Nagel point of the triangle

by Paris Pamfilos

The bottom line for mathematicians is that the architecture has to be right. In all the mathematics that I did, the essential point was to find the right architecture. It's like building a bridge. Once the main lines of the structure are right, then the details miraculously fit. The problem is the overall design.

C.L. Dodgson, College Math. J. 25(1994)

Contents

1 Nagel point of the triangle

This is defined as the intersection point N_a of the lines joining the vertices $\{A,B,C\}$ with the contact points $\{A'',B'',C''\}$ of the opposite sides with the corresponding *"excircles"* of the triangle $t = ABC$. That this point exists can be easily proved by applying *"Ceva's theorem"* (see file **Ceva's theorem**)

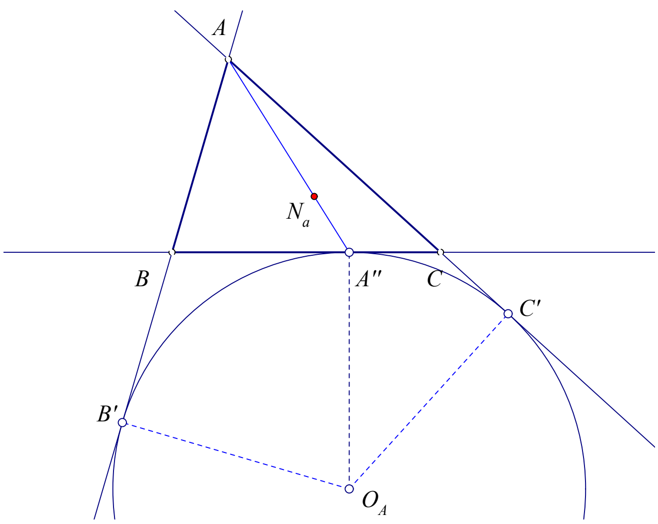

Figure 1: Nagel's point characteristic ratios $\frac{A''B}{A''C} = -\frac{s-c}{s-b}$

Theorem 1. *The lines* $\{AA'', BB'', CC''\}$ *intersect at a point.*

This follows by measuring the ratio and applying Ceva's theorem

$$\frac{A''B}{A''C} = -\frac{BB'}{CC'} = -\frac{s-c}{s-b} \quad \Rightarrow \quad \frac{A''B}{A''C} \cdot \frac{B''C}{B''A} \cdot \frac{C''A}{C''B} = -\frac{(s-c)(s-a)(s-b)}{(s-b)(s-c)(s-a)} = -1,$$

where $\{a = |BC|, b = |CA|, c = |AB|\}$ the side-lengths and $s = (a+b+c)/2$ the half-perimeter of the triangle, $s = |AB'| = |AC'|$ (See Figure 1).

2 Barycentric coordinates of the Nagel point

For the calculation of the barycentrics (see file **Barycentric coordinates**) of N_a we use the known signed ratio

$$r = \frac{A''B}{A''C} = -\frac{s-c}{s-b} \quad \Rightarrow \quad A'' = \frac{1}{1-r}(B - rC) \quad \Rightarrow \quad A'' = \frac{1}{a}((s-b)B + (s-c)C).$$

Analogously we obtain $B'' = \frac{1}{b}((s-c)C + (s-a)A)$. The Nagel point is the intersection of the lines $N_a = AA'' \cap BB''$, the coefficients of which are expressed by the vector products

$$\begin{aligned} AA'' &: \ A \times A'' = (0 : -(s-c) : (s-b)), \\ BB'' &: \ B \times B'' = ((s-c) : 0 : -(s-a)). \end{aligned}$$

The barycentrics of N_a result by taking again the vector product

$$AA'' \times BB'' = (s-a : s-b : s-c).$$

3 The Nagel line of the triangle

This is the line containing the *incenter*, the *centroid* and the *Nagel point* and, as Bottema says, it is *"a counterpart of the Euler line"* [Bot07, p.83]. The justification for this is given by the following theorem.

Theorem 2. *The Nagel point* N_a *is the incenter of the "anticomplementary" triangle* $t' = A'B'C'$ *of* ABC. *The points* $\{I, G, N_a\}$ *are collinear and* $|GN_a| = 2|IG|$.

Consider the homothety f with center G and ratio -2. This maps the triangle t onto t' and the incircle κ of t onto the incircle κ' of t'. The proof amounts to show that the line A_1I' passes through the vertex A (See Figure 2). Here $\{I, I', A_2, A_1\}$ denote respectively the incenters of t, t' and the contact points of κ with $\{BC, DE\}$, where DE is the parallel to BC tangent to κ.

The triangle $A_1I'A_2$ has G for centroid. Hence the line $A_2A_3 = f(I'A_1)$. Because of $BA_2 = CA'' = s-b$, next ratios in the similar triangles $\{ADE, A'CB, A'D'E'\}$ are equal:

$$\frac{A_1E}{A_1D} = \frac{A_2B}{A_2C} = \frac{A_3E'}{A_3D'}.$$

Hence A_2A_3 passes through A' and consequently its homothetic $I'A_1$ passes through A. By the similarity of triangles $\{ADE, ABC\}$ line AI' passes also through the contact point A'' of the excircle with BC. Thus, the cevian AA'' containing the Nagel point passes through I' and analogously the other cevians do the same. Hence I' coincides with the Nagel point N_a of ABC.

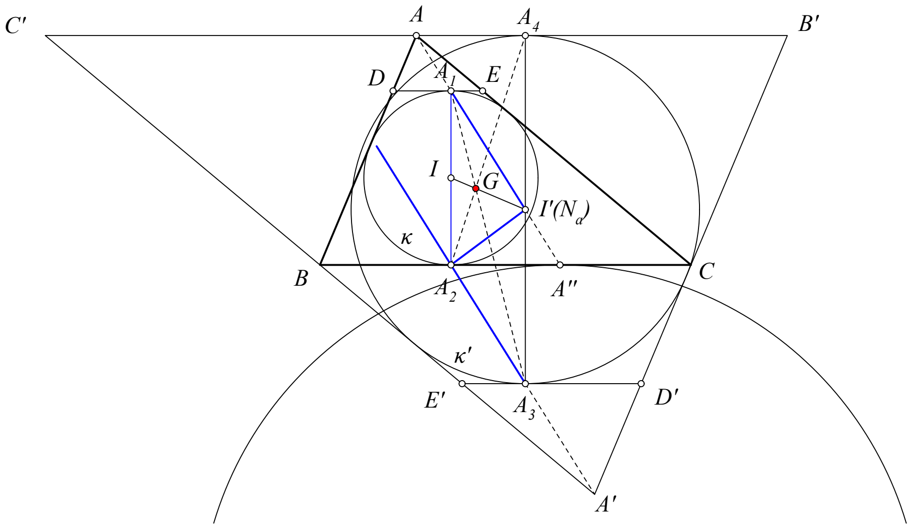

Figure 2: The cevians of the Nagel point

4 Alternative construction of the Nagel point

The following method ([Ask03, p.13]), rediscovered in [Hoe07]) gives another way to construct the Nagel point using only the incircle and not the excircles. For this draw tangents to the incircle parallel to the sides. Then join the contact points of these parallels with the opposite vertices. The three cevians thus created concure at the Nagel point (See

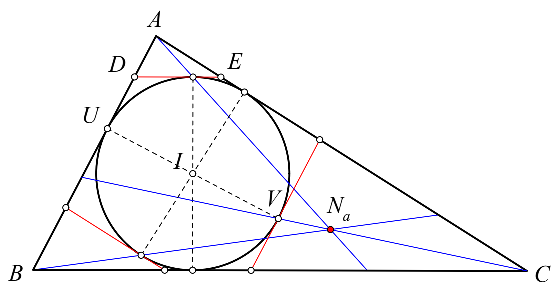

Figure 3: Alternative definition of the Nagel point

Figure 3).

The proof follows directly from the similarity of triangles $\{ADE, ABC\}$ alluded to also in the previous section.

5 Other Nagel-like points

Below is drawn an extension of the figure 1 defining the Nagel point. In this there are seen three additional Nagel-like points $\{N_1, N_2, N_3\}$, resulting as intersections of three cevians to the contact points with the *"tritangent circles"* of the triangle.

1. If point U is the contact of incircle with side AB, then its antipode V is on the line CN_a.
2. The extension of CB_1 passes through the antipode of A_1.

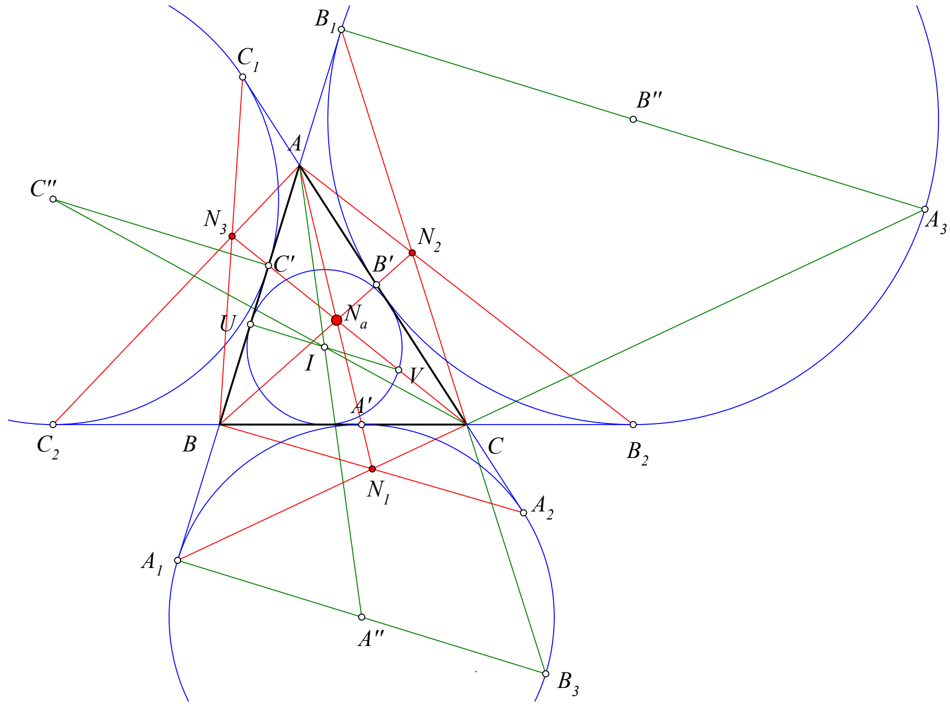

Figure 4: Other Nagel-like points $\{N_1, N_2, N_3\}$

Nr-1 is a consequence of the homothety of the incircle to the excircle opposite to C. The homothety has center at C and this implies that the end-points of parallel radii of the two circles are aligned with C. This is the case with $\{C', V\}$.

Nr-2 holds for, essentially, the same reason. This time C is the (anti) homothety center of the two excircles with centers A″, B″, hence again end-points of anti-parallel radii are collinear with C. This is the case with $\{B_1, B_3\}$.

Analogous properties, of course, hold also for the other vertices of the triangle and the corresponding excircles and contacts.

6 Connection with the de Longchamps point

The *"de Longchamps"* point of the triangle is the symmetric of the orthocenter w.r. to the circumcenter of the triangle. Next theorem relates it to the Nagel-like points of the triangle.

Theorem 3. *The three Nagel-like points $\{N_1, N_2, N_3\}$ joined to respective excenters $\{A'', B'', C''\}$ define three concurring lines at the "de Longchamps" point D_e of the triangle.*

Thus D_e is the center of perspectivity of the two triangles $\{ABC, N_1N_2N_3\}$ and coincides with the (see file **De Longchamps point**). (See Figure 5).

As we did in section 2, we compute here also the barycentrics of the contact points from the known ratios, the symbol $\cong$ denoting here the equality of vectors up to a non-

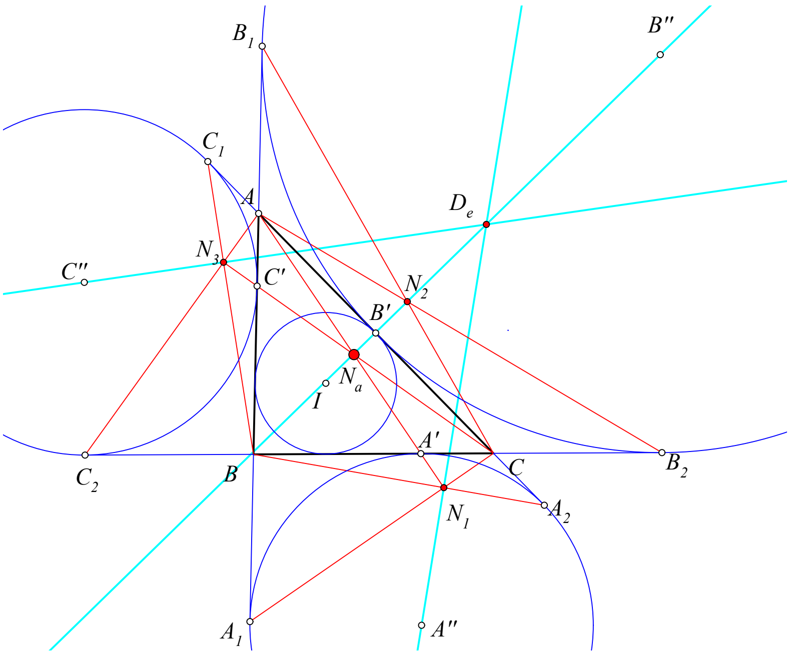

Figure 5: de Longchamps point related to $\{N_1, N_2, N_3\}$

zero scalar factor:

$$\begin{aligned}
\frac{A_1A}{A_1B} &= r = \frac{s}{s-c} \quad \Rightarrow \quad A_1 = \frac{1}{1-r}(A - rB) \;\cong\; ((c-s)A + sB),\\
\frac{A_2C}{A_2A} &= r = \frac{s-b}{s} \quad \Rightarrow \quad A_2 = \frac{1}{1-C}(A - rA) \;\cong\; (sC + (b-c)A),\\
A_1C &: \quad A_1 \times C = (-s \;:\; c-s \;:\; 0),\\
A_2B &: \quad A_2 \times B = (s \;:\; 0 \;:\; s-b) \quad \Rightarrow\\
N_1 &= ((s-b)(c-s) \;:\; s(s-b) \;:\; s(s-c)).
\end{aligned}$$

The collinearity of $\{A, N_a, N_1\}$ and the similar to it triples, suggested by figure 5, results from the obviously vanishing determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ s-a & s-b & s-c \\ (s-b)(c-s) & s(s-b) & s(s-c) \end{vmatrix} = 0.$$

Similar to the previous arguments lead to the barycentrics

$$A'' = (-a \;:\; b \;:\; c), \qquad B'' = (a \;:\; -b \;:\; c), \qquad C'' = (a \;:\; b \;:\; -c).$$

Taking into account that the de Longchamps point has barycentrics

$$D_e = (-3a^4 + 2a^2(b^2+c^2) + (b^2+c^2)^2, \dots)$$

and computing the determinant of the barycentrics vectors of the points $\{D_e, N_1, A''\}$ we find that this vanishes, hence the three points are collinear. This proves that the three lines $\{A''N_1, B''N_2, C''N_3\}$ pass through the de Longchamps point D_e of the triangle ABC.

Bibliography

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