A short proof of Morley's theorem

Paris Pamfilos
University of Crete, Greece
pamfilos@math.uoc.gr

Abstract

A short and elementary proof of Morley's theorem.

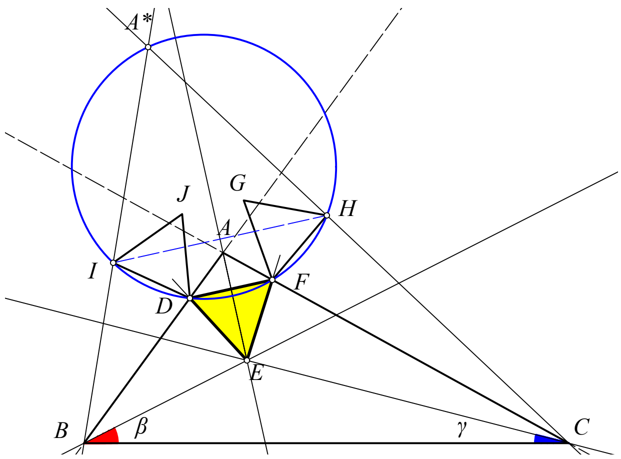

Figure 1: Bisector property

Morley's theorem can be given a short proof based on a trivial property of the bisectors of a triangle ABC with base angles 2β and 2γ. In fact, the following property is obvious.

Lemma 1 *From the incenter E of triangle ABC and on both sides of AE draw two lines inclined to it by 30° and intersecting the other sides, respectively at D, F (See Figure 1). Then DEF is equilateral and DF is orthogonal to AE.*

Having that, reflect BE and DEF on AB to obtain BI and the equilateral DIJ. Do the same on the other side, i.e. reflect CE and DEF on CF to obtain CH and the equilateral FHG. By the symmetry w.r. to AE the quadrilateral $IDFH$ is an isosceles trapezium with three equal sides, hence cyclic, and its angle $\widehat{IHF}$ is easily computed:

$$\begin{aligned}\widehat{IDF} &= 60^\circ + 2(90^\circ - \alpha/2) = 60^\circ + 180^\circ - \alpha = 60^\circ + 2\beta + 2\gamma \quad \Rightarrow \\ \widehat{IHF} &= 120^\circ - 2\beta - 2\gamma \quad \Rightarrow \quad \widehat{IHD} = 60^\circ - \beta - \gamma.\end{aligned}$$

This, essentially, finishes the proof, since the inscribed angle viewing the arc (IDH) will be then of measure

$$180^\circ - 3\beta - 3\gamma,$$

which shows that the intersection A^* of lines BI and CH will be on the circlumcircle κ of $IDFH$ and A^*D, A^*F will be the trisectors of A^*. This obviously implies that for a triangle A^*BC with base angles 3β and 3γ the adjacent trisectors intersect at the vertices of an equilateral triangle, as required by the theorem of Morley ([Cox61, 24]).

References

[Cox61] H Coxeter. *Introduction to Geometry.* John Wiley and Sons Inc., New York, 1961.