Projective Line

A file of the Geometrikon gallery by Paris Pamfilos

Exclusive of the abstract sciences, the largest and worthiest portion of our knowledge consists of aphorisms: and the greatest and best of men is but an aphorism.

S. Coleridge, Aids to Reflection XXVII

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1 Projective line

The standard model of the *"projective line"* consists of the classes $[x_1, x_2]$ of non-zero vectors of $\mathbb{R}^2$ modulo non-zero multiplicative constants. For every non-zero vector $a = (a_1, a_2)$ of $\mathbb{R}^2$, the symbol $[a] = [a_1, a_2]$ denotes a *"point"* of the projective line and the pair $a = (a_1, a_2)$ is called a *"representative"* of the point. Two representatives $\{a, a'\}$ define the same point if and only if $a' = ka$, with a non-zero real number k.

The most important examples of projective lines are the usual lines of the euclidean plane to which we add an additional point, called point at infinity. This additional point makes the line closed and is called *"projectification"* of the line (or *"one-point-compactification"*). The projectification is illustrated by figure 1. In this we consider the points of line

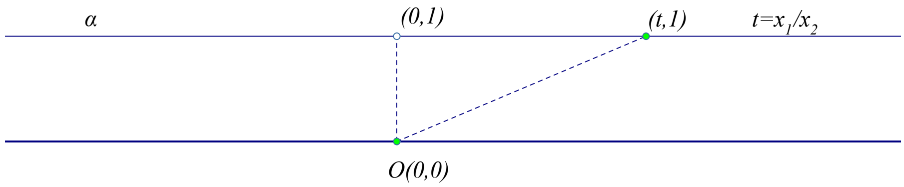

Figure 1: Projective line represented by (a line + a point at infinity)

$\alpha : y = 1$, parallel to the $x-$axis. Each line represented by $[x_1, x_2]$, with non-zero x_2, intersects α at the point $(t, 1)$ with $t = x_1/x_2$. The line parallel to α which is the $x-$axis is represented by $[x_1, 0]$ considered as an additional point of α. The set A^* of all lines of the plane through the fixed point A, generalizes this basic model of projective line and is called the *"pencil"* of lines through A. Figure 2 shows that a pencil A^* is like a circle. The

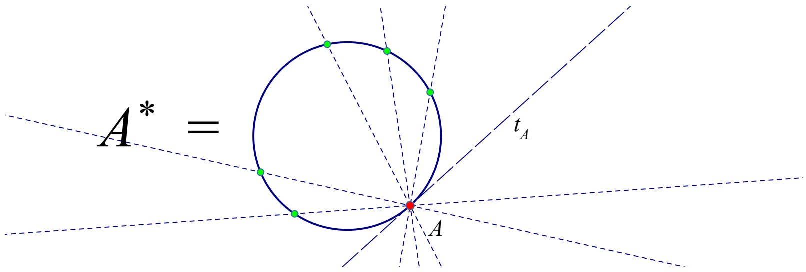

Figure 2: A pencil of lines is like a circle

idea is to pass a circle κ through A and associate to each line through A its intersection with κ. The only line that has no associate is the tangent t_A at A. The remedy for this is to associate to t_A the point A. This makes the projective line via its *"pencil model"* A^* a *"closed"* set and *"homeomorphic"* to a circle.

2 Homogeneous coordinates

The pairs $\{(x_1, x_2)\}$ defining line $[x_1, x_2]$, considered as a *"point"* of the projective line are called *"homogeneous projective coordinates*. They are defined modulo a non-zero multiplicative constant, since (kx_1, kx_2) defines the same point. Obviously another system (x'_1, x'_2) of coordinates of $\mathbb{R}^2$ (with the same origin) is related to (x_1, x_2) by an invertible matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{defines a map} \quad \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \qquad (1)$$

And for the corresponding quotients $t' = x'_1/x'_2, t = x_1/x_2$ we get the so called *"homographic relation"* (see file **Homographic relations**).

$$t' = \frac{at+b}{ct+d}. \tag{2}$$

Notice that a non-zero multiple $A' = kA$ of the matrix defines the same transformation between the homogeneous coordinates.

3 Projective base

Three distinguished points $\{A, B, C\}$ on a projective line determine a so-called *"projective base"* and through it a corresponding *"homogeneous coordinates system"*. The mechanism can be explained in terms of bases of the vector space $\mathbb{R}^2$.

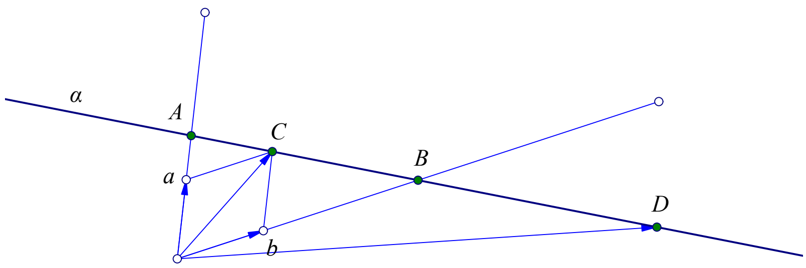

Figure 3: Projective base defined through three collinear points

The points $\{A = [a], B = [b], C = [c]\}$ of the projective line determine three lines represented by corresponding vectors $\{a, b, c\}$. The vectors are selected so that $c = a + b$. This condition uniquely determines $\{a, b, c\}$ from the data $\{A = [a], B = [b], C = [c]\}$, up to a multiplicative constant.

The homogeneous coordinate system results by using the base $\{a, b\}$ of $\mathbb{R}^2$ and writing vor every point D on this line its coordinates with respect to this base $D = [d]$, where $d = d_1 a + d_2 b$. Thus, to D we correspond (d_1, d_2), and it is clear that a non-zero multiple $k(d_1, d_2)$ defines the same point D on the projective line, therefore we often write

$$D = d_1 A + d_2 B, \tag{3}$$

using the symbols for the points $\{A, B\}$ rather than the symbols $\{a, b\}$ for the vectors representing them.

Obviously in this coordinate system the points $\{A, B, C\}$ have correspondingly the coordinates $\{(1, 0), (0, 1), (1, 1)\}$. Point C is called *"coordinator"* or *"unit"* of the projective base $\{A, B, C\}$. It is used to calibrate a basis defined by the other two points. The calibration is done by projecting some vector c, parallel to the other lines determined by $\{A, B\}$ and using the base of R^2 resulting from these projections $\{a, b\}$ (See Figure 3). Having this calibration, the point P of the line can be written as a linear combination

$$P = xA + yB \tag{4}$$

Passing from this equation to the vector equation, there is an ambiguity, which we eliminate using the *"unit"* C of the base. In fact, we could select two other vectors $a' = \mu a$, and $b' = \nu b$ representing the same points $\{A, B\}$ and write for the same point P :

$$P = [x'a' + y'b'] = x'A + y'B = xA + yB = [xa + yb]. \tag{5}$$

Then, we would obtain two different pairs $\{(x', y'), (x, y)\}$ for the same point and this is not good, except if it happens to be $(x', y') = k(x, y)$ with a $k \neq 0$. But this follows from the requirement to have

$$C = A + B. \tag{6}$$

Thus if we select the vectors $\{a, b\}$ to represent $\{A, B\}$ and then two other $\{a', b'\}$ to represent the same base points, the requirement $C = A + B$ implies the vectorial relation

$$c = \lambda(a + b) = \lambda'(a' + b') = \lambda'(\mu a + \nu b) \quad \Rightarrow \quad \lambda = \lambda'\mu = \lambda'\nu \quad \Rightarrow \quad \mu = \nu, \tag{7}$$

and we are salvaged. Thus, you have a certain freedom to select two vectors $a, b \in \mathbb{R}^2$ to represent your base points $\{A, B\}$, but you must always respect the rule $a + b = kc$ with a $k \neq 0$, so that if you select two others $\{a', b'\}$, this rule will imply that $\{a' = ra, b' = rb\}$ with $r \neq 0$.

4 Relation between euclidean and projective base

In this section we consider a line ε in the euclidean plane, defined by two position vectors $\{a, b\}$ and also a point on this line $x = x_a a + x_b b$, where the variables $\{x_a, x_b\}$ satisfy $x_a + x_b = 1$ (See Figure 4). Often the running point x on the line determined by the po-

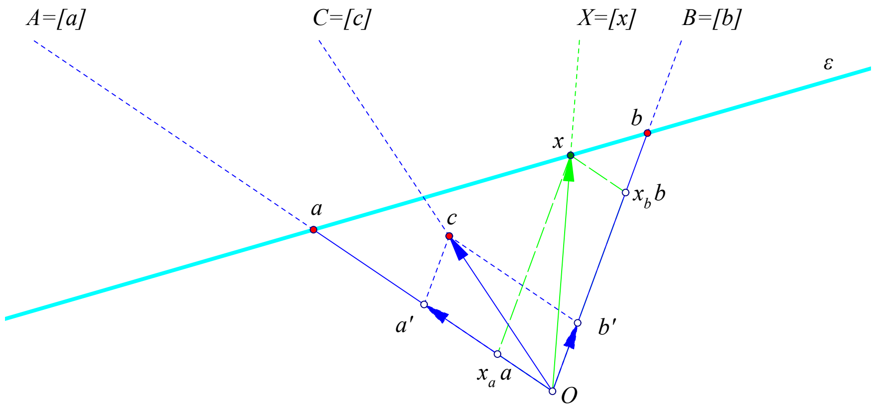

Figure 4: Euclidean base $\{a, b\}$ versus projective base $\{A, B, C\}$

sition vectors $\{a, b\}$ is described parametrically by

$$x = a + (b - a)t \quad \Rightarrow \quad x = (1 - t)a + tb \quad \Rightarrow \quad x_a = (1 - t), \;\; x_b = t, \tag{8}$$

which for the euclidean signed ratio of the segments defined by x imply

$$\frac{xa}{xb} = \frac{t}{t - 1} = -\frac{x_b}{x_a}. \tag{9}$$

We consider also the points $\{A = [a], B = [b], X = [x],\}$ defined by these vectors on the same *"extended"* projective line. We want to find the expression of the projective homogeneous coordinates of X relative to the projective base $\{A, B, C\}$, where $C = [c]$ for an arbitrary other vector different from $\{a, b\}$. We can represent c in the vectorial base a, b through

$$c = c_a a + c_b b = a' + b'.$$

Thus if

$$x \quad = \quad x'_a a' + x'_b b' \quad = \quad x'_a c_a a + x'_b c_b b \quad = \quad x_a a + x_b b,$$

the projective coordinates of X are $\{\lambda x'_a, \lambda x'_b\}$, so that $X = (\lambda x'_a)A + (\lambda x'_b)B$ and from the previous equalities we obtain

$$x_a = x'_a c_a, \;\; x_b = x'_b c_b.$$

Thus, if we know the euclidean (cartesian) coordinates $\{x_a, x_b\}$ of x relative to the vectorial base $a, b,$ then the projective coordinates $\{x'_a, x'_b\}$ of $X = [x]$ relative to the projective base $\{A, B, C\}$ are

$$X = x'_a A + x'_b B \quad \text{with} \quad x'_a = \lambda \frac{x_a}{c_a}, \qquad x'_b = \lambda \frac{x_b}{c_b}, \quad \text{and} \quad \lambda \neq 0. \tag{10}$$

Thus, taking into account the equation 9, we obtain the relation between the ratio of the projective coordinates and the euclidean signed ratio of distances xa/xb.

$$\frac{x'_a}{x'_b} = \frac{c_b}{c_a} \cdot \frac{x_a}{x_b} = -\frac{c_b}{c_a} \cdot \frac{xb}{xa}. \tag{11}$$

5 Change of projective bases

Having two projective bases $\{A, B, C\}$, $\{A', B', C'\}$ of the line α, we can express the same point P of the line as linear combination of the base points:

$$P = xA + yB = x'A' + yB', \quad \text{where} \quad [a] = A, [a'] = A', [b] = B, [b'] = B. \tag{12}$$

Using a bit of linear algebra and the matrix K for the basis-change in $\mathbb{R}^2$ we obtain

$$\text{with} \quad K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}, \; \begin{cases} a' = k_{11}a + k_{21}b, \\ b' = k_{12}a + k_{22}b, \end{cases} \quad \Rightarrow \quad \begin{cases} \lambda x = k_{11}x' + k_{12}y', \\ \lambda y = k_{21}x' + k_{22}y'. \end{cases} \tag{13}$$

Thus, we obtain the rule, that *by homogeneous bases change the coordinates transform by a matrix and corresponding linear equations*:

$$\lambda \begin{pmatrix} x \\ y \end{pmatrix} = K \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \mu K^{-1} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{14}$$

Taking the quotients of coordinates in equation 13 we see that they are related by a simple function

$$t = \frac{x}{y} \quad \text{and} \quad t' = \frac{x'}{y'} \quad \Rightarrow \quad t = \frac{k_{11}t' + k_{12}}{k_{21}t' + k_{22}}. \tag{15}$$

Such relations between two variables are called *"homographic"*, so that we can state the following trivial theorem.

Theorem 1. *The quotients x/y of homogeneous coordinates for the same point in two different bases of coordinates are related by a homographic relation as in equation 15.*

We note that this relation is *"symmetric"*, in the sense that solving it for t', we find that t' is expressible through t by a function of the same kind. In section 8 we discuss again this relation, which plays a central role in projective geometry.

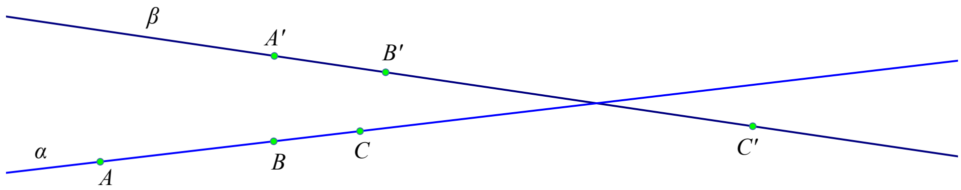

Figure 5: Line projectivity through a linear map between corresponding coordinates

6 Line projectivities or homographies

A bijective map $f : \alpha \to \beta$ between two projective lines $\{\alpha, \beta\}$ is called a *"projectivity"* or *"homography"*, when its representation in homogeneous coordinates is a linear map. This means that selecting any projective bases $\{A, B, C\}$ on α and $\{A', B', C'\}$ on β and using the corresponding coordinate systems (x, y) and (x', y'), the transformation $P' = f(P)$ is described in the respective homogeneous coordinates through linear equations:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} px + qy \\ rx + sy \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{16}$$

Since the change of homogeneous coordinates in each projective line is done also through linear maps, a change of homogeneous coordinates replaces the preceding matrix U with a matrix $U' = YUX$, where $\{X, Y\}$ are also invertible matrices related to the coordinate changes in α and β. Thus the linearity requirement, though it involves a coordinate system on each line, is independent of the specific systems used. This means that if the representation of the map f in a couple of homogeneous coordinate systems in $\{\alpha, \beta\}$ is expressed by a matrix, then the representation of the same map in other homogeneous coordinates systems in the two lines is also expressed by an appropriate matrix.

Using a bit of linear algebra, we can even see that for an arbitrary projectivity, we can find appropriate bases on the lines $\{\alpha, \beta\}$, so that the corresponding representation of the projectivity w.r. to those bases is the *"identity matrix"*. To do this, it suffices to choose initially arbitrary bases and represent the projectivity by an invertible matrix U. Then, change the basis on β only, which means that in the new base the representation will be of the form $U' = YU$, where Y is the matrix of the coordinate change, which can be any invertible matrix. Using this freedom, we can then select $Y = U^{-1}$, which implies that in the new base the projectivity is represented by $U' = YU = U^{-1}U = I$, i.e. the identity matrix.

7 The fundamental theorem for line projectivities

Theorem 2. *There is a unique projectivity between lines $\{\alpha, \beta\}$ for which we prescribe that three pairwise different but otherwise arbitrary points $\{A, B, C\}$ should map correspondingly to three other arbitrarily chosen points $\{A', B', C'\}$.*

Proof. The proof of the existence of such a projectivity is fairly simple. We use the given triples of points to define corresponding bases on the two lines. Then to the variable point $P = xA + yB$ on α we associate the point $P' = xA' + yB'$ on the line β. This way we define a projectivity $f : P \mapsto P'$ representehd in the two selected bases by the identity matrix and obviously satisfying the requirements $f(A) = A', f(B) = B'$ and $f(C) = C'$.

The uniqueness of a projectivity with such prescriptions follows also easily. In fact, if g is a second projectivity with $g(A) = A', g(B) = B', g(C) = C'$, then selecting for bases in

α, β correspondingly the triples $\{A, B, C\}$ and $\{A', B', C'\}$, we verify that in these bases the two maps are represented by (multiples of) the identity matrix, hence they coincide. In fact, assume that g is represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, the coordinates of A are $(1, 0)$ and must map to A' with coordinates also $(1, 0)$, hence $(1, 0) = \lambda(a, c) \Rightarrow c = 0$. Analogously $(0, 1) = \mu(b, d) \Rightarrow b = 0$. Hence the matrix is diagonal of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. But the unit point $C = A + B$ with coordinates $(1, 1)$ must map to (a, d), which is assumed also to be $C' = A' + B'$ with coordinates $(1, 1)$. Thus $(a, d) = \nu(1, 1)$ and, finally, the matrix representing g must be a mutltiple of the identity, as claimed. □

8 Homographic relations

Working with examples of extended lines in the euclidean plane it is often useful to switch from homogeneous projective coordinates on the line to euclidean coordinates and signed ratios of distances on the euclidean plane. The basic relations for this interplay are expressed with equation 11, resulting by considering ratios of homogeneous coordinates and not the coordinates themselves. In fact, if (x, y) are the projective coordinates of a point X of a projective line, referred to the basis $\{A, B, C\}$, so that $X = xA + yB$, then we can also write

$$X = xA + yB = A + (y/x)B = A(x/y) + B, \tag{17}$$

since the coordinates can be multiplied by a non-zero constant. We assume here that neither of $\{x, y\}$ is zero. This can be applied also to projectivities or homographies between projective lines, which, in respective homogeneous coordinates for the two lines, are described by a matrix as in equation 16. From these equations, passing from the coordinates to respective ratios, we obtain

$$\text{for } t' = x'/y' \quad \text{and} \quad t = x/y \quad \Rightarrow \quad t' = \frac{pt + q}{rt + s}. \tag{18}$$

Two variables such as $\{t, t'\}$ satisfying such a relation are said to *"satisfy a homographic relation"*. Often also the homographic relation is encountered in the symmetric form resulting from the previous one after elimination of the denominator:

$$att' + bt + ct + d = 0. \tag{19}$$

Thus, we arrive at the trivial theorem

Theorem 3. *The map $f : \alpha \to \beta$ is a homography or projectivity if for any homogeneous projective coordinate systems for $\{\alpha, \beta\}$ it has the representation of equation 18, i.e. the quotients of coordinates satisfy a homographic relation.*

Using equation 11 to replace quotients of projective coordinates with ratios of euclidean signed distances, we obtain the corresponding relations

$$-\frac{c'_{b'}}{c'_{a'}} \cdot \frac{x'b'}{x'a'} = \frac{p\left(-\frac{c_b}{c_a}\frac{xb}{xa}\right) + q}{r\left(-\frac{c_b}{c_a}\frac{xb}{xa}\right) + s} \quad \Rightarrow \quad \frac{x'b'}{x'a'} = \frac{p^*\left(\frac{xb}{xa}\right) + q^*}{r^*\left(\frac{xb}{xa}\right) + s^*}, \tag{20}$$

with

$$p^* = p\frac{c_b}{c_a} \cdot \frac{c'_{a'}}{c'_{b'}}, \quad q^* = -q\frac{c'_{a'}}{c'_{b'}}, \quad r^* = -r\frac{c_b}{c_a}, \quad s^* = s. \tag{21}$$

Thus, the ratios of signed distances obey the same rule and we have

Theorem 4. *The map $f : \alpha \to \beta$ between two (extended) lines of the euclidean plane is a homography if and only if the quotients of the signed distances $\{t' = x'a'/x'b',\ t = xa/xb\}$ from the respective basis position vectors on the lines satisfy a homographic relation*

$$t' = \frac{s^* t + r^*}{q^* t + p^*}.$$

Notice that the homographic relations for the two sets of quotients, projective and euclidean, are not the same and their connection is given by equations 20 and 21. This theorem is used below in section 11 to obtain an important homography between two tangents of a circle.

9 Homographic relations, the group properties

For the general homographic relation, which can be expressed by a function $x' = f(x)$ of the form

$$x' = \frac{ax + b}{cx + d} \quad \text{with} \quad ad - bc \neq 0, \tag{22}$$

for which we can immediately make three simple observations. The first that it is connected with a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{replaceable with} \quad B = kA = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix},$$

since using B instead of A gives the same function $x' = f(x)$. The second observation is that the relation is *"symmetric"* in the sense, that its inverse, resulting by solving w.r. to x : $x = f^{-1}(x')$ has the same form

$$x = \frac{-dx' + b}{cx' - a} \quad \text{with} \quad B = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix} \quad \text{satisfying} \quad A \cdot B = (bc - ad) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{23}$$

The third that the composition of two successive such relations is again a relation of the same form

$$x' = \frac{ax + b}{cx + d} \quad \text{and} \quad x'' = \frac{a'x' + b'}{c'x' + d'} \quad \Rightarrow \quad x'' = \frac{a''x + b''}{c''x + d''}, \tag{24}$$

where the last function is expressible through the product of matrices

$$\begin{pmatrix} a'' & b'' \\ c'' & d' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

From these observations follows easily that the set of all homographic relations is a *"group"* known as *"the projective linear group $PGL(2, \mathbb{R})$"* or $PGL(2, \mathbb{C})$ if we work in the complex domain.

10 Line perspectivities

The simplest and most prominent example of line projectivity is the *"line perspectivity"*, by which points on two projective lines $\{\alpha, \beta\}$ correspond through a map $P' = f(P)$, such that all lines $\{PP'\}$ pass through a fixed point R, called the *"center of perspectivity"*. Figure 6 illustrates such a map. To verify the linearity condition required for projective maps, we take R to be the origin of coordinates and the projective bases $\{A, B, C\}$ on α and

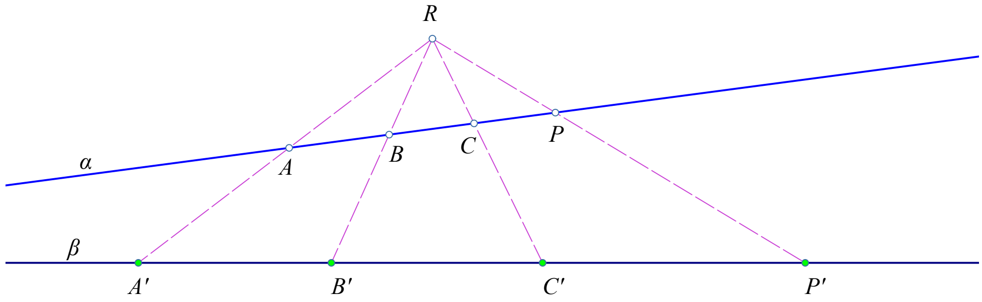

Figure 6: Perspectivity between two lines

$\{A' = f(A), B' = f(B), C' = f(C)\}$ on β. With respect to this bases the map f is described through the identity transformation

$$\begin{pmatrix} x'(P') \\ y'(P') \end{pmatrix} = \begin{pmatrix} x(P) \\ y(P) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(P) \\ y(P) \end{pmatrix}, \tag{25}$$

which is the simplest invertible linear map.

11 Circle and tangents homography

A relatively simple and important homography is one created by a circle and two different fixed tangents of it. The whole story relays on a theorem for the inradius r of triangles XOY according to which

$$r^2 = \frac{|OA||AX||BY|}{|OA|+|AX|+|BY|} = \frac{a(x-a)(y-a)}{a+(x-a)+(y-a)} \quad \Rightarrow \quad y = (a^2+r^2)\frac{x-a}{ax-(a^2+r^2)},$$

where $a = |OA|$ and the coordinates along the fixed tangents $\{OA, OB\}$ have their origin

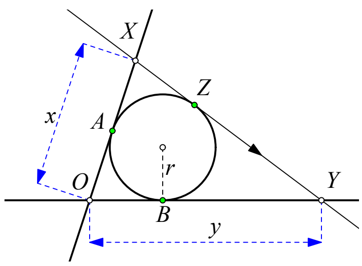

Figure 7: Homography defined by two tangents

at O (See Figure 7). The last relation is a homographic one between the two fixed tangents of the circle at its points $\{A, B\}$. The homographic relation $Y = f(X)$ is defined by intersecting these two tangents with a third tangent XY at the variable point Z of the same circle. We formulate this result as a theorem.

Theorem 5. *Fixing two tangents $\{\varepsilon, \varepsilon'\}$ of a circle and intersecting them with a variable tangent η at points $\{X = \eta \cap \varepsilon, X' = \eta \cap \varepsilon'\}$ defines a homography $f : \varepsilon \to \varepsilon'$ with $X' = f(X)$.*

12 Cross ratio or anaharmonic ratio

To define the *"cross-ratio"* or *"anharmonic ratio"* of four points $\{P, Q, U, V\}$ on a projective line ε one can again use homogeneous coordinates w.r.t. to some homogeneous coordinate system $\{A, B, C\}$ for ε, w.r.t. to which $\{P = p_1 A + p_2 B,\ \ Q = q_1 A + q_2 B,\ \ldots\}$, in short

$$P(p_1, p_2),\ \ Q(q_1, q_2),\ \ U(u_1, u_2),\ \ V(v_1, v_2).$$

Then define the corresponding quotients

$$p = p_1/p_2,\ \ q = q_1/q_2,\ \ u = u_1/u_2,\ \ v = v_1/v_2,$$

and finally define the *"cross ratio"* through

$$(PQ; UV) = \frac{p-u}{q-u} : \frac{p-v}{q-v}. \tag{26}$$

The important point is again that this definition, although it uses a system of homogeneous coordinates, it defines a number independent of the particular coordinates used. In fact, changing to another system of homogeneous coordinates, according to equation 15, involves a homographic relation between the corresponding quotients of the coordinates $\{(x, y), (x', y')\}$ of the same point referred to the two systems. Thus $\{p, q, u, v\}$ change to

$$p' = f(p),\ \ q' = f(q),\ \ u' = f(u),\ \ v' = f(v) \quad \text{with} \quad f(x) = \frac{ax+b}{cx+d}, \tag{27}$$

which replaced in equation 26 and doing a bit of calculation leads to the claimed independence of the value of $(PQ; UV)$ from the particular coordinate system used:

$$(PQ; UV) = \frac{p-u}{q-u} : \frac{p-v}{q-v} = \frac{p'-u'}{q'-u'} : \frac{p'-v'}{q'-v'}.$$

The same reasoning can be applied to *"line-projectivities or homographies"* of the line, since for these the quotients of coordinates are also homographically related, according to theorem 3. We have thus the trivial but important

Theorem 6. *A projectivity or homography $Y = f(X)$ of a line preserves the cross ratio of four points i.e.*

$$\text{if} \quad P' = f(P),\ \ Q' = f(Q),\ \ U' = f(U),\ \ V' = f(V) \quad \text{then} \quad (P'Q'; U'V') = (PQ; UV).$$

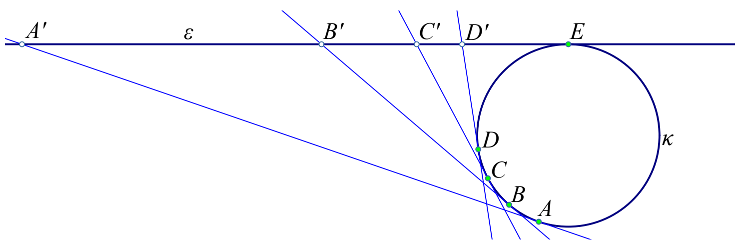

Figure 8: Four tangents cut by a fifth in constant cross ratio

This, combined with theorem 5 leads to the following theorem (See Figure 8).

Theorem 7. *Four fixed tangents of a circle at the points $\{A, B, C, D\}$ intersect any fifth tangent ε at a variable point E at four points $\{A', B', C', D'\}$ having constant cross ratio $(A'B'; C'D')$.*

In the file **Cross ratio** we show that the cross ratio $(A'B'; C'D')$ on the line ε is equal to the cross ratio $(AB; CD)$ on the circle. The fact that it is possible to define the notion of cross ratio *for points of a circle* is also discussed in that file.

13 Projectivities defined by cross ratios

We can use cross ratios to define projectivities and prepare the ground for an inverse of theorem 6.

Theorem 8. *Given two triples of the line $\{P, Q, R\}$ and $\{P', Q', R'\}$, the equality*

$$(P'Q'; R'Y) = (PQ; RX) \quad \text{defines a homography} \quad Y = f(X) \quad \text{of the line.}$$

Proof. In fact, fixing a coordinate system and defining $\{p, q, \ldots p', q' \ldots\}$ as in the preceding section, and setting $x = x_1/x_2, y = y_1/y_2$ for the quotients of coordinates of the variable $\{X, Y\}$, we obtain

$$(P'Q'; R'Y') = (PQ; RX) \quad \Leftrightarrow \quad \frac{p' - r'}{q' - r'} : \frac{p' - y}{q' - y} = \frac{p - r}{q - r} : \frac{p - x}{q - x},$$

in which $\{p, q, \ldots p', q', \ldots\}$ are constants. Solving this equation for y we obtain indeed a relation of the form

$$y = \frac{ax + b}{cx + d} \quad \text{with} \qquad a = E'q' - Ep', \quad b = Eqp' - E'pq', \quad c = E' - E, \quad d = Eq - E'p,$$

$$\text{where} \qquad E = \frac{p - r}{q - r} \quad \text{and} \quad E' = \frac{p' - r'}{q' - r'}.$$

This, by theorem 3, proves that f is a homography as claimed. □

Theorem 9. *The converse of theorem 6 is true. If a map $f : \alpha \to \alpha$ of a line α onto itself preserves the cross ratio, then it is a projectivity of α.*

Proof. Consider three points $\{P, Q, R\}$ and their images $\{P' = f(P),\ Q' = f(Q),\ R' = f(R)\}$ via f. By assumption for any fourth point X and its image via $f, Y = f(X)$ we will have

$$(P'Q'; R'Y) = (PQ; RX).$$

But this equation, according to the preceding theorem defines a homography $Y = g(X)$. It is easily seen that we have also $\{g(P) = P', g(Q) = Q', g(R) = R'\}$. Hence the homographies $\{f, g\}$ coincide at the three points, thus coincide everywhere according to the fundamental theorem for projectivities of section 7. □

14 Cross ratio in euclidean coordinates

In section 4 we saw the interplay between homogeneous and cartesian coordinates. Using these relations we can express the cross ratio also in cartesian coordinates. In fact, the relation is very simple, since referring the line to two position vectors $\{a, b\}$ on this line and using a third vector $c = c_a a + c_b b$ we have for any point on that line the relation

$$t' = k/t, \quad \text{where} \quad k = -\frac{c_b}{c_a} \quad \text{and} \quad t = \frac{xa}{xb},$$

latter being the signed ratio of distances of the position vector of the running point x of the line from $\{a, b\}$. Using the previous formulas and applying the same reasoning used as in the proof of equation 27 we find the next simple rule.

Theorem 10. *The cross ratio* $(PQ;UV)$ *of the four points* $\{P,Q,U,V\}$ *of a line is equal to*

$$(PQ;UV) = \frac{p-u}{q-u} : \frac{p-v}{q-v},$$

where $\{p = pa/pb, q = qa/qb, \ldots\}$ *are the ratios of distances of the points, from the position vectors* $\{a,b\}$ *defining the line.*

By this theorem we obtain an even simpler form for the expression of the cross ratio, if we adopt for four collinear points $\{A,B,C,D\}$, the homogeneous base to be defined by $\{A,B\}$ and some other vector for *"unit"*. Then the corresponding ratios of signed distances $\{AA/AB = 0,\ BA/BB = \infty, \ldots\}$ lead to the expression

$$(AB;CD) = \frac{AC}{BC} : \frac{AD}{BD} = \frac{CA}{CB} : \frac{DA}{DB}. \tag{28}$$

This formula gives the way to *define* the cross ratio in the euclidean plane, an aspect which we discuss in the file **Cross ratio**.

Related material

1. **Cross ratio**
2. **Homographic relation**
3. **Projective plane**

Any correction, suggestion or proposal from the reader, to improve/extend the exposition, is welcome and could be send by e-mail to: pamfilos@uoc.gr