Conway triangle symbols

A file of the Geometrikon gallery by Paris Pamfilos

For the sake of these different types, scientific truth should be presented in different forms, and should be regarded as equally scientific, whether it appears in the robust form of vivid colouring of a physical illustration, or in the tenuity and paleness of a symbolical expression.

J.C. Maxwell, Address to the British Association

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1 Definition, first properties

Conway triangle symbols are called the expressions

$$S_A = \frac{1}{2}(b^2 + c^2 - a^2), \quad S_B = \frac{1}{2}(c^2 + a^2 - b^2), \quad S_C = \frac{1}{2}(a^2 + b^2 - c^2). \tag{1}$$

They come from a triangle ABC with side-lengths $\{a, b, c\}$ and angles lying respectively opposite to them: $\{\alpha, \beta, \gamma\}$ [Yiu13a, p.33]. They are equivalent to the expressions

$$S_A = bc\cos(\alpha), \quad S_B = ca\cos(\beta), \quad S_C = ab\cos(\gamma). \tag{2}$$

Their importance stems from the fact that they represent the *"inner product of displacement vectors"* $U - U'$, where $\{U = (u, v, w), U' = (u', v', w'), \ldots\}$ are vectors of *"absolute barycentric"* coordinates, or *"barycentrics"*. Thus,

$$|UU'|^2 = S_A(u - u')^2 + S_B(v - v')^2 + s_C(w - w')^2 \tag{3}$$

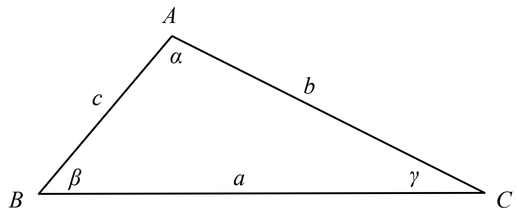

Figure 1: Triangle ABC with side-lengths $\{a, b, c\}$ and opposite angles $\{\alpha, \beta, \gamma\}$

is the square of the distance of the points in the plane, which are represented by the coordinates vectors $\{U, U'\}$ and

$$UU' \cdot PP' = S_A(u-u')(p-p') + S_B(v-v')(q-q') + S_C(w-w')(r-r') \tag{4}$$

represents the inner product of the *displacement vectors*

$$U - U' = (u-u', v-v', w-w') \quad \text{and} \quad P - P' = (p-p', q-q', r-r'),$$

where $\{U, U', P, P'\}$ are vectors of *absolute barycentrics*, satisfying per definition

$$u + v + w = u' + v' + w' = p + q + r = p' + q' + r' = 1.$$

Alternatively the inner product and the square distance can be represented using the side-lengths themselves:

$$|UU'|^2 = -a^2(v-v')(w-w') - b^2(w-w')(u-u') - c^2(u-u')(v-v'). \tag{5}$$

For more general *barycentrics* $U = (u, v, w)$ with $u + v + w \neq 1$ we come to *"absolute barycentrics"* by dividing with

$$s_U = u + v + w \quad \text{giving the absolute barycentric:} \quad U^* = \frac{1}{s_U} U = \frac{1}{s_U}(u, v, w). \tag{6}$$

From their definition, comes not to surprise that these symbols are ubiquitous in subjects of *"triangle geometry"* dealt with barycentrics ([Yiu13a], [Yiu13b], [Kim18]) (see file **Barycentric coordinates**).

2 Identities resulting directly from definition

Next identities are easy consequences of the definition ([Yiu13a, p.33]). In these the expression

$$S = 2\Delta = ab\sin(\gamma) = bc\sin(\alpha) = ca\sin(\beta), \tag{7}$$

is twice the area Δ of the triangle of reference ABC and obviously hold the relations:

$$S_A = S\cot(\alpha), \qquad S_B = S\cot(\beta), \qquad S_C = S\cot(\gamma), \tag{8}$$

$$S_B + S_C = a^2, \qquad S_C + S_A = b^2, \qquad S_A + S_B = c^2, \tag{9}$$

$$S_B - S_C = c^2 - b^2, \qquad S_C - S_A = a^2 - c^2, \qquad S_A - S_B = b^2 - a^2, \tag{10}$$

$$S_A^2 = b^2c^2 - S^2, \qquad S_B^2 = c^2a^2 - S^2, \qquad S_C^2 = a^2b^2 - S^2, \tag{11}$$

$$S_AS_B + S_BS_C + S_CS_A = S^2, \tag{12}$$

$$a^2S_A + b^2S_B + c^2S_C = 2S^2, \tag{13}$$

$$a^2(S_A^2 - S_BS_C) + b^2(S_B^2 - S_CS_A) + c^2(S_C^2 - S_AS_B) = 0. \tag{14}$$

$$S_AS_B + c^2S_C = S_BS_C + a^2S_A = S_CS_A + b^2S_B = S^2, \tag{15}$$

3 Connection with the Brocard angle

The Brocard angle ω of the triangle is defined in two ways, corresponding to the two orientations of the triangle ABC. One way to define it is to select the positive orientation and draw lines from the vertices making the same angle ϕ with the sides (See Figure 2-I).

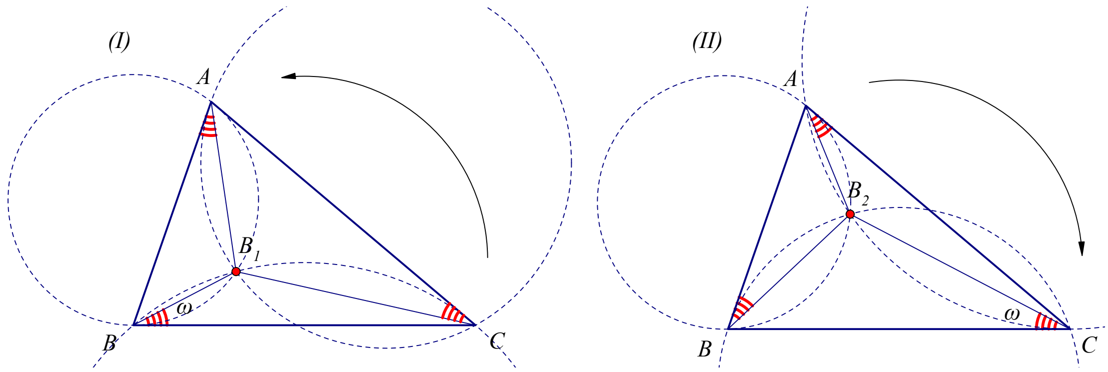

Figure 2: Brocard angle ω of the triangle ABC

In general the three resulting lines do not concur. It is proved though that, for a given triangle ABC, there is precisely one angle $0 < \omega < 60°$, for which this happens indeed and the three lines concur at a point B_1. Selecting the negative orientation in the same triangle (See Figure 2-II) and doing the same work produces again, up to orientation, the *same* angle ω, and a second point B_2 of concurrence of the three lines. The two points are called *"Brocard points"* of the triangle and the angle ω satisfies the remarkable relation to the sides of the triangle, connecting it with the Conway triangle symbols:

$$\cot(\omega) = \cot(\alpha) + \cot(\beta) + \cot(\gamma) \quad \Rightarrow$$
$$S_\omega := S\cot\omega = S_A + S_B + S_C = \frac{1}{2}(a^2 + b^2 + c^2). \tag{16}$$

The importance of the symbol S_ω, defined in the previous line, lies in the possibility to express with it and the symmetric functions of $\{a, b, c\}$ and $\{S_A, S_B, S_C\}$ all possible *"cyclic invariant expressions"* involving $\{a, b, c\}$ and $\{S_A, S_B, S_C\}$. Next *delirium calculantis* gives a flavour of this fact. The sums appearing in the equations are considered over the cyclic permutations of the letters $\{a, b, c\}$ and $\{A, B, C\}$. Thus, considering the additional symmetric functions in the side-lengths of the triangle

$$2s = a + b + c \quad \text{and} \quad R = \frac{abc}{2S},$$

latter proved easily to be the *"circumradius"*, and s called the *"half-perimeter"* of the triangle, we have:

$$ab + bc + ca = \frac{1}{2}((a+b+c)^2 - (a^2+b^2+c^2)) = 2s^2 - S_\omega. \tag{17}$$
$$\begin{aligned} a^3 + b^3 + c^3 &= \left(\sum a^2\right)\left(\sum a\right) - \sum ab(a+b+c-c) \\ &= (2S_\omega)(2s) - (2s^2 - S_\omega)(2s) + 3abc \\ &= (2s)(3S_\omega - 2s^2) + 6RS. \end{aligned} \tag{18}$$

$$\begin{aligned}
\sum a(b+c)^2 &= \sum a(b+c+a-a)^2 = \sum a((2s)^2 - 4sa + a^2) \\
&= (2s)^3 - 4s\sum a^2 + \sum a^3 \;=\; 2s(2s^2 - S_\omega) + 6RS. && (19)\\
\sum ab(a+b) &= \sum ab(a+b+c-c) = 2s\sum ab - 3abc \\
&= 2s(2s^2 - S_\omega) - 6RS. && (20)\\
\sum a(b-c)^2 &= \sum a(b+c)^2 - \sum a(4bc) \\
&= 2s(2s^2 - S_\omega) - 18RS. && (21)\\
\sum aS_A &= \frac{1}{2}\sum a(b^2+c^2+a^2-2a^2) = 2sS_\omega - \sum a^3 \\
&= 4s(s^2 - S_\omega) - 6RS. && (22)
\end{aligned}$$

$$\begin{aligned}
\sum a^4 &= \sum a^2(S_B+S_C) = \sum S_A(b^2+c^2) = \sum S_A\left(\sum a^2 - a^2\right) \\
&= 2(S_\omega^2 - S^2). && (23)\\
\sum a^2b^2 &= \frac{1}{2}\left(\left(\sum a^2\right)^2 - \sum a^4\right) \;=\; S_\omega^2 + S^2. && (24)\\
\sum S_A^2 &= \left(\sum S_A\right)^2 - 2\sum S_AS_B \;=\; S_\omega^2 - 2S^2. && (25)
\end{aligned}$$

$$\begin{aligned}
\sum aS_A^2 &= \sum a(b^2c^2 - S^2) = abc\sum bc - S^2\sum a \\
&= 2RS(2s^2 - S_\omega) - 2sS^2. && (26)\\
\sum a^2b^2S_C &= \frac{1}{2}\sum a^2b^2(a^2+b^2+c^2-2c^2) = S_\omega\sum a^2b^2 - 3a^2b^2c^2 \\
&= S_\omega(S_\omega^2 + S^2) - 12R^2S^2. && (27)\\
\sum a^2S_A^2 &= \sum a^2(b^2c^2 - S^2) = 3a^2b^2c^2 - S^2\sum a^2 \;=\; 2S^2(6R^2 - S_\omega). && (28)\\
\sum a^2S_BS_C &= 2S^2(6R^2 - S_\omega) \quad \text{as expected from (14)} \quad . && (29)\\
S_AS_BS_C &= \frac{1}{3}(3S_AS_BS_C) = \sum S_A(S^2 - a^2S_A) = S^2\sum S_A - \sum a^2S_A^2 \\
&= S^2(S_\omega - 4R^2). && (30)\\
\sum bcS_BS_C &= \frac{1}{2}\left(\left(\sum aS_A\right)^2 - \sum a^2S_A^2\right) = \frac{1}{2}\left(\left(4s(s^2 - S_\omega) - 6RS\right)^2 - 2S^2(6R^2 - S_\omega)\right) \\
&= 8sRS(S_\omega - s^2) + S^2(2s^2 + 12R^2 - S_\omega). && (31)
\end{aligned}$$

$$\begin{aligned}
\sum a^6 &= \sum a^2(S_B+S_C)^2 = \sum a^2(S_B^2 + S_C^2 + S_A^2 - S_A^2 + 2S_BS_C) \\
&= (S_\omega^2 - 2S^2)(2S_\omega) + 2S^2(6R^2 - S_\omega) \\
&= 2(S_\omega^3 + 3S^2(2R^2 - S_\omega)). && (32)\\
\sum S_A^3 &= \sum S_A(b^2c^2 - S^2) = \sum b^2c^2S_A - S^2\sum S_A \\
&= S_\omega^3 - 12R^2S^2. && (33)\\
\sum S_A(S_B^2 + S_C^2) &= \sum S_A\left(\sum S_A^2 - S_A^2\right) = (S_\omega^2 - 2S^2)\sum S_A - \sum S_A^3 \\
&= 12R^2S^2 - 2S^2S_\omega. && (34)\\
\sum a^4S_A &= \sum (S_B+S_C)^2S_A = \sum (S_B^2 + S_C^2 + 2S_BS_C)S_A \\
&= 6S_AS_BS_C + \sum a^2S_A^2 \\
&= 4S^2(S_\omega - 3R^2). && (35)
\end{aligned}$$

$$\begin{aligned}
\sum a^3 S_A^2 &= \sum a^3(b^2c^2 - S^2) = a^2b^2c^2 \sum a - S^2 \sum a^3 \\
&= S^2(2s(2s^2 - 3S_\omega + 4R^2) - 6RS). && (36)\\
\sum a^4 S_A^2 &= \sum a^4(b^2c^2 - S^2) = a^2b^2c^2 \sum a^2 - S^2 \sum a^4 \\
&= 2S^2(S^2 + (4R^2 - S_\omega)S_\omega). && (37)\\
\sum S_A^2 S_B^2 &= \sum (S^2 - c^2 S_C)^2 = \sum (S^4 - 2S^2c^2S_C + c^4 S_C^2) \\
&= 3S^4 - 2S^2 \sum c^2 S_C + \sum c^4 S_C^2 \\
&= 3S^4 - 4S^4 + S^2(S^2 + 8R^2 S_\omega - 2S_\omega^2) \\
&= 2S_\omega S^2(4R^2 - S_\omega) + S^4. && (38)\\
\sum a^4 S_A^3 &= \sum a^4 S_A(b^2c^2 - S^2) = a^2b^2c^2 \sum a^2 S_A - S^2 \sum a^4 S_A \\
&= 4S^4(5R^2 - S_\omega). && (39)
\end{aligned}$$

4 Playing with the formulas, GH, GI

Here and in the next sections, we apply the previous formulas to calculate the distances of various remarkable *"triangle centers"* ([Kim18]) expressed in barycentric coordinates (see file **Barycentric coordinates**). The result of the computation takes the form of an expression in terms of $\{s, R, S, S_\omega\}$ introduced in the preceding section.

As a first example we calculate the square of the distances $\{GH^2, GI^2\}$ between the *centroid* G, the orthocenter H and the *incenter* I of the triangle. From this and the relation $|HG| = 2|GO|$ we obtain also the distance $|GO|$ from the circumcenter O.

$$\begin{aligned}
G &= (1:1:1), \qquad G^* = \frac{1}{3}(1:1:1) \quad \text{(absolute barycentrics)},\\
H &= (S_BS_C : \ldots), \qquad H^* = \frac{1}{S^2}(S_BS_C : \ldots),\\
I &= (a:b:c), \qquad I^* = \frac{1}{2s}(a:b:c).
\end{aligned}$$

$$\begin{aligned}
GH^2 = \sum S_A\left(\frac{1}{3} - \frac{1}{S^2}S_BS_C\right)^2 &= \frac{1}{9S^4}\sum S_A\left(S^4 - 6S^2S_BS_C + 9S_B^2S_C^2\right)\\
&= \frac{1}{9S^4}\left(S^4\sum S_A - 6S^2\sum S_AS_BS_C + 9S_AS_BS_C\sum S_BS_C\right)\\
&= \frac{1}{9S^4}\left(S^4\sum S_A - 9S^2S_AS_BS_C\right)\\
&= \frac{1}{9S^4}\left(S^4S_\omega - 9S^4(S_\omega - 4R^2)\right)\\
&= 4\left(R^2 - \frac{2}{9}S_\omega\right) \quad \Rightarrow && (40)
\end{aligned}$$

$$|GO| = \frac{1}{2}|GH| \quad \Rightarrow \quad GO^2 = R^2 - \frac{2}{9}S_\omega \quad \Rightarrow \quad OH^2 = 9R^2 - 2S_\omega. \qquad (41)$$

$$\begin{aligned}
GI^2 &= \sum S_A\left(\frac{1}{3} - \frac{a}{2s}\right)^2 = \frac{1}{36s^2}\sum S_A(2s - 3a)^2\\
&= \frac{1}{36s^2}\left(4s^2\sum S_A - 12s\sum aS_A + 9\sum a^2S_A\right)\\
&= \frac{1}{9s}(s(6s^2 - 5S_\omega) - 18RS). && (42)
\end{aligned}$$

For the four quantities, anticipating a bit the fundamentals of section 6 and using the expressions in terms of the *"inradius"* r of triangle ABC :

$$S = 2sr, \qquad S_\omega = \frac{1}{2}(a^2+b^2+c^2) = s^2 - r^2 - 4rR,$$

we obtain the formulas:

$$GI^2 = \frac{1}{9}(S_\omega + 6r(r-2R)) = \frac{1}{9}(s^2+5r^2-16Rr), \quad (43)$$

$$HO^2 = 9R^2 - 2S_\omega = 2r^2 + 8Rr + 9R^2 - 2s^2, \quad (44)$$

$$HG^2 = 4\left(R^2 - \frac{2}{9}S_\omega\right) = \frac{4}{9}HO^2, \quad (45)$$

$$GO^2 = R^2 - \frac{2}{9}S_\omega = \frac{1}{9}HO^2. \quad (46)$$

5 Euler's theorem, Gerretsen's inequalities

The distance $|IO|$ of the incenter $I(a:b:c)$ from the circumcenter $O(a^2S_A,\dots)$ is connected with Euler's theorem, formulated below.

$$O = (a^2S_A : \dots), \qquad O^* = \frac{1}{2S^2}(a^2S_A : \dots),$$

$$\begin{aligned} IO^2 &= \sum S_A\left(\frac{a}{2s} - \frac{1}{2S^2}a^2S_A\right)^2 = \frac{1}{4s^2S^4}\sum S_A\left(aS^2 - sa^2S_A\right)^2 \\ &= \frac{1}{4s^2S^4}\left(S^4\sum a^2S_A - 2sS^2\sum a^3S_A^2 + s^2\sum a^4S_A^3\right) \\ &= \frac{1}{4s^2S^4}\left(S^4(2S^2) - 2sS^2(S^2(2s(2s^2-3S_\omega+4R^2)-6RS)) + s^2(4S^4(5R^2-S_\omega))\right) \\ &= \frac{R}{S}(9RS + 4s(S_\omega - s^2)). \end{aligned} \quad (47)$$

Anticipating a bit the fundamentals of the next section and expressing $\{S, S_\omega\}$ in terms of $\{s, r\}$ in this wonderful formula

$$S = 2sr, \qquad S_\omega = s^2 - r^2 - 4rR, \quad (48)$$

where, r is the *"inradius "* of the triangle ABC, we come, after a drastic simplification, at *"Euler's theorem"*, saying that:

$$IO^2 = R(R-2r). \quad (49)$$

In the file **Tritangent circles** we discuss this theorem from the much more elegant synthetic aspect, contrasting this computational derivation, whose real purpose was to test the calculations of section 3.

Here we have an instance indicating the contrast of the *"generality of the method"* to the *"elegance of the particular case"*. The computations with barycentrics supply a general method to calculate the distance, in principle, of any pair of remarkable points related to the triangle ABC i.e. *"triangle centers"*. The particular case though, as is here the case with the distance $|IO|$, can be alternatively handled with geometric tools and give a much deeper insight in the geometry of the triangle.

It is the genius of Euler ([BS07], [Col07]) to foresee that these two points are connected by such a simple relation as the one expressed by equation (49) and avoid some other candidates, such as, for example, the next one of $|HI|$ depending on all three $\{r, R, s\}$.

$$\begin{aligned}
HI^2 &= \sum S_A \left(\frac{S_B S_C}{S^2} - \frac{a}{2s}\right)^2 \\
&= \sum S_A \left(\frac{S_B^2 S_C^2}{S^4} - \frac{a S_B S_C}{s S^2} + \frac{a^2}{4s^2}\right) \\
&= \sum \frac{S_A S_B^2 S_C^2}{S^4} - \sum \frac{a S_A S_B S_C}{s S^2} + \sum \frac{a^2 S_A}{4s^2} \\
&= \frac{S_A S_B S_C}{S^4} \sum S_B S_C - \frac{S_A S_B S_C}{s S^2} \sum a + \frac{1}{4s^2} \sum a^2 S_A \\
&= \frac{S_A S_B S_C}{S^4}(S^2) - \frac{S_A S_B S_C}{s S^2}(2s) + \frac{1}{4s^2}(2S^2) \\
&= S_\omega - 4R^2 - 2(S_\omega - 4R^2) + \frac{1}{2s^2}(S^2) \\
&= \frac{1}{2s^2}(S^2 - 2s^2(S_\omega - 4R^2)) \\
&= 4R(R + r) + 3r^2 - s^2. \qquad (50)
\end{aligned}$$

Combining equations (43) and (50), we come to the well known *"Gerretsen's inequalities"* ([WZ03]), which together with *"Blundon's inequalities"*, discussed in the file **Fundamental invariants**, are two important *"triangle inequalities"*([ea69]) for s^2.

$$16Rr - 5r^2 \;\leq\; s^2 \;\leq\; 4R(R + r) + 3r^2 \,. \qquad (51)$$

6 Connection with the fundamental invariants

The *"fundamental invariants"* of the triangle are traditionally considered to be the *"half-perimeter"* $s = \frac{a+b+c}{2}$, the *" inradius"* r i.e. the radius of the inscribed circle, and the *" circumradius"* R i.e. the radius of the circumcircle of the triangle, somewhat more extensively discussed in the file **Fundamental invariants**. There, among other things, are proved also the next identities, in which the sums extend over the cyclic permutations of $\{a, b, c\}$. For general trigonometric relations between the elements of a triangle see [Lon93, p.135] and [Hob18, p.155]. The basic relations are given by the next two lines.

$$\Delta \;=\; sr \;=\; \frac{abc}{4R} \quad \text{and} \quad S = 2\Delta \;= 2sr \;=\; \frac{abc}{2R}. \qquad (52)$$

$$S_\omega = S_A \;+\; S_B \;+\; S_C \;=\; \frac{1}{2}(a^2 + b^2 + c^2) \;=\; s^2 - r(4R + r), \qquad (53)$$

Replacing $\{S, S_\omega\}$ in the formulas of section 3, we find the corresponding expressions in terms of the *fundamental invariants* $\{s, r, R\}$, as is the case with the next examples:

$$\begin{aligned}
bc + ca + ab &= s^2 + r(4R + r), & (54)\\
a^2 + b^2 + c^2 &= 2(s^2 - r(4R + r)), & (55)\\
a^3 + b^3 + c^3 &= 2s(s^2 - 6rR - 3r^2), & (56)\\
a^4 + b^4 + c^4 &= 2((s^2 - r^2 - 4Rr)^2 - 4r^2 s^2), & (57)\\
ab(a + b) + bc(b + c) + ca(c + a) &= 2s(s^2 - 2rR + r^2), & (58)\\
a(b - c)^2 + b(c - a)^2 + c(a - b)^2 &= 2s(s^2 + r^2 - 14Rr), & (59)
\end{aligned}$$

$$
\begin{aligned}
aS_a \; + \; bS_b \; + \; cS_c \quad &= 4sr(r+R), && (60)\\
S_A^2 + S_B^2 + S_C^2 &= (s^2 - r(4R+r))^2 - 8s^2r^2, && (61)\\
a^2S_A^2 \; + \; b^2S_B^2 \; + \; c^2S_C^2 &= 8s^2r^2(6R^2+4Rr+r^2-s^2) && (62)\\
S_AS_BS_C &= S^2(s^2-(r+2R)^2), && (63)\\
\sum aS_A^2 &= 4sr(rR(4R+r)+s^2(R-2r)), && (64)\\
\sum S_A^2S_B^2 &= 8s^2r^2[s^2(4(r+R)^2-s^2) \; - \; r(r+2R)^2(r+4R)]. && (65)
\end{aligned}
$$

7 Orthic axis = Radical axis of circumcircle and Euler circle

Theorem 1. *The radical axis of the circumcircle and the Euler circle of the triangle ABC is the trilinear polar of the orthocenter H, called "orthic axis " of the triangle.*

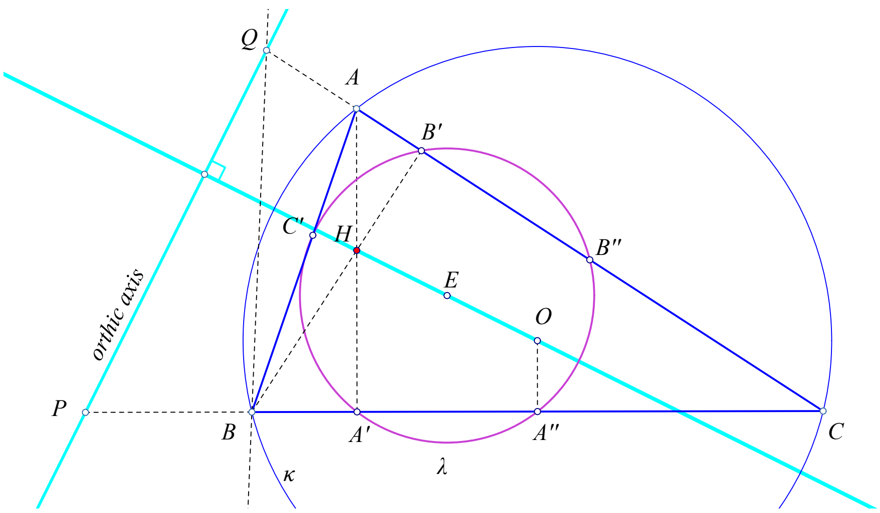

Figure 3: The radical axis of $\{\kappa, \lambda\}$ is the trilinear polar of H

The trilinear polar passes through the harmonic conjugates $\{P = A'(BC), Q = B'(CA)\}$ (See Figure 3). It suffices to show that the power of these points relative to the two circles is the same. This is done by a typical calculation, starting from the cosine rule, for P and setting $x = PB$. The corresponding calculation for Q being similar:

$$
\begin{aligned}
A'B = \frac{1}{a}S_B, \qquad & A'C = \frac{1}{a}S_C,\\
\frac{PB}{PC} \quad &= \quad \frac{A'B}{A'C} = \frac{S_B}{S_C} \quad \Rightarrow \quad x = PB = a\frac{S_B}{S_C - S_B}\\
\text{power w.r.} \quad \kappa \quad &= \quad PB \cdot PC = x(x+a) = \cdots = \frac{a^2S_CS_B}{(S_C-S_B)^2},\\
\text{power w.r.} \quad \lambda \quad &= \quad PA' \cdot PA'' = \cdots = \frac{a^2S_BS_C}{(S_B-S_C)^2}.
\end{aligned}
$$

8 GHI triangle, Feuerbach point

Euler considered the problem of constructing the triangle ABC from the corresponding triangle HIO. This is equivalent to the problem of constructing ABC from the triangle

GHI or GIO, since each of these triangles determines the other two. The problem for GIO is discussed in the file **Fundamental invariants**. Here we discuss some relations con-

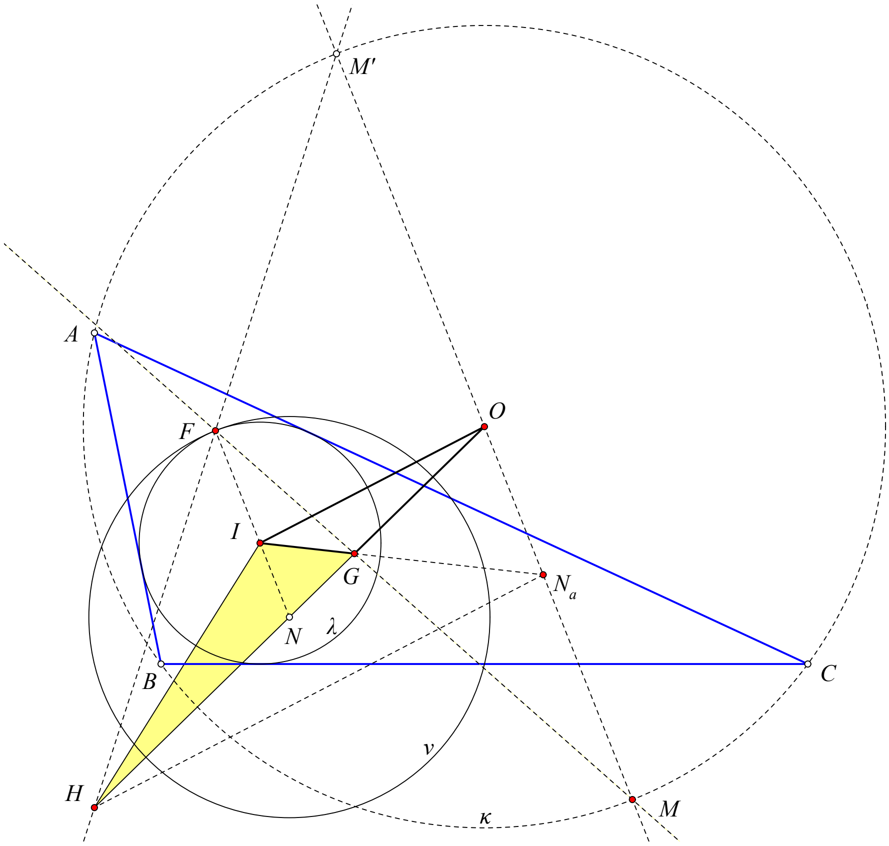

Figure 4: Triangle centers $F = X(11)$, $N_a = X(8)$, $M = X(100)$, $M' = X(104)$

nected with these three triangles, suggested by figure 4. In this $F = X(11)$ is the *"Feuerbach point"* (discussed in the file **Tritangent circles**) of contact of the incircle λ with the Euler circle ν of ABC. Point $N_a = X(8)$ is the *"Nagel point"* of intersection of the cevians to the contact points with the excircles of the triangle and $M = X(100)$, $M' = X(104)$ are the diametral points of the diameter ON_a of the circumcircle κ of ABC. The following theorem summarizes the relevant properties.

Theorem 2. *With the definitions and conventions adopted so far, the following are valid properties:*

1. *Points $\{I, G, N_a\}$ are collinear and $GN_a = 2IG$.*
2. *Lines $\{IO, HN_a\}$ are parallel and $HN_a = 2IO$.*
3. *Lines $\{IF, ON_a\}$ are parallel.*
4. *Point F is the middle of HM'.*
5. *Points F, G, M are collinear and $GM = 2FM$.*

Nr-1 is proved in the files **Barycentric coordinates** and **Nagel point of the triangle**.

Nr-2 follows trivially since $GH = 2OG$.

Nr-3 follows from the fact that the center N of the Euler circle ν is the middle of HO, hence $|GO|/|GN| = |GN_a|/|GI| = 2$.

Nr-4 follows from the fact that H is the homothety center of the two circles $\{\nu, \kappa\}$ with homothety ratio 2.

Nr-5 is a consequence of the previous *nr*.

9 Third degree equation for the symbols

Using Vieta's rules, expressing the coefficients of polynomial equations

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0,$$

by means of symmetric functions of the roots ([Tur47, p.66])

$$\sum_{i_1<\dots<i_k} x_{i_1} \dots x_{i_k} = (-1)^k \frac{a_{n-k}}{a_n}, \quad \text{for} \quad k = 1, \dots, n,$$

and the corresponding formulas for the symmetric functions of the symbols $\{S_A\}$, we arive at the cubic equation satisfied by these symbols:

$$x^3 - S_\omega x^2 + S^2 x - S^2(S_\omega - 4R^2) = 0. \tag{66}$$

Transforming the equation into the *"reduced"* form $y^3 + py + q = 0$ and requiring that the roots of this equation are real ([Tur47, p.121]), we arive at the condition $4p^3 + 27q^2 < 0$, which translates in this case to the biquadratic equation:

$$S^4 + 2[S_\omega^2 - 18R^2 S_\omega + 54R^4]S^2 + S_\omega^3(S_\omega - 4R^2) < 0. \tag{67}$$

Considering this as a quadratic equation w.r. to S^2, we see that the corresponding discriminant is found to be

$$D = 4R^2(9R^2 - 2S_\omega)^3 = 4R^2(OH^2)^3.$$

Thus, in order to satisfy the inequality (67), the quantity S^2 must be between the roots of the corresponding quadratic i.e. it must satisfy the inequalities:

$$U - 2R|OH|^3 < S^2 < U + 2R|OH|^3, \quad \text{where} \quad U = S_\omega^2 - 18R^2 S_\omega + 54R^4. \tag{68}$$

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Related topics

1. **Barycentric coordinates**
2. **Cross Ratio**
3. **Fundamental invariants**.
4. **Nagel point of the triangle**
5. **Projective line**
6. **Tritangent circles**

Any correction, suggestion or proposal from the reader, to improve/extend the exposition, is welcome and could be send by e-mail to: pamfilos@uoc.gr