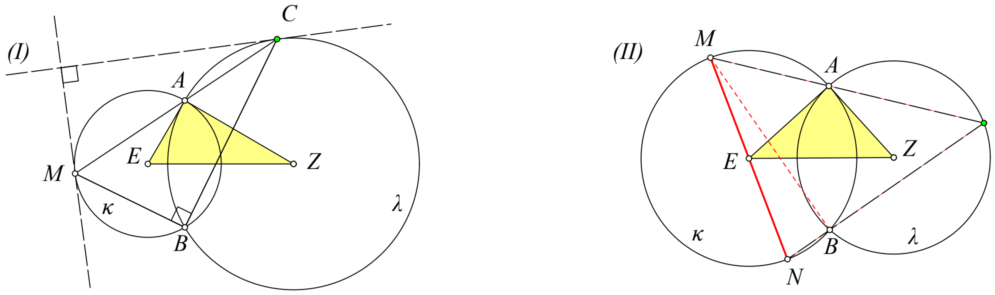

Figure 8: Similar triangles AEZ, BMC Diametrically opposite points M, N

Exercise 3. *Show that the circles $\{\kappa, \lambda\}$, intersecting at points $\{A, B\}$ are orthogonal, if and only if one of the following conditions holds:*

1. *Every line through point A defines points $\{M, C\}$ on circles $\{\kappa, \lambda\}$ such that the angle $\widehat{CBM}$ is a right one.*
2. *For every point C of λ the lines $\{CA, CB\}$ intersect again the circle κ at diametrically opposite points $\{M, N\}$.*
3. *The tangents to the circles at $\{M, C\}$ intersect orthogonally.*

Hint: The triangles EAZ and BMC are similar even when the circles are not orthogonal (see figure 8-(I)). (2) draw BM and use (a) (see figure 8-(II)).

Theorem 2. *The circles μ, which are simultaneously orthogonal to the non concentric circles $\{\kappa, \lambda\}$, have their centers on the radical axis of $\{\kappa, \lambda\}$.*

If the circles $\mu(S, \rho)$ and $\kappa(O, \rho')$ are orthogonal, then their radii at one of their intersection points A will be orthogonal (see figure 9-(I)), therefore line SA will be tangent to κ. The same will happen also with circles μ and λ. Consequently the tangents from point S towards the given circles will be equal and point S will lie on the radical axis of κ and λ.

Corollary 3. *A circle μ orthogonal to two other cirlces $\{\kappa, \lambda\}$ is simultaneously orthogonal also to every circle of the pencil generated by $\{\kappa, \lambda\}$.*

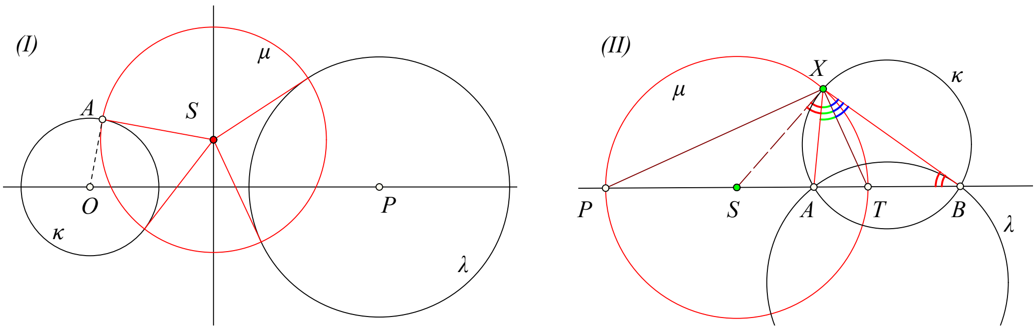

Figure 9: Orthogonal to two circles Orthogonal to intersecting pencil

If μ is orthogonal to $\{\kappa, \lambda\}$ its center will be contained in the radical axis of$\{\kappa, \lambda\}$, which is also the radical axis of the pencil produced by $\{\kappa, \lambda\}$. Consequently μ will have the same power relative to all the circles of this pencil. Thus, if $\mu(S, r)$ intersects a third circle ν of the pencil at A, then the radius SA of μ will also be tangent to ν, therefore the two circles will be orthogonal.

6 Orthogonal pencils

By means of corollary 3 pencils *"go in pairs"*. Each pencil defines a second, consisting of *all circles which are orthogonal to all circles of the first pencil*. In the case of an intersect-

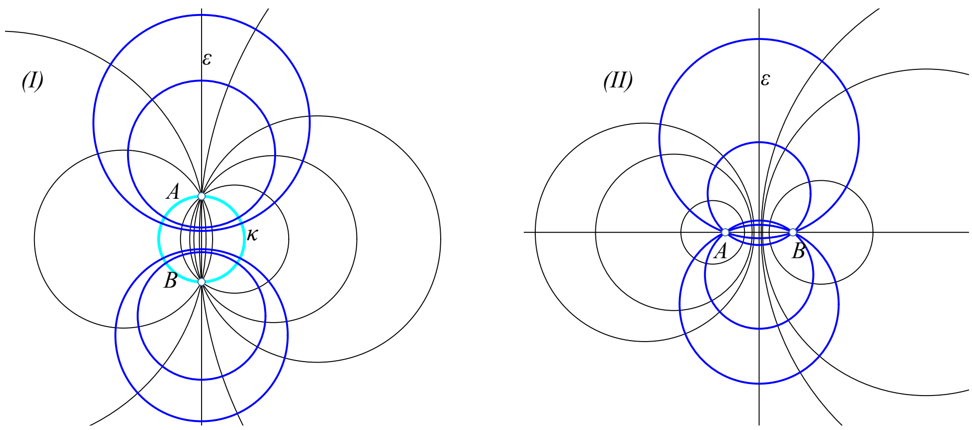

Figure 10: Pencils orthogonal to intersecting/non-intersecting pencils

ing pencil, the orthogonal one is a non-intersecting (see figure 10-(I)) and in the case of non-intersecting the orthogonal is an intersecting pencil (see figure 10-(II)). Thus, orthogonality interchanges these two types of pencils. The radical axis of one becomes line of centers of the other.

In the case of a tangential pencil, its orthogonal consists of all circles passing through the same contact point A of the first pencil, but this time tangent there at the line which is orthogonal to the common tangent of the first pencil. Thus, in this case the orthogonal pencil of a tangential pencil is a tangential pencil too (see figure 11).

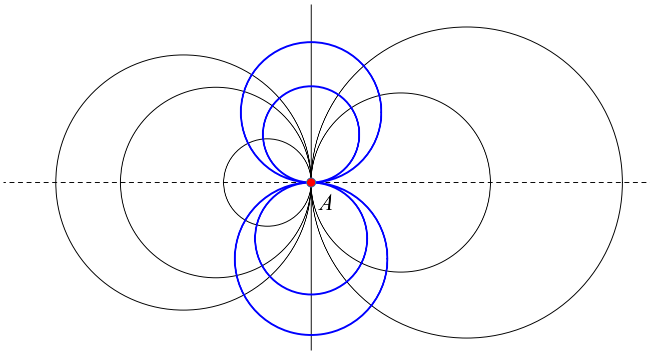

Figure 11: Pencil orthogonal to a tangential one

7 Three exeptional types of pencils

Next figure displays the three last types of pencils consisting, in the first case, of lines passing from a fixed point (see figure 12-(I)). These lines are considered as very large

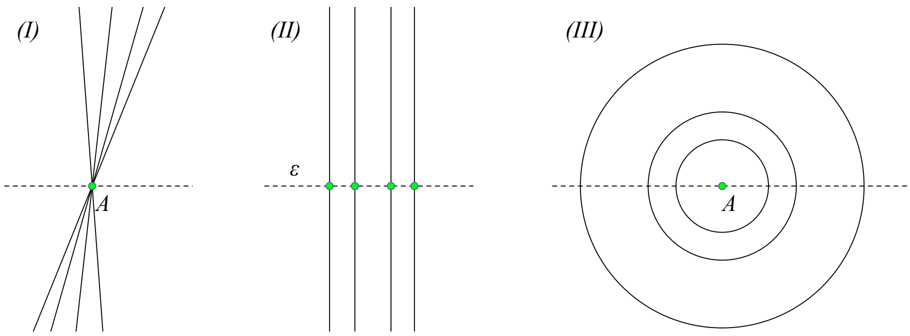

Figure 12: Pencils resulting from the previous as limiting cases

circles of an intersecting pencil, whose one of the common points B is at infinity. The pencil in this case coincides with a *"pencil of lines"*. The next type results from a tangential pencil, whose contact point A is at infinity, hence the circles become parallel lines (see figure 12-(II)). The third pencil consists of concentric circles (see figure 12-(III)) and can be considered as limiting case of a non-intersecting pencil, in which the point B tends to coincide with point A. Some times these pencils are called *"exceptional"*, since they lack a real radical axis and line of centers.

Including these exceptional cases in the set of all pencils we can formulate the following theorem (see file **Inversion**).

Theorem 3. *An inversion w.r.t. to a circle κ maps a pencil of circles to another pencil of circles.*

The theorem follows from the property of inversions to map the set $\mathcal{K}$ of all circles and all lines of the plane into $\mathcal{K}$ itself ([Joh60, p.44]). As will be seen below, using an appropriate inversion, it is possible to invert a usual pencil of circles to an exceptional one and vice-versa.

In the case of *"exceptional"* pencils, the orthogonals are seen in figure 13. In accordance with the main types from which they result as limiting cases, here we have again an interchange of types $(I) \leftrightarrow (III)$ and an orthogonal of the same type in figure 13-(II).

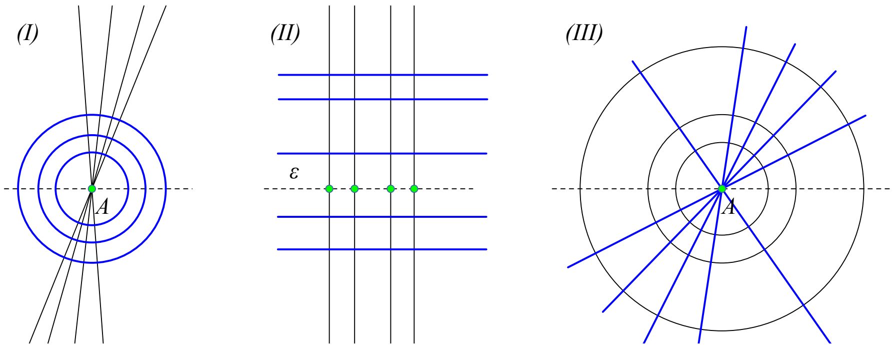

Figure 13: Pencils orthogonal to exceptional pencils

8 Reduction to exceptional cases

Using an appropriate *"inversion transformation"* we can transform the main types of pencils and the tangential one to corresponding exceptional types. For the intersecting type we can apply an inversion with center at one of the common points $\{A, B\}$ of the pencil, and an arbitrary radius. Figure 14 shows the inversion on circle $\lambda(B, |AB|)$ of a pencil of

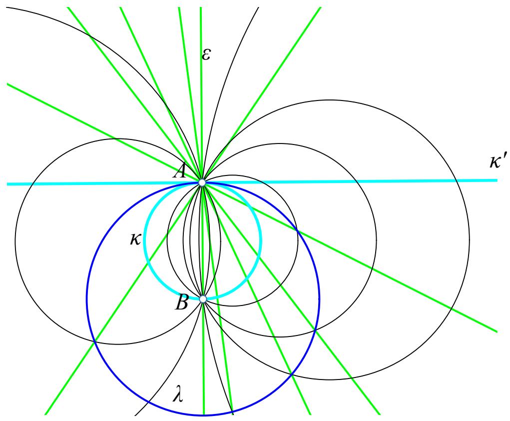

Figure 14: Intersecting pencil inverted to pencil of lines

intersecting type. The inverted of the circle members are the lines through A. Thus, by this inversion the pencil transforms to the pencil of lines through A. Each circle transforms to the line passing through A and its second intersection with λ. The smallest circle of the pencil κ transforms to the line κ' tangent to κ at A.

Figure 15 shows the inversion on circle $\lambda(B, |AB|)$ of a pencil of non-intersecting type. The inverted of the circle members are the concentric circles with center at A. Each circle member of the pencil transforms to the circle with center at A and passing through its intersection points with λ. The radical axis ε of the pencil transforms to such a circle ε' coinciding with the symmetric of λ w.r. to ε. The reason for this behaviour is the fact that *inversions preserve angles*. In fact, the orthogonal to the original pencil is of intersecting type, containing members such as the circle κ in figure 15. Such circles transform by the inversion to lines through A, like line κ' in figure 15. Consequently the original pencil transforms to the orthogonal of the pencil of lines through A, which is the pencil of concentric circles.

Finally figure 16 shows the inversion on a circle $\lambda(B)$ of a pencil of tangential type. The inverted of the circle members are lines parallel to the common tangent ε of the original pencil.

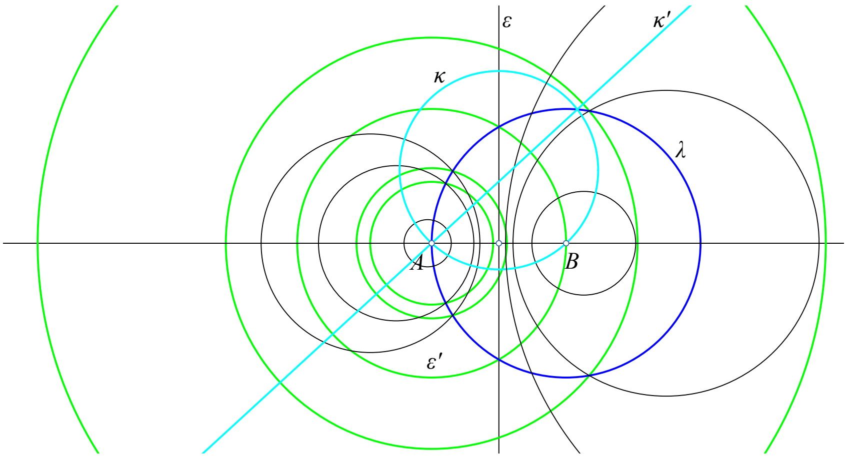

Figure 15: Non-intersecting pencil inverted to pencil of concentric circles

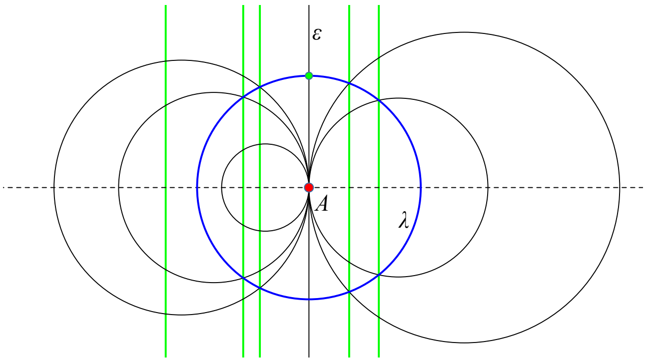

Figure 16: Tangential pencil inverted to a pencil of parallel lines

A typical application of this reduction to the exceptional cases is the following theorem known as *"Haruki's lemma"*

Theorem 4. *The intersection points of two pairs of circles $\{(\alpha, \beta), (\gamma, \delta)\}$, the first belonging to a circle pencil and the second pair belonging to its orthogonal pencil, lie by four on eight circles.*

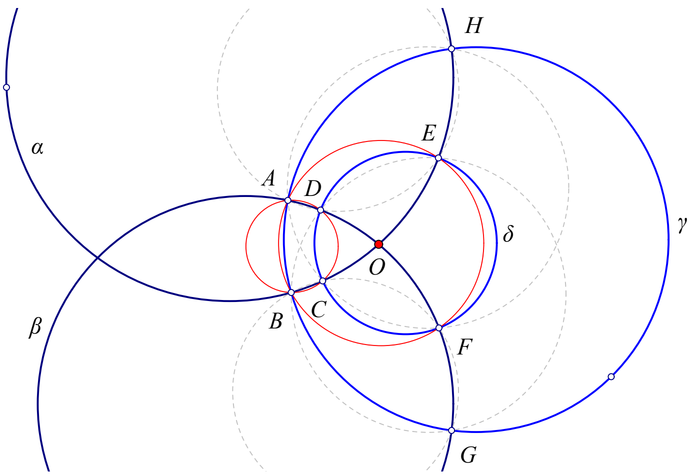

Figure 17: Intersections of pairs of orthogonal circles

Proof. Figure 17 shows two such pairs and two, out of the eight, circles passing through intersection points of the circle pairs. The theorem becomes trivial if we perform an inversion w.r. to a circle of arbitrary radius but center at the base point O of the intersecting pencil. The configuration transforms by this inversion to the one shown in figure 18, for

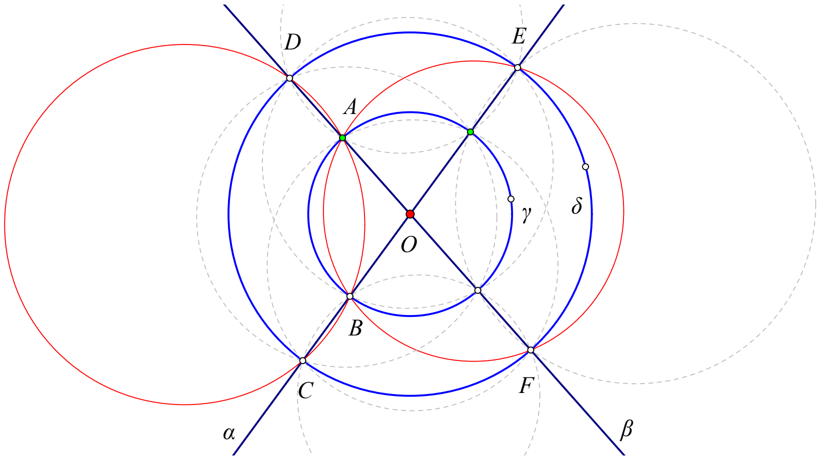

Figure 18: Inverted intersections of pairs of orthogonal circles

which the proof is trivial. □

Figure 19 shows the form obtained by the configuration of the previous theorem in the case of orthogonal pencils of tangential type. The proof for this case, in which there

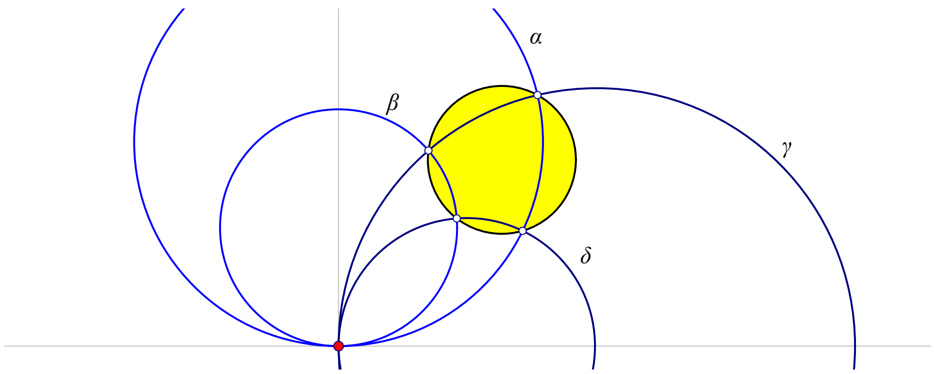

Figure 19: Intersections of pairs of orthogonal circles of tangential type

is one only circle, uses a similar argument to the previous one.

9 Two prominent pencils

There are two much studied pencils of circles connected, the first with a triangle and the second with a quadrangle. The first is the pencil generated by the three *"Apollonian circles"* $\{\kappa_A, \kappa_B, \kappa_C\}$ of the triangle ABC. The circles are defined by the property of their points $\{X\}$ to have constant ratio of distances from the vertices (see figure 20):

$$\kappa_A = \left\{ X : \frac{XB}{XC} = \frac{AB}{AC} \right\}, \qquad \kappa_B = \left\{ X : \frac{XC}{XA} = \frac{BC}{BA} \right\}, \qquad \kappa_C = \left\{ X : \frac{XA}{XB} = \frac{CA}{CB} \right\}.$$

The three circles intersect at two points, the *"isodynamic points"* of the triangle and the line they define is called *"Brocard axis"* of the triangle. It passes through the *"symmedian*

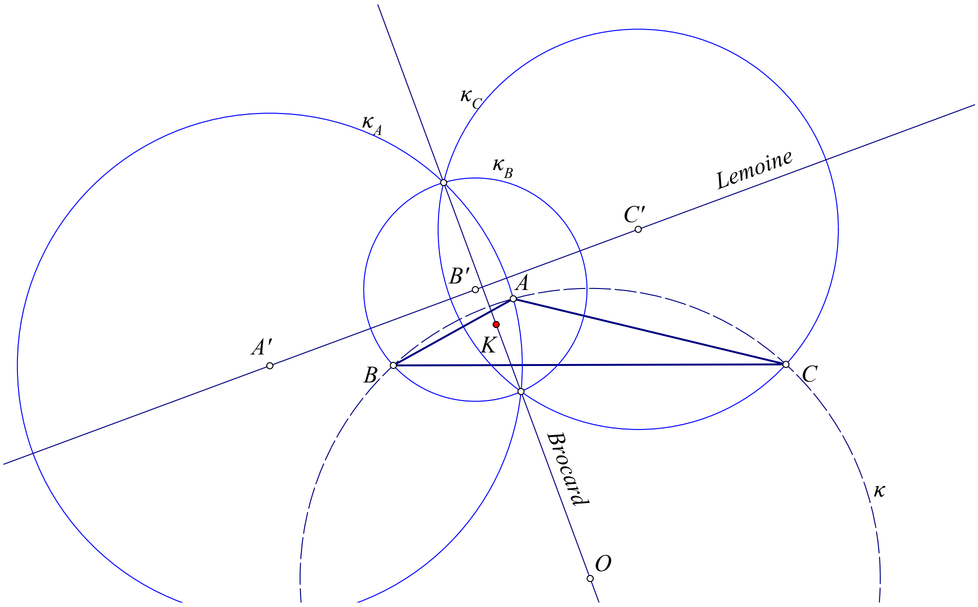

Figure 20: The *"Apollonian circles"* pencil of the triangle

point" K and the *"circumcenter"* O of the triangle. The centers of the circles lie on the *"Lemoine axis"* of the triangle, which is orthogonal to the Brocard axis. The Lemoine axis is the *"trilinear polar"* of the symmedian point K and also the *"polar"* of K w.r. to the circumcircle κ. The three circles are also orthogonal to the circumcircle κ. More on this can be found in the file **Apollonian circles of a triangle and isodynamic points**.

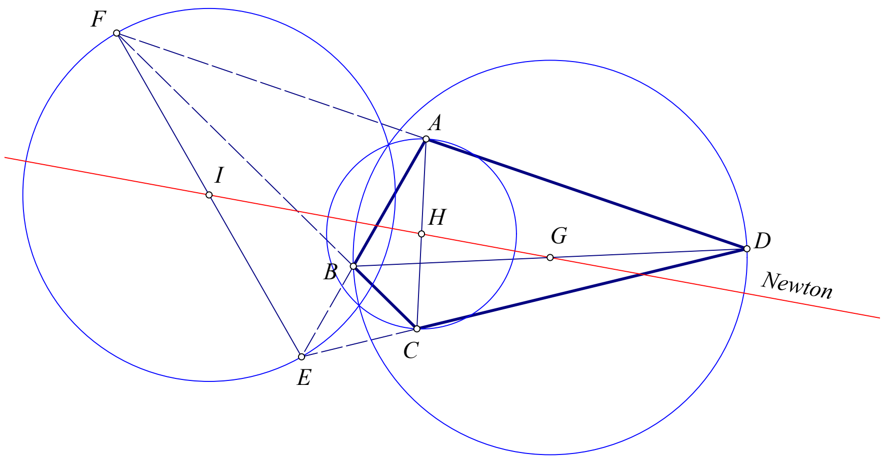

Figure 21: The *"Newton"* pencil of the quadrangle

The second prominent pencil (see figure 21) is connected with the *"diagonals"* of a complete quadrangle. It is generated by the three circles having as diameters the three *"diagonals"* $\{AC, BD, EF\}$ of the quadrangle. The centers of these circles lie on the *"Newton line"* of the quadrangle joining the middles of the diagonals of the quadrangle.

10 Polars and poles w.r. to members of a pencil

Theorem 5. *The polars ε_P of a point P w.r. to the member-circles κ of a pencil $\mathcal{P}$ pass through the point P', which is the diametral of P w.r.t. the circle λ of the orthogonal pencil of the pencil $\mathcal{P}$ passing through P (see figure 22).*

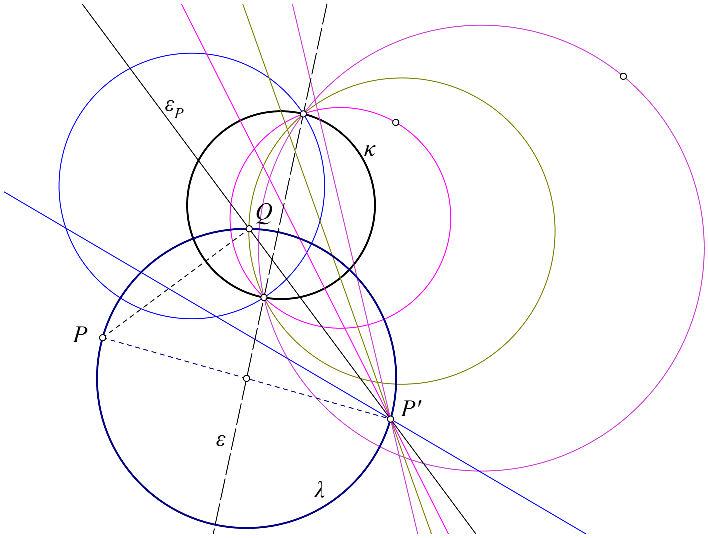

Figure 22: The polars of P pass all through P'

Proof. The proof follows by considering the projection Q of P on the polar ε_P of P w.r. to the circle κ. This is the inverse of P w.r. to κ and every circle passing through $\{P, Q\}$ is orthogonal to κ. In particular the circle λ passing through $\{P, Q\}$ and with center on the radical axis ε of the pencil $\mathcal{P}$ defines the unique circle of the orthogonal pencil passing through P. The polar ε_P is orthogonal to PQ at Q and passes through the diametral point P' of λ, which is the same for all members $\kappa \in \mathcal{P}$. □

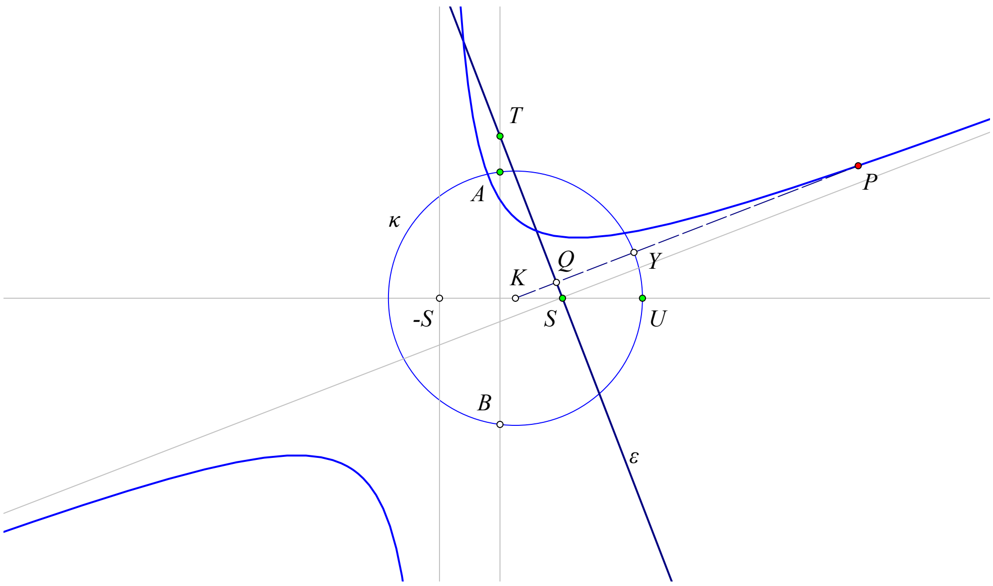

Figure 23: The poles P of line ε w.r. to circle-members $\{\kappa\}$

Considering the poles of a fixed line ε w.r. to the circle-members $\{\kappa\}$ of a pencil we find them defining certain hyperbolas and in some cases parabolas and degenerate conics.

Theorem 6. *The poles of a fixed line w.r.t. the circle-members of a pencil lie on a conic, which is a hyperbola or parabola or a degenerate conic (see figure 23).*

Proof. Here we discuss the case of an intersecting pencil, leaving the other cases as exercises. The pencil we consider has base points $\{A, B\}$ lying symmetrically w.r.t. the origin on the y-axis (see figure 23). All circles $\kappa(K, r)$ of this pencil pass through $\{A(0, a), B(0, -a)\}$. The circles are easily seen to have centers $K(k, 0)$ and be defined by equations of the form:

$$x^2 + y^2 - 2kx - a^2 = 0 \quad \text{with} \quad k = \frac{u^2 - a^2}{2u}, \quad r = |KU| = \frac{u^2 + a^2}{2|u|}, \tag{6}$$

where we denote by $U(u, 0)$ a running point on the x-axis, defining the circle $\kappa(K, r)$. Assuming also the line to be given by an equation of the form

$$\frac{x}{s} + \frac{y}{t} = 1, \tag{7}$$

we find, using the relation $KQ \cdot KP = r^2$ and some simple calculations, the expression satisfied by the poles P of ε w.r. to κ :

$$s \cdot x^2 - t \cdot xy - st \cdot y + sa^2 = 0, \tag{8}$$

which represents in general $(st \neq 0)$ a hyperbola with an asymptote orthogonal to the line ε at S and the other asymptote parallel to the y-axis and passing through $-S$. The exclusion of *ellipses* was expected, since, as the center K of κ approaches S, the corresponding pole P tends to infinity, hence the geometric locus of points $\{P\}$ is not bounded. □

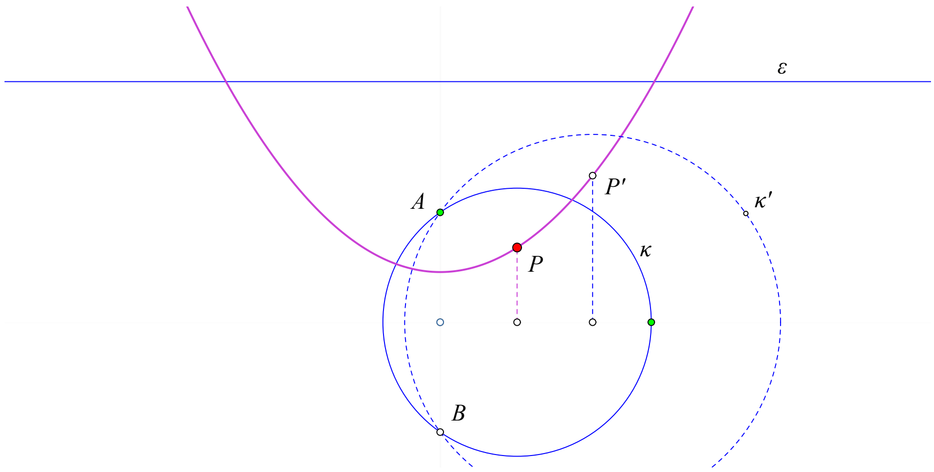

Figure 24: Parabola generated when the line ε is parallel to the line of centers

Figure 24 shows the case of a line ε parallel to the line of centers of the pencil of circles. In this case the poles P of ε w.r.t. to the member-circles lie on a parabola. The focus of the parabola in this figure is the base point A of the pencil.

Next sections show some other instances of the intimate relation of circle pencils with hyperbolas.

11 Quadratic transformation defined by a pencil

Returning to theorem 5 and the corresponding figure 22, we consider a pencil $\mathcal{P}$ and the associated to it transformation $f : P \mapsto P'$, which to every point P associates the common point P' of all polars of P w.r. to the circle members of the pencil. The recipe of construction of P' involves the orthogonal pencil $\mathcal{Q}$ of $\mathcal{P}$. For each point P we select the unique circle member $\lambda \in \mathcal{Q}$, and define P' as the diametral point of P w.r. to λ. It is obvious that this is an *"involutoric"* transformation, i.e. it sastisfies $f^2 = e$, the symbol e denoting the identity transformation. In other words, the transformation f behaves like a reflection, being identical with its inverse transformation i.e. satisfying $f(P') = P$. Next theorem gives a more precise description of it ([Pon65, I,p.43]).

Theorem 7. *The transformation f is a "quadratic" one, transforming lines to conics of all kinds except ellipses.*

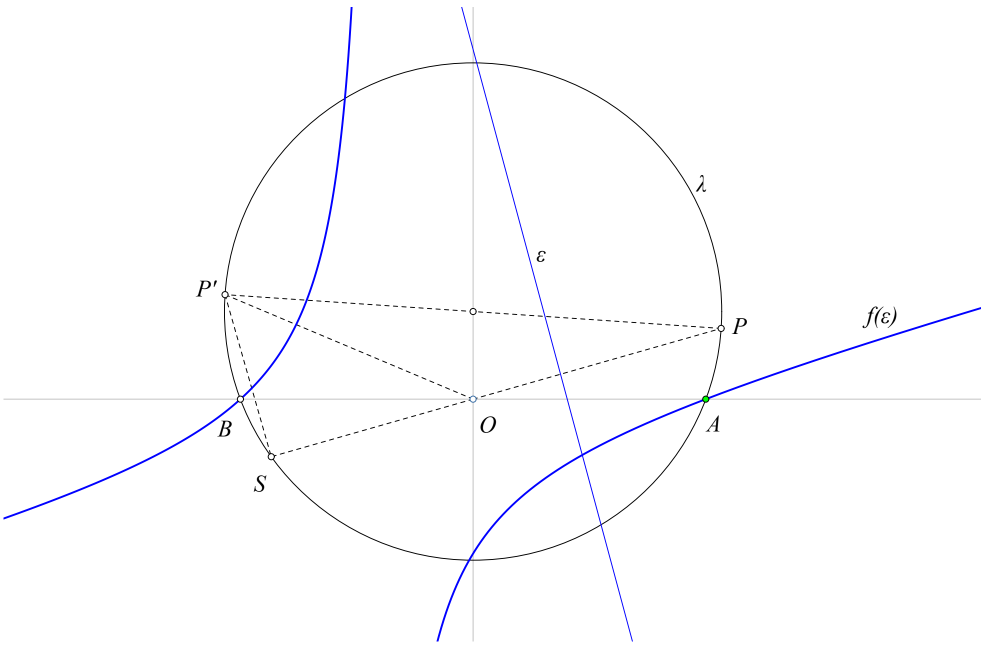

Figure 25: The quadratic transformation $f : P \mapsto P'$

Proof. We handle here the case of a pencil $\mathcal{P}$ of non-intersecting type, leaving the other cases as exercises. In this case the circle members $\{\lambda\}$ of the orthogonal pencil $\mathcal{Q}$ pass through two fixed points $\{A, B\}$, which can be taken on the x-axis symmetric w.r. to the origin (see figure 25). A short calculation shows then that the transformation is given by

$$P(p_1, p_2) \mapsto P'(p_1', p_2') = f(P) \quad \text{with} \quad \left(p_1' = -p_1\,,\; p_2' = \frac{p_1^2 - a^2}{p_2} \right), \tag{9}$$

where $A(a, 0)$. It is easily seen from the formula and also from the figure, that the coordinate vectors have a constant inner product

$$P \cdot P' \;=\; -a^2.$$

Assuming the line ε represented in parametric form

$$\varepsilon \;:\; x = \alpha t + \beta, \quad y = \gamma t + \delta,$$

taking $f(x,y)$ and eliminating the parameter t, we find the equation satisfied by

$$f(\varepsilon)\ :\qquad \alpha\cdot x^2+\gamma\cdot xy+(\beta\gamma-\alpha\delta)\cdot y-a^2\alpha=0, \tag{10}$$

representing, for $\gamma=0$ a parabola and for $\gamma\neq 0$ a hyperbola (see file **The quadratic equation in the plane**). □

Remark 1. The name *"quadratic"* stems from the expression (9), which in homogeneous coordinates is described by the equations

$$x' = -xy, \tag{11}$$
$$y' = x^2-a^2z^2, \tag{12}$$
$$z' = yz, \tag{13}$$

which on the right have *quadratic polynomials*. The transformation belongs to the more general group of *"Cremona transformations"*, studied in the context of *"algebraic geometry"* ([ea70, p.19], [Ode16, p.329]). The transformation is well defined and 1-1 everywhere except for the points of three lines, which are the x-axis and the two parallels to the y-axis at $x=\pm a$. On these two lines the transformation is not 1-1, but sends each line to a point. The three points in homogeneous coordinates

$$(a,0,1),\quad (-a,0,1),\quad (0,1,0),$$

which are the solutions of the system of equations $x'=y'=z'=0$, and represent the intersections of the three exceptional lines are called *"fundamental points"* of the transformation. All the conics-images of lines under this transformation pass through these three points.

Remark 2. An analogous quadratic transformation can be defined more generally for a *"pencil of conics"* through four fixed points (see figure 26). The *fundamental points* in this case are the three diagonal points $\{A,B,C\}$ of the complete quadrangle of the four fixed points. The corresponding quadratic transformation f is defined as before: the

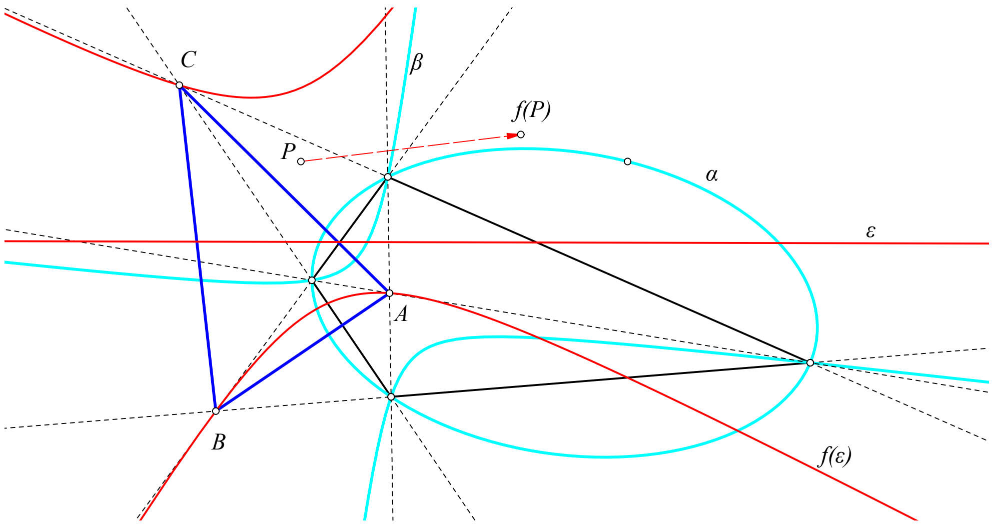

Figure 26: More general quadratic transformation $P\mapsto f(P)$

polars of P w.r.t. to the conics of the pencil pass through the same point $P'=f(P)$. The transformation, referred to the *"projective coordinate system"* of the triangle of these three points, takes then the form of the usual *"isogonal transformation"*

$$x' = yz, \quad y' = zx, \quad z' = xy.$$

The images $f(\varepsilon)$ of lines $\{\varepsilon\}$ are again conics passing through the three *fundamental points* $\{A, B, C\}$. These are the *"triangle conics"* or *"circumconics"* of the triangle ABC ([Yiu13, p. 109]), forming the so called *"homaloidal net"* of conics in the context of Cremona quadratic transformations ([ea09, p.294]).

Remark 3. From equation (9) follows that lines ε parallel to the y-axis map under f to degenerate conics consisting of two lines: the x-axis and the symmetric to ε w.r. to the y-axis. This refines the type of the quadratic transformation f to the type of *"de Jonquieres transformations"*, which are Cremona transformations of the plane preserving a pencil of lines ([Des09, p.51]).

Remark 4. For a member circle $\kappa \in \mathcal{P}$ the quadratic transformation maps it to a curve $\kappa' = f(\kappa)$ of degree four. Figure 27 displays such an example for a circle κ of the pencil of non-intersecting type with limit points $\{A(a,0), B(-a,0)\}$ lying on the x-axis symmetric w.r. to the origin. The circle can be parameterized by its diametral points $\{U(u,0), V(v,0)\}$ satisfying $uv = a^2$. A short computation shows that the curve κ' satisfies the equation

$$y^2(ux + a^2)(x + u) + u(x^2 - a^2)^2 = 0.$$

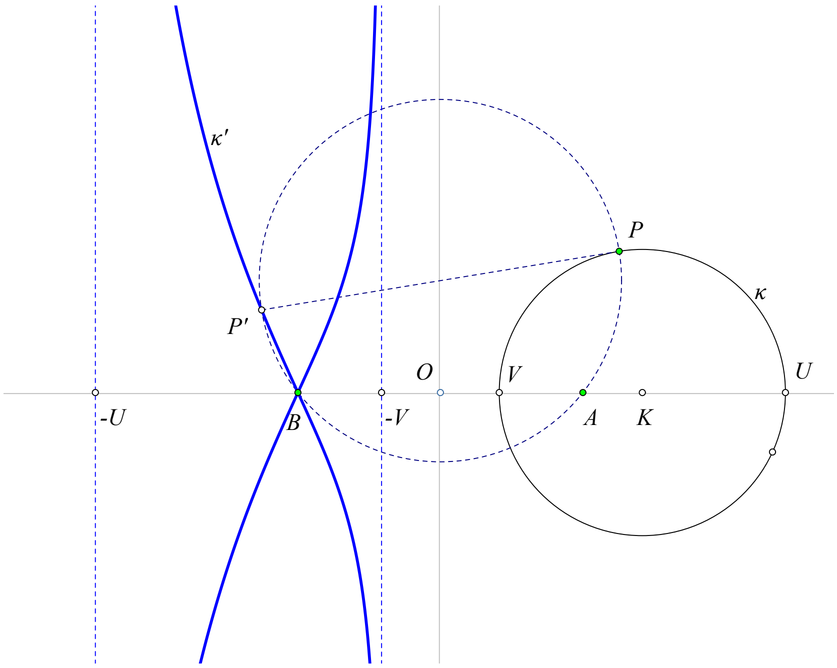

Figure 27: Transform $f(\kappa)$ of $\kappa \in \mathcal{P}$

12 Hyperbola from a pencil of circles

Next properties supply an alternative way to define a hyperbola using a pencil of circles passing through two fixed *base points*. The appropriate configuration consists of two lines $\{OA, OB\}$ and two points $\{F, F'\}$, taken symmetrically relative to O and also lying on a

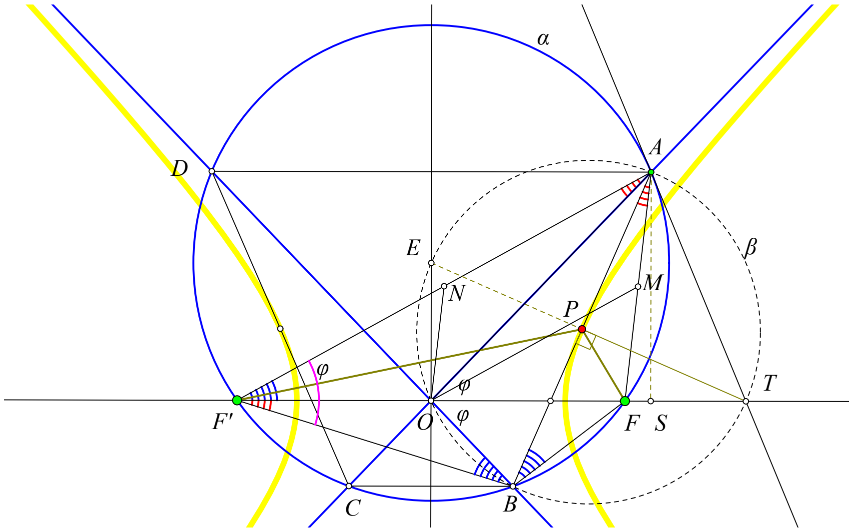

Figure 28: Hyperbola defined by a pencil of circles $\{\alpha\}$

bisector of the angle of the two lines. We then consider all circles $\{\alpha\}$ passing through $\{F, F'\}$ and intersecting the lines along the chords $\{AB, CD\}$ (see figure 28).

Theorem 8. *Under the previous conventions, the following are valid properties.*

1. *The triangles $\{AF'O, BOF', ABF\}$ are similar.*
2. *The triangles $\{F'BA, OFA\}$ are similar.*
3. *The middle P of AB moves on a hyperbola with focal points $\{F, F'\}$.*
4. *The line AB is tangent at P to the previous hyperbola.*

Proof. *Nrs 1-2* follow by observing the angles inscribed in the circle α.

Nr-3. Define first $\{x = AF, y = AF', d = PF, e = PF', m = AO\}$. Then from the similarity of the triangles we have

$$\frac{d}{x} = \frac{ON}{m}, \ \frac{e}{y} = \frac{OM}{m} \ \Rightarrow \ e - d = \frac{1}{m}(yOM - xON) = \frac{1}{2m}(y^2 - x^2).$$

But the difference $y^2 - x^2 = F'S^2 - FS^2 = 2FF' \cdot OS$. Hence

$$\frac{1}{2m}(y^2 - x^2) = (FF' \cdot OS)/m = FF' \cos(\phi).$$

This shows that P is on the hyperbola with focal points $\{F, F'\}$ and major axis $2a = FF' \cos(\phi)$. Since $2c = FF'$, we have also that $b = c \sin(\phi)$ and the two lines $\{OA, OB\}$ are asymptotes of the hyperbola.

Nr-4 follows by showing that PB is a bisector of the angle $\widehat{F'PF}$. In fact, by the equality of the angles $\widehat{AOB} = 2\phi = \widehat{AEB}$, where E is the circumcenter of AFF', follows that points $\{A, E, O, B, T\}$ are concyclic on a circle β and TE is a diameter of it, which passes through the middle P of AB. Then, point T is the pole of AB and $P(FF'; BT)$ is a harmonic pencil with two orthogonal rays $\{PB, PT\}$. Hence these rays are bisectors of the angle of the other two. □

Next theorem and corresponding figure 29 summarizes again the same properties stressing the role of the angles involved.

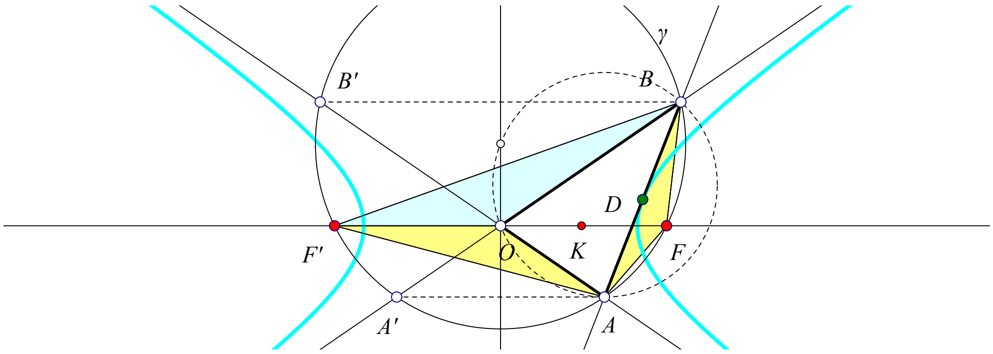

Figure 29: Property of the general asymptotic triangle

Theorem 9. *The asymptotic triangle OAB defines three triangles $\{FAB, OBF', OF'A\}$, which are similar and the angle $\widehat{AFB}$ is constant and equal to $F'OB = (360° - \omega)/2$, where ω is the angle between the asymptotes .*

Corollary 4. *There is a unique hyperbola having for asymptotes the diagonals $\{AB', A'B\}$ of an isosceles trapezium $AA'B'B$ and passing through the middles of the non-parallel sides.*

From the similarity of triangles $\{AOF', F'OB\}$, follows that the distance of O from the sides $\{F'A, F'B\}$ of the triangle $F'AB$ is proportional to the lengths of these sides, something that characterizes the *"symmedian line"* $F'K$ of a triangle, hence the property. This proves next theorem.

Theorem 10. *All triangles ABF', created from the asymptotic triangle OAB, have the angle $\widehat{F'}$ constant and equal to half the angle of the asymptotes. In addition FF' is the symmedian line from F', hence the symmedian point K of this triangle lies on line FF', which is also a bisector of the angle $\widehat{AOB}$. Analogously, line FF' is also a symmedian of the triangle AFB.*

Theorem 11. *A hyperbola, whose asymptotes $\{OA, OB\}$ make an angle ω, is generated by rotating an angle of measure $\widehat{AF'B} = \omega/2$ or $\widehat{AFB} = 180° - \omega/2$, about their fixed vertices $\{F, F'\}$, which are points of a bisector of the fixed angle ω. The rotating angle intersects the fixed angle ω at points $\{A, B\}$ and the hyperbola is the locus of the middle D of AB. Lines AB are tangents to this hyperbola.*

13 Rectangular hyperbolas related to pencils of circles

Theorem 12. *The circles having for diameter chords of a rectangular hyperbola parallel to the direction α, define a "pencil" $\mathcal{D}$. Their centers lie on the conjugate diameter ε of α and their radical axis ζ is the orthogonal diameter to ε through the center O of the hyperbola (see figure 30). The orthogonal to this pencil $\mathcal{D}'$ is created analogously by the orthogonal to α direction β of parallel chords.*

Proof. We sketch the proof in the case of the rectangular hyperbola $xy = 1$. The general rectangular hyperbola is similar to this one and the property in question is invariant under similarities. Assume that the direction is determined by a vector $\alpha = (a_1, a_2)$. A typical point on the conic is of the form $X(t) = (t, 1/t)$ and the other point of the conic on the line $X(t) + \lambda \cdot \alpha$ is easily seen to be

$$X'(t) = X(t) + \lambda \cdot \alpha \quad \text{with} \quad \lambda = \frac{a_1 + a_2 t^2}{a_1 a_2 t}.$$

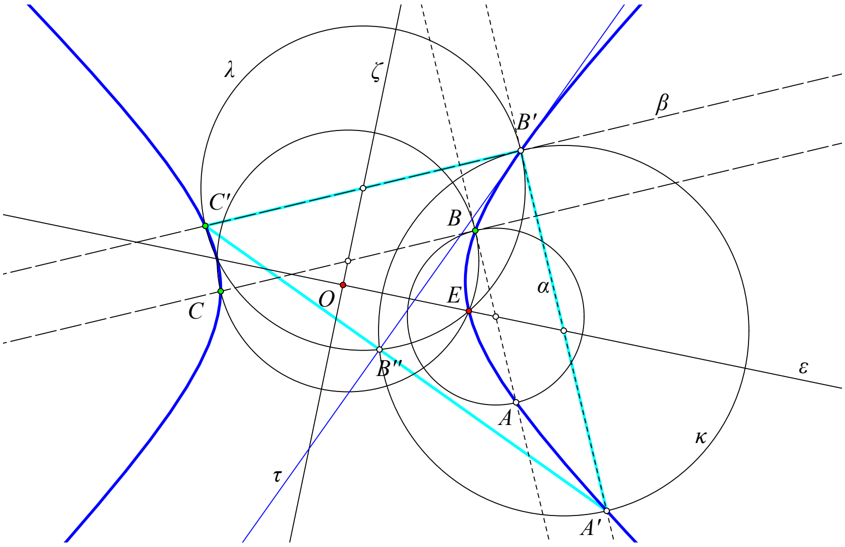

Figure 30: Pencil on parallel chords

Setting $X_0 = (X + X')/2$ for the center and $r^2 = (X - X')^2/4$ for the radius, the variable circles with diameters $A'B'$ are described by the equation:

$$f_t(x, y) \;=\; (x - X_0(t))^2 + (y - Y_0(t))^2 - r^2(t) \;=\; 0.$$

The radical axis of two such circles $\{\kappa, \kappa'\}$, characterized by the parameters $\{t, t'\}$ is given by the equation

$$f_t(x, y) - f_{t'}(x, y) \;=\; \frac{(a_1 x - a_2 y)(t' - t)(a_2 t t' + a_1)}{a_1 a_2 t t'} \;=\; 0 \qquad \Leftrightarrow \qquad a_1 x - a_2 y \;=\; 0\,,$$

which is independent of the values $\{t, t'\}$, showing that the circles belong to a pencil. The direction of the radical axis given by (a_2, a_1) and its orthogonal direction $(a_1, -a_2)$ is the one of the line of centers. All other claims are easily proved from these remarks. Notice that point E in figure 30 is a limit point of the pencil and $|OE|$ is the length of the tangent from O to every member-circle of the pencil. In this figure the radical axis ζ does not intersect the hyperbola and the pencil $\mathcal{D}$ is of non-intersecting type. The orthogonal pencil $\mathcal{D}'$ consists of member-circles λ on diameters with endpoints on different branches. □

In figure 30 notice the line τ, which is the radical axis of the pair of orthogonal circles $\{\kappa, \lambda\}$ and coincides with the altitude from B' of the right triangle $A'B'C'$. The theorem has the following short of converse.

Theorem 13. *Given a pencil of circles of intersecting or non-intersecting type and a direction α, the diametral points of the diameters of member-circles, which are parallel to α generate rectangular hyperbolas.*

Proof. We handle the case of non-intersecting pencils. The other case can be reduced to this one by considering the orthogonal to the given pencil and applying the last claim of theorem 12.

Let $\{E, E'\}$ denote the limit points of the pencil of non-intersecting type. Drawing the line α from E realizing the given direction, we have a system of lines determining a rectangular hyperbola (see figure 31). In fact, the line $\beta = EE'$ joining the limit points of the given pencil and the parallel lines $\{\alpha, \alpha'\}$ through these points determine the bisectors

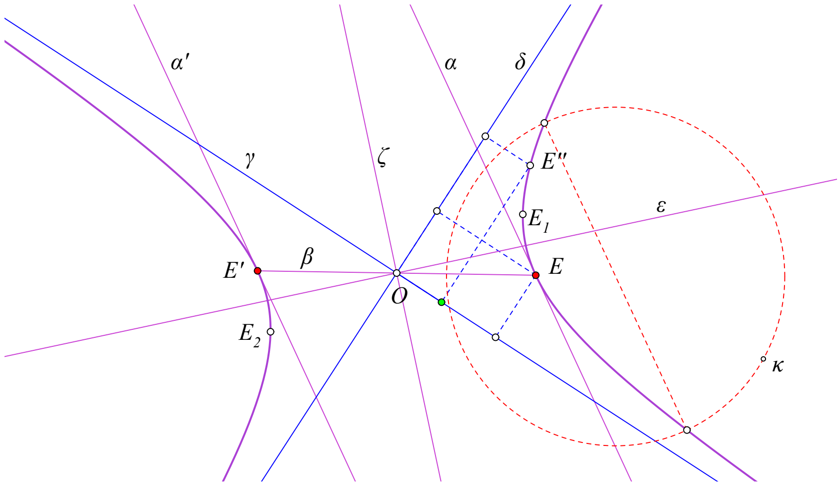

Figure 31: Rectangular hyperbola generated by a pencil of circles $\{\kappa\}$

$\{\gamma, \delta\}$ of the angles of lines $\{\alpha, \beta\}$. There is then a unique rectangular hyperbola having aysmptotes the lines $\{\gamma, \delta\}$ and passing through E. This hyperbola passes through the symmetrics $\{E_1, E', E_2\}$ of E w.r.t. the bisectors $\{\varepsilon, \zeta\}$ of the angles of lines $\{\gamma, \delta\}$. It is also required to pass through a fifth point E'', whose projections on the axes $\{\gamma, \delta\}$ define a parallelogram with area equal to the area of the corresponding parallelogram defined by the point E.

Having the hyperbola, we consider its intersections with lines parallel to α. By theorem 12 we know that the circles having diametral points these intersections build a pencil with limit points $\{E, E'\}$, hence coinciding with the given pencil. □

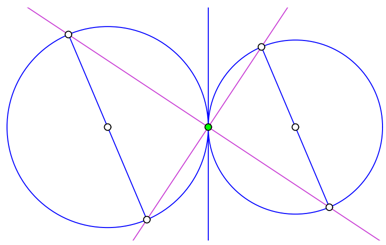

Figure 32: Tangential pencil and diameters in a fixed direction

Exercise 4. *Show that the corresponding proposition for a tangential pencil produces a pair of orthogonal lines. More precisely, the diametral points of the member-circles of a tangential pencil defining a line with fixed direction lie on a degenerate conic consisting of two orthogonal lines (see figure 32).*

Theorem 14. *To the circles of a pencil of intersecting or non-intersecting type, tangents at a fixed given direction are drawn. The geometric locus of contact points P is a rectangular hyperbola.*

Proof. We handle the case of intersecting type, leaving the other one as an exercise (see remark 5 below). Consider a circle κ with center K of the pencil with base points $\{A, B\}$ (see figure 33). We adopt for coordinate axes the lines $\{Ox, Oy\}$ inclined to the medial line OK of AB by half the fixed angle $\phi = \widehat{TKP}/2$, defined by the orthogonal KP to the fixed

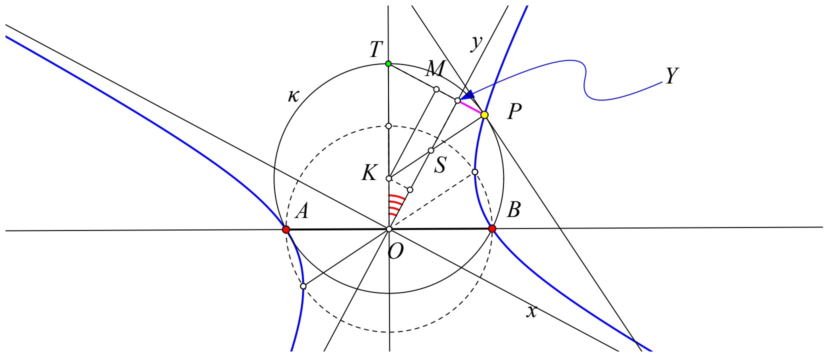

Figure 33: Parallel tangents to circles of a pencil

direction of the tangents. Setting $\{x = YP, y = OY\}$, we have

$$\begin{aligned} \frac{x}{SY} &= \frac{TY}{y} \Rightarrow xy = TY \cdot SY = TY \cdot YP\cot(\phi) = TY(TP - TY)\cot(\phi) \\ \text{but} \quad & TY(TP - TY)\cot(\phi) = (OT\sin(\phi))(2KT\sin(\phi) - OT\sin(\phi))\cot(\phi) \\ &= OT(2KT - OT)\cos(\phi) = OA^2\cos(\phi), \end{aligned}$$

thereby proving the claim. □

Remark 5. Figure 34 results from figure 33 by considering the other intersection point P' of the locus-hyperbola for the fixed direction of the tangent line η of the circle α of the intersecting pencil $\mathcal{D}$ with base points $\{A, B\}$. The circle β with diameter PP' belongs to the orthogonal pencil $\mathcal{D}'$ and its tangent ε at P' has also fixed direction, namely the orthogonal

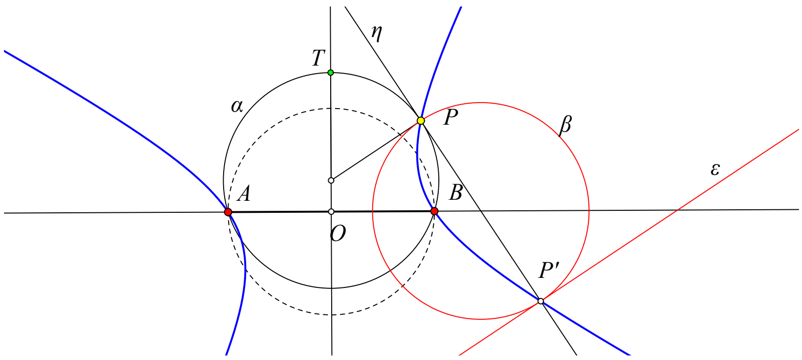

Figure 34: Parallel tangents to circles of a non intersecting pencil

to that of η. Thus the same hyperbola is the locus of tangent points of tangents in the fixed direction ε to the circles of the non-intersecting pencil $\mathcal{D}'$ with *"limit points"* $\{A, B\}$.

Remark 6. It is trivial to see that the analogous problem for a *"tangential pencil"* of circles leads to a pair of orthogonal lines, which are identical with the asymptotes $\{Ox, Oy\}$ of theorem 14 if the common tangent to the pencil is AB and the common contact point is O (see figure 33).

14 Some exercises related to pencils of circles

In the first exercise we consider two circles $\{\lambda, \nu\}$ of an intersecting pencil with base points $\{D, E\}$ (see figure 35). The lines $\{AD, AE\}$ for $A \in \nu$ define triangles $\{ADE, ABC\}$, where

$\{B,C\}$ the second intersections of these lines with the circle λ. The following properties are easily verified:

1. The triangles $\{ABC, ADE\}$ are similar through an *"anti-similarity"* f_A with center A and axis the bisector AN of $\widehat{A}$ passing through the middle N of the arc END of ν.
2. The segment BC has constant length and $r = BC/ED$ is the constant ratio of the similarity f_A, for all positions of $A \in \nu$. The circumcircle κ of $\triangle\ ABC$ has constant radius.
3. The orthocenter H' of $\triangle\ ADE$ moves on a circle ν' of the pencil, equal to the circumcircle ν of $\triangle\ ADE$. The orthocenter H of $\triangle\ ABC$ moves on the image-circle $\nu'' = f_A(\nu')$ via the similarity.
4. The angles $\{\widehat{HAB}, \widehat{H'AD}\}$ are equal and AH is orthogonal to BC. Since the orthocenter H' and the circumcenter K' of $\triangle\ ADE$ are isogonal conjugate points, line AH passes through K'.

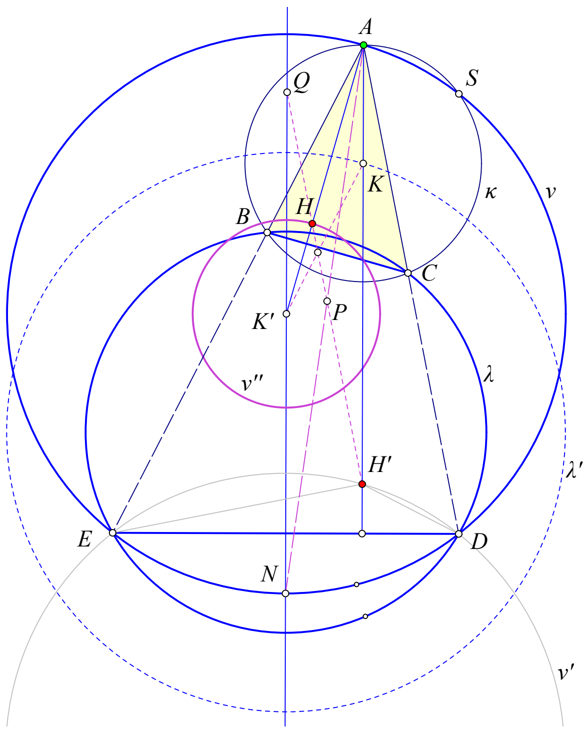

Figure 35: Variable triangle ABC between two member circles $\{\lambda, \nu\}$

5. From the same equality of angles follows that the circumcenter K of $\triangle\ ABC$ is on the line AH'.
6. The segment AH' between the equal circles $\{\nu, \nu'\}$ of the pencil has constant length, hence also the segment $AH = f_A(AH')$ has constant length. It follows that the circle ν'' is concentric with the circle ν.
7. The ratio $K'H/HA$ is constant and HH' passes through a fixed point Q on $K'N$.
8. The segment KK' is divided by the line QH' in a constant ratio.
9. The second intersection S of the circumcircles of the triangles $\{ABC, ADE\}$ is the direct-similarity center of the oriented segments $\{BC, ED\}$.

Figure 36 shows another property of the preceding configuration.

Exercise 5. *The line* $\zeta = AS$ *envelopes a conic sharing with the circle* ν *two common parallel tangents at the points* $\{L, N\}$.

Hint: This follows immediately from the exercise 6 below, by observing that the perpendicular to ζ at A meets LN at a fixed point M whose distance $|K'M| = |KA|$ is equal to the constant radius of the circle κ.

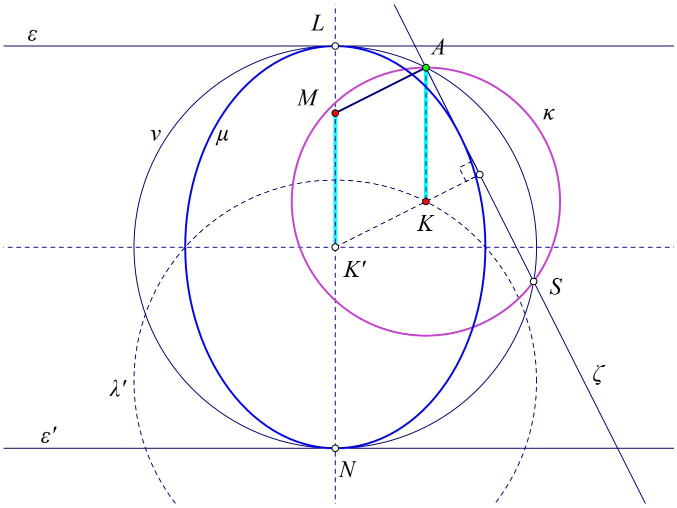

Figure 36: The conic enveloping the lines $\zeta = AS$

Exercise 6. *Let* M *be a fixed point and* ν *a fixed circle. For every point* $A \in \nu$ *consider the line* ζ_A *orthogonal to* MA *at* A. *The lines* $\{\zeta_A\}$ *envelope a conic* μ.

Hint: The solution of this reduces to the following one representing a standard generation of a conic.

Exercise 7. *Let* M *be a fixed point and* ν *a fixed circle. For every point* $A \in \nu$ *consider the orthogonal bisector line* λ_A *of the segment* MA. *The lines* $\{\lambda_A\}$ *envelope a conic* ν *homothetic to the conic* μ *of the preceding exercise w.r.t.* M *by the ratio* $1/2$ *(see figure 37).*

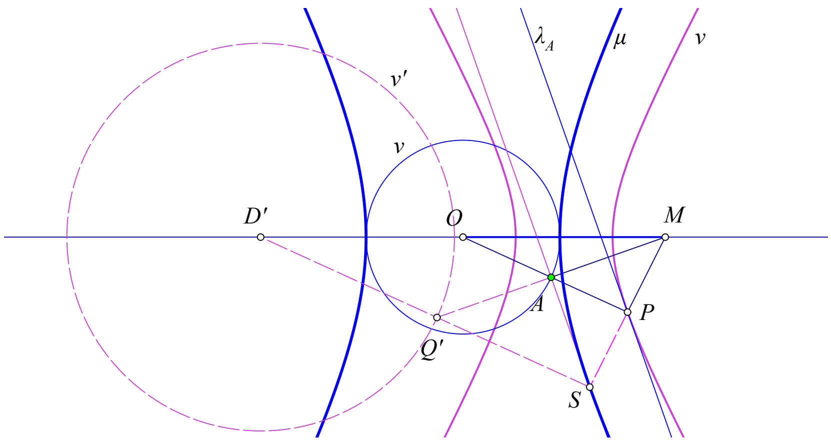

Figure 37: Two homothetic conics

Hint: Assume M to be external to the circle κ and extend OA to its intersection P with λ_A. Then $||PO| - |PM|| = |OA|$ is the constant radius of ν. This implies that ν is a hyperbola with focal points $\{M, O\}$. The intersection S of MP with the parallel to λ_A from A

has the same relation with the circle ν' as P with ν. Here ν' is the homothetic of ν w.r.t. to M at the ratio 2.

Exercise 8. *Given two circles $\{\kappa, \lambda\}$ the polars λ_P of points $P \in \kappa$ w.r.t. λ envelope a conic (see figure 38).*

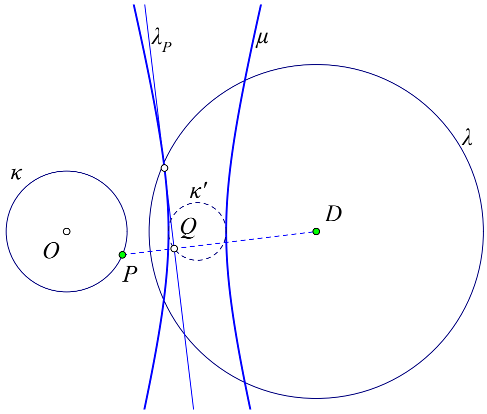

Figure 38: The conic enveloping the polars $\{\lambda_P\}$

Hint: Figure 38 shows such a conic enveloped by the lines $\{\lambda_P.\}$ The line λ_P is orthogonal to PD at the inverse Q of P w.r.t. λ, which moves on the circle κ' inverse of κ w.r.t. λ. Thus we have the configuration of exercise 6.

On the occasion of the preceding figure notice that the contact point P' of the line λ_P is the pol w.r.t. λ of the tangent t_P of κ at P (see figure 39). The inverse P'' of P' w.r.t. λ is on t_P coinciding with the projection of D on t_P. Thus, the inverse of the hyperbola μ w.r.t. λ is the *"pedal"* of the circle κ w.r.t. D i.e. the locus of projections of D on the tangents of the circle κ. This is one of the traditional definitions of the *"limacon"*, which is a curve of degree four ([Loc61, p.47], [Law72, p.113], [Ode16, p.406]). Since D is one of the focal points of μ this shows that *"the inverse of a conic w.r.t. a circle with arbitrary radius and center at a focal point is a limacon"*. The figures below show the corresponding

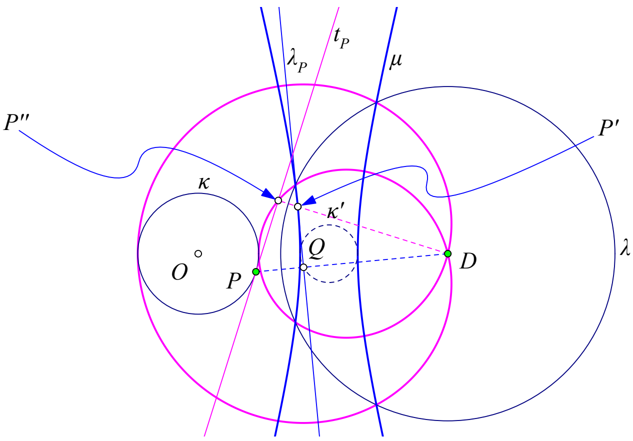

Figure 39: The inversion of the hyperbola μ to a limacon

shape of the limacon in the case of a parabola and an ellipse inverted w.r.t. an arbitrary circle centered at a focal point D of the conic. While the inversion of the hyperbola is a limacon with a self-intersection at D, the parabola produces analogously a limacon with a singular point (cusp) at D which is traditionally called a *"cardioid"*. Finally the ellipse produces a limacon without singular points.

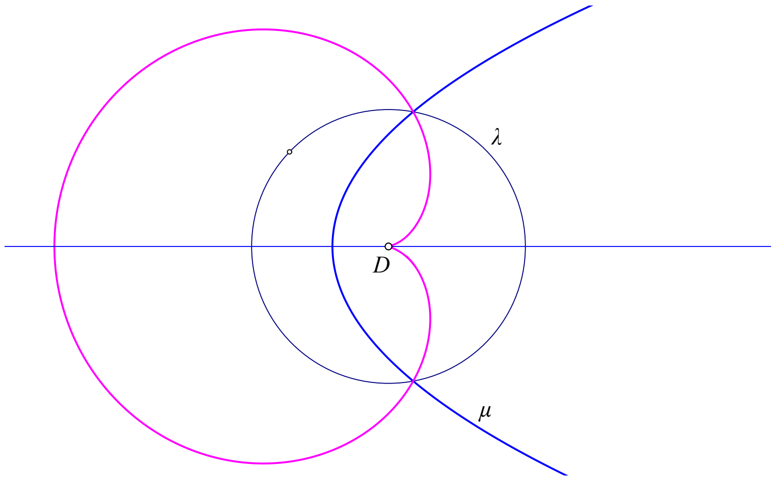

Figure 40: The inversion of the parabola μ to a limacon (cardioid)

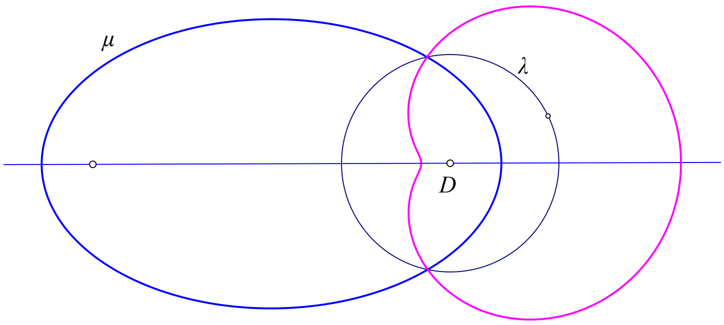

Figure 41: The inversion of the ellipse μ to a limacon

Remark 7. Finishing this discussion we should notice the definition of a *"homographic relation "* on a line ε through its intersections with the members of a pencil of circles. This is discussed in the file **Homographic relations**.

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Related material

1. **Abridged notation**
2. **Apollonian circles of a segment**
3. **Barycentric coordinates**
4. **Cross Ratio**
5. **Homographic relations**
6. **Inversion**
7. **Apollonian circles of a triangle and isodynamic points**
8. **The quadratic equation in the plane**

Any correction, suggestion or proposal from the reader, to improve/extend the exposition, is welcome and could be send by e-mail to: pamfilos@uoc.gr