Menelaus' theorem

by Paris Pamfilos

Most of the best work starts in hopeless muddle and floundering, sustained on the "smell" that something is there.

Littlewood's Miscellany Academic Life p. 144

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1 Menelaus' theorem

Menelaus theorem deals with *"signed ratios"* of segments, which are properly defined in *"affine geometry"* (see file **Affine geometry**). In euclidean geometry the theorem has the following formulation.

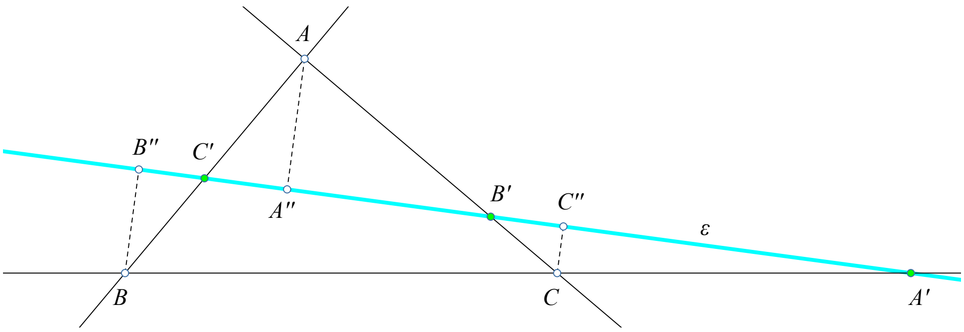

Figure 1: Menelaus' theorem

Theorem 1. *The points $\{A' \in BC, B' \in CA, C' \in AB\}$ on the sides of the triangle ABC are on a line, if and only if the following condition holds.*

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1. \tag{1}$$

The necessity of the condition results by projecting the vertices on the line ε and using the similar triangles (See Figure 1):

$$\frac{A'B}{A'C} = \frac{BB''}{CC''}, \qquad \frac{B'C}{B'A} = \frac{CC''}{AA''}, \qquad \frac{C'A}{C'B} = \frac{AA''}{BB''}. \tag{2}$$

The result follows by multiplying the sides of the equations and simplifying.

The sufficiency can be proved using the proved part of the necessity. In fact, assuming that the condition of the theorem is valid for the three points $\{A', B', C'\}$ and defining the line $\varepsilon = B'C'$ and its intersection $A'' = \varepsilon \cap BC$, we show that $A'' = A'$. This is so because by the first part of the theorem

$$\frac{A''B}{A''C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1 \quad \text{and by assumption} \quad \frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1. \quad \Rightarrow \quad A'' = A. \tag{3}$$

2 Menelaus applications

Consider a parallelogram $EFGH$ and two points $\{I, J\}$ on two opposite sides. The intersection points K, L of the two triangles $\{HJG, EIF\}$ define a line $\varepsilon = KL$, passing through the center O of the parallelogram ([Ant95, p.77], [Pap96, II,p.62]).

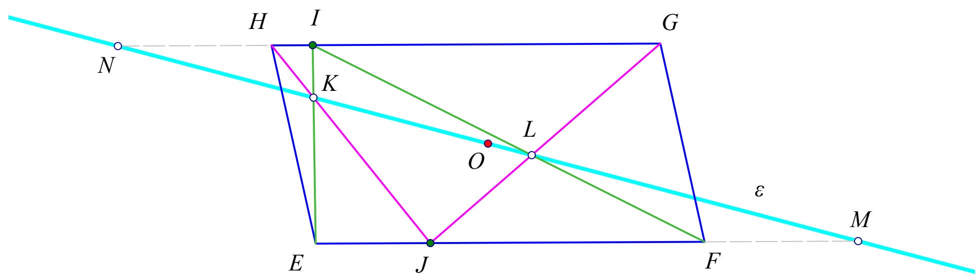

Figure 2: Menelaus' theorem application

To see this Apply Menelaus' theorem two times for the line ε and the two triangles :

$$EIF \quad \Rightarrow \quad \frac{MF}{ME} \cdot \frac{KE}{KI} \cdot \frac{LI}{LF} = 1, \qquad HJG \quad \Rightarrow \quad \frac{NH}{NG} \cdot \frac{LG}{LJ} \cdot \frac{KJ}{KH} = 1. \tag{4}$$

Divide the sides of the equations and note that

$$\frac{KE}{KI} = \frac{KJ}{KH}, \qquad \frac{LI}{LF} = \frac{LG}{LJ} \qquad \Rightarrow \qquad \frac{MF}{ME} = \frac{NH}{NG}. \tag{5}$$

Later implies that $MF = NH$. Hence line ε passes through the center O of the parallelogram.

Remark-1 Essentially the exercise is a special case of the famous *"theorem of Pappus"* on the collinearity of three intersection points (see file **Pappus' theorem**). The general case of this theorem reduces to the present particular case by using a special *"projectivity"* which sends two intersection points to the points at infinity corresponding to the parallel sides of the present parallelogram.
In the next exercise we consider a parallelogram $ABCD$ and an arbitrary point E. Then, we draw from E parallels to the sides to build parallelograms $\{AHEI, FCGE\}$ (See Figure 3). Then, the diagonals $\{HI, FG, BD\}$ of these parallelograms and the original one intersect at a point J.

To see this consider point J as the intersection of the two lines $J = BD \cap FG$ and apply Menelaus' theorem to triangle BCD and its secant FG :

$$\frac{GD}{GC} \cdot \frac{FC}{FB} \cdot \frac{JB}{JD} = 1, \quad \text{noticing} \quad \frac{GD}{GC} = \frac{HA}{HB}, \quad \frac{FC}{FB} = \frac{ID}{IA} \quad \Rightarrow \quad \frac{ID}{IA} \cdot \frac{HA}{HB} \cdot \frac{JB}{JD} = 1.$$

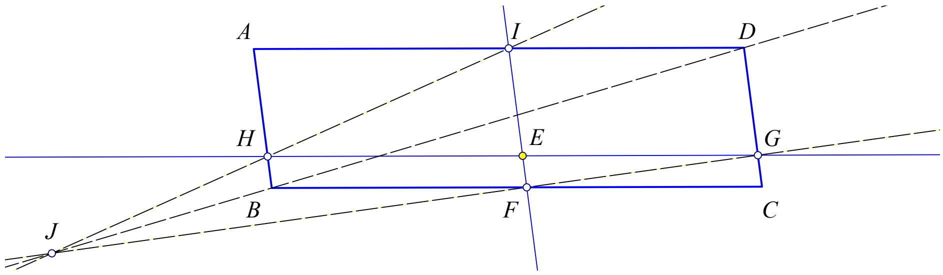

Figure 3: Menelaus' theorem application

This, by applying the Menelaus theorem to triangle ABD, implies that HI passes through J too.

Next exercise formulates a generalization of the previous property of parallelograms. In fact, consider an arbitrary quadrangle $q = ABCD$ and an arbitrary point E. Join point E to two intersections $\{K, L\}$ of opposite sides of q and consider also the quadrilateral $q' = FGIH$, defined by the intersections of lines $\{EK, EL\}$ with the sides of the original quadrilateral (See Figure 4). Then, the pairs of opposite sides of q' intersect on the diagonals of q.

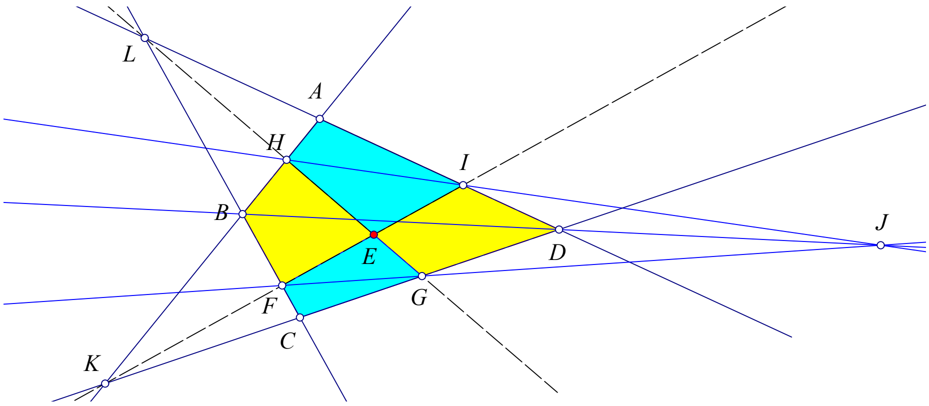

Figure 4: Menelaus' theorem application

Figure 4 shows only one pair (HI, FG) of opposite sides of q', intersecting at J, the other pair being (HF, GI). The property follows by transforming the quadrilateral to a parallelogram via a *"projectivity"* and applying the results of the previous exercise.

3 Menelaus projective

The condition of Menelaus' theorem can be generalized to cross ratios (see file **Cross ratio**) by introducing an arbitrary line ε, which together with the other four lines forms a set in general position, i.e. such that no three lines of the set pass through the same point. Figure 5 shows such a case. The condition of Menelaus is then equivalent to

$$(ABDJ) \cdot (BCFI) \cdot (CAEG) = 1.$$

The proof ([Gre54, p.354]) can be given as in the projective case of Ceva's theorem (see file **Ceva's theorem**). For this, define the *"projective transformation"* which maps the side-lines of the triangle, each to itself and the line ε to the line at infinity ε_∞. Then the cross ratios are preserved and their values are transferred to corresponding equal *"ratios"* in the case

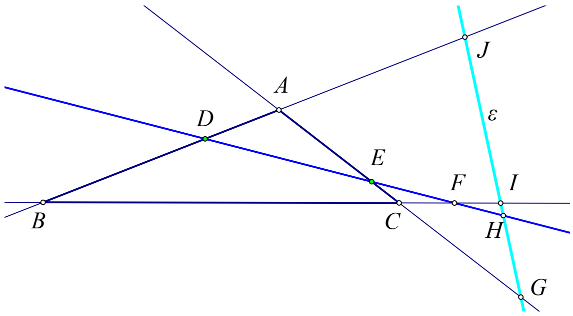

Figure 5: Menelaus' theorem, projective version

of line at infinity, e.g. $(ABDJ)$ becomes DA/DB when J is at infinity. The theorem then reduces to Menelaus theorem in its usual form.

4 Menelaus from Ceva

Here we prove this theorem by deducing it from Ceva's theorem discussed in the file **Ceva's theorem**. Since in that file we proved Ceva's theorem by deducing it from Menelaus, the two theorems are equivalent. The proof, originally given by Silvester ([Sil00]), may

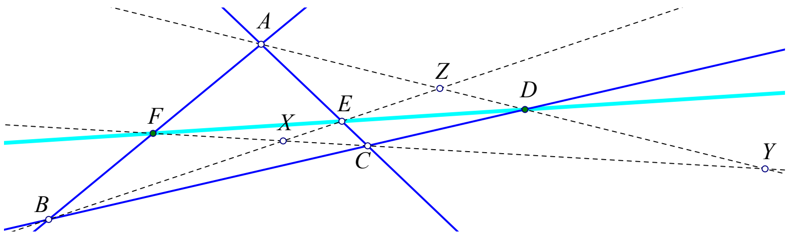

Figure 6: Menelaus from Ceva

proceed as follows: Apply Ceva's theorem to triangles and respective *"Cevians"*:

$$\begin{array}{llll} BCE & \text{and lines through} & F: BA, CX, ED & \Rightarrow & (AC/AE)(XE/XB)(DB/DC) = 1, \\ CAF & \text{and lines through} & D: CB, AY, FE & \Rightarrow & (BA/BF)(YF/YC)(EC/EA) = 1, \\ ABD & \text{and lines through} & E: AC, BZ, DF & \Rightarrow & (CB/CD)(ZD/ZA)(FA/FB) = 1, \\ BEF & \text{and lines through} & C: BD, EA, FX & \Rightarrow & (DE/DF)(AF/AB)(XB/XE) = 1, \\ CFD & \text{and lines through} & A: CE, FB, DY & \Rightarrow & (EF/ED)(BD/BC)(YC/YF) = 1, \\ ADE & \text{and lines through} & B: AF, DC, EZ & \Rightarrow & (FD/FE)(CE/CA)(ZA/ZD) = 1, \end{array}$$

Multiply the equations to find:

$$(DB/DC)^2(EC/EA)^2(FA/FB)^2 = 1.$$

But $(DB/DC)(EC/EA)(FA/FB) \neq -1$, otherwise by Ceva's theorem $\{AD, BE, CF\}$ would be concurrent (or parallel). Hence the previous relation yields

$$(DB/DC)(EC/EA)(FA/FB) = 1.$$

5 Applications of Menelaus' theorem II

Exercise 1. *Points $\{A', B', C'\}$ are respectively on the sides $\{BC, CA, AB\}$ of the triangle ABC and divide them respectively into ratios*

$$\frac{A'B}{A'C} = \kappa, \qquad \frac{B'C}{B'A} = \lambda, \qquad \frac{C'A}{C'B} = \mu.$$

Line AA' intersects $B'C'$ at D. Show that D divides the respective segments into the following ratios:

$$\frac{DC'}{DB'} = \frac{\kappa\mu(1-\lambda)}{\mu - 1}, \qquad \frac{DA}{DA'} = \frac{\mu(\kappa - 1)}{\kappa\lambda\mu - 1}.$$

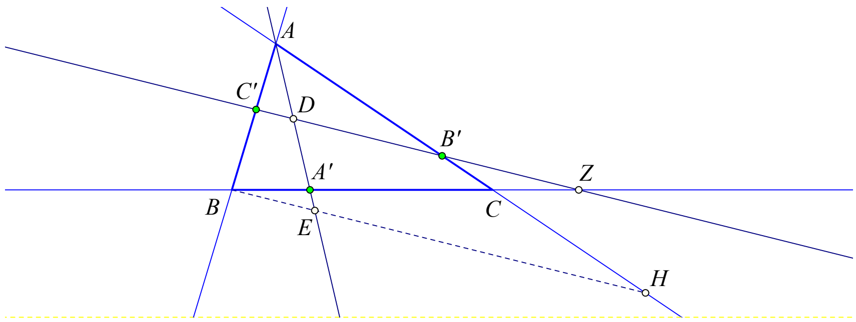

Figure 7: Calculation of segment ratios

Hint: Draw the parallel from B to $B'C'$ which intersects AC at H (See Figure 7). Calculate first the ratio $\frac{AC}{AH}$ and next the ratio $\frac{DC'}{DB'} = \frac{EB}{EH}$, where E is the point at which AA' intersects BH.

$$\begin{aligned}
\frac{B'A}{B'H} = \mu \quad &\Rightarrow \quad B'H = \frac{1}{\mu}B'A = \frac{1}{\mu(1-\lambda)}CA \\
HA = HB' + B'A \quad &= \quad -\frac{1}{\mu(1\lambda)}CA + \frac{1}{1-\lambda}CA = \frac{\mu - 1}{\mu(1-\lambda)}CA. \\
HC = HA + AC \quad &= \quad \frac{\mu\lambda - 1}{\mu(1-\lambda)}CA \quad \Rightarrow \quad \frac{HC}{HA} = \frac{\mu\lambda - 1}{\mu - 1}.
\end{aligned}$$

The ratio $\frac{EB}{EH}$ is calculated from the theorem of Menelaus applied to the triangle BCH with secant AA':

$$\frac{EB}{EH} \cdot \frac{AH}{AC} \cdot \frac{A'C}{A'B} = 1 \quad \Rightarrow \quad \frac{EB}{EH} = \frac{AC}{AH} \cdot \frac{A'B}{A'C} = \frac{\kappa\mu(1-\lambda)}{\mu - 1}.$$

From the theorem of Menelaus we also find the ratios

$$\frac{ZB}{ZC} = \frac{1}{\mu\lambda} \Rightarrow ZB = \frac{1}{\mu\lambda - 1}BC, \qquad \frac{ZB}{ZA'} = \frac{\kappa - 1}{\kappa\lambda\mu - 1}.$$

Finally, one more application of the theorem of Menelaus on the triangle ABA' with secant $C'B'$ gives:

$$\frac{C'A}{C'B} \cdot \frac{ZB}{ZA'} \cdot \frac{DA'}{DA} = 1 \quad \Rightarrow \quad \frac{DA}{DA'} = \frac{C'A}{C'B} \cdot \frac{ZB}{ZA'} = \frac{\mu(\kappa - 1)}{\kappa\lambda\mu - 1}.$$

Exercise 2. *On the sides AB and AC of the triangle ABC we consider respectively points C' and B', such that $|AB'| = |AC'|$. Show that the median AA' intersects the line segment $B'C'$ at D, in such a way as to have $\frac{|DC'|}{|DB'|} = \frac{|AC|}{|AB|}$.*

Exercise 3. *Show that for every point O, not lying on the side-lines of the triangle ABC, and the intersection points $\{A', B', C'\}$ respectively of $\{OA, OB, OC\}$ with $\{BC, CA, AB\}$, holds:*

$$\frac{OA'}{AA'} + \frac{OB'}{BB'} + \frac{OC'}{CC'} = 1, \qquad \frac{AO}{AA'} + \frac{BO}{BB'} + \frac{CO}{CC'} = 2.$$

Exercise 4. *Given is a triangle ABC and two points D, E. Point Z moves onto BC and the lines DZ, EZ intersect AB, AC respectively at points I and J. Show that the intersection point H of DJ, EI moves on a fixed line $B'C'$.*

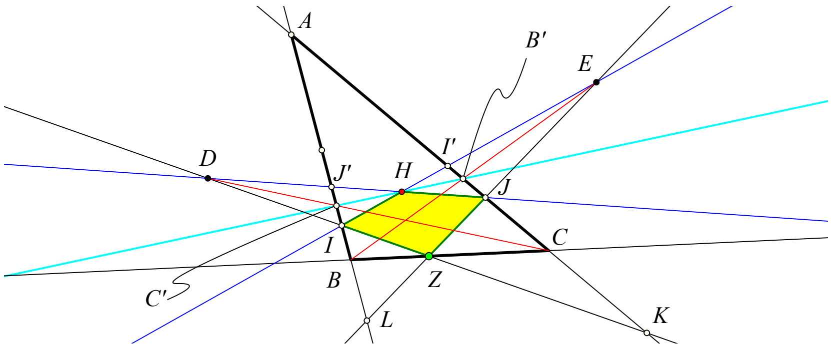

Figure 8: Application of the cross ratio

Hint: Apply the theorem of Menelaus on ABC twice, for the secant lines ZD and ZJ (See Figure 8). The following equalities result

$$\frac{IA}{IB} \cdot \frac{ZB}{ZC} \cdot \frac{KC}{KA} = 1, \quad \frac{ZB}{ZC} \cdot \frac{JC}{JA} \cdot \frac{LA}{LB} = 1 \quad \Rightarrow \quad \frac{IA}{IB} \cdot \frac{KC}{KA} = \frac{JC}{JA} \cdot \frac{LA}{LB}.$$

The last equality is written equivalently

$$(ABIL) = \frac{IA}{IB} : \frac{LA}{LB} = \frac{KA}{KC} : \frac{JA}{JC} = (ACKJ).$$

However, according by the *preservation of cross ratio on pencils* (see file **Cross ratio**)

$$(ABIL) = (AB'I'J) \text{ and } (ACKJ) = (AC'IJ') \quad \Rightarrow \quad (AB'I'J) = (AC'IJ').$$

The last equality has as a consequence the concurrence of the lines $\{B'C', II', JJ'\}$ at a point.

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