Cross Ratio

A file of the Geometrikon gallery by Paris Pamfilos

> . . . it is hard to communicate understanding because that is something you get by living with a problem for a long time. You study it, perhaps for years, you get the feel of it and it is in your bones. You can't convey that to anybody else. Having studied the problem for five years you may be able to present it in such a way that it would take somebody else less time to get to that point than it took you but if they haven't struggled with the problem and seen all the pitfalls, then they haven't really understood it.
>
> *Interview with M. Atiya. Math. Intelligencer 6(1984)*

Contents

(Last update: 09-02-2021)

1 Cross Ratio

The cross ratio of four points $\{A, B, C, D\}$ of a line ε of the *"euclidean plane"* is defined by the quotient of two signed ratios of distances :

$$(AB, CD) = \frac{CA}{CB} : \frac{DA}{DB}. \tag{1}$$

As can be seen in the file **Projective line**, this definition is compatible with the *"projective"* one given there. Notice also that the whole number system and the cross ratio can be defined by simple geometric constructions based on the *"axioms of projective geometry"* (see [VY10, I,p.141]). Analogously to this definition we can select another ordering of the letters and define different similar quotients of ratios. There result $4! = 24$ permuted symbols, giving by 4 the same values as in the following table.

$$\begin{array}{lllllllll}
(AB, CD) & = & (BA, DC) & = & (DC, BA) & = & (CD, AB) & = & \lambda \\
(AB, DC) & = & (BA, CD) & = & (CD, BA) & = & (DC, AB) & = & 1/\lambda \\
(AC, BD) & = & (CA, DB) & = & (DB, CA) & = & (BD, AC) & = & 1-\lambda \\
(AC, DB) & = & (CA, BD) & = & (BD, CA) & = & (DB, AC) & = & 1/(1-\lambda) \\
(AD, BC) & = & (DA, CB) & = & (CB, DA) & = & (BC, AD) & = & (1-\lambda)/\lambda \\
(AD, CB) & = & (DA, BC) & = & (BC, DA) & = & (CB, AD) & = & \lambda/(1-\lambda)
\end{array}$$

Identifying the point with its line coordinate relative to an arbitrary coordinate system of

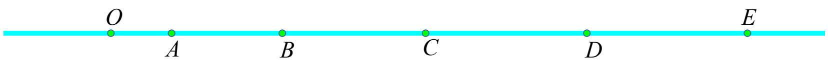

Figure 1: Four points on a line

the line, i.e. fixing two points $\{O, E\}$ on the line and measuring the location of an arbitrary point X by the signed ratio $x = OX/OE$ (may be negative), the points $\{A, B, C, D\}$ define the respective coordinates $\{a, b, c, d\}$ and the cross ratio can be expressed through the *"cross ratio of four numbers"*

$$(AB, CD) = \frac{a-c}{b-c} : \frac{a-d}{b-d}. \tag{2}$$

The expression on the right is independend of the location of points $\{O, E\}$ i.e. independent of the special line coordinate system. In fact, changing the coordinate system to some other (defined by two other $\{O', E'\}$), the new coordinate x' is connected to the old x by a relation of the form $x = m \cdot x' + n$, for appropriate constants $\{m, n\}$. The fact is that by substituting this expression of x into the formula we get the same number expressed in the other coordinates i.e.

$$\frac{a'-c'}{b'-c'} : \frac{a'-d'}{b'-d'} = \frac{a-c}{b-c} : \frac{a-d}{b-d}. \tag{3}$$

More generally the cross ratio of four numbers, defined by equation 3 is invariant under the *"broken linear transformations"* or *"Moebius transformations"* (see file **Projective line**):

$$x = f(x) = (m \cdot x' + n)/(p \cdot x' + q) \quad \text{with} \quad mq - np \neq 0. \tag{4}$$

This makes it possible to generalize the cross ratio for four points $\{A, B, C, D\}$ lying on a conic. Relations between two variables expressed through equation 4 are called *"homographic relations"* and the functions f are called *"homographies"*. Latter are subject of a further study in the files **Projective line** and **Homographic Relation**.

The definition allows for a number to be taken at infinity. For example, taking $D(d)$ at infinity $(d = \infty)$, the cross ratio reduces to the signed ratio

$$(AB, C\infty) = (ab, c\infty) = (a - c)/(b - c) = CA/CB. \tag{5}$$

Taking further $c = 0$, this reduces to a/b. Thus, the ratio of two numbers is the cross ratio $(ab, 0\infty)$. Using a homography we can transform a cross ratio (ab, cd) to an *equal* one of the form $(a'b', c'\infty)$. This often simplifies proofs involving cross ratios. An example is the following [Ber87, I,p.138].

Exercise 1. *The following equation is valid for five arbitrary points* $\{A, B, C, U, V\}$ *on a line*:

$$(AB, UV)(BC, UV)(CA, UV) = 1. \tag{6}$$

The property becomes trivial by setting $V = \infty$:

$$(AB, U\infty)(BC, U\infty)(CA, U\infty) = \frac{UA}{UB} \cdot \frac{UB}{UC} \cdot \frac{UC}{UA} = 1. \tag{7}$$

It is astonishing how many geometric properties depend on the cross-ratio. In fact, it can be proved that the cross ratio, considered as a function of four collinear points (variables), is the unique invariant of the projective geometry. Hence one can expect to find it behind every incidence relation between points and lines.

Exercise 2. *Show that, given three different points* $\{A, B, C\}$ *on the line* ε, *the position of a fourth point* X *on* ε *is completely determined through the cross ratio* $k = (AB, CX)$.

2 Properties of cross-ratio of four numbers

Next list of properties of the cross ratio gives, among other things, the reason for the validity of the relations between the 24 possible orderings of the table of the preceding section. The first two properties are discussed in the file **Projective line**. The rest is proved by easy calculations.

1. Given three pairwise different real numbers $\{x_1, x_2, x_3\}$ the *"cross ratio function"* $y = f(x) = (x_1x_2, x_3x)$ defines an invertible *"homographic relation"*, whose graph is a rectangular hyperbola.

2. Given three pairs of real numbers $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ in general position, the equation $(y_1y_2, y_3y) = (x_1x_2, x_3x)$ solved for y defines uniquely a homographic relation $y = f(x)$, such that $\{f(x_1) = y_1, f(x_2) = y_2, f(x_3) = y_3.\}$

3. Every permutation, which is product of two *"transpositions"* of the letters $\{a, b, c, d\}$ leaves the cross ratio invariant i.e.

   $$(ab, cd) = (ba, dc) = (cd, ab) = (dc, ba).$$

   Hence from the $4!$ in total permutations of the 4 letters, the various resulting cross ratios obtain only 6 different values (seen in the table of section 1).

4. $(ab, dc) = (ab, cd)^{-1}$ and $(ac, bd) + (ab, cd) = 1$. From these follow the last equalities in the rows of the table of section 1.

5. The four numbers a, b, c, d are pairwise different if and only if the cross ratio (ab, cd) has a value different from $1, 0, \infty$.

A special, but important case of cross ratio is that, which results from two points (A, B) and two others (C, D), which are harmonic conjugate to the first and for which we say that they are *"harmonic pairs"* and denoted by $(A, B) \sim (C, D)$. This, by definition means, that the signed ratios are the same, except for the sign,

$$\frac{CA}{CB} = \lambda, \quad \frac{DA}{DB} = -\lambda \quad \Rightarrow \quad \frac{CA}{CB} : \frac{DA}{DB} = -1.$$

In this case the 6 values of the various cross ratios reduce to three : $\{-1, \ 1/2, \ 2\}$ and we say also that $\{C, D\}$ are *"harmonic conjugate to* $\{A, B\}$*"*. We say also that D is *"harmonic conjugate to* C*"* w.r. to $\{A, B\}$ and denote this by the symbol $D = C(AB)$. Obviously for fixed $\{A, B\}$ this relation between $\{C, D\}$ is symmetric

$$D = C(AB) \quad \Leftrightarrow \quad C = D(AB).$$

3 Cross ratio expressed through angles

Take four points $\{A, B, C, D\}$ on a line ε and a fifth point E outside ε. Then the cross ratio has an interpretation in terms of the angles at E of lines $\{EA, EB, EC, ED\}$ (See Figure 2).

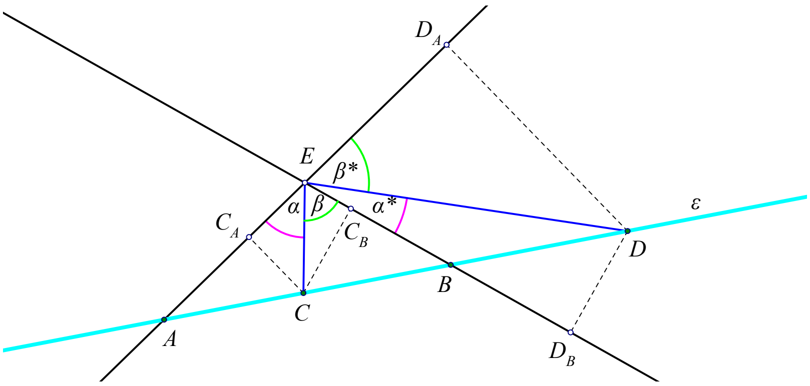

Figure 2: Cross ratio (AB, CD) in terms of angles

We speak about the *"pencil of lines"* at E and denote it with $E(ABCD)$ and also denote the corresponding cross ratio with $E(AB, CD)$. Projecting the points $\{C, D\}$ on the lines $\{EA, EB\}$ and doing some simple calculations suggested by figure 2, we find the relations

$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{AC}{AD} \cdot \frac{BD}{BC} = \frac{C_A C}{D_A D} \cdot \frac{D_B D}{C_B C} = \frac{EC \sin(\alpha)}{ED \sin(\beta^*)} \cdot \frac{ED \sin(\alpha^*)}{EC \sin(\beta)}$$

Thus, finally

$$(AB, CD) = \frac{\sin(\alpha)}{\sin(\alpha^*)} : \frac{\sin(\beta)}{\sin(\beta^*)}. \tag{8}$$

The dependence of $E(AB, CD)$ only from the angles, shows that keeping the four lines $\{EA, EB, EC, ED\}$ fixed and varying the line ε, the cross ratio defined by the four intersections on ε does not change its value. Also if we vary the position of E so that the angles between the four lines remain fixed, then the cross ratio on any line ε defined by the four intersections has a constant value. These simple observations lead to the following theorems.

Theorem 1. *A pencil of four lines $E(ABCD)$ intersects on any line ε four points whose cross ratio $E(AB, CD)$ is constant and independent of the particular position of the intersecting line ε (See Figure 3).*

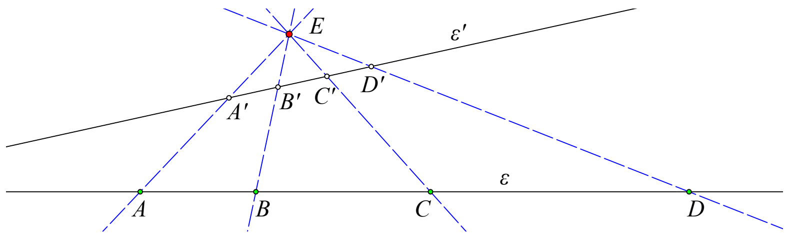

Figure 3: Cross ratio of a pencil of four lines

Theorem 2. *Fixing four points $\{A,B,C,D\}$ on a circle κ and varying a fifth point $E \in \kappa$ the pencil $E(ABCD)$ defines on any line ε intersecting the pencil a cross ratio $E(AB,CD)$ independent (i) from the particular position of E on the circle and (ii) independent also from the particular line intersecting the pencil (see figure 4).*

The first theorem allows us to associate a number to each quadruple of lines passing through a common point denoted by $E(AB,CD)$. This is the cross ratio, which the four lines define on any intersecting them line. According to the theorem, this cross ratio is independent of the special intersecting line and, consequently, defines some characteristic of the four lines. We call this number *"Cross ratio of four lines"* which pass through the

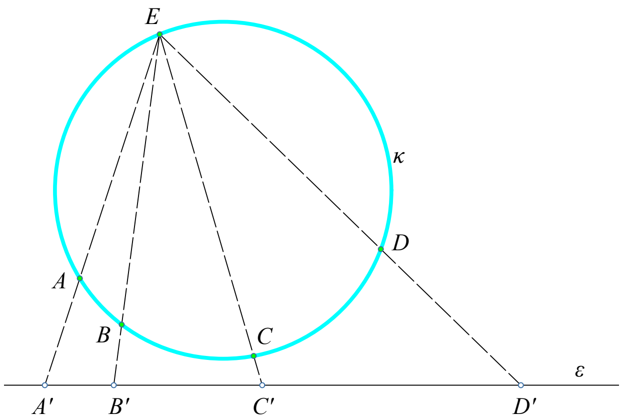

Figure 4: Cross ratio of four points on a circle

same point, or *cross ratio of the pencil of four lines*. A pencil consisting of four lines whose cross ratio equals $E(AB,CD) = -1$ is called a *"harmonic pencil "*.

Analogously, last theorem is important in that it allows to speak about the *"cross ratio (AB,CD) of four points on a circle"*. One has only to take an arbitrary fifth point E on the circle and an arbitrary line ε and apply the last theorem. Then define (AB,CD) through $(A'B',C'D')$, where $\{A',B',C',D'\}$ are the intersections of the line ε with the lines $\{EA,EB,EC,ED\}$ (See Figure 4). Obviously the relation of *"harmonic pairs"* and points $\{C,D\}$ *"harmonic conjugate"* to $\{A,B\}$ extends to pairs of points on the circle.

Theorem 3. *Consider four given lines $\{\eta_1,\eta_2,\eta_3,\eta_4\}$ passing through a fixed point E and expressed as a linear combination of two other fixed lines $\{\mu,\nu \;:\; \eta_i = \mu + \lambda_i \nu \;, i = 1,2,3,4\}$ through E. Then, the cross ratio of the four lines is given by*

$$(\eta_1\eta_2,\eta_3\eta_4) \;=\; \frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3} : \frac{\lambda_1 - \lambda_4}{\lambda_2 - \lambda_4}.$$

Proof. If the lines are $\{\mu : ax + by + c = 0\ ,\ \nu : a'x + b'y + c' = 0\}$, then

$$\eta_i = \mu + \lambda_i \nu = (a + \lambda_i a')x + (b + \lambda_i b')y + (c + \lambda_i c') = 0.$$

Intersecting these lines with the x-axis $\varepsilon : y = 0$ leads to the equations for the correspondig x:

$$(a + \lambda_i a')x + (c + \lambda_i c') = 0 \quad \Rightarrow \quad x_i = -\frac{c + \lambda_i c'}{a + \lambda_i a'}\ .$$

The cross ratio of the four lines is then $\frac{x_1 - x_3}{x_2 - x_3} : \frac{x_1 - x_4}{x_2 - x_4}$, which by substitution of the $\{x_i\}$ and simplification leads to the stated formula. □

4 Harmonic pencils of lines

Corollary 1. *The cross ratio of four lines $\{EA, EB, EC, ED\}$ is equal to the signed ratio $\frac{CA}{CB}$ of the three points $\{ABC\}$, which the three first lines excise on a line ε, which is parallel to the fourth line $\varepsilon' = ED$.*

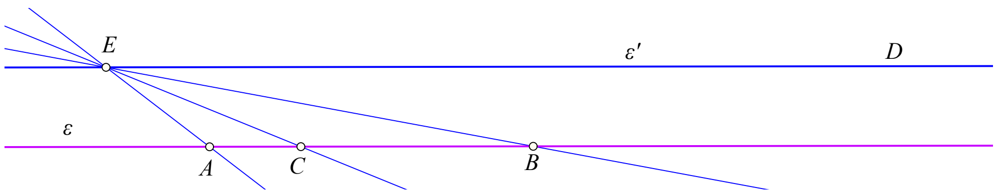

Figure 5: Cross ratio of four lines on a parallel of ε'

Proof. The proof follows from the comment made above, according to which, the cross ratio (AB, CD) is reduced to the simple signed ratio $\frac{CA}{CB}$, in the case where the fourth point D tends to infinity (See Figure 5). □

Corollary 2. *A pencil of four lines $E(ABCD)$ is harmonic, if and only if a parallel ε to the fourth line ED intersects the others at three points $\{A, B, C\}$ of which one (C) is the middle of the line segment (AB) which is defined by the other two.*

Next proposition and the one after that examine the two main examples of harmonic pencils.

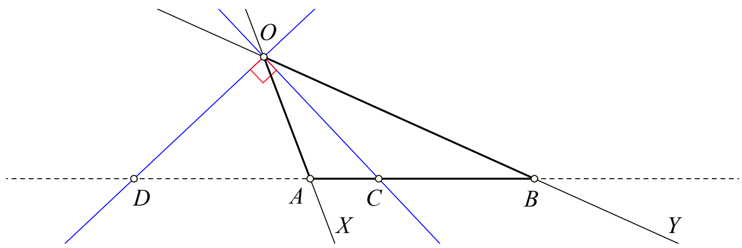

Figure 6: Harmonic pencil of an angle and its bisectors

Theorem 4. *In every triangle OAB the two sides OA, OB and the bisectors OC, OC of the angle at O make a harmonic pencil.*

Proof. It is well known that in the triangle OAB with bisectors OC and OD, holds

$$\frac{|CA|}{|CB|} = \frac{|DA|}{|DB|} = \frac{|OA|}{|OB|} \quad \Rightarrow \quad \frac{CA}{CB} : \frac{DA}{DB} = -1.$$

□

Theorem 5. *Show that if in the harmonic pencil $O(XYZW)$ the two lines $\{OX, OY\}$ are orthogonal, then these coincide with the bisectors of the angle $\widehat{ZOW}$.*

Proof. Assume that the intersection points of the pencil with a line ε satisfy $(XY, ZW) = -1$. Consider the harmonic conjugate OW' of OZ relative to OX (See Figure 7). Then

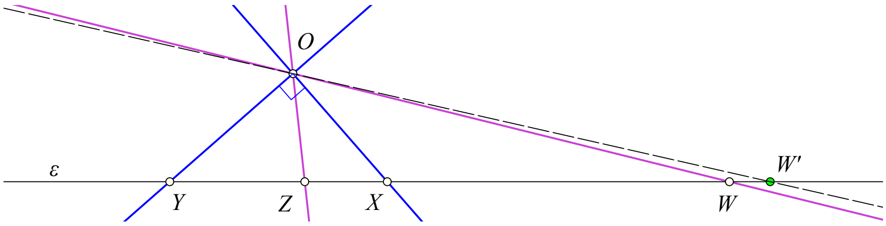

Figure 7: Harmonic pencil with bisectors

the corresponding point W' on ε will satisfy also $(XY, ZW') = -1$, therefore it will be coincident with point W. □

Theorem 6. *In every triangle OEB the two sides $\{OE, OB\}$, the median OM and the parallel to the base EB from O make a harmonic pencil.*

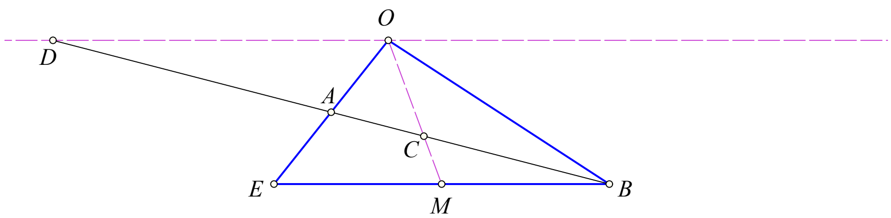

Figure 8: Harmonic pencil of median and opposite side

Proof. If we draw in triangle OEB the median OM and we extend the other median BA until it intersects the parallel to the base at D (see figure 8), this forms a pencil of lines $O(ABCD)$ and holds

$$-\frac{CA}{CB} = \frac{DA}{DB} = \frac{1}{2},$$

from which follows that the pencil is harmonic (alternatively apply corrolary 2). □

Next is a key figure containing several harmonic pairs and being thus useful in many applications in euclidean, affine and projective geometry.

Theorem 7. *In the complete quadrilateral of figure 9 the pairs $(A, B) \sim (G, H)$ are harmonic.*

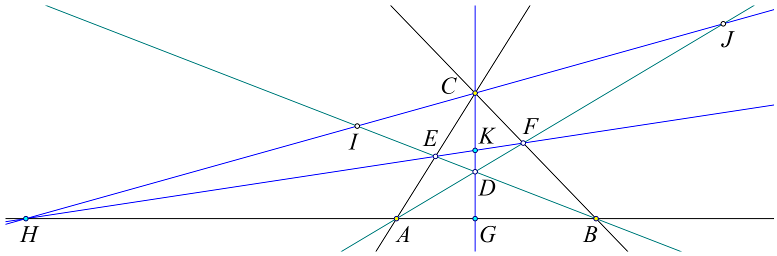

Figure 9: Harmonic quadruples and pencils in a complete quadrilateral

Proof. Apply Menelaus' theorem to triangle ABC with secant HK and Ceva's theorem to the same triangle relative to the point D :

$$\text{Menelaus:}\quad \frac{HA}{HB}\cdot\frac{FB}{FC}\cdot\frac{EC}{EA}=1,\quad \text{Ceva:}\quad \frac{EC}{EA}\cdot\frac{GA}{GB}\cdot\frac{FB}{FC}=-1.$$

The result follows by dividing these relations side by side. □

Corollary 3. *All the quadruples of points lying on each line of figure 9 define harmonic pairs.*

Proof. From the pencil $C(ABGH)$ we find the relations $(E,F)\sim(KH)$, $(E,B)\sim(D,I)$ and $(A,F)\sim(D,J)$. From the pencil $A(EBDI)$ follows analogously $(K,G)\sim(D,C)$. □

5 Coincidences and cross ratios of pencils

Here we examine two coincidence relations, of three points on a line and three lines on a point, connected with the cross ratio of pencils.

Theorem 8. *Assume that the two line pencils $O(\alpha\beta\gamma\delta)$, $P(\alpha\beta'\gamma'\delta')$ have the same cross ratio and common the line α, then the other corresponding lines intersect at three points $B=\beta\cap\beta'$, $C=\gamma\cap\gamma'$, $D=\delta\cap\delta'$ which are collinear.*

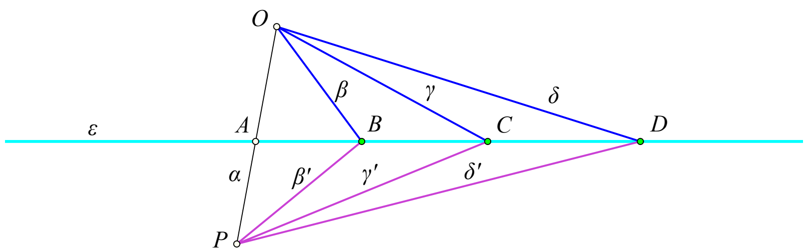

Figure 10: Pencils with common line and the same cross ratio

Proof. Consider the line $\varepsilon=BC$ and the intersection point of A with α (See Figure 10). On ε the two pencils will define the same cross ratio. Then if $D'=\varepsilon\cap\delta$, $D''=\varepsilon\cap\delta'$, then from the equality of the cross ratios $(AB,CD')=(AB,CD'')$, which the two pencils on ε define, follows that $D'=D''$, therefore also the third intersection point $D'=D''=D$ of δ, δ' will be on ε. □

Theorem 9. *Assume that on two lines $\{\alpha,\beta\}$ intersecting at point O, are defined respectively points $\{A,B,C\}$ and $\{A',B',C'\}$, such that the cross ratios (OA,BC), $(OA',B'C')$ are equal. Then the lines $\{AA',BB',CC'\}$ pass through a common point or are parallel (See Figure 10).*

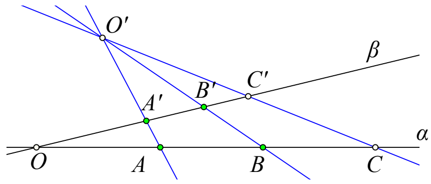

Figure 11: Common cross ratio $(OA, BC) = (OA', B'C')$

Proof. Consider the case in which $O' = AA' \cap BB'$ is a real point of the plane and draw the line $O'C$ intersecting the line β at C'' for which $(OA, BC) = (OA', B'C'')$. From the assumed $(OA, BC) = (OA', B'C')$ we conclude $C' = C''$. Analogous is the proof with parallel lines, O' is at infinity. □

Exercise 3. *Given is a triangle OBC and three points $\{X, Y, E\}$ on its base BC. From the point E are drawn lines intersecting the sides $\{OC, OB\}$ respectively at points $\{X', Y'\}$. Show that the lines $\{XX', YY'\}$ intersect at a point Z contained in a fixed line, which passes through O.*

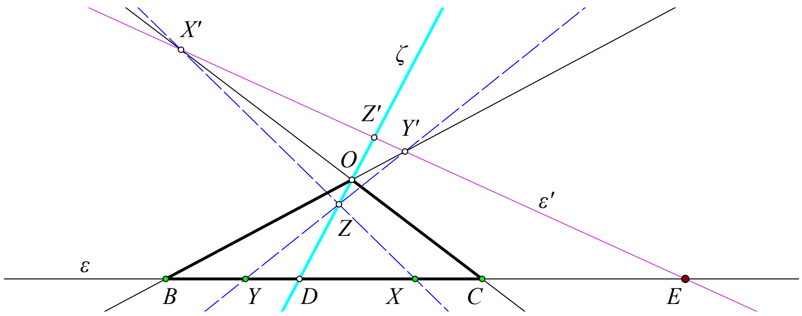

Figure 12: Position determination from cross ratio

Hint: The line pencils $O(BCDE)$ and $Z(YXDE)$ define the same cross ratio $(Y'X', Z'E)$ on line $X'Y'$, hence also on line BC (See Figure 12). Using a system of coordinates on the line BC the equation $(BC, DE) = (XY, DE)$ contains only one unknown, the coordinate x of D.

Another aspect of the last figure offers its connection with the following problem.

Problem 1. *Given the four points $\{B, C, D, E\}$ on line ε and the variable point $X \in \varepsilon$ find the point Y, such that the cross ratios are equal:*

$$(YX, DE) = (BC, DE).$$

Hint: A geometric solution of the problem can be given by taking an arbitrary point O and considering the pencil $\{O(BCDE)\}$ and an arbitrary line ε' through E (See Figure 12). The pencil defines on ε' the same cross ratio: $O(BC, DE) = (Y'X', Z'E)$. The required point $Y : (YX, DE) = (BC, DE)$ defining the same cross ratio will have corresponding lines $\{XX', YY', DZ'\}$ passing through the same point Z (theorem 9). From the given data we determine $Z = XX' \cap DZ'$ and drawing line $Y'Z$ we find $Y = Y'Z \cap \varepsilon$.

6 Cross ratio on a circle

By a theorem discussed in the file **Projective line**, a variable tangent ε of a circle κ intersects on four other fixed tangents at the points $\{A, B, C, D\}$ of the circle, points $\{A', B', C', D'\}$

in ε, whose cross ratio $(A'B', C'D')$ on ε is constant and independent of the particular position of ε.

It is interesting to note a consequence of this theorem, when the variable tangent ε takes the position of the tangent at one of the four points $\{A, B, C, D\}$, like point A

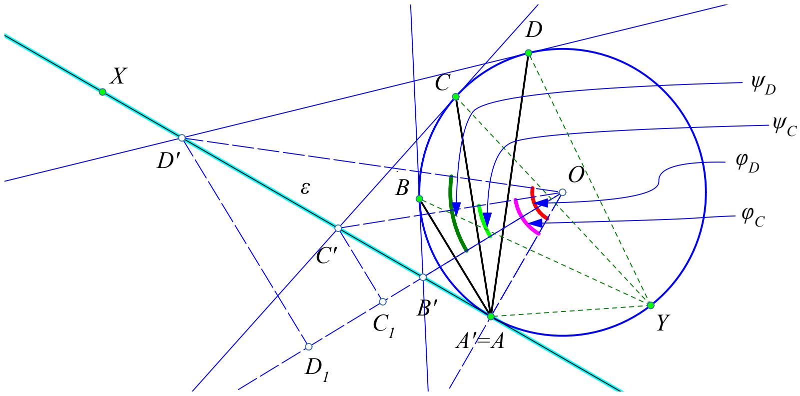

Figure 13: Cross ratio (AB, CD) measured on the tangent at A

say (See Figure 13). Then, projecting points $\{C', D'\}$ to $\{C_1, D_1\}$ onto line OB', the ratio $(A'B', C'D')$ becomes equal to

$$\frac{C'A'}{C'B'} : \frac{D'A'}{D'B'} = \frac{C'A'}{D'A'} \cdot \frac{D'B'}{C'B'} = \frac{OC' \sin(\phi_C)}{OD' \sin(\phi_D)} \cdot \frac{OD' \sin(\psi_D)}{OC' \sin(\psi_C)} = \frac{\sin(\phi_C)}{\sin(\phi_D)} \cdot \frac{\sin(\psi_D)}{\sin(\psi_C)}.$$

Taking the points $X \in AD'$ and Y on the circle and comparing angles, we find that the last product of ratios is equal to

$$\frac{\sin(\widehat{XAC})}{\sin(\widehat{XAD})} \cdot \frac{\sin(\widehat{BAD})}{\sin(\widehat{BAC})} = \frac{\sin(\widehat{AYC})}{\sin(\widehat{AYD})} \cdot \frac{\sin(\widehat{BYD})}{\sin(\widehat{BYC})}.$$

This shows that the cross ratio defined by the four tangents on a fifth tangent ε is equal to this expression, which is independent of the position of Y on the circle. The result is that the cross ratio $(A'B', C'D')$ is equal to the cross ratio of the four lines $\{YA, YB, YC, YD\}$. By theorem 2 this is also the cross ratio (AB, CD) of the *four points on the circle.* We obtain thus the following theorem.

Theorem 10. *The cross ratio $(A'B', C'D')$ defined by the intersections with a tangent ε of four tangents of the circle at the points $\{A, B, C, D\}$ is equal to the cross ratio (AB, CD) on the circle.*

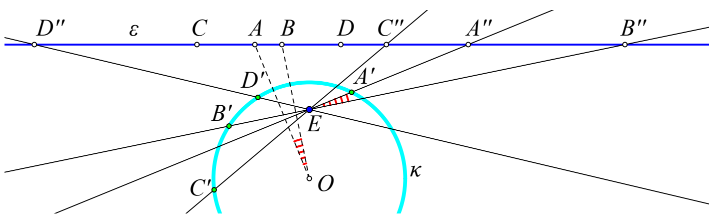

Figure 14: Cross ratio of polars

Theorem 11. *Four points on a line ε define the same cross ratio (AB, CD) as their polars $E(A'B', C'D')$ relative to the circle κ.*

Proof. The polars pass through the *"pole"* E of ε relative to κ (See Figure 14). The proof results by the equality of angles $\widehat{AOB} = A''\widehat{O}B'', \ldots$ and the discussion in 3. Notice the equalities of the cross ratios defined by the polars on ε.

$$(AB,CD) = E(A'B',C'D') = (A''B'',C''D'').$$

□

7 Cross Ratio on a conic

The cross ratio for points $\{U,V,X,Y\}$ on a conic κ is defined by the same recipe used for circles. For this take an arbitrary line ε and an additional point C on the conic and define the *"cross ratio of four points on the conic (UV,XY)"* to be equal to the cross ratio $(U'V',X'Y')$ (See Figure 15). This is independent of the position of $C \in \kappa$ and also in-

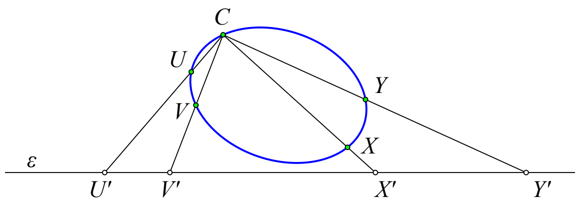

Figure 15: Cross of four points on a conic

dependent from the particular line ε used in the definition. To see this one can use a *"projectivity"* mapping the conic onto the circle, which reduces the general case to that of the circle.

By the same method of *"projection"* of a circle on an arbitrary conic is proved the analogous to theorem 10 for conics:

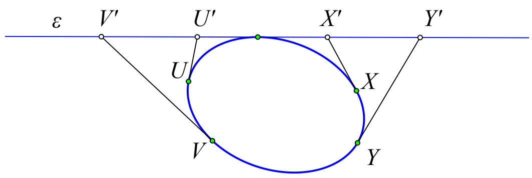

Figure 16: Cross of four points on a conic and tangents at these points

Theorem 12. *The cross ratio $(U'V',X'Y')$ defined by the intersections with a tangent ε of four tangents of the conic at the points $\{U,V,X,Y\}$ is equal to the cross ratio (UV,XY) on the conic (See Figure 16).*

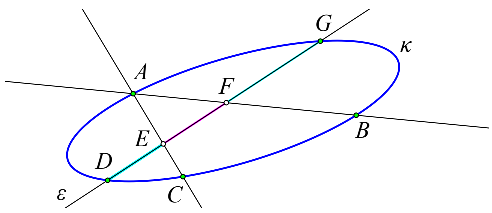

Figure 17: Ratio $(DE \cdot FG)/EF$ independent of position of $A \in \kappa$

Next property, known as *"Haruki's lemma"* is a direct implication of the definition of the cross ratio for points on a conic (See Figure 17).

Theorem 13. *Given the fixed points $\{B, C\}$ on the conic κ, the moving point $A \in \kappa$ and the chord DG of κ, the ratio $(ED \cdot GF)/EF$ defined by the lines $\{AB, AC\}$ on the chord, is constant and independent of the position of $A \in \kappa$.*

Proof. By the definition of the cross ratio, $(CG, DB) = (EG, DF) = (ED \cdot GF)/(GD \cdot EF)$ is constant and independent of the location of $A \in \kappa$. But GD is constant. □

8 Cross Ratio Formularium

In the following table we list the expressions of the cross ratio, relative to various coordinate systems, *"projective"* and *"euclidean"*. For the projective case and the appropriate for it *"homogeneous coordinates"* we refer to the file **Projective line**.

1. For projective coordinates w.r. to the base $\{A, B, C\}$ and $P = p_1 A + p_2 B, \ldots,$ $p = p_1/p_2,\ q = q_1/q_2 \ldots$ $\qquad (PQ, UV) = \frac{p-u}{q-u} : \frac{p-v}{q-v}.$
2. Since $\{a = \infty, b = 0, c = 1\ :\}$ $\quad (AB, UV) = \frac{v_1}{v_2} : \frac{u_1}{u_2} \qquad (AB, CV) = v = \frac{v_1}{v_2}.$
3. For euclidean signed distance ratios: $\qquad (AB, XY) = \frac{XA}{XB} : \frac{YA}{YB}.$
4. For $\{x = (1-s)a + sb,\ \ y = (1-t)a + tb\},$ $\qquad (AB, XY) = \frac{s}{s-1} : \frac{t}{t-1}.$
5. Measuring signed distances from the point O of the line, the point $X(x)$ satisfying $(AB, OX) = k$ has $x = \frac{ab(1-k)}{a-kb}$. In particular, for the harmonic conjugate X_0 of O : $x_0 = \frac{2ab}{a+b}$ is the *"harmonic mean"* of $\{A(a), B(b)\}$.
6. For oriented angles between lines $\{[a], [b], \ldots\}$ $\quad (AB, CD) = \frac{\sin([c][a])}{\sin([c][b]} : \frac{\sin([d][a])}{\sin([d][b])}.$
7. For $\{f(X) = 0, g(X) = 0\}$ representing lines $\{OA, OB\}$ $\quad (AB, CX) = \frac{f(C)}{f(X)} : \frac{g(C)}{g(X)}.$

In the last case we assume that lines $\{[a] = OA, [b] = OB\}$ are given by equations correspondingly $\{f(x, y) = px + qy + r = 0,\ \ g(x, y) = p'x + q'y + r' = 0.\}$ For any point of the plane $P = (p_1, p_2)$ denote by $f(P)$ the number $f(p_1, p_2)$, which is a multiple of the distance of P from the line $[f] : f(x, y) = 0$, i.e. $f(P) = k \cdot d(P, [f])$, with k independent of the particular P. Thus, we have the following expression and its consequence:

$$\frac{f(C)}{f(X)} = \frac{d(C, [f])}{d(X, [f])} \quad \Rightarrow \quad (AB, CX) = \frac{d(C, [a])}{d(X, [a])} : \frac{d(C, [b])}{d(X, [b])}\ .$$

9 Complex Cross Ratio

The *"complex cross ratio"* of four points in the complex plane is defined to be:

$$(AB, CD) = \frac{A - C}{B - C} : \frac{A - D}{B - D}, \tag{9}$$

where points are identified with complex numbers ([Sch79, p.35]).

Theorem 14. *The main properties of the complex cross ratio are the following:*

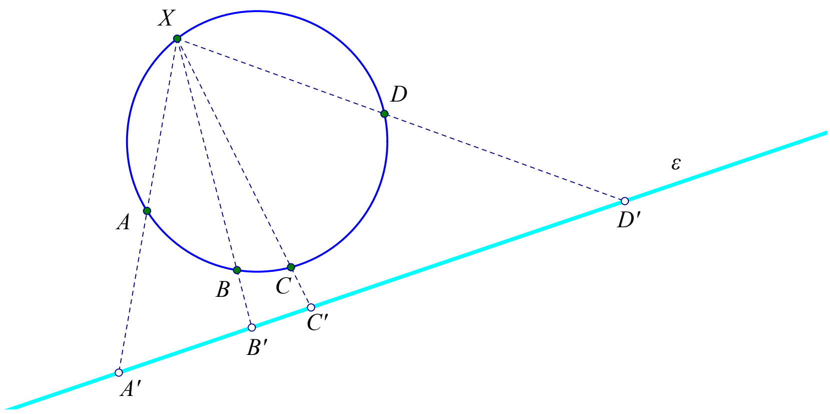

Figure 18: Cross ratio of four complex numbers on a circle

1. (AB, CD) *is real if and only if the points are all four on a circle or on a line.*
2. *Assuming the points are on a circle, project them on a line* ε *from a point* X *of the circle. Let* $\{A', B', C', D'\}$ *be the corresponding projections. Then*
$$(AB, CD) = (A'B', C'D').$$
3. *A "Moebius transformation"*
$$x' = f(x) = \frac{ax+b}{cx+d} \quad \textit{with} \quad a,b,c,d \in \mathbb{C} \quad \textit{and} \quad ad-bc \neq 0, \tag{10}$$
preserves the complex cross ratio, i.e. if $\{p' = f(p), q' = f(q), \ldots\}$, *then*
$$(p'q', u'v') = (pq, uv). \tag{11}$$
4. *Given two triples of complex numbers* $\{(p,q,r),(p',q',r')\}$, *the equation*
$$(p'q', r'x') = (pq, rx) \tag{12}$$
defines the uniqe complex Moebius transformation $x' = f(x)$ *mapping* $\{p,q,r\}$ *correspondingly to* $\{p',q',r'\}$.

Nr-1. For the proof write the complex numbers $\{A-C, B-C, \ldots\}$ in polar form:
$$A-C = r_{CA}\cdot e^{i\phi_{CA}},\;\; B-C = r_{CB}\cdot e^{i\phi_{CB}},\;\; A-D = r_{DA}\cdot e^{i\phi_{DA}},\;\; B-D = r_{DB}\cdot e^{i\phi_{DB}},$$
implying
$$(AB,CD) = \frac{r_{CA}\cdot e^{i\phi_{CA}}}{r_{CB}\cdot e^{i\phi_{CB}}} : \frac{r_{DA}\cdot e^{i\phi_{DA}}}{r_{DB}\cdot e^{i\phi_{DB}}} = \frac{r_{CA}}{r_{CB}} : \frac{r_{DA}}{r_{DB}} \frac{e^{i(\phi_{CA}-\phi_{CB})}}{e^{i(\phi_{DA}-\phi_{DB})}} = \frac{r_{CA}}{r_{CB}} : \frac{r_{DA}}{r_{DB}} \cdot \frac{e^{i(ACB)}}{e^{i(ADB)}},$$
where $\{(ACB), (ADB)\}$ denotes the oriented angle-measures. This quotient is real if and only if these angles are equal or complementary and inverse oriented, which shows the first claim.

Nr-2. The second claim follows from the previous one, since
$$\frac{r_{CA}}{r_{CB}} = \frac{\sin(\widehat{AXC})}{\sin(\widehat{BXC})}, \quad \frac{r_{DA}}{r_{DB}} = \frac{\sin(\widehat{AXD})}{\sin(\widehat{BXD})} \quad \Rightarrow \quad (AB,CD) = \frac{\sin(\widehat{AXC})}{\sin(\widehat{BXC})} : \frac{\sin(\widehat{AXD})}{\sin(\widehat{BXD})},$$
which according to theorem 1 is the expression of the cross ratio $(A'B', C'D')$.

Nr-3+nr-4 are proved as suggested for the real case in the file **Projective line**.

10 Circle parametrization through the cross ratio

Consider three fixed points $\{A, B, C\}$ of the plane, identified with complex numbers. Then the equation

$$(AB, CX) = t \quad \text{with variable} \quad t \in \mathbb{R}, \tag{13}$$

solved for X, defines a parametrization of the circumcircle of the triangle ABC by the real parameter t ([Sch79, p.37]). Figure 19 illustrates this parametrization for a circle with

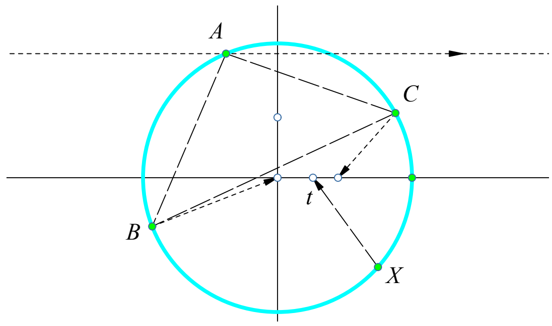

Figure 19: Parametrization of the circle through the cross ratio

center at the origin. Points $\{A, B, C\}$ map respectively to $\{\infty, 0, 1\}$ of the complex plane.

11 Further properties of the cross ratio

In the following list all points appearing in the formulas are on the same line ε and symbols such as $\{AB, CD, ...\}$ represent signed distances of the points, except in the case of the symbol (AB, CD) representing the cross ratio of the four collinear points. The formulas are to be found in [Cha52, ch.II] and [Pap96, ch.V].

1. If $(AB, CD) = (AB, C'D')$ then also $(AB, CC') = (AB, DD')$.
2. One can write $(AB, CD) = \dfrac{1/AB - 1/AD}{1/AB - 1/AC}$.
3. Given three points $\{A, B, C\}$ on line ε, an arbitrary number k, and taking three other points $\{A', B', C'\}$ on ε, such that

$$\begin{aligned} &(BC, AA') = (CA, BB') = (AB, CC') = k, \quad \text{implies} \\ &(B'C', AA') = (C'A', BB') = (A'B', CC') = k. \end{aligned}$$

4. For arbitrary $M \in \varepsilon$: $(AB, CD) = \dfrac{MB/AB - MD/AD}{MB/AB - MC/AC}$.
5. For two quadruples of line ε satisfying $(AB, CD) = (A'B', C'D') \quad \Rightarrow$

$$\frac{AB \cdot CD}{A'B'} + \frac{AC \cdot DB}{A'C'} + \frac{AD \cdot BC}{A'D'} = 0, \tag{14}$$

$$\frac{A'B' \cdot C'D'}{AB} + \frac{A'C' \cdot D'B'}{AC} + \frac{A'D' \cdot B'C'}{AD} = 0, \tag{15}$$

Nr-1 and *nr-2* follow using the definition $(AB, CD) = (AC/BC) : (AD/BD)$.

Nr-3 follows by noticing the relations

$$(CB, A'A) = (CA, BB') = (CC', AB) = k,$$

which follow from the assumed equalities and the equalities in section 1.
nr-4 follows from simple calculations using *"Euler's formula"* for four points on a line

$$PA \cdot BC + PB \cdot CA + PC \cdot AB = 0,$$

by replacing P with M and considering it also for $C = D$.

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Related topics

1. **Barycentric coordinates**
2. **Homographic Relation**
3. **Projective line**
4. **Projective plane**

Any correction, suggestion or proposal from the reader, to improve/extend the exposition, is welcome and could be send by e-mail to: pamfilos@uoc.gr