Pedals

A file of the Geometrikon gallery by Paris Pamfilos

Practice yourself, for heaven's sake, in little things;
and thence proceed to greater.

Epictetus, Discourses I, 18

Contents

(Last update: 23-11-2021)

1 Definition and first properties

Consider the triangle ABC and a point P on its plane. Let $\{A', B', C'\}$ be the orthogonal projections of P on the sides of the triangle. Triangle $A'B'C'$ is the "***pedal of*** P ***w.r.t the triangle*** ABC" (see figure 1-(I)). We say also the pedal $A'B'C'$ is *inscribed* in $\triangle ABC$. More general we say $A'B'C'$ *"is inscribed"* in ABC if the vertices of $A'B'C'$ lie on corresponding sides of ABC.

A basic property of the pedal is the one suggested by figure 1-(II), in which for the moment and until we explicitly remove this restriction, *we assume that P is an inner point of the triangle*. If we turn the segments $\{PA', PB', PC'\}$ by the same oriented angle φ and con-

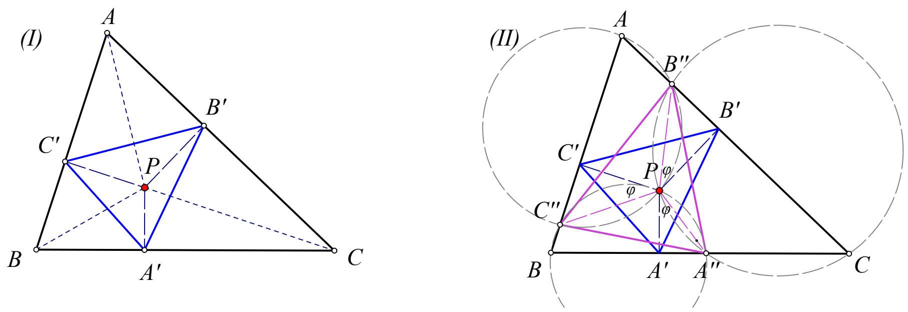

Figure 1: The pedal $A'B'C'$ of P w.r.t ABC

sider the new positions $\{A'', B'', C''\}$ on the sides, then the created right angled triangles $\{PA'A'', PB'B'', PC'C''\}$ are similar and the triangle $A''B''C''$ is similar to the pedal $A'B'C'$. Thus, a point P defines the pedal triangle and also an infinitude of similar to it triangles having their vertices on corresponding sides of the triangle of reference. Obviously the pedal triangle is the *"smallest"* among all these similar triangles and there is no *"biggest"*, since, for φ tending to $\pi/2$ the segment PA'' tends to become parallel to BC and its length tends to infinity. For all these triangles we say that they *"pivot"* about P, the pedal being a special one. In this respect P can be called *"pivot center"* or simply *"pivot"*.

We should notice that any one of the triangles $A''B''C''$ pivoting about P determines this point as intersection of the three circles:

$$(A''B''C) \ , \ (B''C''A) \ , \ (C''A''B) \ .$$

In fact, by the well known theorem of Miquel ([Joh60, p.131]) any three points $\{A'', B'', C''\}$ on respective sides of the triangle ABC define the preceding three circles passing through a common point P (see figure 1-(II)). Drawing the perpendiculars $\{PA', PB', PC'\}$ to respective sides we obtain the pedal of P with the same angles as $A''B''C''$ (exercise).

2 The pedals of points in various regions

In figure 1 and for P lying inside the triangle it is important to notice the division in triangles imposed by P on the two triangles $\{ABC, A'B'C'\}$. The triangles $\{PBC, PCA, PAB\}$ and their angles are related to corresponding triangles and angles of the pedal. In fact, the quadrangles $\{PA'CB', PB'AC', PC'BA'\}$ are cyclic and consequently the following angles are equal:

$$\widehat{PA'B'} = \widehat{PCB'} \ , \ \widehat{PB'A'} = \widehat{PCA'} \quad \text{and} \quad \widehat{A'PB'} = \pi - \widehat{C},$$

analogous relations holding for the cyclic permutations of the letters $\{A, B, C\}$. An easy angle chasing argument shows also that the angles of the two triangles are related:

$$\widehat{BPC} = \widehat{A} + \widehat{A'} \ , \ \widehat{CPA} = \hat{B} + \widehat{B'} \ , \ \widehat{APB} = \widehat{C} + \widehat{C'} \ . \tag{1}$$

Figure 2 shows the pedal of a point P' on the angular domain of angle $\widehat{C}$ and external to

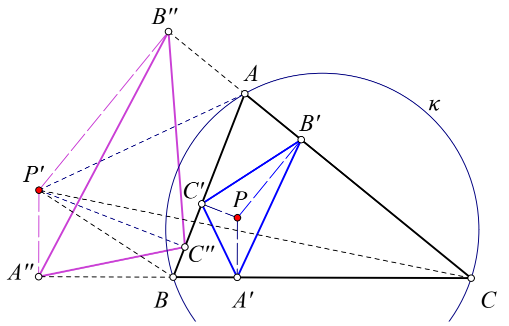

Figure 2: The pedal $A''B''C''$ for P external to ABC

the circumcircle κ of $\triangle ABC$. Here for the angles at P' we must consider the difference of the angles:

$$\begin{aligned} \widehat{BP'A} &= \widehat{BA''C''} + \widehat{C''B''A} = \hat{B} - \widehat{BC''A''} + \widehat{A} - \widehat{AC''B''} = \widehat{A} + \hat{B} - (\pi - \widehat{C''}) \\ &= \widehat{C''} - \widehat{C}, \ \text{and analogously} \ \ \widehat{BP'C} = \widehat{A} - \widehat{A''} \ , \ \widehat{CP'A} = \hat{B} - \widehat{B''}. \end{aligned} \tag{2}$$

The triangle ABC together with its circumcircle κ divides the plane in 10 different regions and the relation of the angles $\{\widehat{BPC}, \widehat{CPA}, \ldots\}$ to the angles of ABC and those of the pedal $A'B'C'$ of P depends on the region in which P is located. Figure 3 shows the various possibilities. In this appear 5 Points in typically different domains together with their

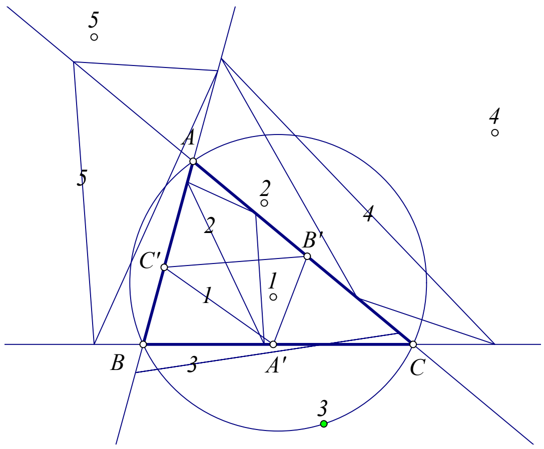

Figure 3: The pedals for various positions of the pivot P

1. $\widehat{B1C} = \widehat{A} + \widehat{A'}$ and cyclic permutations on $\{A, B, C\}$.
2. $\widehat{A2C} + \widehat{B'} + \hat{B} = 2\pi$ but $\widehat{B2C} = \widehat{A'} + \widehat{A}$ and $\widehat{A2B} = \widehat{C'} + \widehat{C}$.
3. $A'B'C'$ degenerate carried on the Wallace-Simson line of 3.
4. $\widehat{A4C} = \widehat{B'} - \hat{B}$ but $\widehat{B4C} = \widehat{A} - \widehat{A'}$ and $\widehat{A4B} = \widehat{C} - \widehat{C'}$.
5. $\widehat{B5C} = \widehat{A} - \widehat{A'}$ but $\widehat{B5A} = \widehat{C'} - \widehat{C}$ and $\widehat{A5C} = \widehat{B'} - \hat{B}$.

Table 1: Angle relations

corresponding pedals both labeled by the same number. The table accompanying the figure gives the corresponding to each case angle relations. In these it is assumed that the pedal has $\{A' \in BC, B' \in CA, C' \in AB\}$ or their extensions and the angles are positively oriented (counterclockwise).

This distinction of cases can be eliminated using the idea of *"directed angles"* introduced by Johnson ([Joh60, p.13]). For two lines, the *directed angle* $\angle(\alpha,\beta)$ is defined to be the minimal positive angle ϕ by which we must rotate α in order to identify it with or make it parallel to β. Analogously, the *directed angle* $\angle(ABC)$ is the minimal positive angle ϕ by which rotating line BA we get line BC (see figure 4). Obviously for positively

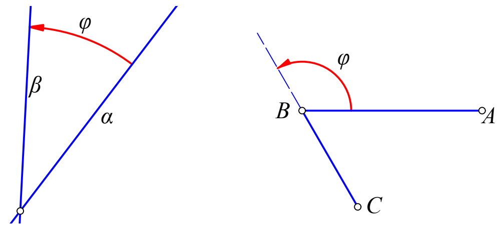

Figure 4: The *"directed angle"* of two lines and an angle $\angle ABC$

oriented angles less than π the *directed* coincide with the usual angles and for negatively oriented the *directed* coincide with the supplementary angle. Also the identity of *directed angles* is defined by considering *"equal"* two directed angles differing by a multiple of π. Thus two directed angles of $120°$ sum up to $60°$. Using this we have the rule: *"the geometric locus of points B viewing AC at a fixed directed angle $\phi = \angle(ABC)$ is a circle κ"* (see figure 5-(I)). Notice that this includes the whole circle. No distinction is made between

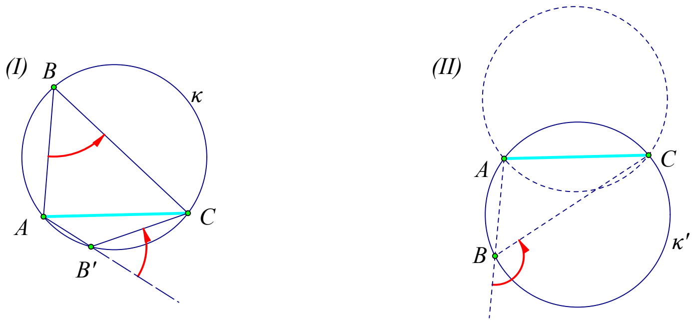

Figure 5: Circle: locus of point B with constant $\angle(ABC)$

the two arcs determined by the chord AC. Figure 5-(II) shows the related to the κ locus of points viewing AC at the angle $\angle(ABC) = \pi - \phi$, coinciding with the reflection κ' of κ in AC.

Using this definition of angle, Johnson shows also ([Joh60, p.133]), that all the above cases of table 1 reduce to the simple one:

$$\angle(BPC) = \angle(BAC) + \angle(B'A'C'). \tag{3}$$

Figure 6 shows the application of this rule to the first sub-case of our case 2. The equivalence of the two formulations results from the relations

$$\widehat{APC} = \angle(APC) \quad , \quad \angle(ABC) = \pi - \hat{B} \quad , \quad \angle(A'B'C') = \pi - \widehat{B'} \; .$$

In any case, concerning pedals and relations of the involved angles of the pedal and the triangle of reference ABC, in order to be precise, we must either examine all the possible cases of table 1 separately or apply the unifying concept of *directed angles*, which however needs to be learned and exercised until to become familiar and apply it correctly. In the

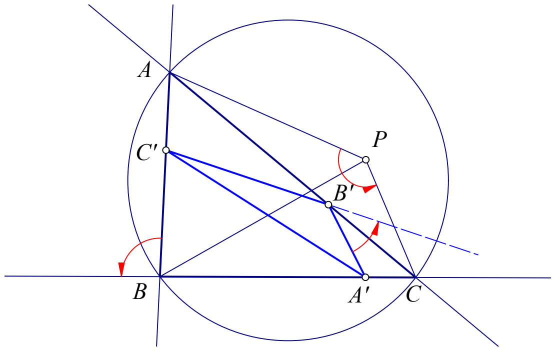

Figure 6: $\angle(APC) = \angle(ABC) + \angle(A'B'C')$

following we use mainly the traditional concept of angle and in situations where, for completeness of the argument, several cases must be examined, we handle one or two and leave the rest as exercises. Occasionally we use also the directed angle concept.

3 How to inscribe, 12 pivots

We come now to the question of how to inscribe into $\triangle ABC$ a triangle $A'B'C'$ similar to $\triangle A_1B_1C_1$ respecting a given *"correspondence"* for the relative location of the vertices.

Figure 7 illustrates an example of such a task, the *"correspondence"* on the vertices being in this case: *A' to lie on AB, B' to lie on BC and C' to lie on CA*. Here is the recipe:

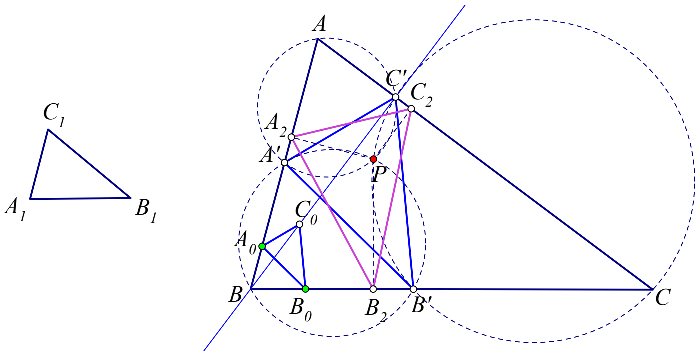

Figure 7: Inscribing a similar to $A_1B_1C_1$ into ABC

1. Select arbitrary $\{A_0 \in AB,\ B_0 \in BC\}$.
2. Draw on A_0B_0 triangle $A_0B_0C_0$ similar to $A_1B_1C_1$.
3. Find the intersection $C' = BC_0 \cap CA$.
4. Draw parallels $\{C'A' \| C_0A_0,\ C'B' \| C_0B_0\}$.

The triangle $A'B'C'$ satisfies all requirements. It is similar to $A_1B_1C_1$, is inscribed in ABC and has its vertices on respective sides of ABC as required. The circles $(AA'C'), (BB'A')$ and $(CC'B')$ intersect at a point P, the pivot of $A'B'C'$. The pedal $A_2B_2C_2$ of P is similar to $A_1B_1C_1$ and satisfies the requirements $A_2 \in AB, B_2 \in BC, C_2 \in CA$ too.

Given the triangles $\{A_1B_1C_1, ABC\}$ and the *correspondence*, *"which vertex of $A_1B_1C_1$ lies on each side of ABC"*, the location of the pivot for the corresponding $A_2B_2C_2 \sim A_1B_1C_1$ (similar triangles) inscribed in ABC can be at a finite number of places determined by the table (1). In figure 7 for example the location of P corresponds to the case (1) of table (1) and P is at the intersection of arcs viewing the sides at the angles: BC at $\widehat{A}+\widehat{B_1}$, CA at $\widehat{B}+\widehat{C_1}$ and AB at $\widehat{C}+\widehat{A_1}$ (see figure 8).

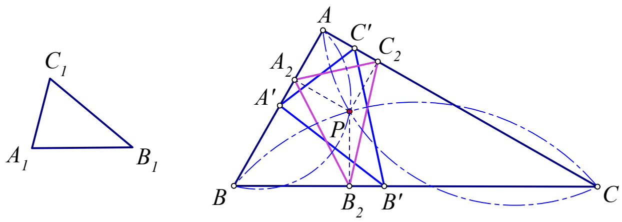

Figure 8: $P \in \widehat{BPC}$ viewing BC at the angle $\widehat{A}+\widehat{B_1}$, etc.

Thus, given the two triangles $\{A_1B_1C_1, ABC\}$, each *correspondence* generates actually two cases related to the two possible orientations of the initially inscribed triangle $A_0B_0C_0 \sim A_1B_1C_1$. In the example of figure 8 the orientation of both triangles $A_0B_0C_0$ and ABC is positive.

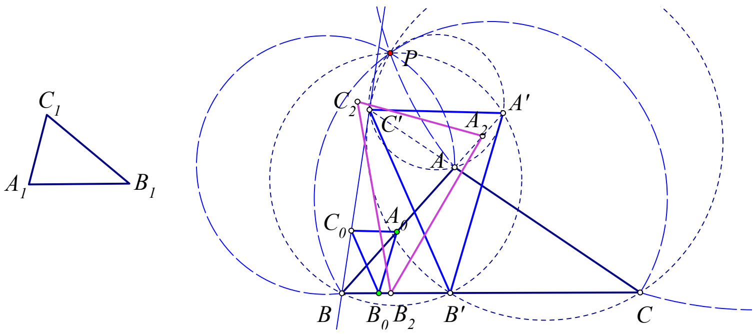

Figure 9: Locating the position of the pivot P for a certain prescription

In the example of figure 9 we use the same recipe of four steps, maintaining the same *correspondence* of the preceding example, but changing the orientation of the inscribed triangle $A_0B_0C_0$, which now is negative, while that of ABC is positive. We obtain thus a pivot P different from the preceding one. The location of P shows that we have the case (5) of table (1). According to this, P is viewing the side BC at $\widehat{A}-\widehat{B_1}$, the side AB at $\widehat{A_1}-\widehat{C}$ and AC at $\widehat{C_1}-\widehat{B}$ and, as seen in figure 9, P is located at the intersection of the

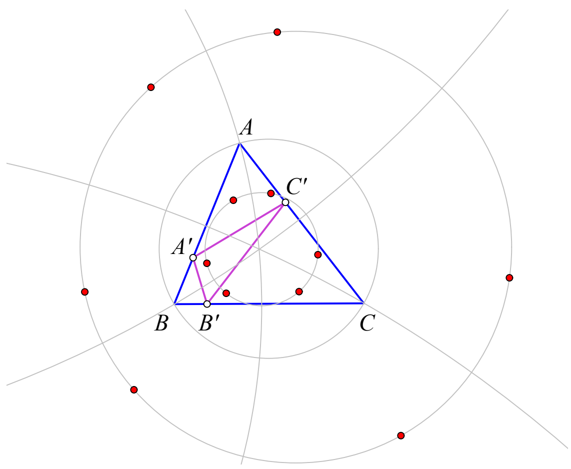

Figure 10: Twelve pivots

circle arcs viewing the corresponding sides at these angles. In general, each *correspondence* together with a definite orientation of $A_0B_0C_0$ determines precisely one pivot. Since there are six possible *correspondences* and for each case two different orientations we come at the following result.

Theorem 1. *With the notation and conventions adopted so far, for given triangles* $\{A_1B_1C_1, ABC\}$ *there are in general twelve different pivots of inscription of* $\triangle A_1B_1C_1$ *into* $\triangle ABC$.

Figure 10 shows the twelve pivots of inscription of $\triangle A'B'C'$ into $\triangle ABC$ and suggests some structure which we'll study in the next sections. Each pivot results from a concrete "***prescription***" consisting of a *correspondence*, dictating which vertex of $\triangle A'B'C'$ lies on a side of $\triangle ABC$, plus an orientation of $\triangle A'B'C'$.

4 Invariance under inversions

Inversions $\{X' = f(X)\}$ relative to a circle $\kappa(O,r)$ transform the set of {lines + circles} onto itself but do not behave in a simple manner when applied to triangles. We have however two useful formulas for the transformation of lengths of segments (see figure 11-(I)) and measures of angles (see figure 11-(II)). The second, in order to be formulated in a general form, needs the use of *"directed angles"* ([Joh60, p.52]).

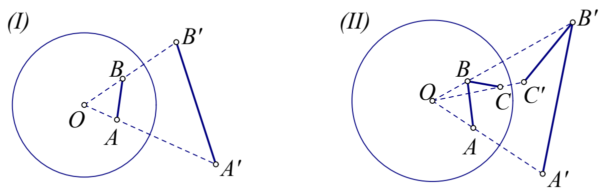

Figure 11: Length and Angle-measures relations by inversions

$$\text{lengths:}\quad |A'B'| = \frac{r^2|AB|}{|OA||OB|}, \qquad \text{angles:}\quad \angle(ABC) + \angle(A'B'C') = \angle(AOC)\,. \qquad (4)$$

The second formula used in the case of figure 11-(II) and translating the *"directed"* to usual angles of positive measure is equivalent to the angle relation: $\widehat{ABC} - \widehat{C'B'A'} = \widehat{AOC}$. The interesting fact in our context is expressed by the following theorem.

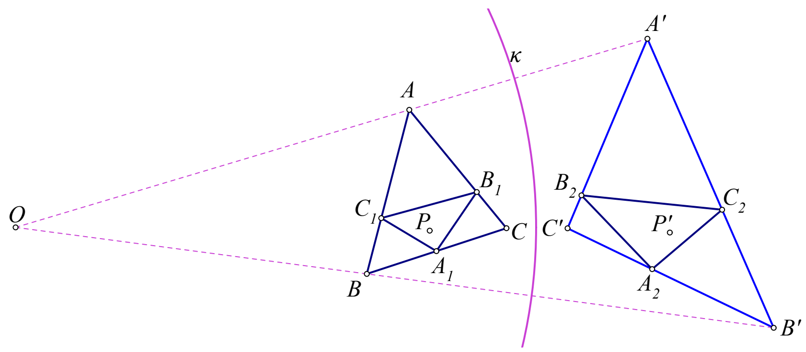

Figure 12: Inverted triangle and pedal of inverted point

Theorem 2. *Let the inversion* $\{f(X) = X'\}$ *relative to the circle* $\kappa(O,r)$ *map the vertices of the triangle* ABC *to the vertices of the triangle* $A'B'C'$ *and point* P *to* P'. *Then the pedal* $\triangle A_2B_2C_2$ *of* P' *w.r.t.* $\triangle A'B'C'$ *is similar to the pedal* $\triangle A_1B_1C_1$ *of* P *w.r.t.* $\triangle ABC$ *(see figure 12).*

Proof. Working with directed angles this is a formal computation. In fact by the angle relations (3) and (4) we have

$$\angle(A_2) = \angle(B'P'C') - \angle(B'A'C') = (\angle(BPC) - \angle(BOC)) - (\angle(BAC) - \angle(BOC))$$
$$= \angle(BPC) - \angle(BAC) = \angle(A_1).$$

Analogous reasoning shows the equality of the two other angles of the triangles. □

Remark 1. Notice that the inversion f does not map the vertices of the pedal $A_1B_1C_1$ to corresponding vertices of the pedal $A_2B_2C_2$. Since, if f were interchanging the triangles $\{A_1B_1C_1, A_2B_2C_2\}$, then it would map the triple $\{A, B_1, C\}$ onto the triple $\{A', B_2, C'\}$, contradicting the fact that f maps lines to circles. Also the triangles $\{ABC, A'B'C'\}$ are not similar. Since, if they were, then, since they are also perspective and oppositely oriented, this would lead to a contradiction (exercise).

Remark 2. Notice that theorem 2 holds also for *"anti-inversions"* i.e. inversions followed by the point symmetry w.r.t. their center.

Theorem 2 has two important in our context specializations concerning inversions that map the set of vertices of a triangle ABC to itself. There are four such inversions: the inversions relative to the *"Apollonian circles"* of $\triangle ABC$ and the inversion relative to the circumcircle of $\triangle ABC$, discussed in the following sections.

5 Invariance under Apollonian inversions

The **Apollonian circles of a triangle** ABC are three circles $\{\kappa_A, \kappa_B, \kappa_C\}$ orthogonal to the circumcircle κ of the triangle closely related with the enumeration of the pivots of inscription of triangle $A_1B_1C_1$ into ABC. The circles intersect at two points, called **Isodynamic points** of the triangle (see figure 13). Thus, they define a pencil $\mathcal{A}$ of intersecting circles called *"Apollonian pencil"* of the triangle and having all its members orthogonal to the circumcircle of the triangle. Some important in our context properties are that the inversion w.r.t. an Apollonian circle, (i) induces, by theorem 2, a permutation of the set of pivots, (ii) interchanges the other two Apollonian circles and (iii) leaves invariant every circle of the pencil $\mathcal{A}'$ which is orthogonal to the Apollonian one. Latter property implies, that if

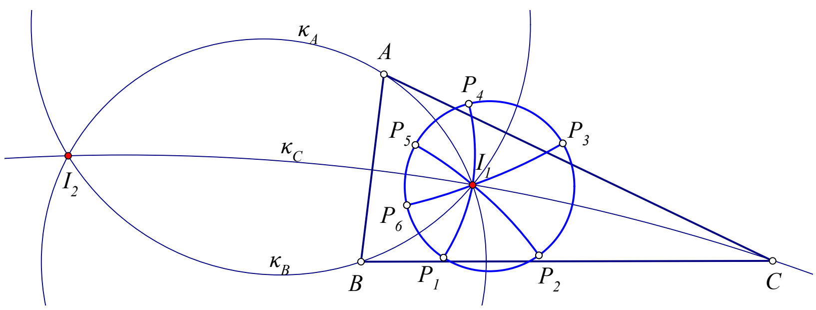

Figure 13: The Apollonian circles $\{\kappa_A, \kappa_B, \kappa_C\}$ and the isodynamic points $\{I_1, I_2\}$

we consider the circle ξ orthogonal to the Apollonian pencil $\xi \in \mathcal{A}'$ and passing through a pivot P_1 of inscription of $\triangle A_1B_1C_1$ into $\triangle ABC$, then the successive application of the inversions w.r.t. Apollonian circles $\{\kappa_A, \kappa_B, \kappa_C, \kappa_A, ...\}$ will generate an *"orbit"* of pivots $\{P_1, P_2, P_3, ...\}$ contained in ξ. The nice property is that this orbit contains precisely six points. This follows directly from the property of the Apollonian circles to intersect pairwise at angles of 60° and the fact that inversions are *conformal* i.e. they preserve angles.

Latter implies that arcs of the Apollonian pencil member circles, such as $\{\widehat{I_1P_1}, \widehat{I_2P_2}\}$ of figure 13, are inclined to the Apollonian circle κ_A at the same angle. By watching these angles at I_1 and taking into account the fact that Apollonian circles intersect at $60°$ we realize that after 6 successive inversions of P_1 w.r.t. the Apollonian circles we come back to the starting point P_1, creating an orbit of six points contained in the circle ξ. Since the pencil $\mathcal{A}'$ which is orthogonal to the Apollonian pencil $\mathcal{A}$ is of non intersecting type, the remaining six pivots of inscription of $\triangle A_1B_1C_1$ into $\triangle ABC$ will lie on another circle $\xi' \in \mathcal{A}'$ of this pencil. Thus, all of the 12 pivots are contained in two non intersecting circles $\{\xi, \xi'\}$ of the pencil A', as suggested by figure 10. We formulate this as a theorem.

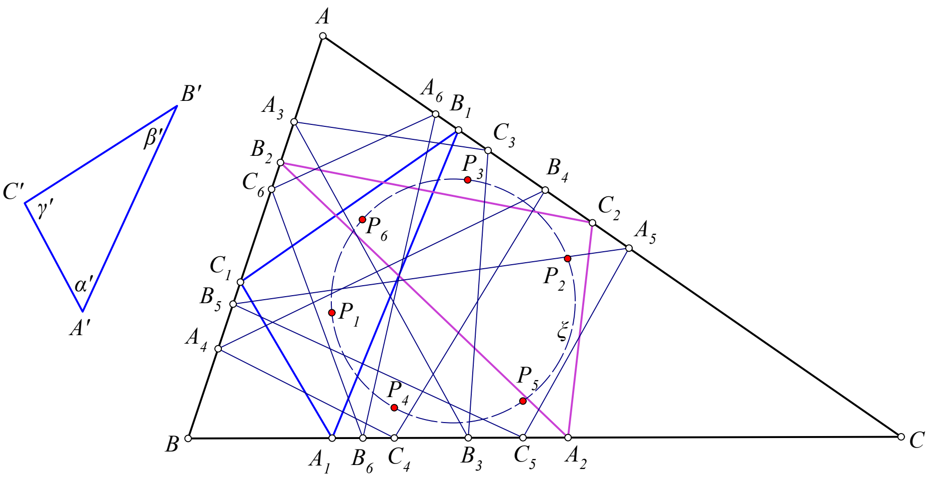

Figure 14: Six pivots of inscription of $A'B'C'$ in ABC

Theorem 3. *The twelve pivots of inscription of $\triangle A_1B_1C_1$ into $\triangle ABC$ are contained by six in two disjoint circles $\{\xi, \xi'\}$ orthogonal to the Apollonian pencil of the triangle.*

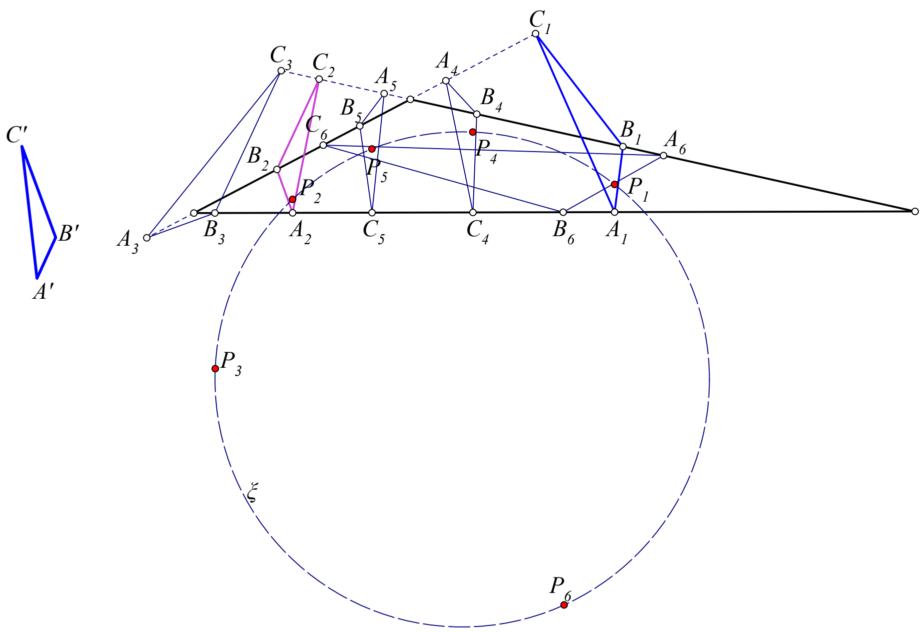

Figure 15: A second configuration of pivots of inscription of $A'B'C'$ in ABC

The pencil $\mathcal{A}'$ which is orthogonal to the pencil $\mathcal{A}$ of Apollonian circles is traditionally called *"Schoute pencil"* of the triangle ([Joh17]). The circles ξ considered above, as well as the circumcircle κ, belong to this pencil. Figures 14 and 15 show two configurations of the six pivots of inscription of $\triangle A'B'C'$ into $\triangle ABC$ contained in a *Schoute circle* ξ and the corresponding to these pedal triangles. The configurations depend on the shapes of the two triangles $\{\triangle A'B'C', \triangle ABC\}$ tacitly assumed in this section to be *generic*. For special shapes of triangles we may obtain fewer pivots as for example in the case of isosceli $A'B'C'$ inscribed in a generic triangle ABC, a case in which the corresponding pivots fall *on* the Apollonian circles and we have a total of 6 instead of 12 pivots. In case $\triangle A'B'C'$ is equilateral we obtain only two pivots, coinciding with the isodynamic points and in case both $\{\triangle A'B'C', \triangle ABC\}$ are equilateral we obtain one only pivot, the center of $\triangle ABC$ (see file **Apollonian circles of the triangle**).

6 Invariance under the circumcircle inversion

By the discussion in the preceding section, we know that there are 12 pivots of inscription of a generic triangle $A'B'C'$ into a generic triangle ABC, grouped by 6 on two Schoute circles $\{\xi, \xi'\}$. On the other side, from theorem 2 we know that the inversion w.r.t. the circumcircle κ of the triangle induces a permutation on the set of pivots and maps the circle ξ to another Schoute circle, necessarily identical with ξ'. This follows from the general property of circle-pencils, according to which, picking a single circle κ of the pencil, the inversion w.r.t. κ introduces a permutation of the circles of the pencil. Combining this with the results of the preceding section we come to the following property formulated again as a theorem.

Theorem 4. *For two generic triangles $\{A'B'C', ABC\}$ the two Schoute circles $\{\xi, \xi'\}$ carrying by 6 the 12 pivots of inscription of $\triangle A'B'C'$ in $\triangle ABC$ are inverse to each other w.r.t. the circumcircle κ of the triangle ABC and the pivots on one of these are the κ-inverses of the pivots on the other circle.*

7 Reduction to the equilateral

Theorem 2, combined with a property of the **Isodynamic points** of the triangle, establish

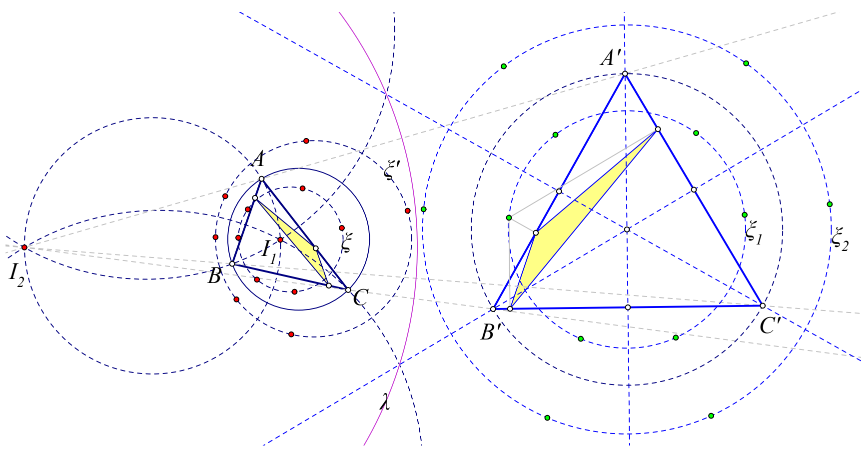

Figure 16: Reduction to the equilateral, the two pedals are similar

a relation of the problem of inscription of $\triangle A'B'C'$ into $\triangle ABC$ to the apparently simpler problem of inscription of $\triangle A'B'C'$ into an equilateral. In fact, it can be proved (see file **Isodynamic points**) that an inversion f w.r.t. a circle λ centered at one of the two isodynamic points, the outer isodynamic point I_2 say, maps the triangle ABC to an equilateral $A'B'C'$ and the Apollonian circles of $\triangle ABC$ to the symmetry axes of $\triangle A'B'C'$. Then, the Schoute pencil of $\triangle ABC$ maps to the pencil of circles centered at the center of the equilateral (see figure 16). The two circles $\{\xi, \xi'\}$ carrying by 6 the 12 pivots of inscription of a triangle $A_1B_1C_1$ into $\triangle ABC$ map via f to two concentric circles $\{\xi_1, \xi_2\}$ carrying by 6 the 12 pivots of inscription of the same triangle $A_1B_1C_1$ into the equilateral $\triangle A'B'C'$. Circles $\{\xi, \xi'\}$ are inverse w.r.t. the circumcircle of $\triangle ABC$ and $\{\xi_1, \xi_2\}$ are inverse w.r.t. the circumcircle of $\triangle A'B'C'$.

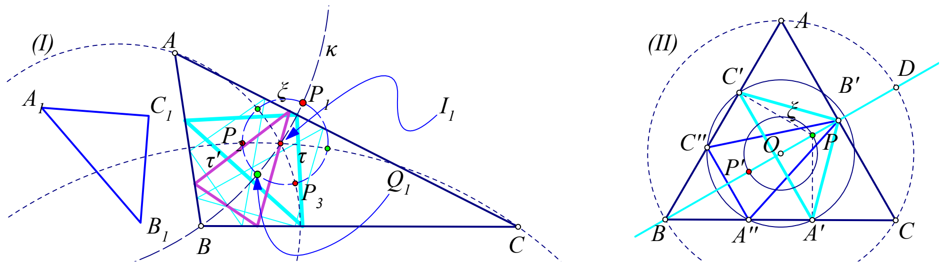

Figure 17: A duality of isosceli

This reduction to the equilateral can be convenient in some cases, as the one concerning a certain *"duality"* I met in studying isosceli inscribed in a triangle. In this we start with an arbitrary triangle ABC and an isosceles $\triangle A_1B_1C_1$ and locate its pivots of inscription in $\triangle ABC$. They are 6, lying by 3 on two Schoute circles $\{\xi, \xi'\}$, coinciding with 3 out of the 6 intersections of these circles with the Apollonian circles of $\triangle ABC$. Figure 17-(I) shows a case, the pivots $\{P_1, P_2, P_3\}$ on circle ξ, their pedal triangles similar to $\triangle A_1B_1C_1$ and the three other points $\{Q_1, Q_2, Q_3\}$ which together with the $\{P_i\}$ define all six intersection of ξ with the Apollonian circles. By our preceding discussion the pedals of Q_i will be also isosceli similar to another fixed isosceles. Thus, fixing $\triangle ABC$ and associating to the pedal $\triangle\tau$ of a point on the Apollonian circle $P_1 \in \kappa$ the pedal $\triangle\tau'$ of the other intersection point $Q_1 \in \kappa$ of κ with the Schoute circle ξ through P_1, we define a *"douality"* between similar types of isosceli, apparently depending on $\triangle ABC$. Using the reduction to the equilateral, we can easily see that this duality is independent of the particular $\triangle ABC$ fixed at the beginning. In fact, from the discussion in this section we know that the same similarity type of $\triangle A_1B_1C_1$ can be inscribed in a fixed equilateral and set in correspondence via an inversion f to its inscription in $\triangle ABC$. The inversion maps the Apollonian circles to corresponding symmetry axes of the equilateral $\{P_1, Q_1\}$ map via f to points on such an axis lying diametrically w.r.t. the center O of the equilateral. In figure 17-(II) we see two such dual triangles. The following relations are easily established.

$$y = |PA'|\,,\; y' = |P'A''| \quad \text{satisfy} \quad y + y' = \frac{a}{\sqrt{3}} \quad \text{with} \quad a = |BC|.$$

$$|A'C'| = \sqrt{3}y\,,\; \text{area}(A'B'C') = \frac{3y(a - \sqrt{3}y)}{2}.$$

The triangles $\{A'B'C', A''B''C''\}$ have the same area. For P varying from O to D the triangle $A'B'C'$ is obtuse at B' and varies from the equilateral to the line segment AC, while its dual $A''B''C''$ varies from the equilateral to the segment OB. By the way, notice that the isosceles τ of figure 17-(I) is the right-angled one, and its dual τ' is the isosceles having ratio altitude/base$= 3/2$, which is the ratio of edge/circumsphere-radius of the *"truncated* *octahedron"* (see figure 18). Also the dual isosceles to the one with angles $\{30°, 120°, 30°\}$

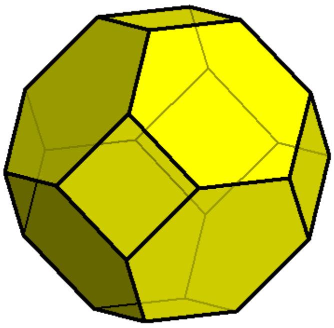

Figure 18: Trunkated octahedron

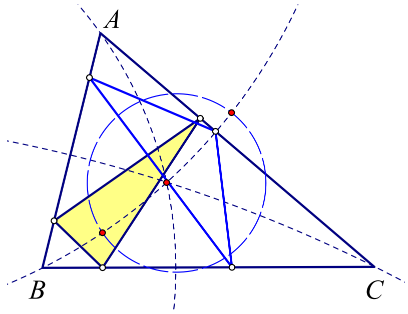

Figure 19: The dual isosceles of the one with apex angle $120°$

is the isosceles with ratio side/base= $\sqrt{7}$ (see figure 19).

8 The dual viewpoint

There is an alternative viewpoint to see our subject. This is to fix the inscribed $\triangle A'B'C'$ and consider circumscribed triangles ABC similar to a fixed one $A_1B_1C_1$ (see figure 20). The various triangles ABC have their vertices on three circles and the whole configura-

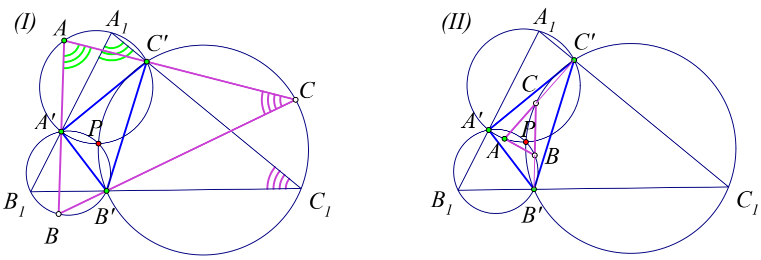

Figure 20: $\triangle ABC$ circumscribing $\triangle A'B'C'$

tion depends on *"which side of $\triangle A'B'C'$ is viewed by an angle of $\triangle ABC$ "*. There are several choices or *"prescriptions"* and for each prescription we obtain again a *"pivot"* P which is the common point of the three circles carrying the vertices of the variable circumscribing triangle ABC. We say *"the pivot of circumscription of $\triangle ABC$ about $\triangle A'B'C'$ "*, briefly *"**circum-pivot**"*. Obviously, fixing a pivoting $\triangle ABC$, such a pivot coincides with a pivot of inscription of $\triangle A'B'C'$ into $\triangle ABC$ as those discussed in the preceding sections and henceforth referred to as *"**in-pivots**"*. Figure 20 shows two instances of the pivoting of $\triangle ABC$ about $\triangle A'B'C'$. Figure 20-(II) shows a triangle ABC lying inside $\triangle A'B'C'$, which nevertheless *circumscribes* $\triangle A'B'C'$, since the vertices of $\triangle A'B'C'$ lie on respective sides (extensions) of $\triangle ABC$. There is no minimal circumscribed, since the vertices of $\triangle ABC$ can be arbitrary close to the pivot P and even become all identical to it. There is though a maximal circumscribed.

Exercise 1. *Referring to figure 21 show that the pivot P is the similarity center of the similarity mapping $\triangle A_1B_1C_1$ onto $\triangle ABC$.*

Exercise 2. *With the notation of this section, the triangle ABC with sides parallel to the triangle $A_0B_0C_0$ of the centers of circles carrying the vertices of the pivoting triangles, is the maximal one w.r.t. perimeter and area (see figure 21).*

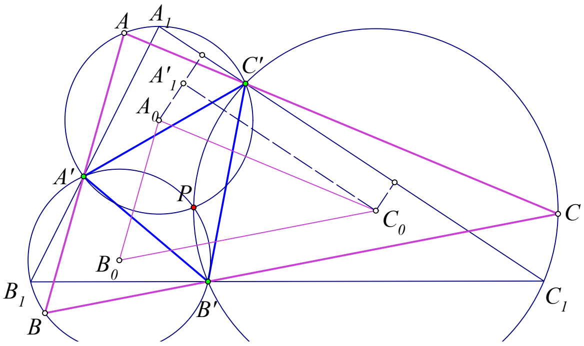

Figure 21: Maximal circumscribed ABC similar to $A_1B_1C_1$

Hint: Project $\{A_0, C_0\}$ on A_1C_1 and form the right angled triangle $A_0C_0A'_1$ (see figure 21). Then $C_0A'_1$ is half the side A_1C_1 and shorter than C_0A_0, since latter is the hypotenuse of the right angled triangle $A_0C_0A'_1$.

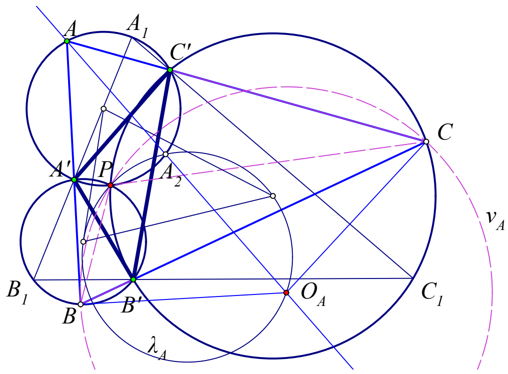

Figure 22: Maximal circumscribed ABC similar to $A_1B_1C_1$

Exercise 3. *With the notation of this section, the following are valid properties (see figure 22):*

1. *The triangles $\{APB, BPC, CPA\}$ are of fixed similarity types and their circumcenters O_C, O_A, O_B describe circles.*
2. *The triangles $\{ABO_A, ACO_A, BCO_A\}$ are also of fixed similarity types.*
3. *The line AO_A passes through a fixed point A_2 of circle $(AA'C')$.*

9 In-pivots and circum-pivots

As we noticed already, fixing the circumscribed triangle ABC, the circum-pivot of ABC about $A'B'C'$ coincides with an in-pivot of $\triangle A'B'C'$ in *"this"* $\triangle ABC$. If however we circumscribe $\triangle ABC$ about $\triangle A'B'C'$ following another *prescription*, letting for example $\triangle A^*B^*C^* \sim \triangle ABC$ circumscribe $\triangle A'B'C'$ with prescription ($\rightarrow$ standing for *"viewing"*):

$$\widehat{A^*} \rightarrow A'B' \;,\; \widehat{B^*} \rightarrow B'C' \;,\; \widehat{C^*} \rightarrow C'A' \quad \text{instead of}$$
$$\widehat{A} \rightarrow A'C' \;,\; \widehat{B} \rightarrow B'A' \;,\; \widehat{C} \rightarrow C'B' \;,$$

then the pivot P of inscription of $\triangle A'B'C'$ in $\triangle ABC$ is different from the pivot P^* of inscription of $\triangle A'B'C'$ in $\triangle A^*B^*C^* \sim \triangle ABC$ (see figure 23). The location of $\{P, P^*\}$ w.r.t. to the immovable $\triangle A'B'C'$ are determined by the intersection of corresponding triples of

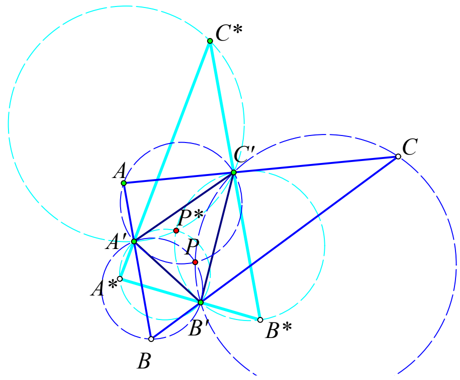

Figure 23: Different circumscribing prescriptions lead to different pivots

circles defined by the angles at which the vertices of $\triangle ABC$ view the sides of $\triangle A'B'C'$, which, of course, depend on the circumscribing prescription chosen which includes the choice of orientation of $\triangle ABC$. In any case the corresponding pivot P is the same for all triangles ABC with vertices respectively on the three associated to the prescription circles through P. Figure 24 shows two circum-pivots $\{\alpha\beta\gamma, \alpha\beta\gamma^*\}$ of $\triangle A_1B_1C_1$ about

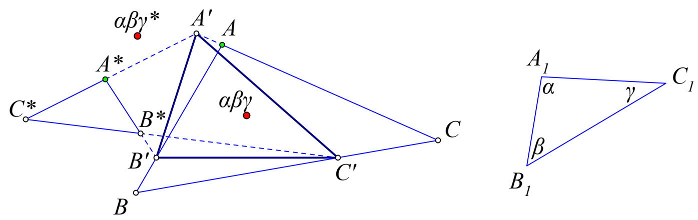

Figure 24: Circumscribing prescriptions differing only by the orientations

$\triangle A'B'C'$. Concerning the coordination {vertex → side} both prescriptions are the same:

$$A \to A'B' \,,\;\; B \to B'C' \,,\;\; C \to C'A' \,,$$

"→" meaning *"vertex viewing side"*, where the angle is that of the corresponding of the similar $\triangle A_1B_1C_1$ or its supplement. The prescriptions though differ regarding the orientation of the two circumscribing similar triangles, $\triangle ABC$ being positive oriented while $\triangle A^*B^*C^*$ is negatively oriented.

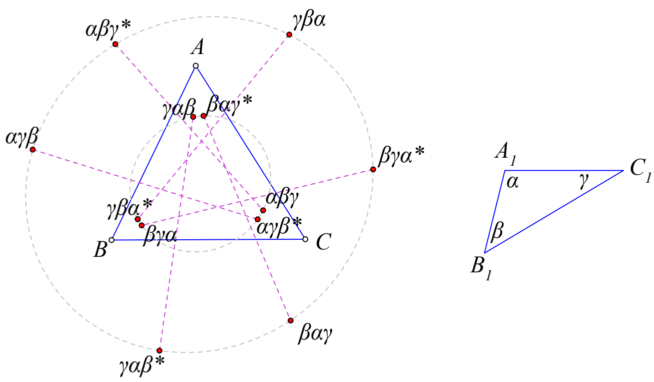

Figure 25: The 12 pivots of circumscription of $\triangle A_1B_1C_1$ about $\triangle ABC$

Thus, each one of the 6 combinations {vertex → side}, together with the corresponding two orientations produces 12 circum-pivots of $\triangle A_1B_1C_1$ about $\triangle A'B'C'$. Regarding the orientation, notice that if we consider the three circles carrying the vertices of $\triangle ABC$ and defining the pivot $\alpha\beta\gamma$, then the other pivot $\alpha\beta\gamma^*$ is obtained as intersection of the reflections of the preceding three circles w.r.t. corresponding sides of $\triangle A'B'C'$. Figure 25 shows a configuration of the 12 circum-pivots of $\triangle A_1B_1C_1$ about $\triangle ABC$. The segments shown join pivots differing only w.r.t. the orientations. The pivots seem to be distributed by 6 on two conics, but this is not true in general.

Exercise 4. *Show that for an equilateral $\triangle ABC$ and an arbitrary $\triangle A_1B_1C_1$ the 12 circum-pivots of $\triangle A_1B_1C_1$ about the equilateral are by 6 on two circles concentric to the circumcircle of the equilateral (see figure 26).*

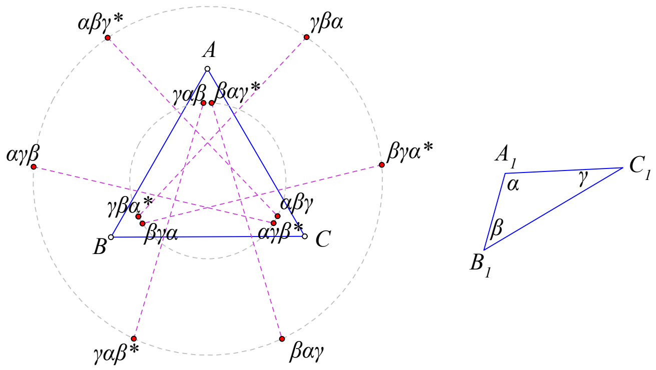

Figure 26: The 12 pivots of circumscription of $\triangle A_1B_1C_1$ about the equilateral $\triangle ABC$

Hint: The three circles carrying the vertices of the circumscribing triangle have the same radii $\{r_1, r_2, r_3\}$ independently of the particular prescription. This produces pivots distributed symmetrically w.r.t. the axes of symmetry of the equilateral. Figure 27 shows two pivots $\{D, E\}$ whose defining triples of circles (carrying their vertices) are symmetric w.r.t respective sides of the equilateral producing oppositely oriented triangles $\{d, e.\}$ Another such couple of triangles will be produced by the reflection of these pivots to an axis of symmetry of the equilateral, such as $D'E'$.

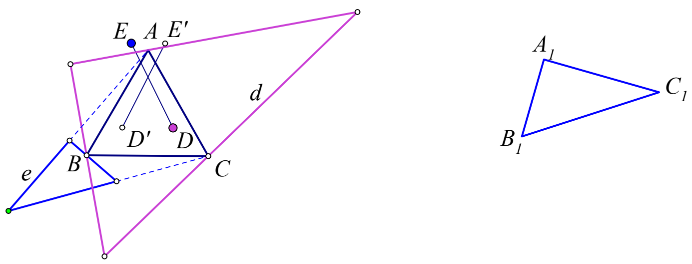

Figure 27: Circumscription pivots and the symmetry of the equilateral

Remark 3. For an equilateral $\triangle ABC$, referring to figure 27 and taking into account the discussion in section 5, we notice that the circum-pivot D coincides with an isodynamic point of the triangle d and analogously the pivot E coincides with an isodynamic point of triangle e.

Next two theorems recapitulate the results of this section.

Theorem 5. *Given a pair of triangles $\{ABC, A_1B_1C_1\}$ there are 12 circum-pivots of $\triangle A_1B_1C_1$ about $\triangle ABC$. If $\triangle ABC$ is equilateral the circum-pivots lie by six on two circles concentric to the circumcircle of the equilateral.*

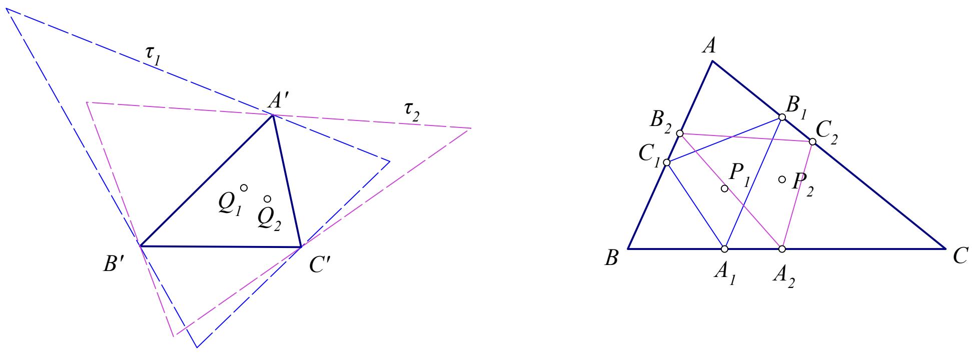

Figure 28: Relation of circum-pivots to in-pivots

Theorem 6. *Given a pair of triangles $\{A'B'C', ABC\}$ the circum-pivots $\{Q_1, Q_2, \ldots\}$ of $\triangle ABC$ about $\triangle A'B'C'$ result from the in-pivots $\{P_1, P_2, \ldots\}$ of $\triangle A'B'C'$ in $\triangle ABC$ as follows. For each P_i we consider its pedal $\triangle A_iB_iC_i$ and the similarity f_i sending it onto $A'B'C'$. Then $Q_i = f_i(P_i)$.*

Figure 28 illustrates the result of application of $\{f_1, f_2\}$ involved in this theorem. The triangles $\{\tau_1, \tau_2\}$ are the images $\{\tau_1 = f_1(A_1B_1C_1), \tau_2 = f_2(A_1B_1C_1)\}$.

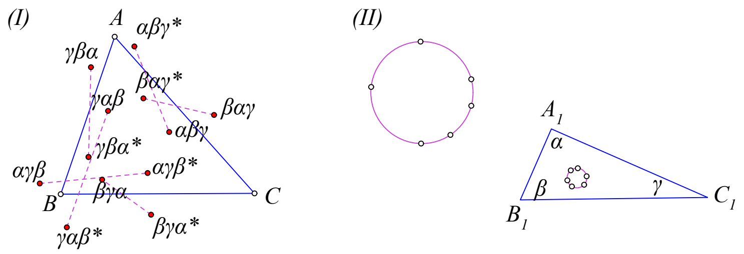

Figure 29: The two kinds of pivots related by 12 similarities $\{f_i\}$

Figure 29 gives a visual impression of the application of the last theorem to all pivots of the two kinds. In part (I) we have the 12 circum-pivots of $\triangle A_1B_1C_1$ about $\triangle ABC$. In part (II) we have the 12 in-pivots of $\triangle ABC$ in $\triangle A_1B_1C_1$.

Remark 4. There is a particular case which deserves special consideration. This is the case for which the number 12 drops to 11. It results by considering the in-pivot O, the circumcenter of the triangle of reference ABC. The pedal of O is similar to $\triangle ABC$ and the inverse relative to the circumcircle of O is at infinity, failing to define a genuine pedal triangle. Thus, the corresponding configuration involves only 11 in-pivots, among them the two *"Brocard points"* of the triangle ABC and the related 11 circum-pivots. Figure 30 shows the corresponding configuration, the Schoute circle κ through O called *"Brocard circle"* and its inverse w.r.t. the circumcircle, which in this case is a line ε called *"Lemoine line"* of the triangle. The points $\{B_1, B_2\}$ are the *"Brocard points"* of the triangle. The figure on the right shows the in-pivots. On the left stands the corresponding configuration of the circum-pivots. Again the segments joining two points indicate circumscribed triangles whose prescriptions differ only with respect to orientation. A more detailed discussion of this configuration can be found in the file **Brocard points**.

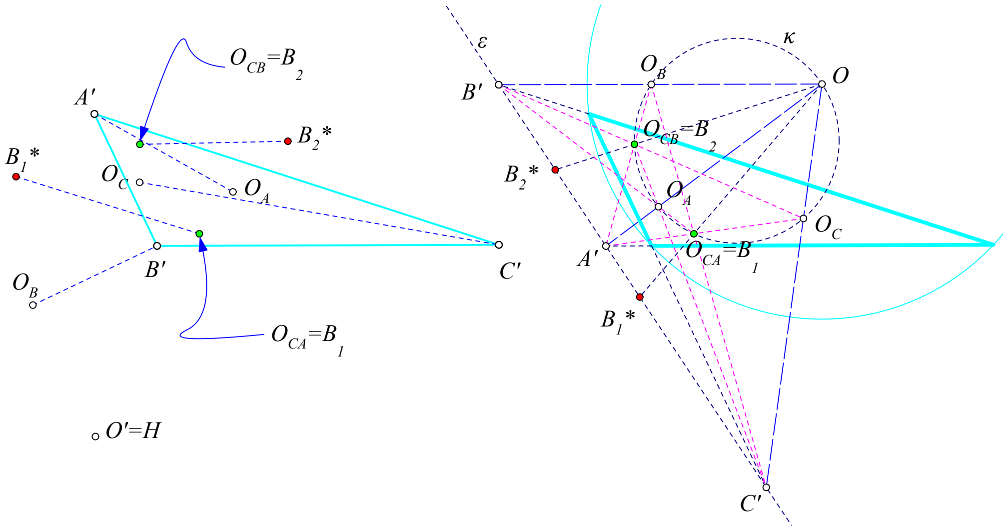

Figure 30: The Brocard configuration

10 Area of pedal

Consider the triangle of reference ABC, a point D and the pedal $D_1D_2D_3$ w.r.t. $\triangle ABC$.

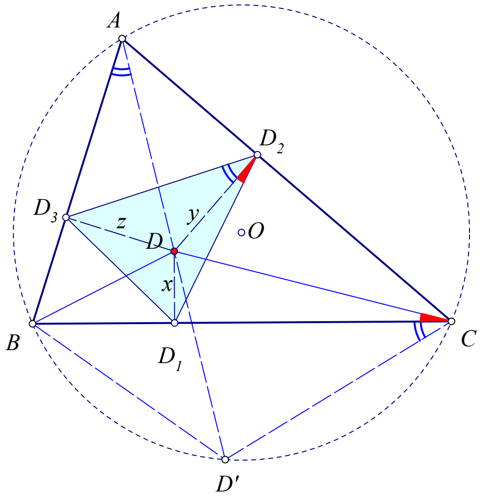

Figure 31: Area of pedal $D_1D_2D_3$

Theorem 7. *The following are valid properties (see figure 31).*

1. *The area* $(D_1D_2D_3) = (R^2 - |DO|^2)\sin(\widehat{A})\sin(\widehat{B})\sin(\widehat{C})/2$, *where* R *is the circumradius of* $\triangle ABC$ *and* O *is the circumcenter.* $R^2 - |DO|^2$ *is the power of* D *w.r.t. the circumcircle.*

2. *For D moving on circles concentric to the circumcircle, the pedal triangles have constant area and vice versa. In particular, for D on the circumcircle the corresponding area is zero, the three projection-points being on the "Wallace-Simson line of D".*

3. *Denoting by (x, y, z) the trilinear coordinates of D, and considering signed (oriented) areas:*
$$(D_1D_2D_3) = \frac{1}{2}(\sin(\widehat{A})yz + \sin(\hat{B})zx + \sin(\widehat{C})xy) \tag{5}$$
is a quadratic form in the trilinear coordinates.

Proof. Nr-1 The formula follows by extending DA to cut the circumcircle at D' and observing that $\widehat{D_1D_2D_3} = \widehat{D'CD}$ (see figure 31). Also $\widehat{DD'C}$ equals angle $\hat{B}$ of the triangle and by the sinus theorem applied to triangle $D'CD$ we have:
$$\sin(\widehat{D'CD})/|DD'| = \sin(\hat{B})/|DC|. \qquad (*)$$
In addition, since BD_1DD_3 is a cyclic quadrangle and DB is a diameter of its circumcircle we have $\{|DB| = |D_1D_3|\sin(\hat{B}) \quad (**)\}$ and analogous equations for the segments DC and DA. Now using $\{(*), (**)\}$ the area of the pedal triangle is:
$$\begin{aligned}(D_1D_2D_3) &= |D_2D_3||D_2D_1|\sin(\widehat{D_1D_2D_3}/2 = |DA|\sin(\widehat{A})|DC|\sin(\widehat{C})\sin(\widehat{D'CD}/2\\ &= |DA||DD'|\sin(\hat{B})\sin(\widehat{C})\sin(\widehat{A})/2 = (R^2 - |DO|^2)\sin(\widehat{A})\sin(\hat{B})\sin(\widehat{C})/2.\end{aligned}$$
Nr-2 Follows immediately from *nr-1*.
Nr-3 Follows by dividing the area of the triangle in the sum:
$$(D_1D_2D_3) = (D_1DD_2) + (D_2DD_3) + (D_3DD_1).$$
□

Remark 5. Equation (7) is general valid, even when point D is outside the triangle, provided we use oriented areas. Next figure illustrates the corresponding proof.

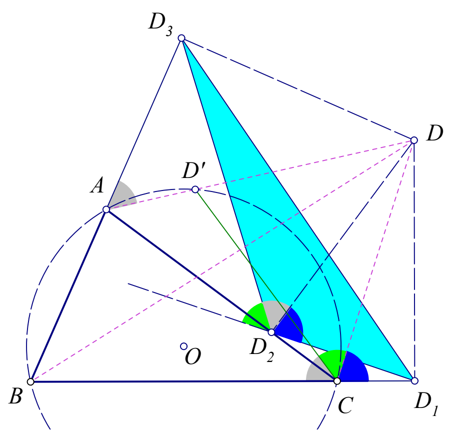

Figure 32: Area of pedal $D_1D_2D_3$ for D external to the circumcircle

Remark 6. By equating the expressions in (7) and (9) we get:
$$\sin(\widehat{A})yz + \sin(\hat{B})zx + \sin(\widehat{C})xy = (R^2 - |PO|^2)\sin(\widehat{A})\sin(\hat{B})\sin(\widehat{C}) \quad \Leftrightarrow$$
$$ayz + bzx + cxy = \frac{(R^2 - |PO|^2)abc}{4R^2}. \tag{6}$$

By taking D on the circle we get the *"equation of the circumcircle"* in trilinears.

$$ayz + bzx + cxy = 0.$$

Equation (6) shows that the quadratic form $f(x,y,z) = ayz + bzx + cxy$, where $\{a,b,c\}$ denote the lengths of the sides of the triangle, is positive inside the circumcircle, zero on the circumcircle and negative outside. Notice that (x,y,z) are not independent, but satisfy the equation $ax + by + cz = 2(ABC)$. Points at infinity satisfy $ax + by + cz = 0$ and fall *far out* of the circumcircle of $\triangle ABC$ where f is negative.

Exercise 5. *Show that for points D satisfying $|OD| = \sqrt{5}R$ the area of the pedal $D_1D_2D_3$ equals the area of the triangle and for D satisfying $|OD| = 3R$ the area of the pedal is the double of the area of the triangle ABC.*

11 Formularium

Here are some formulas relating elements of the pedal $\triangle D_1D_2D_3$ of a point D relative to the $\triangle ABC$. The symbols used are explained through figure 33, in which (x,y,z) are

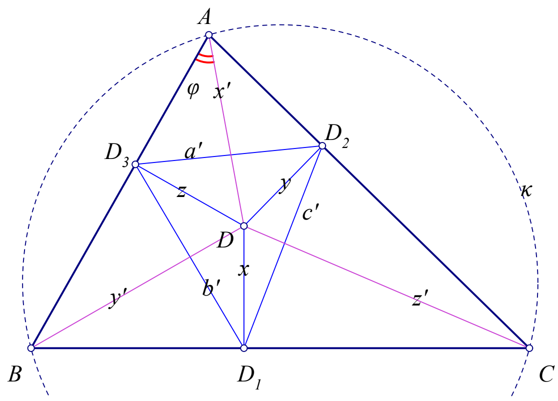

Figure 33: Trilinear relations of the pedal $D_1D_2D_3$ of D relative to $\triangle ABC$

the *"trilinear coordinates"*, short *trilinears* of point D.

$$2(ABC) = ax + by + cz \quad \text{where} \quad (\dots) \text{ denotes area,} \tag{7}$$

$$a' = \sin(\widehat{A})x' = \frac{ax'}{2R} \quad \text{where } R \text{ the circumradius of } ABC, \tag{8}$$

$$a' = \sqrt{y^2 + z^2 + 2yz\cos(\widehat{A})}, \tag{9}$$

$$x' = \frac{a'}{\sin(\widehat{A})} = \frac{\sqrt{y^2 + z^2 + 2yz\cos(\widehat{A})}}{\sin(\widehat{A})}, \tag{10}$$

$$x = \frac{\sqrt{(z'^2 - (a-y')^2)((a+y')^2 - z'^2)}}{2a}, \tag{11}$$

$$\frac{x'^2\sin(2\widehat{A})}{4} = (AD_3DD_2) - 2(D_3DD_2), \tag{12}$$

$$(ABC) - 2(D_1D_2D_3) = \frac{x'^2\sin(2\widehat{A}) + y'^2\sin(2\widehat{B}) + z'^2\sin(2\widehat{C})}{4}, \tag{13}$$

$$(D_1D_2D_3) = \frac{\sin(\widehat{A})yz + \sin(\widehat{B})zx + \sin(\widehat{C})xy}{2}. \tag{14}$$

Equation (7) results by calculating the area of ABC in terms of (x,y,z).
Equations (8), (9) result from sine and cosine rules.
Equation (10) is a consequence of the preceding ones.
Equation (11) is a calculation of the altitude of a triangle. Equations (12), (13) were shown by Steiner as follows: Because the quadrangle AD_3DD_2 is cyclic $\sin(\widehat{D_2DD_3}) = \sin(\widehat{A})$ and the right side of equation (12) is:

$$\frac{1}{2}(AD_3AD_2\sin(\widehat{A}) + DD_3DD_2\sin(\widehat{A})) - DD_3DD_2\sin(\widehat{A}) =$$
$$\frac{1}{2}(AD_3AD_2\sin(\widehat{A}) - DD_3DD_2\sin(\widehat{A})) =$$
$$(\sin(\widehat{A})x'^2/2)(\cos(\phi)\cos(\widehat{A}-\phi) - \sin(\phi)\sin(\widehat{A}-\phi)) = (\sin(\widehat{A})x'^2/2)\cos(\widehat{A}).$$

Equation (13) is a consequence of equation (12) and equation (14) is proved in the preceding section.

12 The third pedal

Theorem 8. *The third pedal τ_3 of a triangle $\tau = ABC$ relative to a point P is similar to τ.*

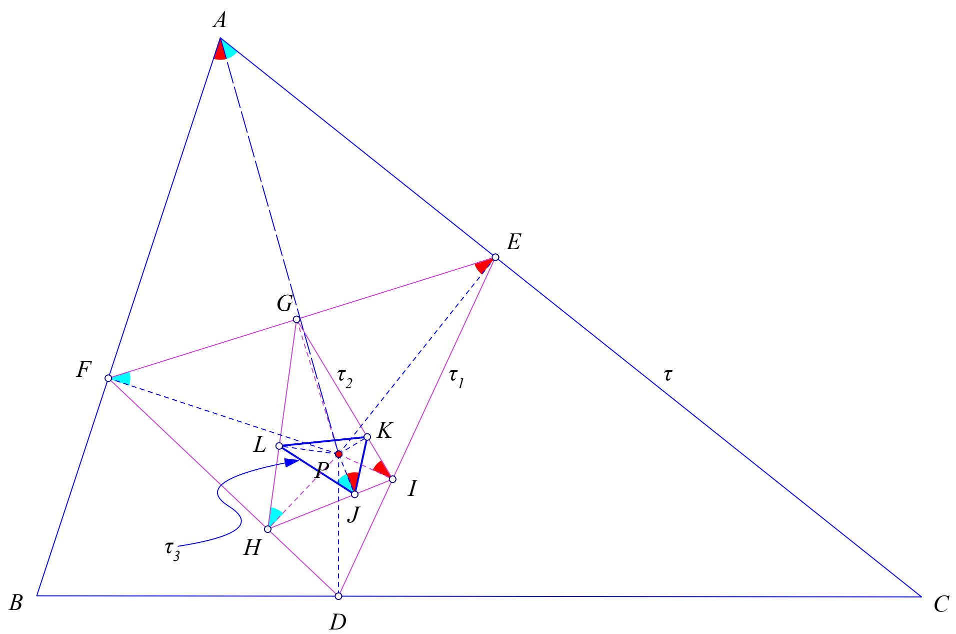

Figure 34: Third pedal τ_3 of $\tau = ABC$ relative to P

Proof. The proof is an easy angle chasing based on the cyclic quadrangles created by the projections of P on the sides of the $\{\tau, \tau_1, ...\}$ as indicated in figure 34 in the case of an inner point of the triangle. Analogous is the proof for other positions of P relative to the triangle. The pedal $\tau_1 = DEF$ is created by projecting point P on the sides of triangle $\tau = ABC$. The pedal $\tau_2 = GHI$ of τ_1 relative to P is created by projecting P on the sides of τ_1 and τ_3 by projecting P on the sides of τ_2. □

Remark 7. The argument of the proof with the angles indicated in figure 34 shows that the triangles defined by P and composing $\triangle ABC$ $\{APB, BPC, CPA\}$ are respectively similar with the corresponding triangles composing the third pedal $\{JPK, KPL, LPJ\}$. This implies that P is the similarity center of the similarity mapping triangle τ onto τ_3.

13 Pedal, cevian and circumcevian triangles

The trilinears of the projections $\{D_1, D_2, D_3\}$ of point D on the sides of the triangle of reference ABC can be expressed by means of the trilinears (x, y, z) of D and the altitudes of the triangle $\{h_A, h_B, h_C\}$:

$$\begin{array}{lll} D_{1x} = 0, & D_{1y} = y + x\cos(\widehat{C}), & D_{1z} = z + x\cos(\hat{B}), \\ D_{2x} = x + y\cos(\widehat{C}), & D_{2y} = 0, & D_{2z} = z + y\cos(\widehat{A}), \\ D_{3x} = x + z\cos(\hat{B}), & D_{3y} = y + z\cos(\widehat{A}), & D_{3z} = 0. \end{array}$$

The *"cevian"* triangle of a point P relative to the triangle ABC has vertices the *"traces"* $\{E_1, E_2, E_3\}$ of lines $\{AP, BP, CP\}$ respectively on the opposite sides (see figure 35). Their trilinears are:

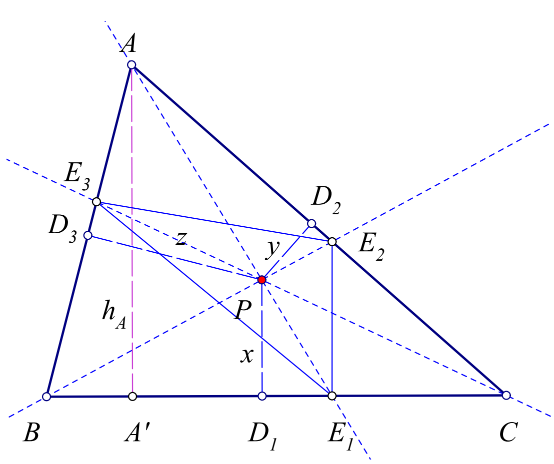

Figure 35: The cevian $\triangle E_1E_2E_3$

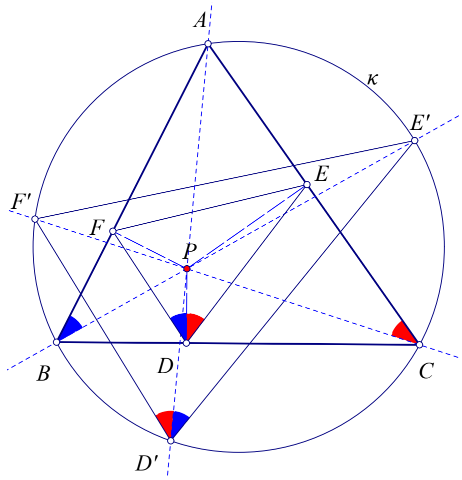

Figure 36: The circumcevian of P

$$\begin{array}{lll} E_{1x} = 0, & E_{1y} = y\dfrac{h_A}{h_A - x}, & E_{1z} = z\dfrac{h_A}{h_A - x}, \\ E_{2x} = x\dfrac{h_B}{h_B - y}, & E_{2y} = 0, & E_{2z} = z\dfrac{h_B}{h_B - y}, \\ E_{3x} = x\dfrac{h_C}{h_C - z}, & E_{3y} = y\dfrac{h_C}{h_C - z}, & E_{3z} = 0. \end{array}$$

The *"circumcevian"* triangle $D'E'F'$ of a point P relative to the triangle ABC is the triangle of second intersections of the lines $\{PA, PB, PC\}$ with the circumcircle κ of $\triangle ABC$ seen in figure 36, which suggest also the proof of the next theorem.

Theorem 9. *The pedal $\triangle DEF$ of a triangle $\tau = ABC$ relative to a point P is similar to the circumcevian triangle $D'E'F'$ of P.*

Corollary 1. *The pedal $\triangle A''B''C''$ of P relative to the circumcevian $\triangle A'B'C'$ is similar to the original triangle of reference ABC (see figure 37).*

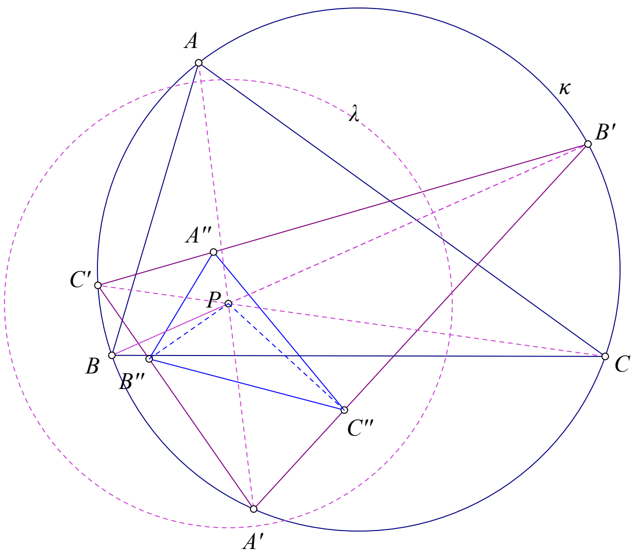

Figure 37: Pedal $\triangle A''B''C''$ of P relative to circumcevian similar to $\triangle ABC$

Proof. Use the relations (1) and see that $\widehat{A'PC'} = \widehat{B''} + \widehat{B'}$. But $\widehat{B'} = \widehat{APC} - \widehat{B}$. Hence we obtain: $\widehat{B''} - \widehat{B} = \widehat{A'PC'} - \widehat{APC} = 0$. □

Remark 8. Notice in figure 37 the circle $\lambda(P, r)$ with $r^2 = PA \cdot PA'$ which anti-inverts the vertices of $\triangle ABC$ to those of $\triangle A'B'C'$ and leaves invariant κ so that the two circles intersect at diametral points of λ.

14 Darboux cubic

The *"Darboux cubic"* ([Gib21]) is the geometric locus of points D for which the pedal $\triangle D_1D_2D_3$ is *"cevian"* i.e. the lines $\{AD_1, BD_2, CD_3\}$ intersect at a point P (see figure 38).

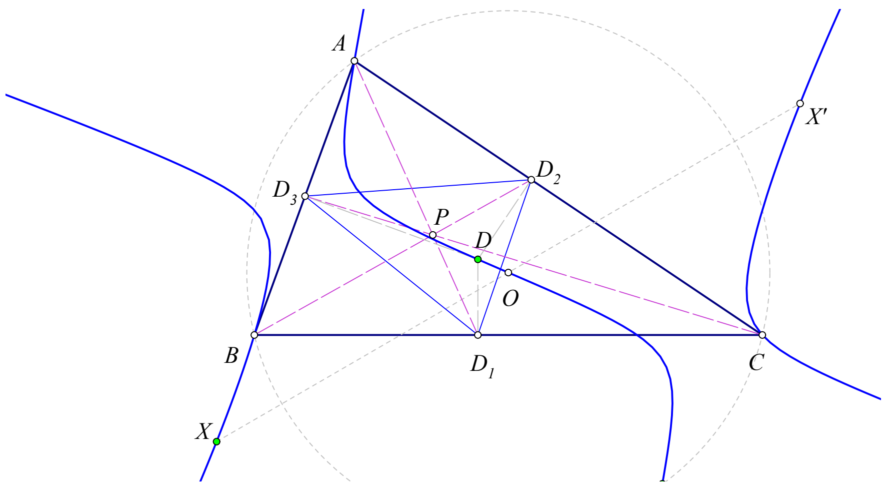

Figure 38: The Darboux cubic of $\triangle ABC$

Theorem 10. *The Darboux cubic expressed in trilinears w.r.t. $\triangle ABC$ is represented by the equation:*

$$[\cos(\widehat{A})\cos(\hat{B}) - \cos(\widehat{C})][z(x^2 - y^2)] + [\cos(\hat{B})\cos(\widehat{C}) - \cos(\widehat{A})][x(y^2 - z^2)] + [\cos(\widehat{C})\cos(\widehat{A}) - \cos(\hat{B})][y(z^2 - x^2)] = 0\,.$$

Proof. We calculate the condition of **Ceva's theorem** for the cevians $\{AD_1, AD_2, AD_3\}$. For this we need the signed ratio D_1B/D_1C (see figure 39). In the following $\{h_A, h_B, h_C\}$ denote the altitudes and $\{a, b, c\}$ the side-lengths of the triangle.

$$\frac{h_B}{D_{1y}} = \frac{a}{D_1C}\,, \quad \frac{h_C}{D_{1z}} = \frac{a}{D_1B} \quad \Rightarrow \quad \frac{D_1B}{D_1C} = -\frac{h_B}{h_C} : \frac{D_{1y}}{D_{1z}}\,,$$

and analogous formulas for the ratios $\{D_2C/D_2A, D_3A/D_3B\}$. Replacing these into the Ceva condition

$$\frac{D_1B}{D_1C} \cdot \frac{D_2C}{D_2A} \cdot \frac{D_3A}{D_3B} = -1,$$

and taking into account the equations of section 13 we obtain:

$$\frac{z + x\cos(\hat{B})}{y + x\cos(\widehat{C})} \cdot \frac{x + y\cos(\widehat{C})}{z + y\cos(\widehat{A})} \cdot \frac{y + z\cos(\widehat{A})}{x + z\cos(\hat{B})} = 1\,,$$

which is equivalent to the requested equation. □

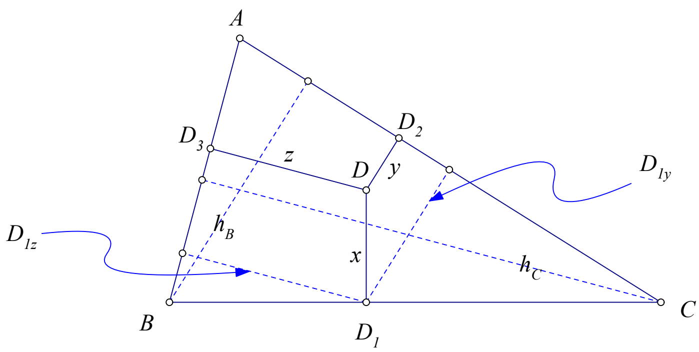

Figure 39: The signed ratio D_1B/D_1C in terms of the trilinears of D_1

Remark 9. The Darboux cubic passes through the vertices of the triangle, the circumcenter $O(\cos(\widehat{A}), \cos(\hat{B}), \cos(\widehat{C}))$ of $\triangle ABC$ and several other *"triangle centers"* ([Kim18]), such as the *"incenter"* $I(1,1,1)$, the *"orthocenter"* $H(\cos(\widehat{A}) - \sin(\hat{B})\sin(\widehat{C}) : \ldots)$ and the *"de Longchamps"* point $L(\cos(\widehat{A}) - \cos(\hat{B})\cos(\widehat{C}) : \ldots)$ in parenthesis standing the trilinears of the respective points and the dots denoting the other two coordinates resulting by cyclic permutations of the letters. The cubic is symmetric w.r.t. the circumcenter O as is easily verified using the rerpresentation of the $O-$symmetry w.r.t. to trilinears, which has in normalized trilinears the same typical form as the point symmetry expressed in cartesian coordinates:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = -\begin{pmatrix} x \\ y \\ z \end{pmatrix} + 2\begin{pmatrix} O_x \\ O_y \\ O_z \end{pmatrix}.$$

We recall that *"normalized"* means that the trilinears measure exact signed distances of the point $D(x, y, z)$ from the sides of the triangle, so that $xa + yb + zc = 2(ABC)$.

15 Pedals of isogonal points

If we draw the circumcircle $\kappa(O)$ of the pedal $\triangle D_1D_2D_3$ of the point D relative to the triangle of reference ABC, the circle intersects triangle's sides a second time at the points $\{E_1, E_2, E_3\}$ and it is easy to see that these points are the projections of a single point E (see figure 40). This follows by observing that the perpendiculars to the sides at

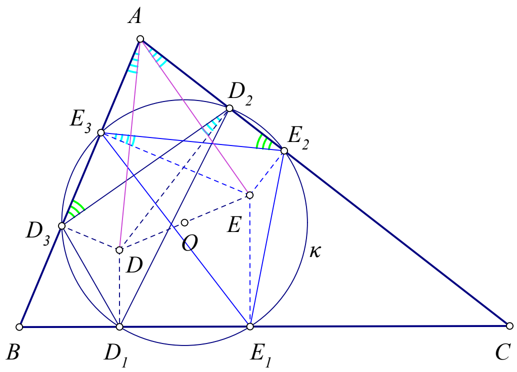

Figure 40: The pedal $\triangle E_1E_2E_3$ of the isogonal E of D

$\{E_1, E_2, E_3\}$ pass through the symmetric E of D w.r.t. O. The points $\{D, E\}$ are *"isogonal conjugate"* w.r.t. $\triangle ABC$ in the sense that the angles $\widehat{DAD_3} = \widehat{EAE_3}$, analogous relations holding w.r.t. the other two vertices of $\triangle ABC$. This follows by observing that triangles $\{AD_2D_3, AE_3E_2\}$ are similar and consequently also the triangles $\{D_3AD, EAE_2\}$ are similar. The argument can be reversed and we arive at the following theorem.

Theorem 11. *The points $\{E, D\}$ are isogonal w.r.t. to the triangle ABC, if and only if their pedals have the same circumcircle κ.*

The theorem establishes a map f in the set of all pedals of $\triangle ABC$ which is *"involutive"* in the sense that it coincides with its inverse $f^2 = id$. It has also a *"fixed point"* in this set, coinciding with the pedal of the incenter of $\triangle ABC$. Next exercises formulate some additional properties of the two pedals of isogonal points.

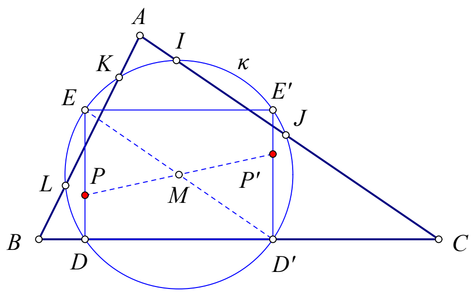

Figure 41: The rectangle defined by isogonals and a side

Exercise 6. *Let $\{D, D'\}$ be the projections on BC of the isogonal points $\{P, P'\}$ and $\{E, E'\}$ the second intersections with the circle κ of the projections of $\{P, P'\}$ on the sides of the triangle respectively with the lines $\{PD, P'D'\}$. Show that $DD'E'E$ is a rectangle.*

Remark 10. Notice that in figure 41 the product $PE = P'D'$ and $PD \cdot P'D' = PD \cdot PE = k$ is the power of P w.r.t. κ which is the same also for the other projections of the isogonal points $\{P, P'\}$ on the sides of $\triangle ABC$. This leads to the simple relation of the trilinear coordinates of $\{P, P'\}$: $xx' = yy' = zz' = k$.

Exercise 7. *With the notation and conventions of this section show the following properties (see figure 42).*

1. *Quadrangles DD_3AD_2 and EE_2AE_3 are cyclic and similar.*
2. *Line AH, H being the intersection of lines $\{D_2D_3, E_2E_3\}$ is orthogonal to DE.*
3. *Line D_3E_2 passes through the pole F of AH with respect to κ the common circumcircle of the pedals $\{D_1D_2D_3, E_1E_2E_3\}$.*
4. *Quadrangles DD_3AD_2 and EE_2AE_3 are perspective with respect to F.*
5. *The perspectivity axis of point-perspective triangles $\{DD_2D_3, EE_3E_2\}$, line HIK, passes through the center O of κ and is orthogonal to AF.*

Analogous statements hold for similar constructs with respect to the other vertices $\{B, C\}$ of triangle ABC.

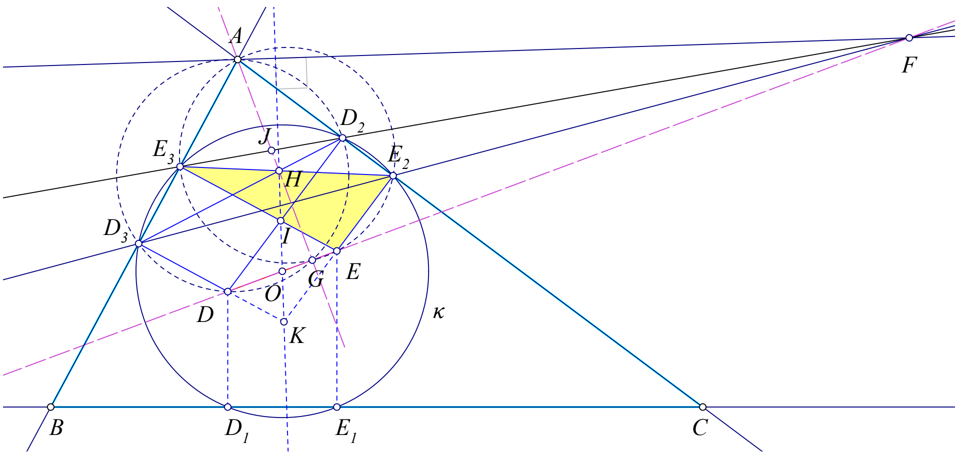

Figure 42: Pedal properties of isogonal points $\{D, E\}$

Hint: Nr-1 Follows by angel chasing argument in figure 40 showing the similarity of the corresponding triangles.
Nr-2 H is on the radical axis of the circumcircles of the similar quadrangles (actually it is the radical center of the three circles drawn). Since $\{AD, AE\}$ are diameters of these circles, their radical axis is orthogonal to line DE.
Nr-3 Considering F to be the intersection point of $\{D_2E_3, D_3E_2\}$ we have a complete quadrilateral and F is the pole of AH with respect to the circle, thus the orthogonal OG to AJ passes through F.
Nr-4 Follows from *nr-3*.
Nr-5 The perspectivity axis is also diagonal of parallelogram $DKEI$, hence passes through the middle O of DE. By an argument like that of *nr-3* we deduce that H is the pole of line AF, hence OH is orthogonal to AF.

Exercise 8. *Show that the pedal $E_1E_2E_3$ of the isogonal point E of a point D w.r.t the isosceles $\triangle ABC$ is similar to the pedal $D_1D_2D_3$ when D is on the circle (BIC) passing through the incenter I of $\triangle ABC$ (see figure 43).*

Figure 44 shows an other property of the triangle related to isogonal points $\{D, E\}$ and their pedals. This is a theorem going back to Steiner ([Ste71, I, pp. 191-120]) and stating, that *"an ellipse tangent to the sides of the triangle has isogonal conjugate focal points*

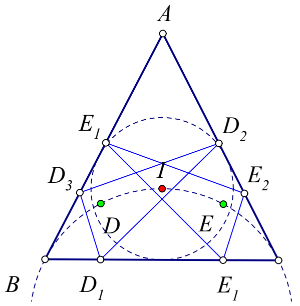

Figure 43: Isogonal points $\{D, E\}$ on (BIC)

$\{D, E\}$". The proof of this results easily from the *"isogonal property"* of conics discussed in the book ([AZ12, p.11]), and according to which, *"the tangents to the ellipse from an external point form with the focals equal angles"* as e.g. in figure 44 the angles $\widehat{BAD} = \widehat{CAE}$. The ellipse has the common circumcircle κ of the two pedals $\{D_1D_2D_3, E_1E_2E_3\}$ as auxiliary circle. Further the points of contact with the sides are the harmonic conjugates $\{A'(D_1E_1), B'(D_2E_2), C'(D_3E_3)\}$ of the corresponding intersections $\{A', B', C'\}$ of the triangle sides with line DE.

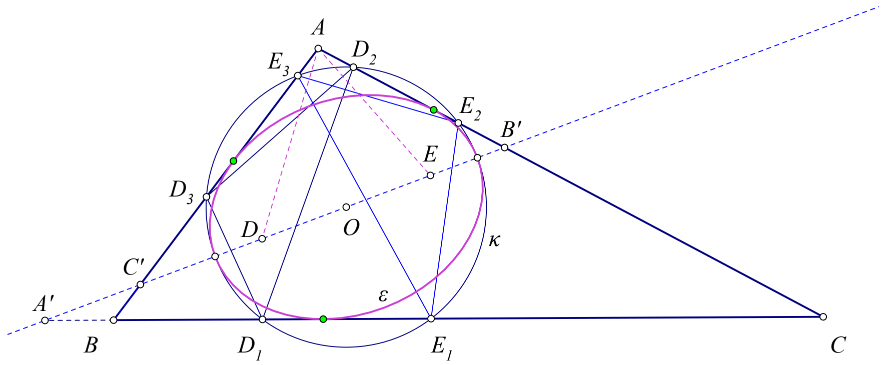

Figure 44: Isogonal points $\{D, E\}$ as focals of an inscribed ellipse

Remark 11. The preceding property suggests an easy method to construct the ellipse inscribed in the triangle with focal points two given isogonal conjugate points $\{D, E\}$. Having these points we can construct the circle κ and its diameter points along line EF which are two points of the conic. Then by the aforementioned conjugates $\{A'(D_1E_1), \ldots\}$ we obtain three more points on the conic and construct it as a conic through 5 points.

Having gone this far, we should recall the nice related property of two special isogonal points, namely the focal points of the *"Steiner in-ellipse"* i.e. the inscribed in the triangle ellipse contacting the sides at their middle. The theorem, has been given an elementary proof that can be found in the paper *"Carlson's proof of Marden's theorem"* anonymously circulating in the internet. The title *Marden's theorem* is erroneous, since the theorem was actually discovered 80 years earlier by Siebeck ([Sie64], [Bog17]). It can be formulated as follows in terms of complex polynomials:

Theorem 12. *Let $f(z)$ be the complex polynomial whose roots are the three vertices of the triangle ABC. Then the roots of the derivative $f'(z)$ are the two focals $\{D, E\}$ of the Steiner ellipse which is inscribed in the triangle (see figure 45).*

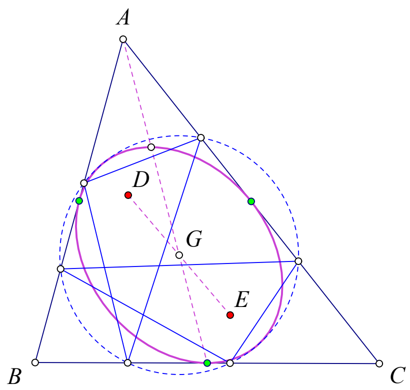

Figure 45: The Steiner in-ellipse of $\triangle ABC$

16 More on pedals of isogonal points

In this section we deal with some additional properties of the pedals of isogonal points.

Theorem 13. *Let the triangles $\{A_1B_1C_1, A_2B_2C_2\}$ be the pedals of isogonal points respectively $\{P, P'\}$. Let also $\{A'B'C', A''B''C''\}$ be corresponding pivoting triangles about $\{P, P'\}$ rotated with respect to the pedals by the angles $\{+\phi, -\phi\}$. Then $\{A'B'C', A''B''C''\}$ have the same circumcircle, whose center S lies on the medial line of PP' (see figure 46).*

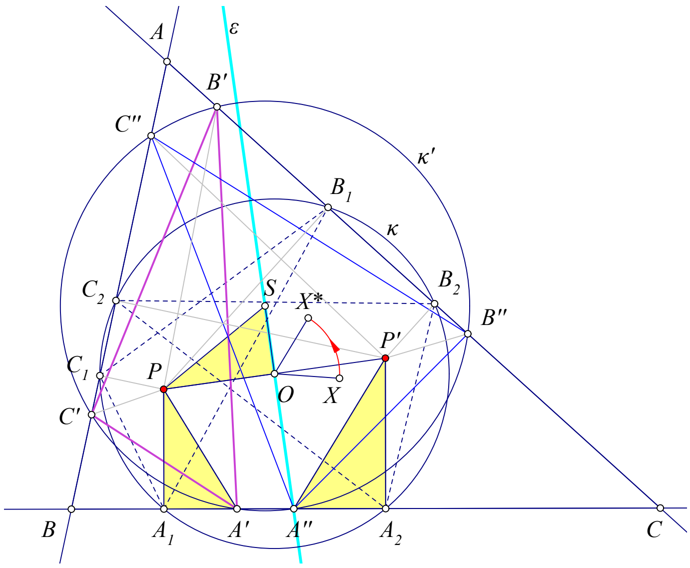

Figure 46: The positions of the circumcenters of pivoting triangles about P

Proof. The proof uses the similarities f_1 mapping $\{A_1B_1C_1\}$ to $A'B'C'$ and f_2 mapping $\{A''B''C''\}$ to $A_2B_2C_2$. These two similarities have their centers respectively at $\{P, P'\}$, their angles are equal and opposite in measure and their stretching ratios are the inverse numbers $\{PA'/PA_1, P'A_2/P'A''\}$. It is easily seen that their composition $f = f_2 \circ f_1$ is the rotation with center the middle O of PP' and angle of measure 2ϕ. This rotation leaves invariant the common circumcircle $\kappa(O)$ of the two pedals. Hence, since f_1 maps κ to the circumcircle κ' of $A'B'C'$, the similarity f_2 maps κ' to κ, showing that the circumcircle of $A''B''C'' = f_2^{-1}(A_2B_2C_2)$ is $f_2^{-1}(\kappa) = f_2^{-1}((f_2 \circ f_1)(\kappa)) = f_1(\kappa) = \kappa'$ as claimed. $\square$

Corollary 2. *The circumcenters S of the pivoting triangles $\{A_1B_1C_1\}$ about the point P lie on the medial line of the segment PP', where P' is the isogonal conjugate of P.*

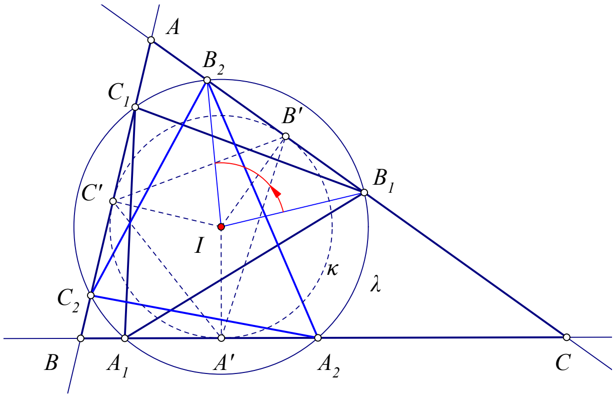

Figure 47: Triangles pivoting about the incenter I of $\triangle ABC$

Figure 47 shows the case of the pedal $\triangle A'B'C'$ of the incenter I, the *"intouch triangle"* and the triangles pivoting about I. Their circumcircles λ are concentric to the incircle κ of $\triangle ABC$ and intersect the sides in two triples of points defining triangles $\{A_1B_1C_1, A_2B_2C_2\}$ equal to each other and similar to the intouch triangle.

The following exercise grew out of an attempt to find, for a generic triangle, all the points P different from the incenter I and the Brocard points, such that the pedals of P and of its isogonal P' are similar, hence equal, since they have the same circumcircle.

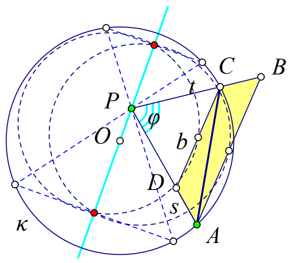

Figure 48: Positions of maximization/minimization of $|AC|$

Exercise 9. *Consider an angle of constant measure $0 < \phi < \pi$ rotating about a point $P \neq O$ inside the circle $\kappa(O)$. Show that the minimal / maximal lengths of the segment AC defined by the intersection of the circle with the sides of the angle are obtained when the diameter OP becomes a bisector of the angle ϕ (see figure 48).*

Hint: With the notation in figure 48, $b = |DC|, t = |PC|, s = |DA|, k = \sin(\phi/2)$, apply the theorem of Ptolemy ([Cou80, p.238]) to the equilateral trapezium and see that $x = |AC|$ satisfies

$$x^2 = 4t^2k^2 + 4tsk^2 + s^2 \quad \Rightarrow \quad \frac{x^2}{t^2} = 4k^2 + 4k^2\frac{s}{t} + \frac{s^2}{t^2}\,.$$

The requested positions occur when $s = 0$. Alternatively, show that the centers of the segments $\{AB, CD\}$ describe respectively two equal circles intersecting at points of OP.

Bibliography

[AZ12] A Akopyan and A Zaslavsky. *Geometry of Conics*. American Mathematical Society, New York, 2012.

[Bog17] B. Bogosel. A geometric proof of the Siebeck-Marden theorem. *Amer. Math. Monthly*, 124:459–463, 2017.

[Cou80] Nathan Altshiller Court. *College Geometry*. Dover Publications Inc., New York, 1980.

[Gib21] B. Gibert. *Cubics in the triangle plane*. https://bernard-gibert.pagesperso-orange.fr/, 2021.

[Joh17] Roger Johnson. Directed Angles and Inversion. *American Math. Monthly*, 24:313–317, 1917.

[Joh60] Roger Johnson. *Advanced Euclidean Geometry*. Dover Publications, New York, 1960.

[Kim18] Klark Kimberling. *Encyclopedia of Triangle Centers*. http://faculty.evansville.edu/ck6/encyclopedia/ETC.html, 2018.

[Sie64] J. Siebeck. Ueber eine neue analytische Behandlungsweise der Brennpunkte. *J. Reine Angew. Math.*, 64:175–182, 1864.

[Ste71] Jacob Steiner. *Gesammelte Werke vol. I, II*. Chelsea Publishing Company, New York, 1971.

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