Inversion

by Paris Pamfilos

'You must always invert', as Jacobi said when asked the secret of his mathematical discoveries. He was recalling what Abel and he had done. If the solution of a problem becomes hopelessly involved, try turning the problem backwards, put the quaesita for the data and vice versa.

E.T. Bell, Men of Mathematics, v. II p. 355

Contents

1 Inversion on a circle

Given a circle $\kappa(O,\rho)$, the *"inversion"* relative to κ or *"κ−inversion"*, is a correspondence between points in the interior and exterior of the circle. To every point X different from the center O, we correspond a point Y on the half line OX, such that $|OX||OY| = \rho^2$. The

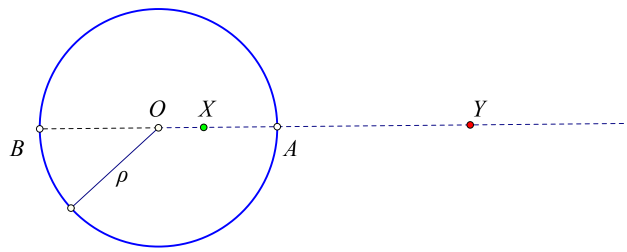

Figure 1: Inverse points X and Y

definition establishes a kind of symmetric relation, somewhat similar to the symmetry relative to a line (reflection). X is the inverse of Y, if and only if Y is the inverse of X. We then often say that such two points are *"inverse points"* relative to the circle κ or that *"they are κ−inverses"*. The points of the circle κ are characterized by being coincident to their inverse. We call these points the *"fixed points"* of the inversion. The circle κ is called the *"circle of inversion"* or *"invariant circle"*, its center is called *"center of inversion"* and ρ^2 is called the *"power of inversion"*.

Exercise 1. *Show that $\{X,Y\}$ are inverse relative to the circle $\kappa(O,\rho)$ precisely when $\{X,Y\}$ are harmonic conjugate w.r. to $\{A,B\}$, latter being the diameter points of κ on line OX.*

Hint: Use coordinates $\{A(\rho),\ B(-\rho),\ X(x),\ Y(y),\ xy = \rho^2\}$ w.r. to O and show the relation $XA/XB = -YA/YB$.

Exercise 2. *Given four points $\{A,B,X,Y\}$ on a line, show that $\{X,Y\}$ are inverse relative to the circle with diameter AB, precisely when $\{A,B\}$ are inverse relative to the circles with diameter XY.*

Hint: Use the symmetry of the relation $XA/XB = -YA/YB$ of the preceding exercise.

Remark 1. Using these coordinates we see that $\frac{AX}{AY} = -\frac{x}{\rho}$. This means that for very large circles (huge ρ) and X near A then $\frac{AX}{AY}$ is approximately -1 i.e. the inversion almost coincides with the *"reflection"* on the line tangent to the circle κ at A. This remark makes plausible why we consider *reflections on lines* as special *inversions*, considering a line as a circle of infinite radius.

Exercise 3. *Referring to figure 1, show that $\{X,Y\}$ are inverse relative to κ if and only if $\{A,B\}$ are inverse relative to the circle κ' with diameter XY.*

Hint: Use the characterization of inverse points of exercise 1.

" Given a circle $\kappa(O,\rho)$, to construct the inverse Y of a given point X relative to κ."

If the point X is external to the circle κ (See Figure 2-I), then the intersection Y of the chord AB of contacts of the tangents from X and the line OX is the wanted point, since from the right triangle OAX, we will have

$$\rho^2 = |OX||OY|.$$

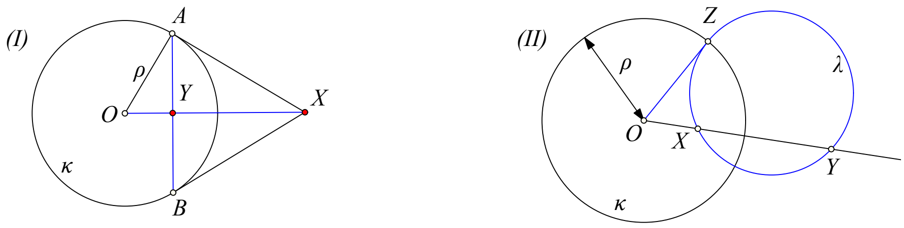

Figure 2: Construction of inverses X, Y Orthogonal λ on κ

The same relation shows that X is also the inverse of Y. Therefore, for internal points of the circle, we perform the reverse procedure: we raise the orthogonal AB on OY and draw the tangents of κ at its intersection points A, B with the circle. X is the intersection point of these tangents. The inverse of an $X \in \kappa$ is X itself.

Theorem 1. *Every circle λ, which passes through two inverse points $\{X, Y\}$ relative to the circle κ, is orthogonal to κ.*

Proof. Indeed, if $\{X, Y\}$ are inverse points relative to κ and Z is an intersection point of the circle λ which passes through X, Y (See Figure 2-II), then, according to the definition of inversion, we will have $|OZ|^2 = \rho^2 = |OX||OY|$, which shows that OZ is tangent to the circle λ. □

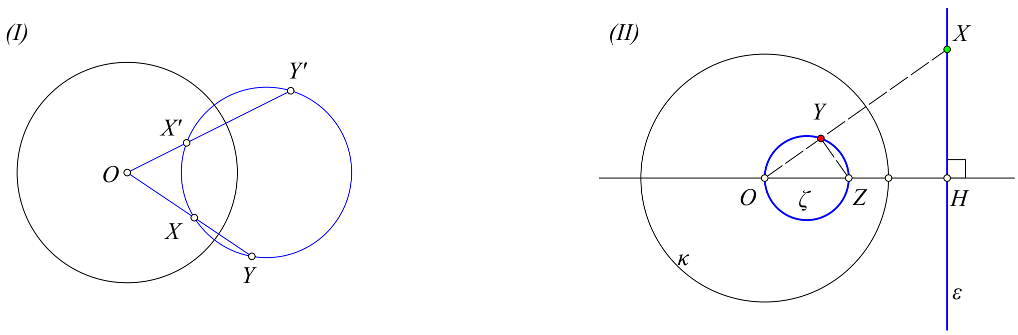

Figure 3: Pairs of inverse points Inverses of line and circle through O

Theorem 2. *Two different pairs of inverse points (X, Y) and (X', Y') relative to the circle $\kappa(O, \rho)$ define four concyclic points.*

Proof. Obviously, since the relation which connects them $\rho^2 = |OX||OY| = |OX'||OY'|$ (See Figure 3-I), means that $\{X, Y, X', Y'\}$ are concyclic. □

Exercise 4. *Given a circle κ, show that for two points $\{A, B\}$ lying both in the interior/exterior of the circle and such that their line does not pass through the center of κ, there is precisely one circle κ' containing them and being also orthogonal to κ.*

2 Inverses of lines and circles

Theorem 3. *If the point X describes a line ε, then its inverse Y relative to circle $\kappa(O, \rho)$ describes a circle ζ passing through the center O of the circle κ. Conversely, if the point Y describes a circle ζ passing through O, then its inverse X relative to κ, describes a line ε.*

Proof. Let H be the projection of O on line ε and Z the inverse of H relative to κ (See Figure 3-II). Pairs (X,Y) and (H,Z) are concyclic points. Consequently, in the quadrilateral $ZHXY$ the angle at Y will be right, as supplementary to the angle at H, which is right. Consequently Y will be on the circle ζ with diameter OZ.

The second claim follows using the same argument. If Y lies on circle ζ with diameter OZ, consider the line ε, which is orthogonal to OZ at H, which is the inverse of Z relative to κ. For an arbitrary point Y of ζ, let X be the intersection of ε with OY. The quadrilateral $XHZY$ is by construction cyclic, because of the right angles at opposite vertices H and Y. Consequently, we will have $|OY||OX| = |OZ||OH| = \rho^2$, therefore point X will be inverse of Y. □

Theorem 4. *If the point X describes the circle λ not passing through O, then its inverse Y relative to circle $\kappa(O,\rho)$ describes also a circle μ not passing through O.*

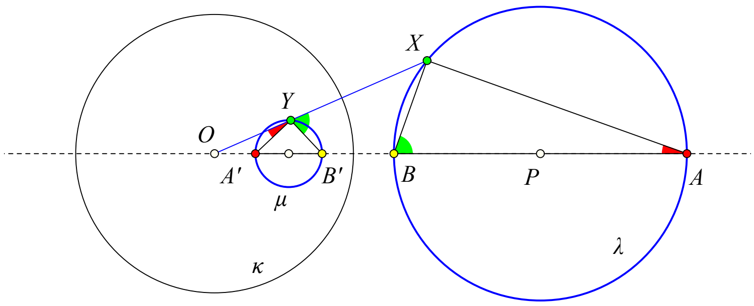

Figure 4: Inverse of circle

Proof. Let AB be the diameter of circle λ on OP, where P is the center of λ (See Figure 4). If A', B' are respectively the inverses relative to κ of A and B, we show also that the inverse Y of the arbitrary point X of circle λ is contained in the circle μ with diameter $A'B'$. Indeed, the quadrilaterals $BB'YX$ and $AA'YX$ are cyclic, therefore equal angles are formed:

$$\widehat{ABX} = \widehat{B'YX}, \quad \widehat{BAX} = \widehat{A'YO}.$$

This implies that the angle $\widehat{A'YB'}$ is right, therefore point Y lies on the circle with diameter $A'B'$. □

Exercise 5. *Show that the location of the center P' and the radius r' of the inverse $\mu(P',r')$ of the circle $\lambda(P,r)$ relative to $\kappa(O,\rho)$ are given respectively by:*

$$OP' = \frac{\rho^2|OP|}{|OP|^2 - r^2} \quad \text{and} \quad r' = \frac{\rho^2 r}{|OP|^2 - r^2}. \tag{1}$$

Remark 2. For the conclusion of theorem 3 we use often the formulation:

"The inverse of a line relative to a circle is a circle through the center of inversion and the inverse of circle through the center of inversion is a line."

Similarly for the conclusion of theorem 4 we use the formulation:

"The inverse of circle not passing through the center of inversion is a circle".

Note the reciprocity of inverse circles. In figure 4 circle μ is the inverse of λ, but also circle λ is inverse of μ. The next proposition underlines the case in which circle λ and μ coincide, in other words the circle coincides with its inverse.

Corollary 1. *The inverses Y of points X of a circle λ are again points of λ, if and only if the circle λ is orthogonal to the circle of inversion $\kappa(O,\rho)$.*

Proof. If the same circle λ contains point X and its inverse Y, then it is orthogonal to the circle of inversion (Theorem 1). If again circle λ is orthogonal to the circle of inversion κ, then for every line ε through O which intersects λ at X and Y, we will have the product $|OX||OY|$, equal to $|OZ|^2 = \rho^2$, where OZ is the tangent to λ from O. □

invariant circles

Remark 3. For the conclusion of corollary 1 we often use the formulation:

"The circles which are orthogonal to the circle of inversion are precisely these, which remain invariant by the inversion."

Corollary 2. *The points $\{X,Y\}$ are inverse relative to circle κ, if and only if there are two circles $\{\alpha,\beta\}$ through $\{X,Y\}$ orthogonal to κ.*

Proof. If the $\{X,Y\}$ are inverse relative to κ, then every circle through them is orthogonal to κ (theorem 1). Conversely if two circles $\{\alpha,\beta\}$ through $\{X,Y\}$ are orthogonal to κ, then the inversion relative to κ leaves these circles invariant hence must interchange the two points. □

Exercise 6. *Given is a circle λ and a point B. Show that all the circles κ which pass through B and intersect λ under a fixed angle ω, are tangent to a fixed circle ν.*

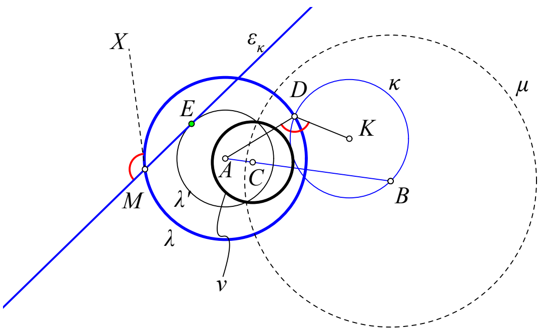

Figure 5: Circles intersecting a fixed circle under fixed angle

Hint: For a point B external to λ, consider the inversion relative to circle μ with center B, which is orthogonal to λ (See Figure 5). Under this inversion the circle λ maps to itself and the circle κ maps to a line ε_κ, which forms the same angle ω with λ. Such a line is tangent to a fixed circle λ', which is concentric with λ. It follows that the wanted circles will be the inverses of the lines ε_κ, which are tangent to λ', therefore the circles κ will be tangent to the inverse ν of λ' relative to μ.

3 Metric relations

lengths comparison

Theorem 5. *Let $\kappa(O,r)$ be a circle and AB a segment. Consider segment $A'B'$ resulting by inverting the end-points of AB relative to κ. Then the segment lengths are related by:*

$$|A'B'| = r^2 \frac{|AB|}{|OA| \cdot |OB|}.$$

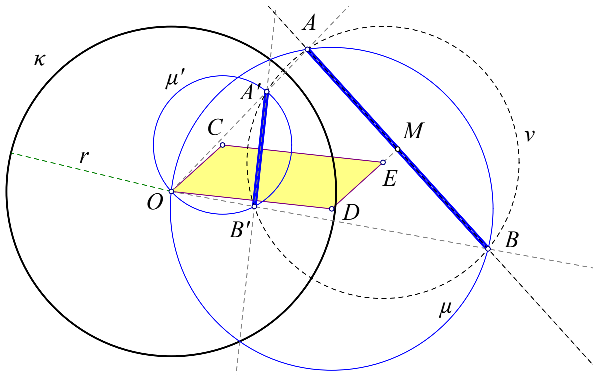

Figure 6: Lengths of *"inverse"* segments

Proof. Triangles $\{OA'B', OBA\}$ are similar (See Figure 6). Thus,

$$\frac{|A'B'|}{|AB|} = \frac{|OA'|}{|OB|} = \frac{|OA'|\cdot|OA|}{|OB|\cdot|OA|} = \frac{r^2}{|OA|\cdot|OB|}.$$

□

Remark 4. Notice that if $\{A', A\}$ coincide (of course coinciding then with a point of κ), then the relation becomes:

$$\frac{|AB'|}{|AB|} = \frac{r}{|OB|}.$$

Exercise 7. *Consider the circles $\{\mu = (OAB), \mu' = (OA'B'), \nu = (ABB'A')\}$ with corresponding centers $\{D, C, E\}$. Show that the radii $\{\rho, \rho'\}$ of the two first satisfy $\rho/\rho' = |AB|/|A'B'|$ and $OCED$ is a parallelogram.*

Hint: Use the similarity of triangles $\{OAB, OB'A'\}$.

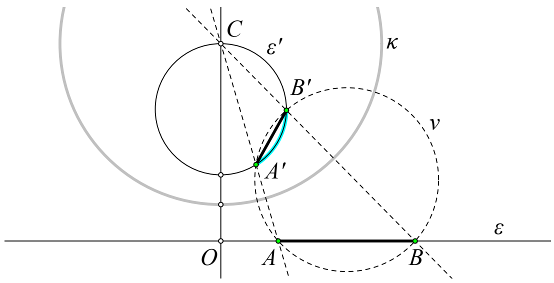

Figure 7: Variable AB on line ε and its inverse

Remark 5. Denoting by f the inversion relative to κ and by ε the line defined by AB, figure 7 underlines the difference between the image $f(AB)$ of the segment $AB \subset \varepsilon$, which is an arc of the circle $\varepsilon' = f(\varepsilon)$ and the chord $A'B'$ of circle ε' defined by the images $\{A' = f(A), B' = f(B)\}$. The theorem deals with the *chord* AB and not with the corresponding arc.

4 Ptolemy's theorem

There are lots of applications of the inversion in various problems of geometry and in particular applications of the metric relation of section 3. In this section we discuss a nice application of this metric relation in the proof of *"Ptolemy's theorem"*, or *"Ptolemy's inequality"* rather ([Ped90, p.90]).

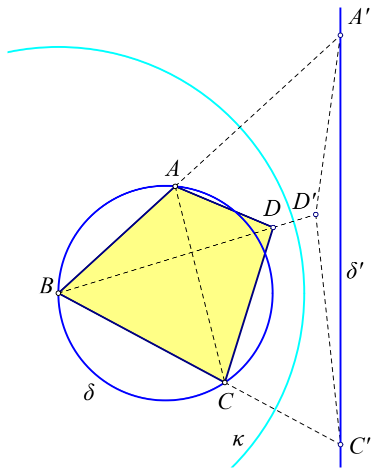

Figure 8: Ptolemy's inequality

Ptolemy's inequality

Theorem 6. *In a convex quadrilateral $ABCD$, the inequality involving the sides and the diagonals*

$$|BC||AD| + |CD||AB| \;\geq\; |BD||AC|$$

is valid. The equality is valid if and only if the quadrilateral has a circumcircle κ (is " cyclic").

Proof. The proof results by inversion on a circle κ, centered at B with arbitrary radius r. Such an inversion maps the circle passing through A, B, C onto a line ε. By theorem 5 the lengths of segments $\{|CD|, |C'D'|\}$, where $\{C', D'\}$ are the κ–inverses of $\{C, D\}$, are related by the formula:

$$|C'D'| = \frac{r^2|CD|}{|BC||BD|} \quad \text{and} \quad |D'A'| = \frac{r^2|DA|}{|BD||BA|}.$$

From the *"triangle inequality"* applied to triangle $A'C'D'$ follows:

$$|A'C'| \;\leq\; |A'D'| + |D'C'| \quad \Leftrightarrow \quad \frac{r^2|AC|}{|BA||BC|} \;\leq\; \frac{r^2|DA|}{|BD||BA|} + \frac{r^2|CD|}{|BC||BD|}, \tag{2}$$

which, multiplying by $|BA||BC||BD|$ and simplifying leads to the claimed inequality. Obviously the collinearity of $\{A', D', C'\}$ is equivalent with the equality on the left side of (2), which in turn is equivalent with $D \in \kappa$. □

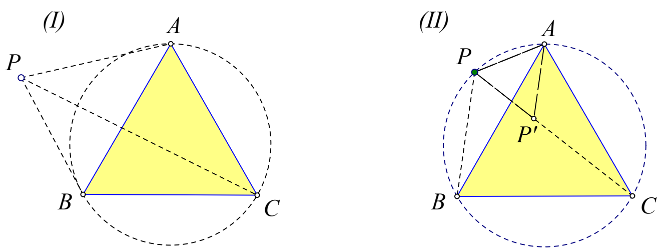

Figure 9: Property of equilaterals derived from Ptolemy's inequality

Exercise 8. *For an equilateral triangle ABC and a point P, the distance from the most distant vertex, C say, satisfies $PA + PB \geq PC$ and equality occurs exactly when P is on the arc AB of the circumcircle of ABC.*

Hint: In fact, by Ptolemy's inequality, applied to the quadrangle $APBC$:

$$PA \cdot BC \ + \ PB \cdot AC \ \geq \ PC \cdot AB,$$

which, because of the equal sides, simplifies to $PA + PB \geq PC$ (See Figure 9-I). To examine the equality, consider P on the arc AB (See Figure 9-II). Take P' so that PAP' is equilateral. Then triangles APB and $AP'C$ are equal and consequently $PA + PB = PP' + P'C$.

5 Inversion is a conformal map

Theorem 7. *Let Y be the inverse of the point X relative to the circle $\kappa(O, \rho)$, with X lying also on circle μ. If the circle or line λ is tangent to the circle μ at X, then the inverse λ' of λ and the inverse μ' of μ are tangent at Y.*

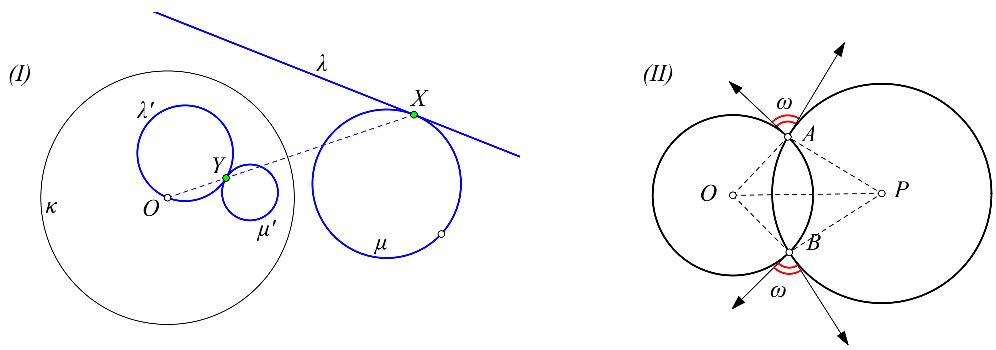

Figure 10: Tangent inverted into tangent — Angle ω of two circles

Proof. In fact, λ and μ are tangent at X, precisely when X is their unique common point (See Figure 10-I). When this happens, then λ' and μ' will also have Y as a common point. If they had another common point Z, then the inverses of λ' and μ' which are λ and μ respectively would have also, besides X, the inverse Ω of Z in common, which contradicts the hypothesis. Therefore λ' and μ' have point Y as their unique common point, therefore they are tangent at this point. □

We define the *"angle between two circles"* $\kappa(O)$, $\lambda(P)$, which intersect at two points A, B, the supplementary ω of the angle $\widehat{OAP} = \widehat{OBP}$, which is the angle of the tangents at their intersection point (See Figure 10-II). Next theorem shows that the inversion preserves these angles. This property is often mentioned with the phrase:

inversion is conformal

"The inversion is a conformal transformation".

For the proof we'll need an auxiliary lemma, which we formulate as an exercise.

Exercise 9. *Construct a circle λ tangent to a given line ε at a given point A and orthogonal to a given circle $\kappa(O, \rho)$.*

Hint: By the orthogonality of $\{\lambda, \kappa\}$ the second intersection B of AO with the required circle λ will satisfy $OA \cdot OB = \rho^2$, hence it is constructible (See Figure 11).

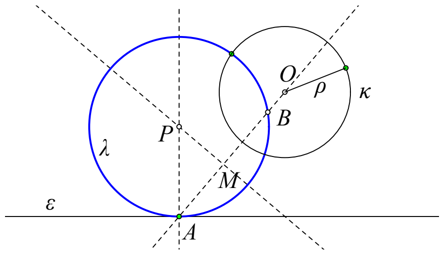

Figure 11: Circle λ tangent to ε at A and orthogonal to κ

Theorem 8. *Let Y be the inverse of point X relative to circle $\kappa(O,\rho)$. If circles λ and μ pass through X and form there the angle ω, then their inverses λ', μ' pass through point Y and form there the same angle ω.*

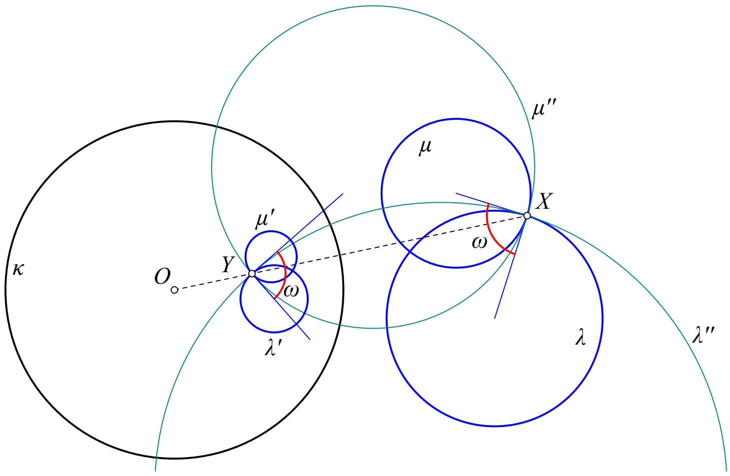

Figure 12: The inversion is a conformal transformation

Proof. Consider the circle μ'', which is tangent to μ at X (See Figure 12) and orthogonal to κ. Consider also the circle λ'', which is tangent to λ at X and orthogonal to κ. Because the circles $\{\lambda, \mu\}$ are tangent to $\{\lambda'', \mu''\}$ respectively at X, the inverse circles $\{\lambda', \mu'\}$ will be tangent at Y of the inverses of λ'' and μ'' respectively. However, from the orthogonality to κ, the inverse of λ'' is λ'' itself and similarly the inverse of μ'' is μ'' itself. Therefore the tangents of λ' and μ' at Y will coincide respectively with their tangents of λ'' and μ'' at Y. The latter however, by symmetry, form at Y the same angle with the one formed by the tangents of λ'', μ'' at X, which by assumption make an angle of measure ω. Therefore the tangents of λ' and μ' at Y will form an angle of the same measure. □

6 Inversion preserves the cross-ratio

The *"cross ratio"* $(ABCD)$ of four *collinear* points is handled in its details in the file **Cross ratio**. Its original definition involves the directed segments and their ratios

$$(ABCD) \;:=\; \frac{CA}{CB} : \frac{DA}{DB}. \tag{3}$$

The definition is extended to four *"concyclic"* points on a circle κ using a fifth arbitrary point P on the circle and an auxiliary line ε. The four lines $\{PA, PB, PC, PD\}$ define with

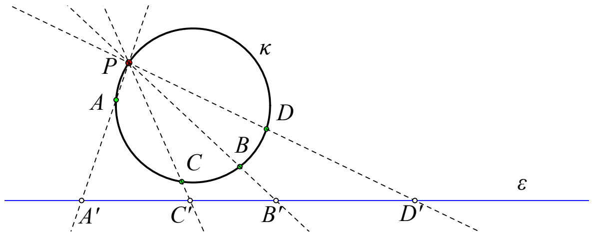

Figure 13: Cross ratio of four concyclic points defined by their projections on a line

their intersections with ε four points $\{A', B', C', D'\}$ (See Figure 13), and the cross ratio is then defined by

$$(ABCD) := (A'B'C'D').$$

It is then proved that this definition does not depend on the particular $P \in \kappa$, as well as on the particular line ε. An important property implied by this definition is the following.

cross-ratio preservation

Theorem 9. *The cross ratio is preserved by inversions, i.e. if $\{A, B, C, D\}$ are points on the circle/line κ, then their images $\{A', B', C', D'\}$ under an inversion satisfy $(A'B'C'D') = (ABCD)$.*

Proof. This derives directly from the definition if the inverse of κ is a line ε. If the inverse of κ is a circle κ', then we consider also the inverse P' of point P. It is then easy to see that the intersections $\{A_0 = PA \cap P'A',\ B_0 = PB \cap P'B',\ C_0 = PC \cap P'C',\ D_0 = PD \cap P'D'\}$

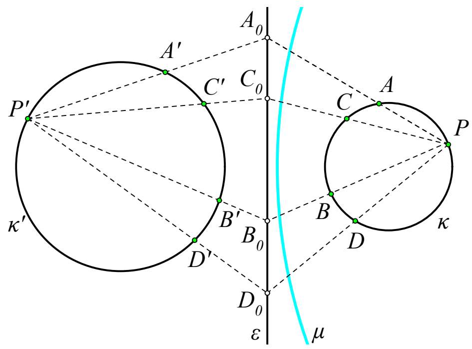

Figure 14: Inversions preserve the cross ratio

are points of the radical axis ε of $\{\kappa, \kappa'\}$ (See Figure 14), which proves the equality of the two cross ratios. □

7 Inversion and pencils of circles

pencils map to pencils

Theorem 10. *For a given pencil of circles $\mathcal{D}$ and a circle $\kappa(O, \rho)$, the inverses of circles λ of the pencil relative to κ build another pencil $\mathcal{D}'$.*

Proof. Assume that the pencil $\mathcal{D}$ consists of all the circles which are orthogonal to two fixed circles μ and ν. Let also μ' and ν' be the inverses of circles μ and ν relative to κ. By Theorem 8, every circle λ orthogonal to μ and ν will have an inverse λ' orthogonal to μ' and ν', therefore it will belong to the pencil $\mathcal{D}'$ of circles which are orthogonal to μ' and ν'. Similarly, every circle λ' orthogonal to μ' and ν' will have an inverse λ which is orthogonal to μ and ν, therefore it belongs to the pencil $\mathcal{D}$ and will have as inverse exactly circle λ' □

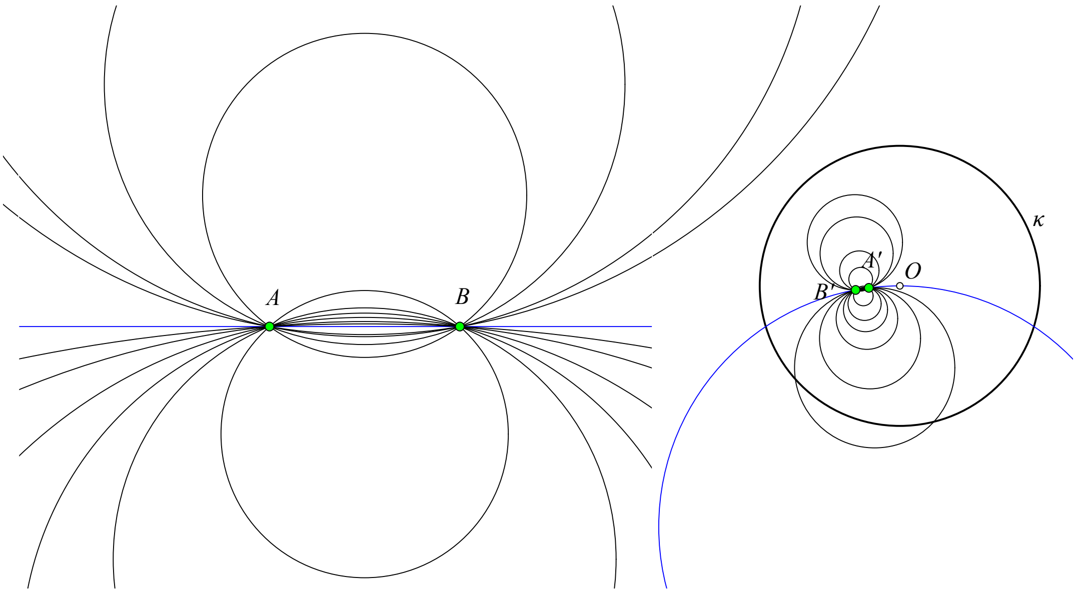

Figure 15: Pencil $\mathcal{D}'$ of the inverses of circles of the pencil $\mathcal{D}$

Remark 6. The pencil $\mathcal{D}'$, ensured by last theorem, is called *"inverse pencil"* of $\mathcal{D}$ relative to the circle κ (See Figure 15).

In the previous theorem, the inversion relative to κ, not only sends the pencil $\mathcal{D}$ onto a pencil $\mathcal{D}'$, but also preserves the quality of the pencil. In other words if the pencil $\mathcal{D}$ is intersecting, then pencil $\mathcal{D}'$ is intersecting too, if it is non intersecting then $\mathcal{D}'$ is also non intersecting etc. This however with a small concession. We must accept that, depending on the relative position of $\mathcal{D}$ and the circle of inversion κ, pencil $\mathcal{D}'$ may be *"non conventional"*, i.e. a pencil of lines through a point, real or at infinity, or a pencil of concentric circles. The next propositions show, for which positions of κ something like that may happen.

map to non-conventional

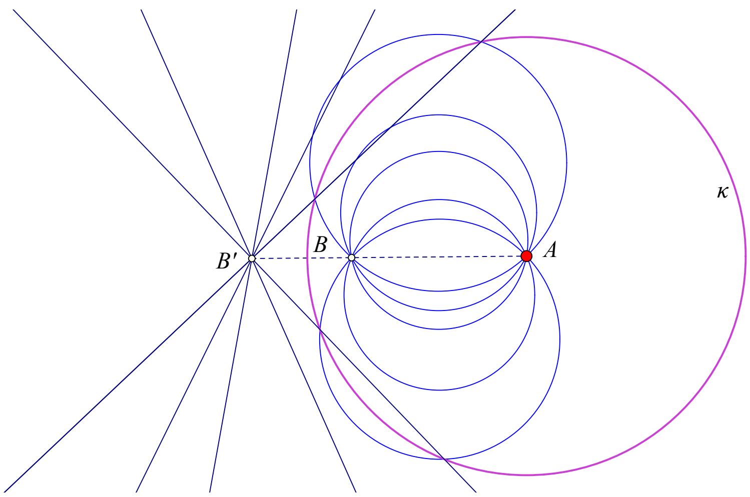

Figure 16: Special inversion of intersecting pencil

Theorem 11. *Let $\mathcal{D}$ be an intersecting pencil and $\kappa(A, \rho)$ be a circle with center at one of the two base points $\{A, B\}$ of the pencil. Then the inverse pencil $\mathcal{D}'$ of $\mathcal{D}$ relative to κ is the non-*

conventional pencil of the lines, which pass through the inverse B' of B relative to κ (See Figure 16).

Proof. Consequence of Theorem 3. Since all the circles of $\mathcal{D}$ pass through A, their inverses will be lines. Since they also pass through B, their inverses will also pass through the inverse B' of B relative to κ. □

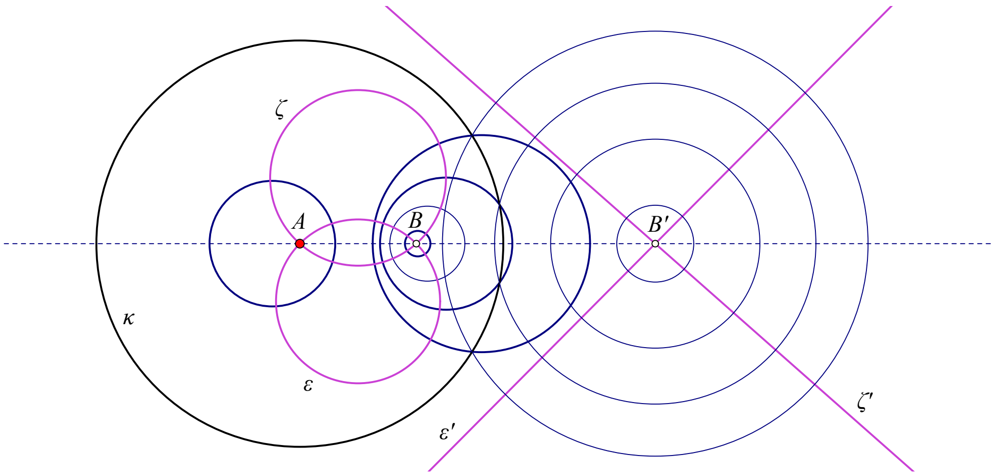

Figure 17: Special inversion of non intersecting pencil

Theorem 12. *Let $\mathcal{D}$ be a non intersecting pencil and $\kappa(A, \rho)$ be a circle with center one of the limiting points A and B of the pencil. Then the inverse $\mathcal{D}'$ of $\mathcal{D}$, relative to κ, is a non conventional pencil of concentric circles with center the inverse B' of B relative to κ (See Figure 17).*

Proof. Consequence of Proposition 3, which we apply to two circles ε and ζ, passing through points A and B, which belong to the orthogonal pencil of $\mathcal{D}$. The inverses ε' and ζ' of these circles are lines through B'. Also because ε, ζ are orthogonal to every circle λ of $\mathcal{D}$, their inverses, which are lines ε' and ζ' will be orthogonal to the inverse λ' of λ (Theorem 8). Therefore λ' will be a circle with center B'. □

Theorem 13. *Let $\mathcal{D}$ be a tangent pencil and $\kappa(A, \rho)$ be a circle with center the base point A of the pencil. Then the inverse $\mathcal{D}'$ of $\mathcal{D}$ relative to κ is the non conventional pencil of the lines, which are parallel to the radical axis ε of $\mathcal{D}$.*

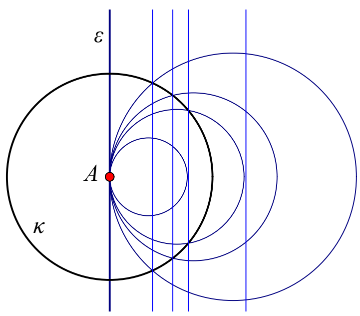

Figure 18: Inversion of tangential pencil

Proof. Consequence of Theorem 3. Since all the circles of $\mathcal{D}$ pass through the point A, their inverses will be lines (See Figure 18-I). The inverse of the radical axis ε is itself. Every circle λ of the pencil has common with the radical axis the unique point A. Consequently the inverse λ' of λ, which is a line, will have no common point with ε, therefore it will be parallel to it. □

Exercise 10. *Show that the inversion f_μ relative to a member-circle μ of the pencil $\mathcal{D}$ leaves the pencil invariant, in the sense that it maps every member-circle of $\mathcal{D}$ to another member-circle of the same.*

Hint: Conceive $\mathcal{D}$ as a set of circles orthogonal to two circles $\{\alpha, \beta\}$, which remain invariant under f_μ.

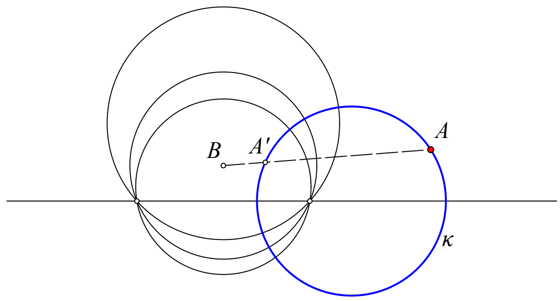

Figure 19: The inverses $\{A'\}$ of A relative to circles of the pencil

Exercise 11. *Show that the geometric locus of the inverses $\{A'\}$ of a fixed point A relative to the member-circles of a pencil $\mathcal{D}$ is the circle κ of the orthogonal pencil $\mathcal{D}'$ passing through A.*

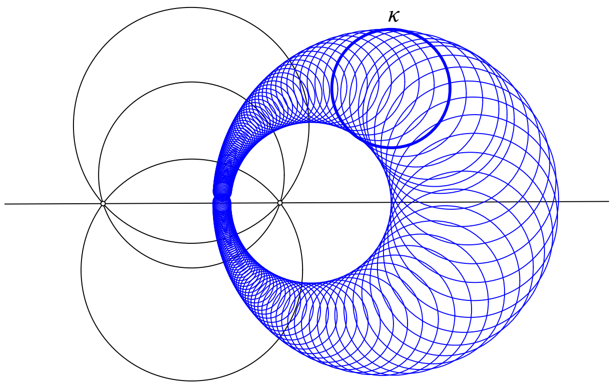

Figure 20: Inverses relative to circles of a pencil

Figure 20 shows the inverses of a circle κ (bold) relative to member-circles of a pencil $\mathcal{D}$ of intersecting type. The figure suggests that all these circles envelope two members of the orthogonal pencil $\mathcal{D}'$. In fact, it can be shown that all these circles are tangent to the two circles of $\mathcal{D}'$ which are tangent to κ. This and some similar questions are handled in the file **Inverting on a family of circles**.

8 Inversion interchanging two circles

inversion interchanging

Theorem 14. *Given two no congruent circles $\lambda(A, \alpha)$ and $\mu(B, \beta)$, there exists an inversion relative to a circle κ, which interchanges the two given circles. In other words the inverse of λ relative to κ is μ and vice versa.*

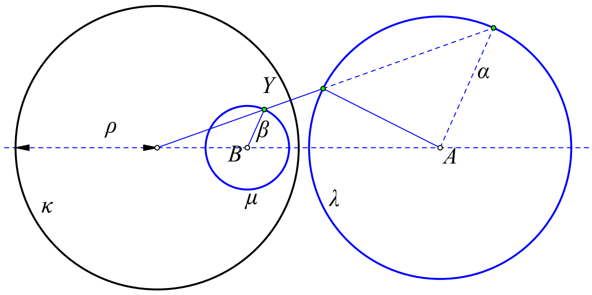

Figure 21: Inversion interchanging two circles

Proof. In the case where the circles λ and μ are mutually external or are externally tangent, the external *"similarity center"* C of the two circles has the desired property of the center of inversion (See Figure 21). Specifically, in this case the product of distances $|CX||CY|$ of two antihomologous points from C is a fixed positive number equal to $|CA||CB| - \alpha\beta$. In this case, the desired circle has as center the external center of similarity C and radius $\rho = \sqrt{|CA||CB| - \alpha\beta}$.

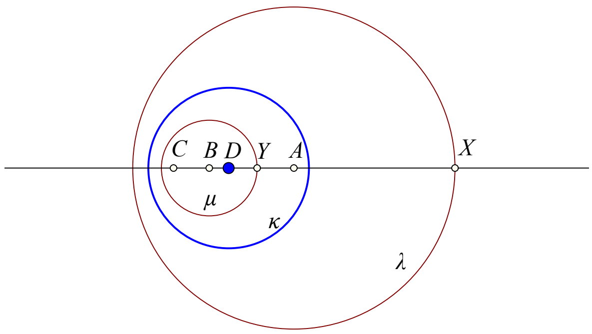

Figure 22: Circle of inversion when λ contains μ

In the case where the circle $\mu(B, \beta)$ is internal to $\lambda(A, \alpha)$ or is internally tangent, the wanted circle has as center the internal *"similarity center"* D of the two circles and radius $\rho = \sqrt{\alpha\beta - |DA||DB|}$ (See Figure 22). Finally, in the case of two circles λ and μ intersecting at two points there exist two circles $\kappa(C, \rho)$ and $\kappa'(D, \rho')$ with radii $\rho = \sqrt{|CA||CB| - \alpha\beta}$, $\rho' = \sqrt{\alpha\beta - |DA||DB|}$ and centers the *"similarity centers"* C (external) and D (internal) of the circles λ and μ (See Figure 23). □

Exercise 12. *Show that a circle κ whose inversion interchanges the circles $\{\lambda, \mu\}$ belongs to the pencil generated by these two circles.*

Hint: Show that a circle ξ orthogonal to $\{\lambda, \mu\}$ remains invariant by this inversion.

Exercise 13. *Show that for given line λ and circle μ there is a circle κ, whose inversion interchanges $\{\lambda, \mu\}$ (See Figure 24).*

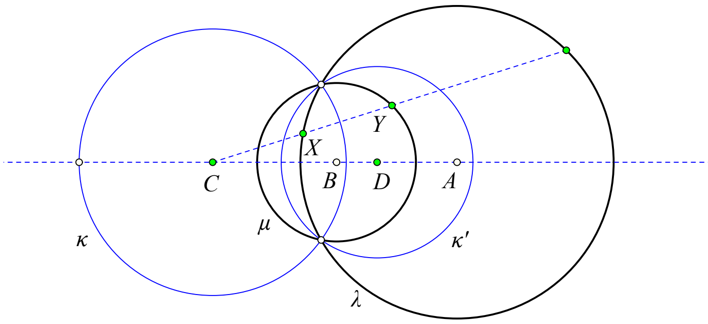

Figure 23: Circles of inversion of two intersecting circles λ and μ

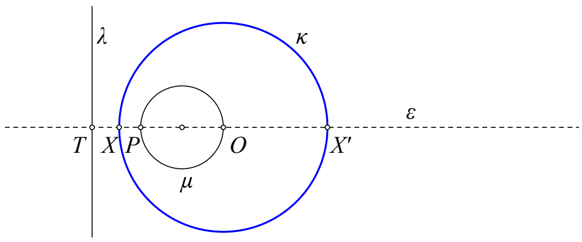

Figure 24: Inverting line λ to the circle μ

9 Inverseness is preserved by inversions

This section's title, coined by Johnson [Joh60, p.55], expresses the essence of the following theorem.

inverseness preserved

Theorem 15. *If the points* $\{X,Y\}$ *are inverse relative to the circle* α *and the inversion relative to circle* κ *maps* α *onto* β *and* $\{X,Y\}$ *to* $\{X',Y'\}$*, then latter are inverse relative to* β.

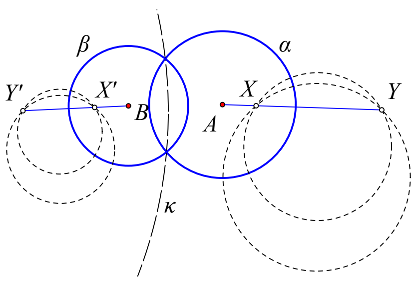

Figure 25: Inverseness is preserved by inversions

Proof. If the points $\{X,Y\}$ are inverse relative to the circle α, take two circles through $\{X,Y\}$ orthogonal to α and invert the whole figure relative to κ. The circle α maps to β and the two circles map to two circles orthogonal to β (section 5). The proof results by applying corollary 2. □

Corollary 3. *If* f_α *is the inversion relative to circle* α *and* g *is an inversion, then the inversion* f_β *relative to the circle* $\beta = g(\alpha)$ *is* $f_\beta = g \circ f_\alpha \circ g^{-1} = g \circ f_\alpha \circ g$.

Proof. In fact, by theorem 15,

$$Y' = f_\beta(X') \quad \Leftrightarrow \quad (g(Y)) = f_\beta(g(X)) \quad \Leftrightarrow \quad f_\alpha(X) = Y = (g^{-1} \circ f_\beta \circ g)(X),$$

which is equivalent to the claimed *"conjugation"* by g of f_α to f_β. □

Corollary 4. *Given a circle α and a line ε, there is an inversion g, such that the inversion f_α relative to α is conjugate to the reflection f_ε relatvie to the line ε :*

$$g \circ f_\alpha \circ g \;=\; f_\varepsilon.$$

This can be also expressed as:

Every inversion is conjugate by another inversion to a reflection.

Exercise 14. *Show that in the case of two intersecting circles λ and μ, the two circles of inversion $\{\kappa,\kappa'\}$ which are ensured by Theorem 14 are orthogonal and pass through the intersection points of λ and μ (See Figure 23).*

Hint: The inversion relative to κ interchanges $\{\lambda,\mu\}$ and maps κ' to a circle κ'' which also will interchange these two circles. Hence κ'' must coincide with κ', which then is invariant under the inversion relative to κ, hence κ' is orthogonal to κ.

Exercise 15. *Show that if the circle κ interchanges the circles $\{\lambda,\mu\}$ and f is the inversion on a circle ξ, then the circle $f(\kappa)$ interchanges the circles $\{f(\lambda), f(\mu)\}$.*

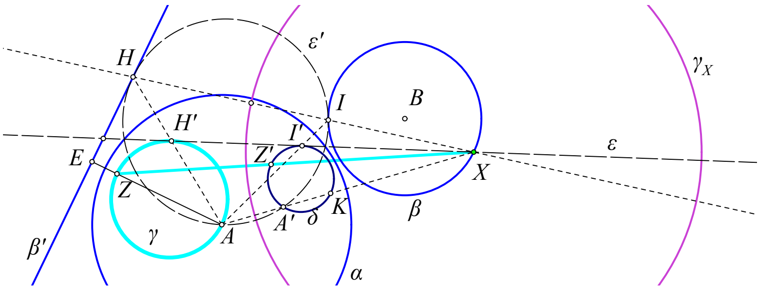

Figure 26: Envelope of lines

Exercise 16. *Consider two external to each other circles $\{\alpha(A,r_\alpha),\beta(B,r_\beta)\}$ and a variable point $X \in \beta$. For each position of X consider the circle γ_X centered at X and orthogonal to α. Consider finally the line $\beta' = f(\beta)$, where f is the γ_X−inversion. Show that all lines $\{\beta\}$ are tangent to a fixed circle ζ with the same center as α and calculate the radius of this circle (See Figure 26).*

Hint: Let ε' be a circle tangent to $\{\beta,\beta'\}$ and passing through A. Consider also the α−inversion g and the inverses $\{\gamma = g(\beta'), \delta = g(\beta), \varepsilon = g(\varepsilon'), H' = g(H), I' = g(I)\}$, where $\{H,I\}$ the contact points of ε' with $\{\beta',\beta\}$, which are γ_X−inverses, hence HI passes through X.

The line $H'I'$, which is tangent at these points to $\{\gamma,\delta\}$, passes also through X. This is seen through the cyclic quadrangle $HII'H'$. By corollary 3 $\{\gamma,\delta\}$ are γ_X−inverses and $I'H'$ passes through X.

To show that the distance AE of line β' from A is constant it suffices to show that for the α−inverse Z, the diameter AZ of γ is constant. Since X is a similarity center of $\{\gamma,\delta\}$ and δ is a fixed circle independent of X, it suffices to show that the ratio XH'/XI' is constant.

For this consider the second intersection A' of XA with ε' and show that AK/AX is constant, where K the second intersection of AX with δ

$$\frac{AK}{AX} = \frac{AK}{(r_\alpha^2/AA')} = \frac{AK\cdot AA'}{r_\alpha^2} = \frac{p_\delta(A)}{r_\alpha^2},$$

where $p_\delta(A)$ the *"power"* of A relative to δ.

10 Inverting to equal circles

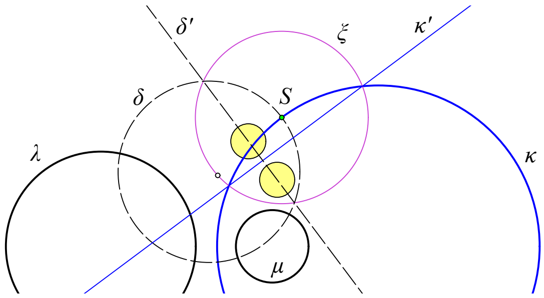

Figure 27: Inversion of two circles λ, μ to two equal circles

inverting to equal

Theorem 16. *The points $\{S\}$ of a circle κ interchanging two given circles or a circle and a line $\{\lambda, \mu\}$ have the following property: Any circle $\xi(S)$ with arbitrary radius inverts $\{\lambda, \mu\}$ to two equal circles (See Figure 27).*

Proof. Consider the inversion f on such a ξ and the circle δ orthogonal to $\{\lambda, \mu\}$ and passing through S. The inversion f maps κ onto a line κ' and δ onto a line δ' orthogonal to κ' and the circles $\{\lambda, \mu\}$ to two circles $\{\lambda', \mu'\}$ orthogonal to line δ', hence having their centers on δ'. By theorem 15, since $\{\lambda, \mu\}$ are inverse relative to κ, the inverses $\{\lambda' = f(\lambda), \mu' = f(\mu)\}$ are inverse relative to $\kappa' = f(\kappa)$, which is a line. Hence they are symmetric relative to κ'. □

Exercise 17. *With the settings of the previous theorem show that we can select the radius of ξ so that one of the equal circles is precisely one of the $\{\lambda, \mu\}$.*

11 Composition of two inversions

The composition $f = f_\beta \circ f_\alpha$ of two inversions on two circles $\{\alpha, \beta\}$ is not an inversion. There are though some properties for f and its conjugate $g \circ f \circ g$ by an appropriate inversion g which may be of interest. For instance, assume that the two circles $\{\alpha, \beta\}$ intersect at two different points $\{O, P\}$. The inversion g relative to the circle $\gamma(P, |OP|)$

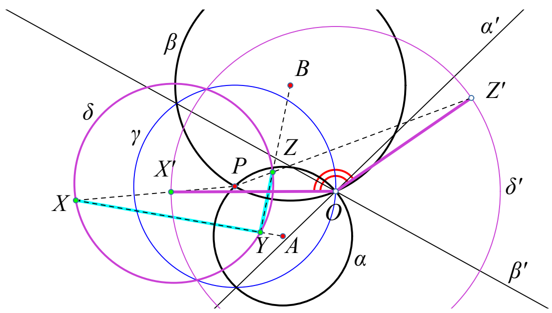

Figure 28: Composition of two inversions relative to intersecting circles

transforms the two circles correspondingly to two lines $\{\alpha', \beta'\}$ through O. It is easily seen that the composition f preserves the circles $\{\delta\}$ which are orthogonal to $\{\alpha, \beta\}$. Thus, a point $X \in \delta$ maps to $Y = f_\alpha(X) \in \delta$ and $Z = f_\beta(Y) \in \delta$. The transformation g

maps the circle δ onto a circle δ' with center at O which contains $\{X' = g(X), Z' = g(Z)\}$ (See Figure 28). The conjugate transformation

$$h = g \circ f \circ g = g \circ (f_\beta \circ f_\alpha) \circ g = (g \circ f_\beta \circ g) \circ (g \circ f_\alpha \circ g) = h_2 \circ h_1 \quad (4)$$

is then a composition of two *reflections* $\{h_1, h_2\}$ respectively on lines $\{\alpha', \beta'\}$. If the circles $\{\alpha, \beta\}$ intersect by an angle ω, the same angle is the one between the lines $\{\alpha', \beta'\}$ and the composition $h = h_2 \circ h_1$ is then a *"rotation"* about O by the angle 2ω.

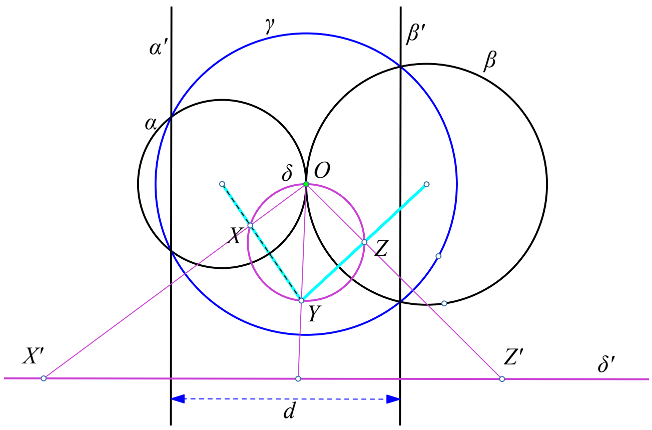

Figure 29: Composition of two inversions relative to tangent circles

In the case the two circles $\{\alpha, \beta\}$ are tangent at a point O, the inversion g on the circle $\gamma(O, r)$ with r arbitrary radius, maps the two circles correspondingly to two parallel lines $\{\alpha', \beta'\}$. It is then again easily seen that the composition f preserves the circles $\{\delta\}$ which are orthogonal to $\{\alpha, \beta\}$. Thus, a point $X \in \delta$ maps to $Y = f_\alpha(X) \in \delta$ and $Z = f_\beta(Y) \in \delta$. The transformation g maps the circle δ onto a line δ' orthogonal to α', β' which contains $\{X' = g(X), Z' = g(Z)\}$ (See Figure 29). The conjugate transformation h of the relation 4 is then a composition of two *reflections* $\{h_1, h_2\}$ respectively on the two parallel lines $\{\alpha', \beta'\}$, i.e. h is a translation by a vector in the direction of δ' and the distance $2d$, where d is the distance of the two parallels $\{\alpha', \beta'\}$. In the case the two circles $\{\alpha, \beta\}$ are non-intersecting,

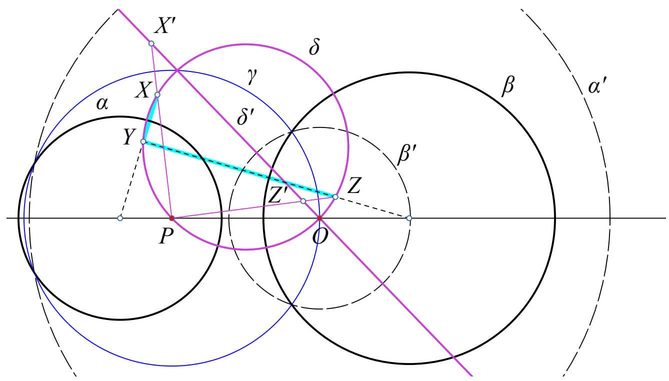

Figure 30: Composition of two inversions relative to non-intersecting circles

the inversion g on the circle $\gamma(P, |OP|)$, where $\{O, P\}$ are the limit points of the pencil they generate, maps the two circles correspondingly to two concentric circles $\{\alpha', \beta'\}$. It is then again easily seen that the composition f preserves the circles $\{\delta\}$ which are orthogonal to $\{\alpha, \beta\}$. Thus, a point $X \in \delta$ maps to $Y = f_\alpha(X) \in \delta$ and $Z = f_\beta(Y) \in \delta$. The transformation g maps the circle δ onto a line δ' passing through the center O of the concentric circles, which contains $\{X' = g(X), Z' = g(Z)\}$ (See Figure 30). The conjugate transformation h of the relation 4 is then a composition of two *inversions* $\{h_1, h_2\}$ respectively on the two concentric circles $\{\alpha', \beta'\}$, i.e. h is a homothety with center at O.

The discussion so far establishes the proof of the following theorem.

composition of two

Theorem 17. *The composition of two inversions* $f = f_\beta \circ f_\alpha$ *on the circles* $\{\alpha, \beta\}$ *is conjugate by an appropriate inversion* g *to a transformation* $h = g \circ f \circ g$*, which is:*

1. *A rotation, if the circles* $\{\alpha, \beta\}$ *intersect at two different points.*
2. *A translation if the circles* $\{\alpha, \beta\}$ *are tangent.*
3. *A homothety if the circles* $\{\alpha, \beta\}$ *are disjoint.*

12 Composition of three inversions

three but same pencil

Theorem 18. *The composition* $f = f_1 \circ f_2 \circ f_3$ *of three inversions relative to circles* $\{\alpha, \beta, \gamma\}$ *of the same pencil of circles* $\mathcal{D}$ *is also an inversion relative to a circle* δ *of* $\mathcal{D}$.

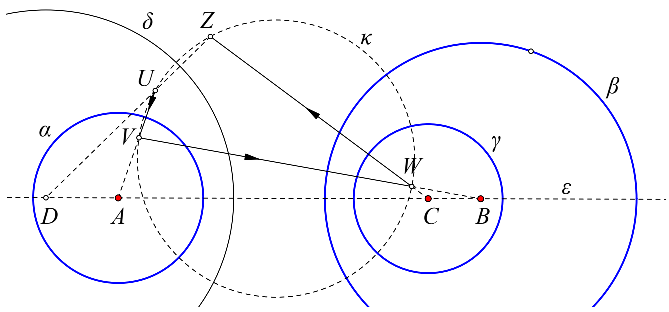

Figure 31: Composition of three inversions relative to circles of the same pencil

Proof. In the case the pencil is a non-conventional of lines through a point, real or at infinity, or a pencil of concentric circles, the theorem is easily proved. In the non conventional case, we know from the discussion in section 7, that there is an inversion g transforming the given pencil $\mathcal{P}$ to a non-conventional $\mathcal{P}'$. Then the three inversions transform to their conjugates $\{h_i = g \circ f_i \circ g\}$, which are reflections on lines through a point, real or at infinity, or inversions on concentric circles and their composition

$$h = (g \circ f_3 \circ g) \circ (g \circ f_3 \circ g) \circ (g \circ f_3 \circ g) = g \circ (f_3 \circ f_2 \circ f_1) \circ g$$

is easily seen to be of the same kind, i.e. either a reflection on a line of of $\mathcal{P}'$ if this consists of lines, or a reflection on a circle of $\mathcal{P}'$ if this consists of concentric circles. The claim follows then from the equivalent to the previous relation:

$$f_3 \circ f_2 \circ f_1 = g \circ h \circ g.$$

□

13 Inversion action on an invariant circle

There is an interesting aspect of an inversion f leaving a circle λ invariant. Such a circle is orthogonal to the circle of inversion $\kappa(O, \rho)$ (theorem 1) and for each $X \in \lambda$ the inverse

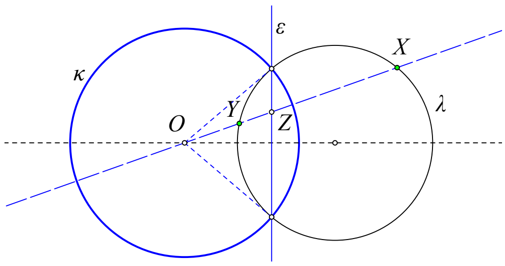

Figure 32: Homography $f^* : X \mapsto Y$ defined by an inversion

$Y = f(X) \in \lambda$ too (See Figure 32). Besides the line XY passes through O and intersects the *radical axis* ε of $\{\kappa, \lambda\}$ at a point Z satisfying

$$(XYZO) = \frac{OX}{OY} : \frac{ZX}{ZY} = -1, \tag{5}$$

i.e. $\{X, Y\}$ are *"harmonic conjugate"* w.r. to $\{O, Z\}$. This is equivalent to the fact that the radical axis in this case (orthogonality of the circles) is also the *"polar"* of O w.r. to λ. Forgetting for the moment the inversion and looking at this relation, we can use it to define a map f^* which makes sense not only for the points of λ but for all the points $\{X \neq O\}$ of the plane . This is the *"harmonic perspectivity"* or *"harmonic homology"* ([Cox87, p.55], [VY10, I,p.223]) defined generally by a line ε and a point $O \notin \varepsilon$ called respectively *"axis"* and *"center"* of the perspectivity. The recipe for such a transformation f^* is, to associate to each X the point Y on the line OX, so that equation-5 holds. We formulate this simple fact through the next theorem.

action on invariant cir.

Theorem 19. *The inversion f relative to the circle $\kappa(O, \rho)$ restricted to the points of a circle λ, orthogonal to κ, coincides there with the harmonic perspectivity f^* with center O and axis equal to the radical axis ε of the circles $\{\kappa, \lambda\}$.*

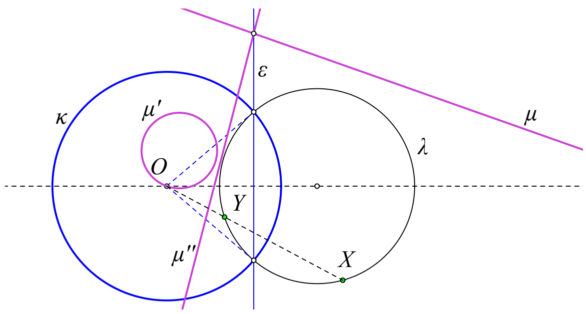

Figure 33: The image $\mu' = f(\mu)$ and the image $\mu'' = f^*(\mu)$

Figure 33 shows how differently act on a line μ the inversion f and the harmonic perspectivity f^*. The first maps μ onto a circle μ' through the center of inversion. The second maps μ onto another line μ'', such that the pair (μ, μ'') is harmonic conjugate w.r. to the line ε and the point O. The two maps have the same effect only when applied to the points $X \in \lambda$. For all other points of the plane their action on them is totally different, the inversion mapping circles and lines onto circles and the perspectivity, as a special kind of *"homography"*, mapping circles, and more generally conics, onto conics and lines onto lines.

The inversion f relative to κ leaves κ point-wise fixed and maps λ onto itself. The

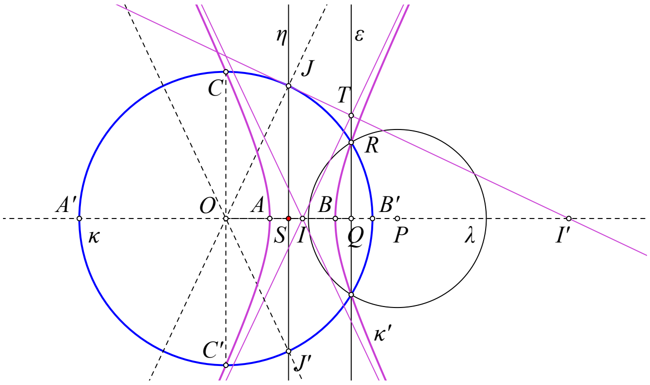

Figure 34: The image $\kappa' = f^*(\kappa)$ is a hyperbola

perspectivity f^*, as we said, coincides with f on the points of λ but for points $X \in \kappa$ has a totally different behavior. Figure 34 shows that $\kappa' = f^*(\kappa)$ is a hyperbola, the characteristics of which can be determined from the two orthogonal circles $\{\kappa, \lambda\}$. Following theorem lists the properties of this figure, which result easily from the very definition of f^* and its fundamental properties:

1. f^* is an *"involutive"* map, i.e. satisfies $(f^*)^2 = e$ (e the identity).
2. Lines map via f^* to lines, a finite line possibly mapping to the line at infinity.
3. The line η at half the distance of O from the axis ε of f^* maps via f^* to the line at infinity and vice versa.
4. A circle, and more general a conic, maps via f^* to a conic.
5. The tangent τ_X of a curve κ at its point X maps to the tangent $\tau_{X'}$ of the curve $f^*(\kappa)$ at the point $X' = f^*(X)$.

Theorem 20. *The image $\kappa' = f^*(\kappa)$ of the circle κ is a hyperbola with the following properties.*

1. *κ' passes through the intersections of $\kappa \cap \lambda$ and the points $\{C, C'\}$ which are diametral of κ with CC' orthogonal to OP.*
2. *If $\{J, J'\}$ are the intersections $\{\kappa \cap \eta\}$, then $\{OJ, OJ'\}$ are parallel to the asymptotes.*
3. *The asymptotes are the images via f^* of the tangents $\{\tau_J, \tau_{J'}\}$ to κ at $\{J, J'\}$.*
4. *These tangents intersect at $I' = f^*(I)$, point I being the center of the hyperbola.*
5. *The angle ω of the asymptotes, not containing any branch, is the same with the angle of the tangents $\{\tau_J, \tau_{J'}\}$ and this angle, depending on the position of λ relative to κ, satisfies $0 < \omega < \pi/3$.*
6. *The line η is a directrix of the hyperbola and point O is the corresponding focal point.*
7. *The eccentricity of κ' is $CO/OS = 2r_0/q$, where r_0 the radius of κ and $q = OQ$.*

Remark 7. The symmetry of the orthogonality relation implies that there is also another hyperbola λ', resulting by applying the preceding discussion after interchanging the roles of $\{\kappa, \lambda\}$. Figure 35 illustrates the case showing the two hyperbolas together with their asymptotes and $\{\eta, \zeta\}$ and the corresponding focal points $\{O, P\}$. An easy calculation using theorem 20 shows that the eccentricities $\{e, e'\}$ of the two hyperbolas satisfy the relation

$$e \cdot e' = \frac{8}{\sin(2\widehat{ROP})} = \frac{8}{\sin(2\widehat{RPO})}.$$

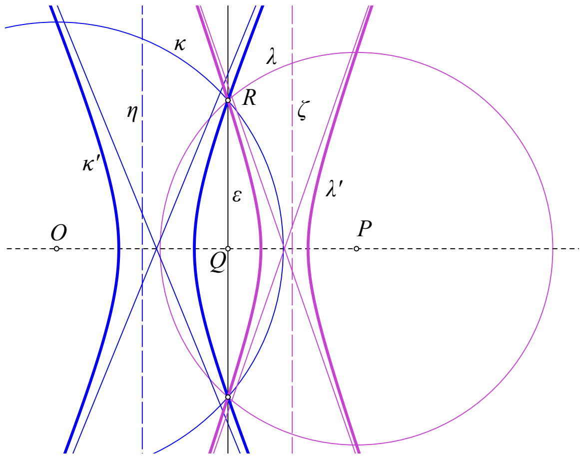

Figure 35: The two hyperbolas naturally related to two orthogonal circles

Remark 8. The restriction on the angle $\omega\ :\ 0<\omega<\pi/3$ stems from the property of the circle κ to be defined by its center O, which is the focal point of the hyperbola and the point J where the corresponding directrix η intersects the parallel OJ to the asymptote (See Figure 36). The triangle OJQ is isosceles and $OQ<OJ$. For $\omega>\pi/3$ the circle

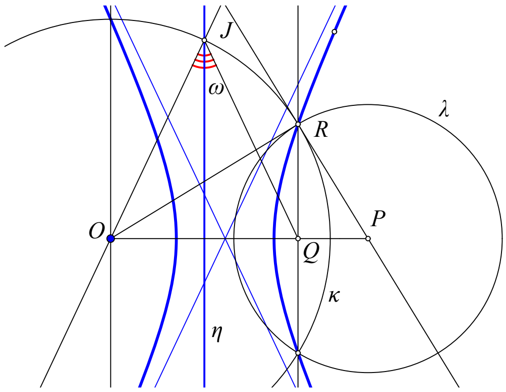

Figure 36: The restriction on the angle ω

$\kappa(O,|OJ|)$ does not intersect the hyperbola in four points as required. The figure shows also the recipe to construct the two orthogonal circles generating the hyperbola by the method described above, provided its asymptotes form an angle $\omega<\pi/3$ in the domain not containing a branch of it.

14 Compositions of two inversions and homographies

Given two circles $\{\alpha(A),\beta(B)\}$, each of the corresponding inversions $\{f_1,f_2\}$ relative to these circles leaves invariant every circle κ orthogonal to both, hence also the composition $f=f_2\circ f_1$ does the same and for each $X\in\kappa$ creates a triangle with vertices $\{X,Y=f(X),Z=f(Z)\}$ inscribed in κ. In section 13 we saw that each of the two inversions $\{f_1,f_2\}$ coincide with *"perspectivities"* with centers respectively $\{A,B\}$ and axes the radical axes $\{\eta_1,\eta_2\}$ of the pairs of circles $\{(\alpha,\kappa),(\beta,\kappa)\}$. The composition f^* of these perspectivities coincides on the points of κ with the composition f of the two inversions and defines a *"homography"* preserving the circle κ. Figure 37 shows the line XZ joining X to its image $Z=f(X)=f^*(X)$. By a well known theorem (see file **Abridged notation**),

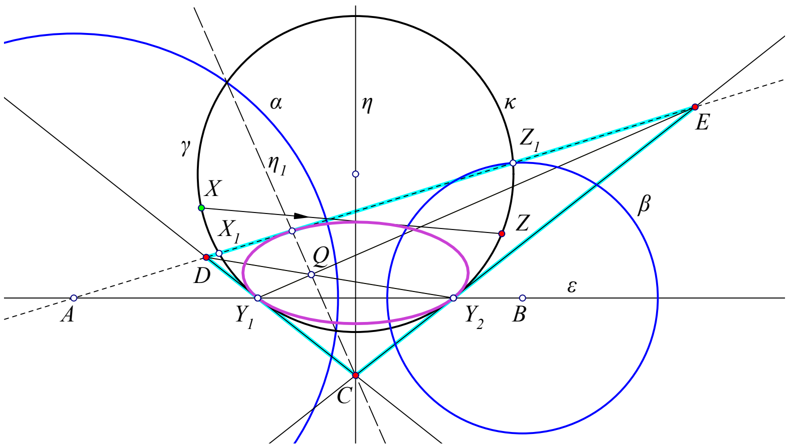

Figure 37: Composition of two inversions $X \mapsto Z$

we know that these lines, for variable $X \in \kappa$, envelope a conic. The figure shows this conic (ellipse) and some other characteristics of the resulting configuration listed in the form of a theorem.

2 inversions and homographies

Theorem 21. *With the preceding notation and definitions and with $\mathcal{P}$ denoting the pencil of circles orthogonal to $\{\alpha, \beta\}$, the following are valid properties*

1. *The composition $f = f_2 \circ f_1$ of the inversions defines a transformation leaving invariant each member-circle $\kappa \in \mathcal{P}$.*
2. *The transformation f restricted on each member κ of the pencil $\mathcal{P}$ defines a homography f^*, whose homography axis is the line AB and whose isolated fixed point is the pole C of AB relative to the circle κ.*
3. *The points $\{X, Z = f(X)\}$ define for $X \in \kappa$ lines XZ enveloping a conic κ' belonging to the (bitangent) family of conics generated by the circle κ and the line AB.*
4. *The conic κ' is invariant w.r. to the harmonic perspectivity with axis AB and center C.*

Notice that the above procedure can be reversed. Given a triangle like CDE and an inscribed conic like κ', such that $|CY_1| = |CY_2|$, one can reconstruct the above figure as follows:

1. Define the circle circle κ tangent at $\{CD, CE\}$ respectively at $\{Y_1, Y_2\}$.
2. Define the circle α centered at the intersection point $A = Y_1Y_2 \cap DE$ and orthogonal to κ.
3. Define the circle β by the requirement to be orthogonal to κ with center B on Y_1Y_2 and passing through Z_1 inverse of X_1 relative to circle α.

The procedure shows that each conic inscribed in a triangle with the particular condition $|Y_1C| = |Y_2C|$ for some vertex C of the triangle and the corresponding tangent adjacent sides, can be produced by two inversions and the resulting envelope of lines $\{XZ\}$.

15 Anti-inversion

antiinversions

A variation of the inversion is the so called *"anti-inversion"*. It is defined, like the inversion on a circle $\kappa(O, r)$ and corresponds to every point $A \neq O$ of the plane a point B on the same

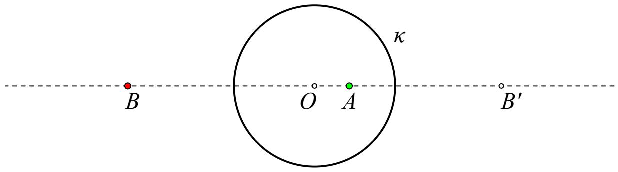

Figure 38: Anti-inversion relative to κ

line OA, such that $|OA||OB| = r^2$ and such that OA, OB are oppositely oriented (See Figure 38). This, using oriented segments, could be expressed by the condition

$$OA \cdot OB = -r^2,$$

and it is easy to see that B is the symmetric, relative to O, of the inverse B' of A relative to the circle κ. The circle κ is called circle of anti-inversion, its center O center of the anti-inversion and its radius r radius of the anti-inversion.

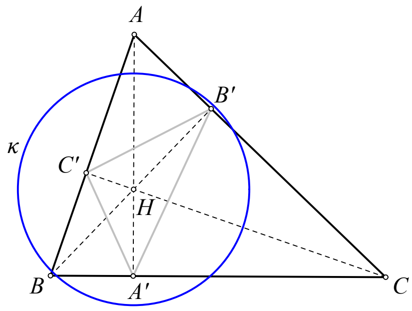

Figure 39: The polar circle κ of an acute-angled triangle

A prominent example commes from the altitudes $\{AA', BB', CC'\}$ of an acute-angled triangle ABC (See Figure 39), which intersect at its *"orthocenter"* H and define oriented segments such that

$$HA \cdot HA' = HB \cdot HB' = HC \cdot HC' = -r^2 = \frac{1}{2}(a^2 + b^2 + c^2) - 4R^2,$$

where $\{a, b, c\}$ are the side-lengths and R the circumradius of the triangle. The circle κ with center H and radius r is called the *"polar circle of the triangle"* (see file **Autopolar triangles**).

Next exercises show, that several (but not all) properties of inversions carry over with slight modifications to anti-inversions. The effort to solve them gives an opportunity to review the theory of inversion. A key-fact for their solution is that the anti-inversion f_κ relative to the circle $\kappa(O)$ is a composition $f_\kappa = g_\kappa \circ s_O = s_O \circ g_\kappa$ of the inversion g_κ relative to κ and the symmetry s_O relative to O.

Exercise 18. *Show that an anti-inversion maps the internal points of the circle to its external points and vice-versa. Also every point of the circle is mapped to its diametrically opposite. Conclude that the anti-inversion admits no fixed points.*

Exercise 19. *If the point X varies on a line ε, then its anti-inverse point Y relative to a circle $\kappa(O, r)$ varies on a circle ζ passing through the center O of κ. If the point Y varies on the circle ζ passing through point O, then its anti-inverse X relative to κ, varies on a line ε.*

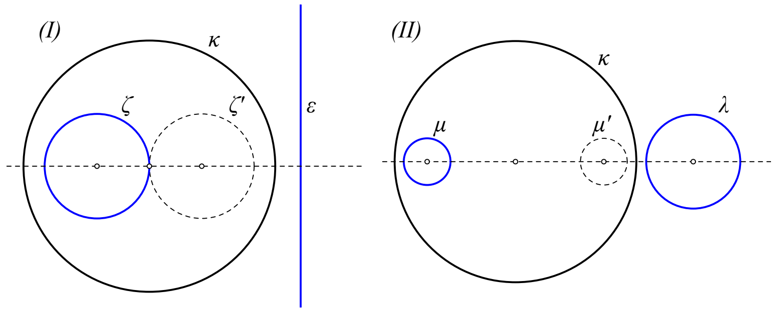

Figure 40: Comparison of inverse- anti-inverse images

Exercise 20. *If the point X varies on a circle λ not passing through O, then its anti-inverse Y relative to the circle $\kappa(O,r)$ varies on a circle μ.*

Figure 40 shows the difference between the *inverse* and *anti-inverse* of a line and a circle relative to circle κ. The two images are symmetric relative to the center of κ.

Exercise 21. *Show that for every pair of anti-inverse points $\{A,B\}$ relative to the circle κ and every circle λ passing through them, circles κ and λ intersect by a diameter of κ and λ remains invariant relative to the κ–anti-inversion. The converse is also true and a circle λ intersecting κ by a diameter of κ remains invariant by the κ–anti-inversion.*

Exercise 22. *Two different pairs of anti-inverse points (X,Y) and (X',Y') relative to the circle κ define four concyclic points of a circle λ intersecting κ by a diameter of the latter.*

Exercise 23. *Let Y be the anti-inverse relative to the circle κ of the point X. If the circles λ and μ pass through X and intesect under the angle ω, then their anti-inverses λ', μ' pass through point Y and intersect under the same angle ω.*

This means that, like the inversions, also the *"anti-inversions are conformal transformations"*.

Exercise 24. *For a given circle pencil $\mathcal{D}$ and a circle κ, the anti-inverses relative to κ, of the circles λ of the pencil, build another pencil of circles $\mathcal{D}'$ of the same type with $\mathcal{D}$.*

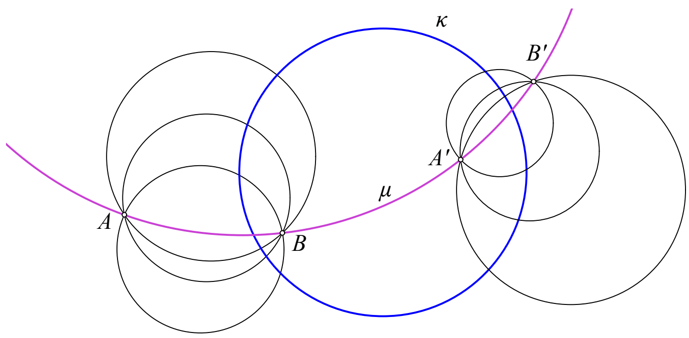

Figure 41: Common circle μ of intersecting pencil $\mathcal{D}$ and anti-inversed $\mathcal{D}'$

Exercise 25. *For a given intersecting pencil of circles $\mathcal{D}$ and a circle κ, show that the pencil $\mathcal{D}'$ of the anti-inverses relative to κ, of the circles λ of the pencil and the original pencil $\mathcal{D}$ have exactly one circle-member μ in common. Circle μ is characterized by the fact that it passes through the basic points of $\mathcal{D}$ as well as through the basic points of $\mathcal{D}'$ (See Figure* 41*).*

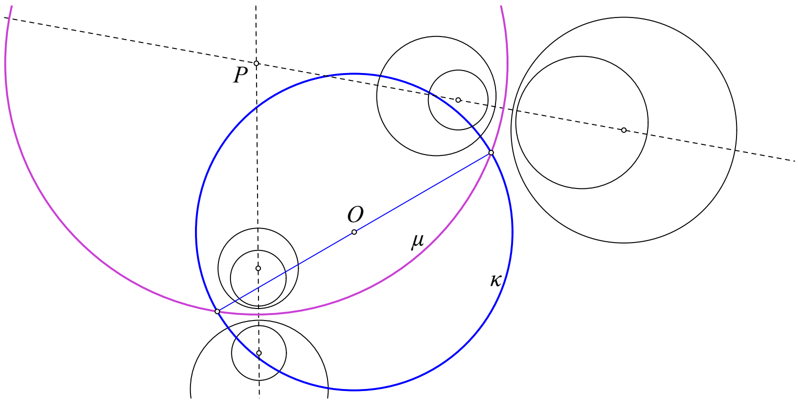

Figure 42: Common circle μ of pencil $\mathcal{D}$ and anti-inversed $\mathcal{D}'$

Exercise 26. *For a given non intersecting pencil of circles $\mathcal{D}$ and a circle κ, show that the pencil $\mathcal{D}'$ of the anti-inverses relative to κ, of the circles λ of the pencil and the original pencil $\mathcal{D}$ have exactly one circle-member μ in common. μ is characterized by the fact that it intersects κ by a diameter of κ.*

Hint: The circle μ has its center at the intersection P of the *lines of centers* of the two pencils (See Figure 42).

Exercise 27. *For a given tangential pencil of circles $\mathcal{D}$ and a circle κ, show that the pencil $\mathcal{D}'$ of the anti-inverses relative to κ, of the circles λ of the pencil and the original pencil $\mathcal{D}$ have exactly one circle-member μ common. What characterizes μ in this case?*

Exercise 28. *Given a circle κ and a line ε, show that there exists a circle ω, relative to which the anti-inversion inerchanges κ with ε.*

Exercise 29. *Show that there is no anti-inversion interchanging two intersecting circles κ and λ.*

Hint: If $\{\kappa, \lambda\}$ are tangent, then the anti-inversion should leave the point of contact fixed, something impossible. If $\{\kappa, \lambda\}$ intersect at two different points, then the anti-inversion should interchange them, therefore the circles should be invariant and cannot map one onto the other.

Exercise 30. *Given two non congruent and non intersecting circles κ and λ, show that there exists exactly one circle ν, relative to which the anti-inversion interchanges κ and λ (See Figure 43). The center of ν is the internal point of similarity D of κ, λ, the radius r of ν satisfies $r^2 = \alpha\beta\left(\frac{|AB|^2}{(\alpha+\beta)^2} - 1\right)$ and the circle of inversion μ of the theorem 14, which, this one as well, interchanges $\{\kappa, \lambda\}$, intersects the circle ν along a diameter of ν.*

Two circles like $\{\kappa, \lambda\}$, as well as the related to them circles $\{\mu, \nu\}$, defined in the previous exercise, make an interesting system with many applications. The next exercise lists the basic properties of this system.

Exercise 31. *Given two non congruent and non intersecting circles $\{\kappa(A, \alpha), \lambda(B, \beta)\}$ and the circles $\{\mu, \nu\}$ defined in the previous exercise (See Figure 43), show that:*

1. *Two pairs of "antihomologous" points $\{(X, X'), (Y, Y')\}$ whose lines $\{XX', YY'\}$ pass through the exterior (resp. interior) center of similarity C (resp. D), are contained in a circle τ (resp. σ), which intersects orthogonally (resp. by diameter of ν) the circle μ (resp. ν).*
2. *Circle τ (resp. σ) is invariant with respect to the inversion (resp. anti-inversion) relative to μ (resp. relative to ν).*

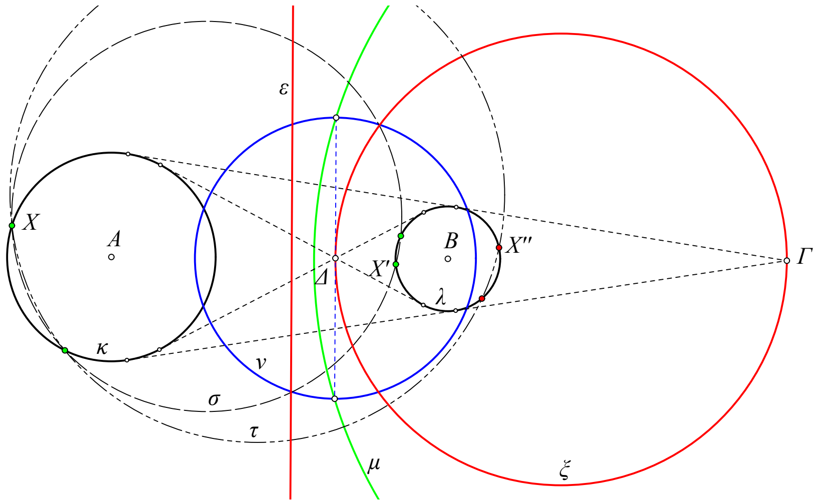

Figure 43: The system of two non intersecting circles

3. *Conversely, every circle τ (resp. σ) which intersects circles $\{\kappa, \lambda\}$ and is invariant with respect to the inversion relative to μ (resp. with respect to the anti-inversion relative to ν) intersects $\{\kappa, \lambda\}$ by pairs of antihomologous points $\{(X, X'), (Y, Y')\}$, whose lines $\{XX', YY'\}$ pass through point C (resp. D).*
4. *A circle tangent to $\{\kappa, \lambda\}$ will be invariant with respect to the inversion relative to μ or it will be invariant with respect to the anti-inversion relative to ν. In the first case the line of the contact points with $\{\kappa, \lambda\}$ will pass through C. In the second case this line will pass through point C.*
5. *The inverse (resp. anti-inverse) of the circle of similitude ξ of $\{\kappa, \lambda\}$ relative to μ (resp. ν), is the radical axis of $\{\kappa, \lambda\}$.*

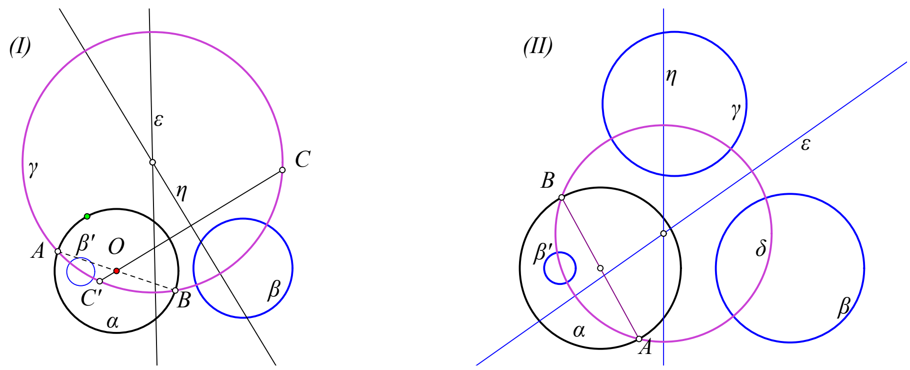

Figure 44: Intersection along diameter ... and orthogonal to β

Exercise 32. *Given are two circles $\{\alpha, \beta\}$ outside each other and the point C outside both. To construct a circle γ intersecting circle α along a diameter AB and orthogonal to β.*

Hint: Consider the anti-inversion f relative to α and the anti-inverse $\beta' = f(\beta)$, as well as the anti-inverse $C' = f(C)$ of the given point (See Figure 44-I). The requested circle γ maps under f to itself, is orthogonal to $\{\beta, \beta'\}$ und passes through points $\{C, C'\}$. It follows that its center must be on the medial line η of CC' and also on the radical axis ε of the two circles $\{\beta, \beta'\}$.

Exercise 33. *Given are the circles $\{\alpha, \beta, \gamma\}$ lying outside each other. To draw a circle δ intersecting α along a diameter AB and orthogonal to the circles $\{\beta, \gamma\}$.*

Hint: Consider the anti-inversion f relative to α and the anti-inverse $\beta' = f(\beta)$ (See Figure 44-II). The requested γ maps under f to itself and is orthogonal to $\{\beta, \beta', \gamma\}$. Consequently its center coincides with the radical center of the three last circles.

16 Three pairwise orthogonal circles

Here we examine the composition of three inversions on three pairwise orthogonal circles, starting with some easy exercises.

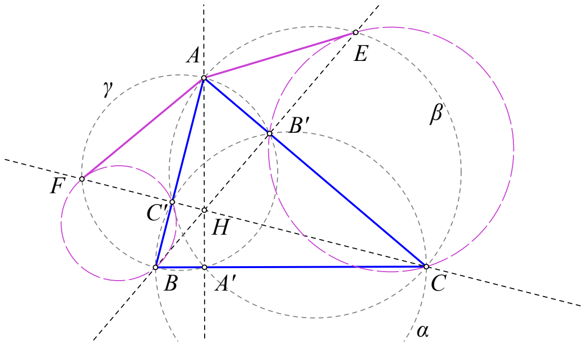

Figure 45: Equal segments $AE = AF$

3 pairwise orthogonals

Exercise 34. *Draw the circles $\{\alpha, \beta, \gamma\}$ with diameters the sides of the triangle ABC intersecting the altitudes correspondingly at the points $\{D, E, F\}$ (See Figure 45). Show the equalities $\{AE = AF, BF = BD, CD = CE\}$.*

Hint: $\{AE, AF\}$ are respectively tangents to the circle $\{(EB'C), (AC'F)\}$ implying the equalities $AE^2 = AB' \cdot AC = AC' \cdot AB = AF^2$ etc...

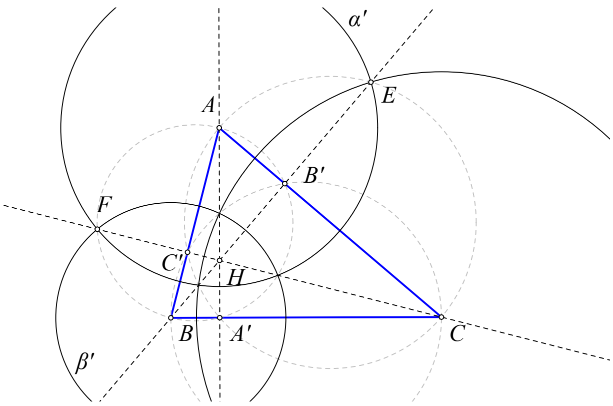

Figure 46: Three paiwise orthogonal circles $\{\alpha', \beta', \gamma'\}$

Exercise 35. *Show that there are precisely three circles $\{\alpha', \beta', \gamma'\}$ pairwise orthogonal and centered at the vertices of the triangle ABC and construct them (See Figure 46). Show that their radical center is the orthocenter of the triangle.*

Hint: Use the previous exercise.

Exercise 36. *Show the converse to the previous exercise, i.e. that three circles* $\{\alpha', \beta', \gamma'\}$ *which are pairwise orthogonal define by their centers the vertices of a triangle* ABC *with the properties of the previous exercise.*

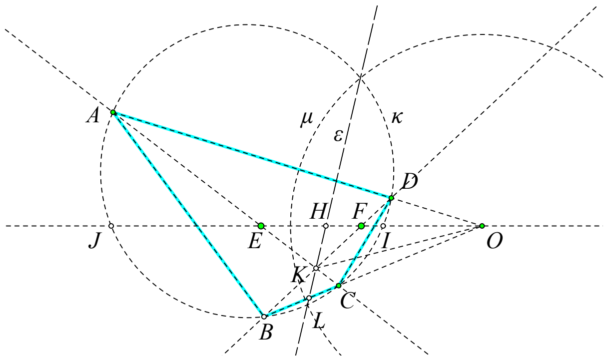

Figure 47: Diagonals passing through fixed points

Exercise 37. *Let* $\{\kappa, \mu(O)\}$ *be two orthogonal circles and* AC *a variable chord of* κ *passing through the fixed point* E. *Let also* $\{B, D\}$ *be the second intersections of lines* $\{OC, OA\}$ *with* κ. *Show that line* BD *passes through a fixed point* F *of line* EO.

Hint: See first that the diagonals of the quadrangle BD and AC intersect at a point K of the radical axis ε of $\{\kappa, \mu\}$ which is also the *"polar"* of O relative to κ. This implies that (B, C) are harmonic conjugate to (L, O) : $(BCLO) = -1$, where $L = BC \cap \varepsilon$. Hence (F, E) are also harmonic conjugate to (H, O) : $(FEHO) = -1$. Since $\{E, H, O\}$ are fixed this relation implies that also F has a fixed position.

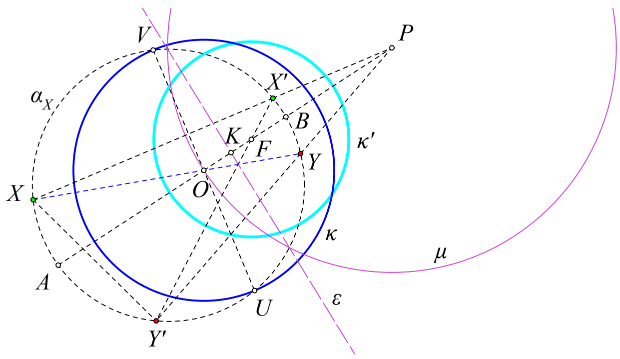

Figure 48: Anti-inversion inverted to anti-inversion

Theorem 22. *Consider the anti-inversion* f *relative to the circle* $\kappa(O, r)$ *and the inversion* g *relative to the circle* $\mu(P, R)$ *passing through* O. *Then two anti-inverse points* $\{X, Y = f(X)\}$ *invert to* $\{X' = g(X), Y' = g(Y)\}$, *which are anti-inverse relative to a circle* $\kappa'(F)$. *The circles* $\{\kappa, \kappa'\}$ *have point* P *as similarity center.*

Proof. See first that $XX'Y'Y$ is a cyclic quadrangle whose circumcircle α_X is orthogonal to μ (See Figure 48). This circle is invariant not only relative to μ but also relative to the κ−anti-inversion. In fact, α_X intersects κ in two diametral points $\{U, V\}$ of κ.

This is seen by watching a point X moving on α_X and taking the other intersection point Y of OX with α_X. If $OX = -OY$ then $\{X, Y\}$ are points of κ and we are done. If

$OX \neq -OY$ and $|OX| < |OY|$ say, then moving X on κ to the position of Y we'll have $|OX| > |OY|$ hence, by continuity, for some intermediate position we'll have $|OX| = |OY|$ implying that $\{X, Y\}$ are diametral points of κ.

Let now $\{A, B\}$ the intersections of OP with α_X. These two points satisfy two relations

$$OA \cdot OB = -r^2 \quad \text{and} \quad PA \cdot PB = R^2,$$

Hence their position is fixed and independent of the particular X. This implies that the radical axis ε of μ, α_X passes through a fixed point $K \in OP$, characterized by the harmonicity relation $(ABKP) = -1$. Applying then exercise 37 we see that the diagonal $X'Y'$ of the quadrangle $XY'YX'$ passes through a fixed point $F \in OP$ satisfying

$$FX' \cdot FY' = FA \cdot FB = -k^2,$$

for a constant k. This shows the first claim and a simple calculation of the constant k proves also the second claim. □

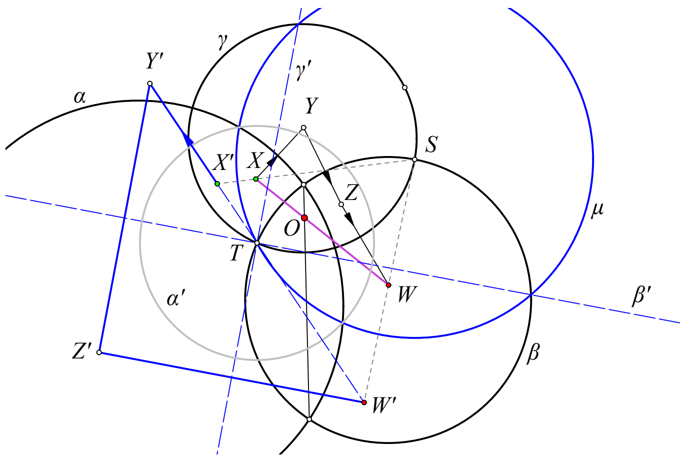

Figure 49: Composition of inversions on three pairwise orthogonal circles

Theorem 23. *The composition $f = f_3 \circ f_2 \circ f_1$ of three inversions respectively on the pairwise orthogonal circles α, β, γ is an anti-inversion relative to a circle with center at the radical center O of the three circles.*

Proof. We prove this by inverting the related figure w.r. to a circle μ with center S one of the intersection points of two of the circles, $\{\beta, \gamma\}$ say (See Figure 49). The radius of μ can be taken arbitrarily and for definiteness we take it to be $|TS|$, where T is the other intersection point of the two selected circles. The inversion g relative to μ maps the circles $\{\beta, \gamma\}$ to two orthogonal lines $\{\beta', \gamma'\}$ through T and the third circle α to a circle α' orthogonal to these lines, hence having its center at T.

Taking conjugates by g we have:

$$h = g \circ f \circ g = (g \circ f_3 \circ g) \circ (g \circ f_2 \circ g) \circ (g \circ f_1 \circ g) = h_3 \circ h_2 \circ h_1$$

and each of the $\{h_i = g \circ f_i \circ g\}$ is an inversion respectively relative $\{\alpha', \beta', \gamma'\}$ (section 9). Since $\{\alpha', \beta'\}$ are two orthogonal lines intersecting at T, the composition $s_T = h_3 \circ h_2$ is the symmetry at point T, hence $h = s_T \circ h_1$ is the anti-inversion on circle α'. Application of theorem 22 implies then that $f = g \circ h \circ g$ is an anti-inversion. According to this theorem the center of the anti-invesion must lie on ST and it is easily checked that f interchanges the intersection points of the other pairs $\alpha \cap \beta$ and $\alpha \cap \gamma$, thereby proving the claim. □

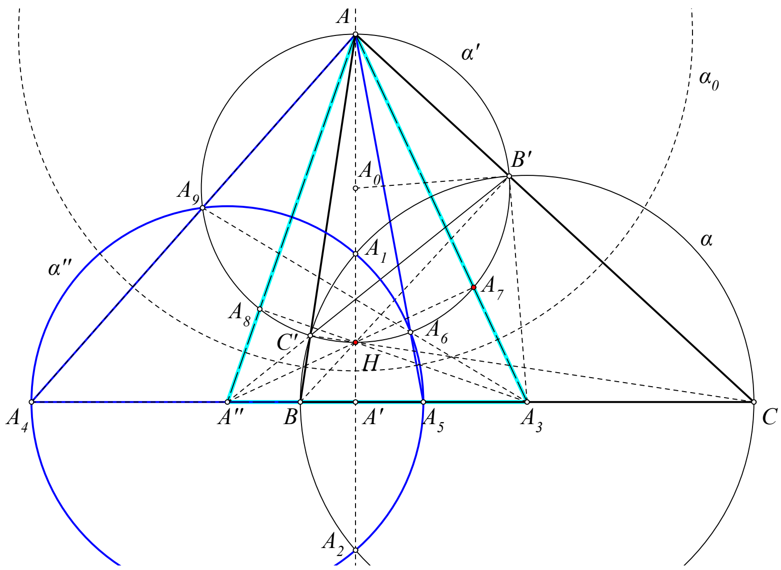

Figure 50: Three mutually orthogonal circles

Exercise 38. *Let $\{AA', BB', CC'\}$ be the altitudes of the triangle ABC intersecting at its "orthocenter" H (See Figure 50). Show the following properties:*

1. *The circles $\{\alpha, \alpha'\}$ with diameters respectively $\{BC, AH\}$ intersect orthogonally at the feet $\{B', C'\}$ of the altitudes.*
2. *If $A'' = BC \cap B'C'$, then the circle α'' centered at A'' and orthogonal to $\{\alpha, \alpha'\}$ intersects these circles in the pairs $\{(A_1, A_2), (A_6, A_9)\}$ the first lying on the altitude AA' and the other two defining a line through the middle A_3 of BC.*
3. *The inversion f_0 on the circle α_0 centered at A and orthogonal to $\{\alpha, \alpha''\}$ maps the line BC onto the circle α'.*
4. *The points $\{A', A_3\}$ are inverse relative to α''.*
5. *If $\{A_7, A_8\}$ are the f_0−inverses of $\{A_3, A''\}$, then $\{(A'', H, A_7), (A_3, H, A_8)\}$ are triples of collinear points.*
6. *The triangle AA_4A_5 is related to the other two $\{ABC, AA''A_3\}$ in the same way ABC is related to $\{AA_4A_5, AA''A_3\}$. The three triangles have AA' as a common altitude and H as common orthocenter.*
7. *Lines $\{A''H, A_3H\}$ are the radical axes respectively of the circle pairs (α, α_0) and (α'', α_0).*

Hint: Nr-1. A_3 and the middle A_0 of AH are diametral points of the Euler circle and $\{A_0B', B'A_3\}$ are orthogonal at B'.

Nr-2. The centers of a circle orthogonal to two other circles is on the radical axis of these two.

Nr-3. Since BC maps to a cirlce through A and $\{B', C'\}$.

Nr-4. Since AA' is the polar of A_3 relative to α''.

Nr-5. $\tau = A_7A_3A'H$ is a cyclic quadrangle, since $\{(A_3, A_7), (A', H)\}$ are pairs of f_0−inverses. Since $\{A', A_3\}$ are α''−inverses the circumcircle of τ is also orthogonal to α'' hence the collinearity of $\{A'', H, A_7\}$. Analogously is proved the collinearity of the other triple. Latter can be also proved by noticing that by the previous argument $\{A''A_7, AA'\}$ are two altitudes of the triangle $AA''A_3$, consequently the third altitude will pass also through their intersection H.

Nr-6. Follows from the previous *nr*.

Nr-7. Apply exercise 35 or prove directly by showing that A'' is on the radical axis of $\{\alpha, \alpha_0\}$, which amounts to the fact that $\{A', A''\}$ are $\alpha-$inverses. This implies that H is the *"radical center"* of the three circles $\{\alpha, \alpha'', \alpha_0\}$, hence A_3H being orthogonal to AA'' is the radical axis of $\{\alpha'', \alpha_0\}$.

17 Inversions and complex numbers

Here I follow Schwerdtfeger [Sch79], to discuss some relations of inversions with the field of complex nubmers $\mathbb{C}$. The equation of the circle $\kappa(K, r)$ in cartesian coordinates

$$|X-K|^2 \;=\; r^2 \quad \text{expands to} \quad x^2+y^2-2k_1x-2k_2y+k_1^2+k_2^2 \;=\; r^2$$

This, representing the points with complex numbers $\{z = x+iy, k = k_1+ik_2\}$, can be written

$$z\cdot\overline{z} - k\cdot\overline{z} - \overline{k}\cdot z + k\cdot\overline{k} - r^2 = 0.$$

The inversion relative to κ leads to an expression with coefficients depending on $\{k, r\}$:

$$z' \;=\; k + \frac{r^2}{|z-k|^2}(z-k) \;=\; k + \frac{r^2}{\overline{z}-\overline{k}} \qquad \Leftrightarrow \qquad z' \;=\; \frac{k\overline{z} + (r^2 - k\cdot\overline{k})}{\overline{z}-\overline{k}}$$

One can describe the reflections on lines with complexes, as a particular kind of inversions, with a similar expression. For this start with the expression of a line with complexes

$$k\cdot\overline{z} + \overline{k}\cdot z - n \;=\; 0,$$

and notice that the corresponding reflection expressed in complexes ([Dea56, p.29]) is given by

$$z' \;=\; \frac{k\cdot\overline{z} - n}{-\overline{k}}.$$

Thus, in both cases, the inversion on a circle or a line (reflection) can be expressed through

$$z' \;=\; f(z) \;=\; \frac{k\cdot\overline{z} - n}{m\cdot\overline{z} - \overline{k}}\;, \tag{6}$$

where the *real* numbers $\{m = 1,\; n = k\cdot\overline{k} - r^2\}$ in the case of the circle and $m = 0$ in the case of the line. The function f satisfies, as expected, $f^2 = e$ (e: the identity transformation) and this, using matrices, is equivalent to

$$M\cdot\overline{M} \;=\; \begin{pmatrix} k & -n \\ m & -\overline{k} \end{pmatrix}\cdot\begin{pmatrix} \overline{k} & -n \\ m & -k \end{pmatrix} \;=\; (k\cdot\overline{k} - m\cdot n)\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

circle matrices

Complex 2×2 matrices M satisfying the more general equation

$$M\cdot\overline{M} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \;=\; \mu\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad \mu \neq 0, \tag{8}$$

are called *"circle-matrices"*. It can be shown ([Sch79, p.80]) that these matrices are characterized by a simple operation that transforms them to *"hermitian matrices"* with non-zero determinant. The operation is a change of signs in one row and interchange of the two rows:

$$M \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\mapsto\; \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \;=\; M_1\,.$$

The *"hermitian"* property of M_1 means, by definition, that *"the transposed matrix equals the conjugate"*: $M_1^t = \overline{M}_1$, which in our case means that $\{b, c\}$ are real and $\{a, -d\}$ are conjugate. In the aforementioned reference it is also shown that , *"every circle-matrix represents an inversion"*, as well as, the fact that the *"group"* of all transformations described by *invertible* complex 2×2 matrices and corresponding relations of the form

define inversions

$$\left\{ z' = \frac{az + b}{cz + d} \quad \text{or} \quad z' = \frac{a\overline{z} + b}{c\overline{z} + d} \right\},$$

are generated by a product (composition) of at most 4 inversions. The first kind (on the left), called *"Moebius transformations"* requiring two or four inversions, and the second (on the right) requiring one or three inversions.

group generated by inversions

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