Projective plane

A file of the Geometrikon gallery by Paris Pamfilos

He who knows only his own side of the case, knows little of that.

J. Mill, On Liberty ch. II

Contents

1 Projective plane, the standard model

The *"standard model"* of the real projective plane, to which, for the time being we restrict our study, is often denoted by $P\mathbb{R}^2$ or P_2. It consists of the *"classes"* $\{X = [x] = [x_1, x_2, x_3]\}$ of vectors of $x \in \mathbb{R}^3$ modulo non-zero multiplicative constants. For every non-zero vector a of $\mathbb{R}^3, A = [a]$ denotes a point of the projective plane and vector $0 \neq a \in \mathbb{R}^3$ is called a *"representative"* of the point. Two representatives $\{a, a'\}$ defining the same point, if and only if, $a' = ka$, with a non-zero real number k.

model $P\mathbb{R}^2$

The most important shapes of the projective plane are its *"lines"* consisting of all points $\{X = [x]\}$, whose representatives x lie on a plane $a_1x_1 + a_2x_2 + a_3x_3 = 0$ of $\mathbb{R}^3$, the plane passing through the origin. Since every plane is defined by a triple of coefficients $[a] = [a_1, a_2, a_3]$ modulo a non-zero multiplicative constant, we see that the set of lines of the projective plane is itself a projective plane. This is called the dual projective plane and denoted by P_2^*.

projective lines

Shapes of particular importance of the projective plane are the *"algebraic curves"* defined by equations $p(x) = 0$, where $p(x) = p(x_1, x_2, x_3)$ is a homogeneous polynomial of three variables i.e. a polynomial satisfying $p(kx) = k^r p(x)$, r being the *"degree"* of the polynomial. The homogeneity of the polynomial implies that $p(x) = 0$ is a condition on the class $[x]$ and not on its representative. Special cases of such curves are the lines,

projective lines

$$ax_1 + bx_2 + cx_3 = 0,$$

for which the degree $r = 1$ and the conics for which the degree $r = 2$:

projective conics

$$p(x_1, x_2, x_3) = ax_1^2 + bx_2^2 + cx_3^2 + 2dx_1x_2 + 2ex_2x_3 + 2fx_3x_1 = 0. \quad (1)$$

The 2's appearing before the last coefficients are for convenience, when using the matrix notation, in which the equation takes the form:

$$p(x_1, x_2, x_3) = (x_1, x_2, x_3) \begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

Two *"lines"* are different if their corresponding vectors of coefficients are independent. This, using the *"vector product"* can be expressed by the condition:

two lines intersection

$$\begin{pmatrix} a'' \\ b'' \\ c'' \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} bc' - b'c \\ ca' - c'a \\ ab' - a'b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The *"class"* of the vector $[a'', b'', c'']$ represents then the intersection point of the two lines and shows the first difference from the euclidean plane, namely,

"Two different lines have always an intersection point."

The vector product can be used to find also the *"line joining"* two points given by their coordinates (a, b, c) and (a', b', c'). The joining line is again expressible through the vector product giving the coefficients and also the equation in terms of the *"triple product"*

two points line

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ x_1 & x_2 & x_3 \end{vmatrix} = 0.$$

A nice introduction to this and related models of the projective plane can be found in the book by Kendig [Ken11, p.35].

2 Projective base and associated projective coordinates

projective base

By its definition, *"A projective base consists of four points of the projective plane which are in general position."* This means: four points $\{A, B, C, D\}$ with corresponding representing them vectors $\{a, b, c, d\}$, such that every triple from them forms a *base* of $\mathbb{R}^3$.

The fourth point D, whose corresponding representative vector can be expressed by the other three

$$d = \mu a + \nu b + \xi c, \tag{2}$$

is called the *"coordinator"* or *"unit"* point of the basis. It is used to fix the scalar multiples of vectors $\{a, b, c\}$ (I often say *"calibrate"*) with respect to which we define coordinates associated to this base. Lines $\{[a], [b], [c]\}$ define the directions and equation (2) is used to define the three vectors (up to multiplicative constant) by projecting d on the three lines and taking as representatives

role of unit D

$$a' = \mu a,\ b' = \nu b,\ c' = \xi c \quad \Rightarrow \quad d = a' + b' + c'.$$

Figure 1 illustrates the case suggesting the way d calibrates the other three vectors. The

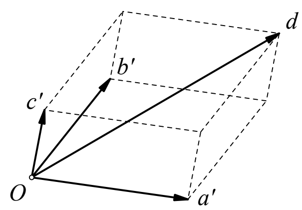

Figure 1: Projecting the diagonal onto three other directions

three vectors determine the lines through O and d sets the diagonal of a parallelepiped, the calibrated vectors being then the edges at O. Considering multiples of the four vectors creates a similar parallelepiped since the directions of the lines remain the same.

Every projective base introduces a corresponding *"homogeneous coordinate system"* by which we associate to *"points"* X of $P\mathbb{R}^2$ *"coordinates"* $(u, v, w) \in \mathbb{R}^3$ and write formal non-sense

$$X = uA + vB + wC. \tag{3}$$

This equation between points, which actually cannot be added, has per definition the meaning comming from the corresponding representative vectors and their analysis to the basis $\{a', b', c'\}$

$$x = ua' + vb' + wc' = u(\mu a) + v(\nu b) + w(\xi c),$$

The numbers $\{u, v, w\}$ being defined up to a non-zero multiplicative constant i.e. satisfying

$$uA + vB + wC = u'A + v'B + w'C \quad \Leftrightarrow \quad (u', v', w') = k(u, v, w) \quad \text{with} \quad k \neq 0.$$

In particular, the points $\{A, B, C, D\}$ defining the base, have correspondingly the coordinates $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$.

In these coordinates a line is represented by a linear equation

$$pu + qv + rw = 0,$$

equations $\{u = 0, \quad v = 0, \quad w = 0\}$ representing respectively the lines $\{BC, CA, AB\}$ described respectively also by combinations with the remaining two coordinates:

$$BC: \ vB + wC, \qquad CA: \ wC + uA \qquad AB: \ uA + vB.$$

Remark 1. The above *"addition of points"* is formal and must be used carefully. For example Y and vY define the same point, but considering another point X and taking the sums $\{X+Y, X+vY\}$ we obtain two different points of the line XY. The equality condition being here also

$$uX + vY = u'X + v'Y \quad \Leftrightarrow \quad (u', v') = k(u, v).$$

The rule is that

rule one

"opperating with such formal sums we are not allowed to incorporate the multiplying coefficients into the symbols and replace vY with Y."

Having two combinations $Z = uX + vY$, $Z' = u'X + v'Y$, the *"addition"*

$$Z + Z' = (uX + vY) + (u'X + v'Y) = (u + u')X + (v + v')Y,$$

is legitimate and represents a point depending, though, on the particular representation of $\{X, Y\}$ by corresponding classes of vectors $\{[x], [y]\}$. Having made a choice for this representation, the point $Z + Z'$ is then the one represented by $[(u + u')x + (v + v')y]$. Representing $\{X, Y\}$ in a different way $\{X = [x'] = [mx], \ Y = [y'] = [ny]\}$ we obtain, in general, a different sum

$$\begin{aligned} Z + Z' &= [(u + u')x' + (v + v')y'] \\ &= [(u + u')(mx) + (v + v')(ny)] \neq [(u + u)x + (v + v)y] \quad \text{if} \quad m \neq n. \end{aligned}$$

Thus, the rule for manipulating such *"linear combinations of points"* of the projective plane is that

rule two

"everything depends on the selected representation $\{[x], [y], \ldots\}$ of points $\{X, Y, \ldots\}$ and we must stick to the selected representation during the whole manipulation."

3 Projective lines

Projective lines are the lines of the projective plane. Their prototype has been handled in the file **Projective line**. Selecting two points $\{A, B\}$ on the particular line η, all other points $\{X \in \eta\}$ can be described by formal sums

$$X = \mu A + \nu B \quad \text{with} \quad \mu, \nu \in \mathbb{R}, \ \mu\nu \neq 0.$$

More general, the collinearity on a line of three pairwise different points $\{X, Y, Z\}$ is expressed by an equation of the form

collinearity condition

$$uX + vY + wZ = 0 \quad \text{with} \quad uvw \neq 0,$$

implying that anyone of them, Z say, can be expressed by the other two points:

$$Z = -\frac{u}{w}X - \frac{v}{w}Y = \frac{u}{w}X + \frac{v}{w}Y,$$

since multiplication of the expression by $k = -1$ does not change Z. For three non-collinear points $\{X, Y, Z\}$ the equation

non collinear

$$uX + vY + wZ = 0 \quad \text{is equivalent to} \quad u = v = w = 0.$$

As a consequence, having three points expressed in a projective base with correspoding coefficients

$$X = xA + yB + zC, \quad X' = x'A + y'B + c'C, \quad X'' = x''A + y''B + z''C, \tag{4}$$

The collinearity of the points is equivalent with the existence of a triple $\{\mu, \nu, \xi\}$ such that

$$\begin{aligned}
&\mu X + \nu X' + \xi X'' = 0 \quad \Leftrightarrow \\
&(\mu x + \nu x' + \xi x'')A \;+\; (\mu y + \nu y' + \xi y'')B \;+\; (\mu z + \nu z' + \nu'' z'')C \;=\; 0 \quad \Leftrightarrow \\
&(\mu x + \nu x' + \xi x'') \;=\; (\mu y + \nu y' + \xi y'') \;=\; (\mu z + \nu z' + \nu'' z'') = 0.
\end{aligned}$$

The last homogeneous system having a non-zero solution only in the case of vanishing determinant leads to the proof of the theorem:

Theorem 1. *Three points expressed as formal combinations of three independent points $\{A, B, C\}$ through equations (4) are collinear, if and only if the determinant of their coefficients vanishes:*

$$\begin{vmatrix} x & x' & x'' \\ y & y' & y'' \\ z & z' & z'' \end{vmatrix} \;=\; 0.$$

Analogous theorem is valid also for three lines expressed in projective coordinates through corresponding equations:

$$\alpha u + \beta v + \gamma w \;=\; \alpha' u + \beta' v + \gamma' w \;=\; \alpha'' u + \beta'' v + \gamma'' w \;=\; 0. \tag{5}$$

Theorem 2. *Three lines expressed through equations (5) are concurrent in a point, if and only if the determinant of their coefficients vanishes:*

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} \;=\; 0.$$

Similar theorems expressed through the vanishing of determinants we obtain for the coefficients of a line through two points and the intersection of two lines:

Theorem 3. *Given two points $\{X = xA + yB + zC\;,\; X' = x'A + y'B + z'C\}$, the coefficients of their line can be expressed by the cross-product of the coordinate vectors:*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} yz' - zy' \\ zx' - xz' \\ xy' - yx' \end{pmatrix}.$$

Theorem 4. *The coordinates of the intersection of two lines expressed in projective coordinates:*

$$\alpha u + \beta v + \gamma w = 0, \quad \alpha' u + \beta' v + \gamma' w = 0$$

can be expressed through the cross-product of the coefficient vectors:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \times \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} \;=\; \begin{pmatrix} \beta\gamma' - \gamma\beta' \\ \gamma\alpha' - \alpha\gamma' \\ \alpha\beta' - \beta\alpha' \end{pmatrix}.$$

Corollary 1. *The condition of collinearity of three points*

$$A' = \beta B + \gamma C, \quad B' = \gamma' C + \alpha' A, \quad C' = \alpha'' A + \beta'' B,$$

Menelaus projectively

on the sides of the triangle ABC is

$$\begin{vmatrix} 0 & \beta & \gamma \\ \alpha' & 0 & \gamma' \\ \alpha'' & \beta'' & 0 \end{vmatrix} = \beta\gamma'\alpha'' + \gamma\alpha'\beta'' = 0.$$

Corollary 2. *The condition of concurrency of three lines through the vertices of the triangle ABC, defined through the points on the opposite sides of the triangle:*

$$A' = \beta B + \gamma C, \quad B' = \gamma' C + \alpha' A, \quad C' = \alpha'' A + \beta'' B,$$

is

Ceva projectively

$$\beta\gamma'\alpha'' - \gamma\alpha'\beta'' = 0.$$

Proof. The coefficients of the lines $\{AA', BB', CC'\}$ are given respectively by:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ -\gamma \\ \beta \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} \alpha' \\ 0 \\ \gamma' \end{pmatrix} = \begin{pmatrix} \gamma' \\ 0 \\ -\alpha' \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} \alpha'' \\ \beta'' \\ 0 \end{pmatrix} = \begin{pmatrix} -\beta'' \\ \alpha'' \\ 0 \end{pmatrix}.$$

The condition of collinearity is equivalent with the vanishing of the determinant:

$$\begin{vmatrix} 0 & -\gamma & \beta \\ \gamma' & 0 & -\alpha' \\ -\beta'' & \alpha'' & 0 \end{vmatrix} = \beta\gamma'\alpha'' - \gamma\alpha'\beta'' = 0.$$

□

Remark 2. The two last corollaries express respectively the theorems of *"Menelaus"* and *"Ceva"* in projective form. The theorems in their usual formulation are handled in the files **Menelaus' theorem** and **Ceva's theorem**.

4 Cross ratio

The ubiquitous in the projective plane notion of *"cross-ratio"* (PQ, UV) or *"anharmonic ratio"* of four points $\{P, Q, U, V\}$ on a line η, expressed as combinations of two points of it $X(x_1, x_2) = x_1 A + x_2 B$:

$$P(p_1, p_2), \; Q(q_1, q_2), \; U(u_1, u_2), \; V(v_1, v_2),$$

cross ratio

is defined by the corresponding quotients

$$p = p_1/p_2, \quad q = q_1/q_2, \quad u = u_1/u_2, \quad v = v_1/v_2, \quad \text{through}$$

$$(PQ, UV) := \frac{p-u}{q-u} : \frac{p-v}{q-v} = \frac{p_1 u_2 - p_2 u_1}{q_1 u_2 - q_2 u_1} : \frac{p_1 v_2 - p_2 v_1}{q_1 v_2 - q_2 v_1}. \tag{6}$$

The important point is, that this definition, although it uses a description of the points of line η in terms of a particular system of coordinates, ultimately it is independent of this particular description, all descriptions producing the same result. In fact, expressing the points of η w.r. to another pair $\{A', B' \in \eta\}$, we obtain new quotients $\{p', q', u', v'\}$, related to the old by a *"homographic relation"* f :

$$p' = f(p), \ q' = f(q), \ u' = f(u), \ v' = f(v) \quad \text{with} \quad f(x) = \frac{ax+b}{cx+d}, \tag{7}$$

which, replaced in equation 6 and doing a bit of calculation, leads to the claimed independence of the value of (PQ, UV) from the selected particular representation of the points of η :

$$(PQ, UV) \ = \ \frac{p-u}{q-u} : \frac{p-v}{q-v} \ = \ \frac{p'-u'}{q'-u'} : \frac{p'-v'}{q'-v'}.$$

A particular case appears often in applications, when $\{U, V\}$ are expressed in terms of $\{P, Q\}$:

$$U = u_1 P + u_2 Q, \quad V = v_1 P + v_2 Q.$$

In this case, setting $\{u = u_1/u_2, \ v = v_1/v_2\}$, the quotient defining the cross section becomes:

$$(PQ, UV) \ = \ \frac{p-u}{q-u} : \frac{p-v}{q-v} \ = \ \frac{\infty - u}{0-u} : \frac{\infty - v}{0-v} \ = \ \frac{v}{u} \ = \ \frac{v_1}{v_2} : \frac{u_1}{u_2}. \tag{8}$$

Four lines through the same point E define on a fifth line ε intersecting them the same

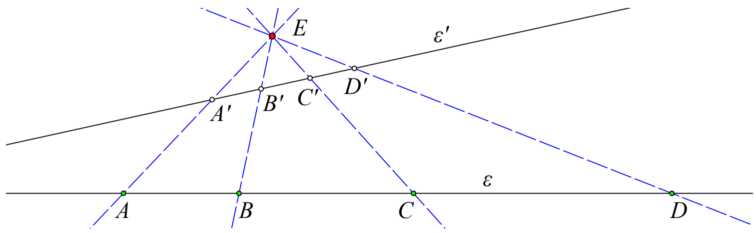

Figure 2: Cross ratio of four lines through the same point

cross ratio, independently of the location of the line ε. This is proved for the case of the euclidean plane in the file **Cross ratio** and the proof for the projective plane is easily reducible to that one. This fact allows the definition of the cross ratio $(\eta_1\eta_2, \eta_3\eta_4)$ of four lines $\{\eta_i, i = 1, ..., 4\}$ passing through the same point in the same way as in the euclidean plane. It is defined to be equal to the cross ratio (AB, CD) of the four intersection points of these lines with an arbitrary line ε. Next theorem has also a proof analogous to the corresponding one of the euclidean plane.

Theorem 5. *The cross ratio* $(\eta_1\eta_2, \eta_3\eta_4)$ *of four lines through the same point* E *of the plane expressed in terms of two fixed lines* $\{\alpha, \beta\}$ *through* E *as linear combinations*

$$\eta_i \ = \ \alpha + \lambda_i \beta \ , \ i = 1, 2, 3, 4 \quad \textit{is} \quad (\eta_1\eta_2, \eta_3\eta_4) \ = \ \frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3} : \frac{\lambda_1 - \lambda_4}{\lambda_2 - \lambda_4} \ .$$

Proof. Represent the lines w.r.t. a system of projective coordinates (u, v, w) in the form

$$\eta_i \ = \ \alpha + \lambda_i \beta \ = \ (a + \lambda_i a')u + (b + \lambda_i b')v + (c + \lambda_i c')w \ = 0,$$

where $\{au + bv + cw = 0, a'u + b'v + c'w = 0\}$ the equations of lines $\{\alpha, \beta\}$. Intersect them with the line $w = 0$ giving for $\{u_i, v_i\}$ the values:

$$(a + \lambda_i a')u_i + (b + \lambda_i b')v_i \ = \ 0 \quad \Rightarrow \quad \frac{u_i}{v_i} \ = \ -\frac{b + \lambda_i b'}{a + \lambda_i a'} \ .$$

The claimed equality follows by substitution of this into the defining formula of the cross ratio of four points on a line and subsequent simplification. □

5 Trilinear polar and pole

The *"trilinear polar"* η_D of a point D w.r. to a triangle ABC, for D not contained on the side-lines of the triangle, is defined as a line created using the three *"cevians"* $\{AD, BD, CD\}$ of the point (See Figure 3). The points may be considered forming a pro-

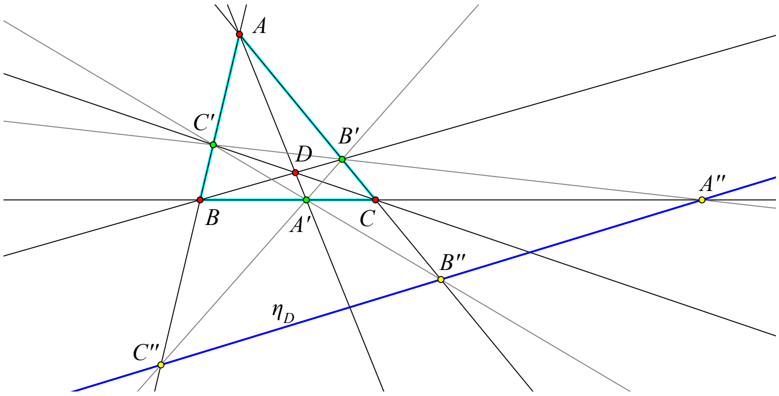

Figure 3: A basic figure of the projective plane

a basic figure

jective base. In figure 3 are seen the intersection points of the corresponding lines:

$$\text{the \textit{"traces"}}: \quad A' = AD \cap BC, \quad B' = BD \cap CA, \quad C' = CD \cap AB \quad \text{and}$$
$$A'' = BC \cap B'C', \quad B'' = CA \cap C'A', \quad C'' = AB \cap A'B'.$$

The coordinates of A' result by considering it as a point of line BC, hence of the form $vB + wC$ and also as a point of the line AD, hence of the form $uA + rD$. Being the intersection of the two lines, we must have

$$\begin{aligned} vB + wC &= uA + rD = uA + r(A + B + D) \quad \Leftrightarrow \\ 0A + vB + wC &= (u + r)A + rB + rC. \end{aligned}$$

Last equation, by our conventions, means that:

$$(0, v, w) = k(u + r, r, r) \quad \text{with} \quad k \neq 0 \quad \Leftrightarrow \quad (u, v, w) = r(-1, 1, 1) \quad \text{with} \quad r \neq 0.$$

Analogous results for the points $\{B', C'\}$ show that we can write:

$$A' = B + C, \qquad B' = C + A, \qquad C' = A + B. \tag{9}$$

The point A'' being on the line BC has the expression $A'' = vB + wC$. Being also on line $B'C'$ it has the expression $A'' = pB' + qC' = p(A + C) + q(A + B)$. Hence we have

$$\begin{aligned} A'' &= vB + wC = p(A + C) + q(A + B) \quad \Leftrightarrow \\ A'' &= 0A + vB + wC = (p + q)A + qB + pC \quad \Leftrightarrow \\ A'' &= p(0, 1, -1) \quad \text{with} \quad p \neq 0. \end{aligned}$$

Hence we can write

$$A'' = B - C, \qquad B'' = C - A, \qquad C'' = A - B. \tag{10}$$

The cross ratio in this case is:

$$(BC, A'A'') = \frac{\infty - 1}{0 - 1} : \frac{\infty + 1}{0 + 1} = -1.$$

Analogous equations we obtain for the other cross ratios and we come to the theorem:

Theorem 6. *For the points* $\{A', B', C', A'', B'', C''\}$ *of the projective base the following pairs are harmonic, i.e. they satisfy the relations:*

$$(BC, A'A'') = (CA, B'B'') = (AB, C'C'') = -1.$$

More generally, an analogous calculation shows that:

harmonic pairs

Theorem 7. *The points* $\{U = pX + qY, \ V = pX - qY\}$ *form harmonic pairs, satisfying the equation* $(XY, UV) = -1$.

Theorem 8. *The points* $\{A'', B'', C''\}$ *are collinear and define the "trilinear polar"* η_D *of* D *w.r. to the triangle. Conversely any line* η_D *not passing through the vertices of the triangle defines a unique point* D, *whose trilinear polar is* η_D. *Point* D *is called the "trilinear pole" of* η_D.

Proof. The first claim follows from $A'' + B'' + C'' = (B - C) + (C - A) + (A - B) = 0$. For the converse claim define the points $\{A'', B'', C''\}$ as intersections of η_D with corresponding sides of the triangle and express them as combinations of the base points:

$$A'' = \beta B + \gamma C, \quad B'' = \gamma' C + \alpha' A, \quad C'' = \alpha'' A + \beta'' B.$$

The assumed collinearity of these points is equivalent (corollary 1) with

$$\beta\gamma'\alpha'' + \gamma\alpha'\beta'' = 0.$$

The harmonic conjugates of $\{A'', B'', C''\}$ w.r. to corresponding vertices of the triangle are

$$A' = \beta B - \gamma C, \quad B' = \gamma' C - \alpha' A, \quad C' = \alpha'' A - \beta'' B$$

and the concurrency of the three lines $\{AA', BB', CC'\}$ is equivalent with (corollary 2):

$$\beta\gamma'\alpha'' - (-\gamma)(-\alpha')(-\beta'') = 0,$$

which is identical with the previous one. □

Remark 3. In the previous arguments we used the fact that a projective base can be considered as a pair (ABC, D) consisting of a triangle and a point not lying on the side-lines of that triangle. In the projective plane we have not a notion of length-ratio of segments and cannot apply the formulation of the theorems of *Ceva* and *Menelaus*, used in the frame of affine geometry, which leads to the proof of the existence of the trilinear polar in the affine plane.

Remark 4. The *"affine plane"* results by deleting a specific line η of the projective plane and maintaining the remaining points. In this plane can be defined the notion of parallelity, and the notions of ratios and middles of segments.. In the projective plane all bases *"look"* equal, whereas in the affine plane we have a *"preferred"* kind of bases, defining the *"barycentric coordinates"* (see file **Barycentric coordinates**). In these bases point D is the centroid of the triangle ABC. In the *"euclidean plane"*, which is an affine plane endowed with a distance-function of pairs of points, besides the *"barycentric coordinates"* we define also some other projective coordinate systems, one of them being the *"trilinear coordinates"*. In these D is the *"incenter"* of the triangle ABC.

Theorem 9. *For any projective base* $\{A, B, C, D\}$ *and any point* $X = \alpha A + \beta B + \gamma C$, *the corresponding trilinear polar* η_X *is described in this base by the equation:*

$$\eta_X \ : \ (\beta\gamma)u + (\gamma\alpha)v + (\alpha\beta)w = 0 \quad \Leftrightarrow \quad \frac{u}{\alpha} + \frac{v}{\beta} + \frac{w}{\gamma} = 0.$$

Proof. The intersection points $\{A' = XA \cap BC\ ,\ B' = XB \cap CA\ ,\ C' = XC \cap AB\}$ are represented correspondingly by $\{\beta B + \gamma C\ ,\ \gamma C + \alpha A\ ,\ \alpha A + \beta B\}$. Their harmonic conjugates on lines $\{BC, CA, AB\}$ are $\{A'' = \beta B - \gamma C\ ,\ B'' = \gamma C - \alpha A\ ,\ C'' = \alpha A - \beta B\}$. The collinearity of the point $U = uA + vB + wC$ with the points $\{A'', B''\}$ spanning the trilinear polar is expressed by the vanishing determinant (corollary 1):

$$\begin{vmatrix} 0 & \beta & -\gamma \\ -\alpha & 0 & \gamma \\ u & v & w \end{vmatrix} = 0\,,$$

which is equivalent to the claimed equations. □

6 Projectification of the euclidean plane

The *"projectification"* of the *euclidean* plane extends it to the *projective* plane by adding a new line. A way to see this in a concrete model is by identifying the euclidean plane ε with the plane

euclidean plane ε

$$\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 :\ x_3 = 1\}$$

of $\mathbb{R}^3$. Every line $\eta = \{k(x_1, x_2, x_3)\}$ of $\mathbb{R}^3$ passing through the origin O and not being parallel to that plane (i.e. not having $x_3 = 0$) intersects ε on a point $[\eta] = (x_1, x_2, 1)$

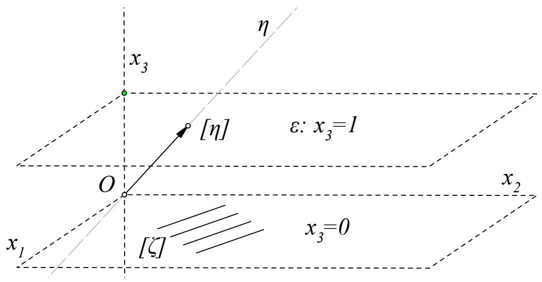

Figure 4: Projectification of ε

(See Figure 4). We identify $[\eta] = (x_1, x_2, 1)$ with this line $\eta = \{k(x_1, x_2, x_3)\}$, which is an element of the projective plane $P\mathbb{R}^2$.

This establishes an identification of the euclidean plane ε with a part E' of the projective plane. The complement $F' = P\mathbb{R}^2 - E'$ consists of all points (=lines of $\mathbb{R}^3$) of the form $\zeta = \{k(x_1, x_2, 0)\}$. This set is a projective line of $P\mathbb{R}^2$. The projectification adds to ε a set of points F, which is isomorphic to this line F'. The added line F is called *"line at infinity"* and consists of *"points at infinity"* which correspond to points of the projective plane of the form $[\zeta] = [\{k(x_1, x_2, 0)\}]$.

line at infinity

The correspondence is given by means of *"directions"* of parallel lines of ε. We consider the class of all lines parallel to a non zero vector $(v_1, v_2, 0)$ as the point at infinity $[\zeta]$ identified with the element $[v_1, v_2, 0]$ of line F'. The projectification leads to a *model* of the projective plane isomorphic to $P\mathbb{R}^3$ and consisting of a union $\overline{\varepsilon} = \varepsilon \cup F$.

projective plane $\overline{\varepsilon}$

Creating this model of the projective plane we have the benefit of embedding all familiar shapes of the euclidean plane into the projective plane. Then we can use the tools of the projective plane to study properties of coincidence of lines and points or even more general intersection questions, to which the projective plane is better suited than the euclidean plane.

The reason for the latter is that all pairs of distinct lines in the projective plane intersect at a point. In the projectification model the intersection point coincides with an ordinary

why projective

point, if the lines intersect inside the euclidean plane. If they don't, then their image in the euclidean plane is that of two parallel lines, thus defining the same point at infinity, which is then considered as their intersection. An ordinary line ε and the line at infinity ε_∞ intersect always at the point at infinity identified with the *"direction"* of that line ε.

Remark 5. With the *projectification*, besides $P\mathbb{R}^2$ of section 1, we have a second model $\overline{\varepsilon}$ of the projective plane in which there is a distinguished line: the *"line at infinity"*. In $P\mathbb{R}^2$ there is no such distinguished line and modifying slightly the procedure of projectification, we could see that deleting any line η_0 from $P\mathbb{R}^2$ we obtain an *"affine"* plane, homeomorphic to ε and consisting of the points of $P\mathbb{R}^2$ not belonging to η_0. Thus, from the point of view of projective geometry the *line at infinity* is like any other line. Only the projectification of the euclidean plane adds an additional distinguished line, consisting of ideal points where two parallels intersect.

7 Homogeneous coordinates

Intimately associated with the procedure of *"extension"* of the euclidean plane to the *projective plane*, described in section 6, is a projective system of coordinates (x_1, x_2, x_3), called *"homogeneous coordinates"*, in which this triple represents the point

$$[\eta] = (x_1/x_3, x_2/x_3, 1) \in \varepsilon, \quad \text{if} \quad x_3 \neq 0,$$

and also represents the point at infinity

$$[\zeta] = \{k(x_1, x_2, 0)\} \quad \text{if} \quad x_3 = 0.$$

Using this coordinate system the equations defining curves of the euclidean plane transfer to equations of the projective plane by replacing the coordinates $\{(x_1, x_2),\}$ correspondingly with $\{(x_1/x_3, x_2/x_3, 1)\}$ which geometrically amounts to a *"projection"* of the shapes onto the plane $x_3 = 1$. Thus, line $ax_1 + bx_2 + c = 0$ projects to

$$a(x_1/x_3) + b(x_2/x_3) + c = 0 \quad \Leftrightarrow \quad ax_1 + bx_2 + cx_3 = 0.$$

Similarly a conic represented by equation

$$ax_1^2 + bx_2^2 + cx_1x_2 + dx_1 + ex_2 + f = 0 \quad \text{projects to}$$
$$a(x_1/x_3)2 + b(x_2/x_3)2 + c(x_1/x_3)(x_2/x_3) + d(x_1/x_3) + e(x_2/x_3) + f = 0 \quad \Leftrightarrow$$
$$ax_1^2 + bx_2^2 + cx_1x_2 + dx_1x_3 + ex2_x3 + fx_3^2 = 0.$$

The inverse procedure consists of setting $\{x_3 = 1\}$ to an equation $p(x_1, x_2, x_3) = 0$ of the projective plane, to obtain the corresponding equation $p(x_1, x_2, 1) = 0$ of the euclidean plane.
Figure 5 shows the *"projective base"* associated to this system of coordinates considered as a particular case of *"projective coordinates"* in the sense of section 2. Here the triangle ABC has the two vertices $\{A, B\}$ at infinity. From the base only the points $\{C, D\}$ are finite points. The line at infinity is characterized by the equation

$$x_3 = 0.$$

Remark 6. The coordinate x_3 is not a *"blessed"* one. We could define analogous *"projectifications"* based on the coordinate x_2 or x_1. We could even do analogous projectifications using any coordinate out of the three (x, y, z) of an arbitrary coordinate system of $\mathbb{R}^3$, all these constructions leading to projective planes isomorphic to $\overline{\varepsilon}$.

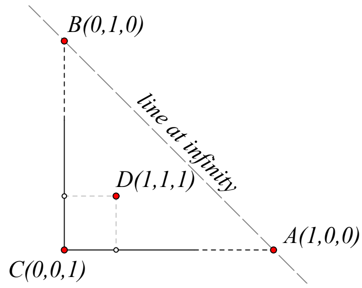

Figure 5: Projective base associated to the *"homogeneous coordinates"*

8 Homogeneous coordinates relation to projective coordinates

Considering the projective plane $\bar{\varepsilon}$ of section 6, we have two kinds of coordinates, the *"homogeneous"*, which now we denote by (x, y, z) and the *"projective coordinates"*, denoted by (u, v, w). These are associated to a *"base"* $\{A, B, C, D\}$ of the plane $\bar{\varepsilon}$, the four points assumed to be on the finite part of the plane, i.e. none of them lying on the line at infinity. Thus, they are represented by vectors:

$$A = \begin{pmatrix} a_1 \\ a_2 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ 1 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 \\ d_2 \\ 1 \end{pmatrix}. \tag{11}$$

The *unit* point D determines the vectors through which we get the projective coordinates:

$$D = \mu A + \nu B + \xi C \quad \Leftrightarrow \quad \begin{pmatrix} d_1 \\ d_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu \\ \nu \\ \xi \end{pmatrix}.$$

This system determines the unknowns (μ, ν, ξ) and the corresponding vectors

$$a = \mu \begin{pmatrix} a_1 \\ a_2 \\ 1 \end{pmatrix}, \quad b = \nu \begin{pmatrix} b_1 \\ b_2 \\ 1 \end{pmatrix}, \quad c = \xi \begin{pmatrix} c_1 \\ c_2 \\ 1 \end{pmatrix}, \tag{12}$$

used to tedermine the *"projective coordinates"* of a point $X = [x]$:

$$x = ua + vb + wc.$$

Written in matrix notation this equation takes the form

projective to homogeneous

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mu a_1 & \nu b_1 & \xi c_1 \\ \mu a_2 & \nu b_2 & \xi c_2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \xi \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \tag{13}$$

giving the transition from the projective coordinates (u, v, w) to the homogeneous coordinates (x, y, z) in terms of the concrete selected representatives of points $\{A, B, C, D\}$. It is interesting here to notice that ordinary points, corresponding to homogeneous coordinates with $z = 1$, have projective coordinates (u, v, w) satisfying the equation:

$$\mu \cdot u + \nu \cdot v + \xi \cdot w = 1,$$

whereas, points at infinity, corresponding to homogeneous coordinates with $z = 0$, have projective coordinates satisfying the equation:

$$\mu \cdot u + \nu \cdot v + \xi \cdot w = 0. \tag{14}$$

line at infinity

Corollary 3. *Equation (14) is the one representing the "line at infinity" in the projective system of coordinates associated to the projective base* $\{A, B, C, D\}$.

The calculation of the converse relation leading from *homogeneous* to *projective* coordinates involves the inversion of the matrices appearing in equation (13) and leads to

homogeneous to projective

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{2\Delta} \begin{pmatrix} \mu^{-1} & 0 & 0 \\ 0 & \nu^{-1} & 0 \\ 0 & 0 & \xi^{-1} \end{pmatrix} \begin{pmatrix} b_2 - c_2 & c_1 - b_1 & b_1c_2 - c_1b_2 \\ c_2 - a_2 & a_1 - c_1 & c_1a_2 - a_1c_2 \\ a_2 - b_2 & b_1 - a_1 & a_1b_2 - a_2b_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{15}$$

in which Δ denotes the oriented area of the triangle on the plane ε defined by the vectors in (11).

9 Changing the system of projective coordinates

The relation of projective coordinates with respect to two coordinate bases $\{A, B, C, D\}$, $\{A', B', C', D'\}$ of the projective plane $\overline{\varepsilon}$ results from the matrix representation of section 8 by multiplying the appropriate matrices:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{2\Delta} \begin{pmatrix} \mu^{-1} & 0 & 0 \\ 0 & \nu^{-1} & 0 \\ 0 & 0 & \xi^{-1} \end{pmatrix} \cdot \begin{pmatrix} b_2 - c_2 & c_1 - b_1 & b_1c_2 - c_1b_2 \\ c_2 - a_2 & a_1 - c_1 & c_1a_2 - a_1c_2 \\ a_2 - b_2 & b_1 - a_1 & a_1b_2 - a_2b_1 \end{pmatrix} \cdot \begin{pmatrix} a'_1 & b'_1 & c'_1 \\ a'_2 & b'_2 & c'_2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu' & 0 & 0 \\ 0 & \nu' & 0 \\ 0 & 0 & \xi' \end{pmatrix} \cdot \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}.$$

In particular, keeping fixed the triangle ABC and changing only D to D' gives for the corresponding transition functions relating the two systems of projective coordinates the simple rule:

change only unit D

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \mu'\mu^{-1} & 0 & 0 \\ 0 & \nu'\nu^{-1} & 0 \\ 0 & 0 & \xi'\xi^{-1} \end{pmatrix} \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}.$$

10 Obtaining equations in a projective coordinate system

We obtain equations in projective coordinate systems by homogenizing the known equations in euclidean cartesian systems and then, transforming to projective system by the transformation rule of section 8. Thus, the line represented in cartesian coordinates by an equation of the form

$$ax + by + c = 0 \quad \text{transforms in homogeneous coordinates to:} \quad ax + by + cz = 0.$$

This, changing the coordinates to any projective system by the invertible matrix M, given by equation 13:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} \qquad \text{leads to an equation of a similar form:} \quad a'u + b'v + c'w = 0.$$

Analogously is seen that a quadratic equation transforms to a homogeneous quadratic equation etc. This procedure is reversible and proves statements like the following:

Theorem 10. *Any linear equation in projective coordinates represents a line and vice-versa, any line of the projective plane is represented in a system of projective coordinates by a linear homogeneous equation.*

conic by quadratic

Theorem 11. *Any quadratic equation in projective coordinates represents a conic and vice-versa, any conic of the projective plane is represented in a system of projective coordinates by a quadratic homogeneous equation.*

11 Example calculation of the radical axis of two circles

The discussion of section 10 can be applied to find the equation in projective coordinates of the radical axis of two circles, whose equations in cartesian coordinates are of the form:

$$x^2 + y^2 + (ax + by + c) = 0 \quad \text{and} \quad x^2 + y^2 + (a'x + b'y + c') = 0.$$

Transforming to homogeneous coordinates and subtracting the resulting equations we come to the equation:

$$z[(a - a')x + (b - b')y + (c - c')z] = 0.$$

radical axis

This is the product of two lines: the line at infinity $z = 0$, and the radical axis in homogeneous coordinates

$$(a - a')x + (b - b')y + (c - c')z = 0.$$

By transforming the circle equation to a general projective system of coordinates (u, v, w) the expression $x^2 + y^2$ transforms to a fixed quadratic one $Q(u, v, w)$ and the remaining terms transform to a product of lines $\eta(u, v, w) \cdot \eta_0(u, v, w)$, of which the second linear term $\eta_0(u, v, w) = 0$ represents the line at infinity. Thus, in the general projective coordinate system, the equation of the circle is written as a sum

$$Q_0(u, v, w) + \eta(u, v, w)\eta_0(u, v, w) = 0, \tag{16}$$

$Q_0(u,v,w) \sim x^2 + y^2$

in which $Q_0(u, v, w)$ is a standard quadratic expression, independent of the particular circle, and $\eta_0(u, v, w)$ is the line at infinity. Also the difference

$$\eta(u, v, w) - \eta'(u, v, w) = 0,$$

of these parts of the equations of two circles represents the *"radical axis"* of these circles ([Lon91, II,p.69]).

Exercise 1. *With the notation of the previous discussion, show that $Q_0(u, v, w)$ can be represented as a quadratic form:*

$$Q_0(u, v, w) = (\mu u, \nu v, \xi w) \begin{pmatrix} a_1^2 + a_2^2 & a_1b_1 + a_2b_2 & a_1c_1 + a_2c_2 \\ a_1b_1 + a_2b_2 & b1^2 + b_2^2 & b_1c_1 + b_2c_2 \\ a_1c_1 + a_2c_2 & b_1c_1 + b_2c_2 & c_1^2 + c_2^2 \end{pmatrix} \begin{pmatrix} \mu u \\ \nu v \\ \xi w \end{pmatrix}. \tag{17}$$

12 The circumcircle of ABC and the general circle

Here we continue the application of the formulas of sections 10 and 11 and determine the form of the equation $f_0(u, v, w) = 0$ of the *"circumcircle"* of the triangle ABC of the projective base $\{A, B, C, D\}$, as well as, the equation $f(u, v, w) = 0$ of the general circle, in terms of the associated projective coordinates. Figure 6 shows the circumcircle κ of

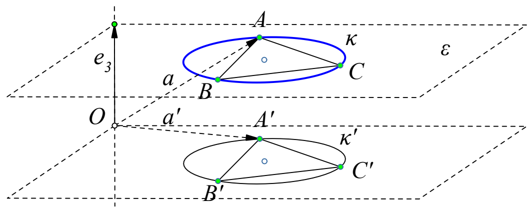

Figure 6: The circumcircle κ of ABC

ABC, as well as, its projection κ' on the plane $x_3 = 0$, which is a circle κ' congruent to κ. From the discussion in section 11 we know that the circle κ in the projective coordinates (u, v, w) associated to the base $\{A, B, C, D\}$ is described by equation (16) with Q_0 given by equation (17). Assuming the product of linear terms in the form

$$\eta(u, v, w) \cdot \eta_0(u, v, w) \;=\; (pu + qv + rw) \cdot (\mu u + \nu v + \xi w),$$

and taking into account that the circle passes through $\{A(1, 0, 0), B(0, 1, 0), C(0, 0, 1)\}$, we find that

$$p = -\mu(a_1^2 + a_2^2), \quad q = -\nu(b_1^2 + b_2^2), \quad r = -\xi(c_1^2 + c_2^2).$$

Introducing this into equation (16) and doing a short calculation, we see that the coefficients of the squares $\{u^2, v^2, w^2\}$ vanish and the equation takes the form

$$\sum \left(2\overline{a} \cdot \overline{b} - \overline{a}^2 - \overline{b}^2\right)(\mu\nu)(uv),$$

where the sum extends over the cyclic permutations of the letters and $\{\overline{a} = (a_1, a_2), \ldots\}$, the dot denoting the inner product. Taking into account that

$$2\overline{a} \cdot \overline{b} - \overline{a}^2 - \overline{b}^2 = -|AB|^2$$

is the negative square length of side AB, we obtain next theorem.

Theorem 12. *The equation of the circumcircle κ of the triangle ABC of the base $\{A, B, C, D\}$ w.r. to the projective coordinates associated to that base is*

circumcircle of ABC

$$\left(\frac{|BC|^2}{\mu}\right) vw \;+\; \left(\frac{|CA|^2}{\nu}\right) wu \;+\; \left(\frac{|AB|^2}{\xi}\right) uv \;=\; 0. \tag{18}$$

Corollary 4. *With the preceding conventions and notation, the equation $f(u, v, w) = 0$ of a circle κ w.r. to the projective coordinates associated to the base $\{A, B, C, D\}$ has the form*

general circle

$$f(u, v, w) \;=\; \left(\frac{|BC|^2}{\mu}\right) vw \;+\; \left(\frac{|CA|^2}{\nu}\right) wu \;+\; \left(\frac{|AB|^2}{\xi}\right) uv$$
$$+\; (pu + qv + rw)(\mu u + \nu v + \xi w) = 0,$$

where $pu + qv + rw = 0$ is the equation of the radical axis of κ and the circumcircle κ_0 of the triangle ABC. In particular, the equation of "concentric" circles to the circumcircle of ABC is of the form

concentric circumcircle

$$\left(\frac{|BC|^2}{\mu}\right) vw \;+\; \left(\frac{|CA|^2}{\nu}\right) wu \;+\; \left(\frac{|AB|^2}{\xi}\right) uv \;+\; \lambda(\mu u + \nu v + \xi w)^2 = 0, \;\; \lambda \in \mathbb{R}.$$

13 Projective transformations or projectivities

Depending on the point of view, one can give different but equivalent definitions of the *"projective transformation"* or *"projectivity"*. A synthetic formulation, analysed in axiomatic foundations of projective geometry ([VY10], [Whi06]) defines it as an invertible map of the projective plane onto itself which *"preserves lines"*, i.e maps lines to lines. Using the analytic apparatus of bases we can define it as a map of the projective plane onto itself, which, w.r to a base $\{A, B, C, D\}$ and corresponding projective coordinates (u, v, w) has a representation by an invertible matrix:

projectivity by a matrix

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \tag{19}$$

two matrices differing by a non-zero constant defining the same projectivity. From the discussion in section 9 follows that the definition is independent of the specific coordinate system. This means, that if the transformation is represented by an invertible matrix in a specific coordinate system, then it does the same w.r. to any other system.

Using the model $\bar{\varepsilon}$ of section 6 we can prove the *"fundamental theorem"* of projective geometry:

fundamental theorem

Theorem 13. *For any couple of quadruples of four points* $\{(A, B, C, D), (A', B', C', D')\}$ *of the projective plane in general position, there is a projectivity* f *mapping the first onto the second* $\{f(A) = A', f(B) = B', \ldots\}$.

Proof. Identify the points $\{A, B, \ldots\}$ with respective representative vectors and consider a triple of constants $\{\mu, \nu, \xi\}$ and the matrix M transforming $\{A, B, C\}$ to $\{\mu A', \nu B', \xi C'\}$. Putting the vectors in the columns of a matrix, this translates into a matrix equation of the form:

$$(\mu A', \nu B', \xi C') = M \cdot (A, B, C) \quad \Leftrightarrow \quad M = (\mu A', \nu B', \xi C') \cdot (A, B, C)^{-1}. \tag{20}$$

The matrix M, depending on $\{\mu, \nu, \xi\}$, is completely determined up to multiplicative constant by requiring for a fourth constant σ to have also

$$\sigma \cdot D' = M\dot{D} \quad \Leftrightarrow \quad (A', B', C')^{-1} D' = \begin{pmatrix} \mu/\sigma & 0 & 0 \\ 0 & \nu/\sigma & 0 \\ 0 & 0 & \xi/\sigma \end{pmatrix} (A, B, C)^{-1} D. \tag{21}$$

This is a diagonal linear system in $\{\mu/\sigma, \nu/\sigma, \xi/\sigma\}$ with solutions of the form

$$\mu = k_1 \sigma, \quad \nu = k_2 \sigma, \quad \mu = k_3 \sigma, \quad \text{for some constants} \quad k_1, k_2, k_3,$$

completely, up to the factor σ, determining the matrix M and proving the theorem. □

Remark 7. In some applications, trying to construct the projectivity from four given points $\{A, B, C, D\}$ and their images $\{A', B', C', D'\}$, it is useful to have in mind the simple recipe contained in formulas (20) and (21):

1. Fix a projective coordinate system and form the matrices with columns the coordinates of the points: $P = (A, B, C)$, $Q = (A', B', C')$.
2. The matrix representing the projectivity has the form $M = QKP^{-1}$, with a diagonal matrix $K = diag(k_1, k_2, k_3)$.
3. The constants $\{k_i\}$ are determined by solving the linear system $QKP^{-1}D = k_4 D'$.

An example application of this recipe is contained in section 18, where we construct a projectivity mapping the unit circle to a parabola.

In the case the projectivity f maps the four points $\{A, B, C, D\}$ to themselves, the matrices $\{(A, B, C), (A', B', C')\}$ are identical and the vectors $\{D, D'\}$ are also identical, implying in the last equations $k_1 = k_2 = k_3 = 1$ and producing as a result for M a multiple of the identity. This proves the corollary:

case of identity

Corollary 5. *A projectivity mapping four points $\{A, B, C, D\}$ in general position to themselves is the identity. Two projectivities coinciding on a set of four points $\{A, B, C, D\}$ in general position, coincide everywhere.*

Proof. For the second claim notice that if two projectivities $\{f_1, f_2\}$ coincide on such a set, then $g = f_2 \circ f^{-1}$ maps the four points onto themselves. Hence by the first part of the corollary it is the identity. □

We say that two shapes $\{S, S'\}$ of the projective plane are *"projectively equivalent"* if there is a projectivity f mapping one to the other $f(S) = S'$. By the preceding discussion we have the corollary:

Corollary 6. *Any two triangles and any two quadrangles of the projective plane are projectively equivalent.*

any triangle ∼equilateral

any quad ∼square

Thus, working in $\bar{\varepsilon}$ of section 6, all triangles are projectively equivalent to the equilateral and all quadrangles are equivalent to the square. This has an interesting consequence regarding the properties of these shapes: To prove a property of triangles/quadrangles, which is preserved by projectivities, it suffices to prove it in the case of the equilateral/square.

An example application of this principle is the following one concerning the *"trilinear polar"*, encountered already in section 3. The traces of the center D of an equilateral triangle ABC, define the medial triangle $A'B'C'$ and the pairs of opposite sides of the two triangles are parallel, hence *"intersect"* at the line at infinity. Taking an arbitrary triangle ABC and an arbitrary point D not lying on its side-lines, we have again the triangle $A'B'C'$ of the traces of D. By the preceding discussion, there is a projectivity f mapping the points $\{A, B, C, D\}$ of the equilateral to the corresponding points of the arbitrary triangle (on the right in figure 7). The projectivity f maps then the side-lines of

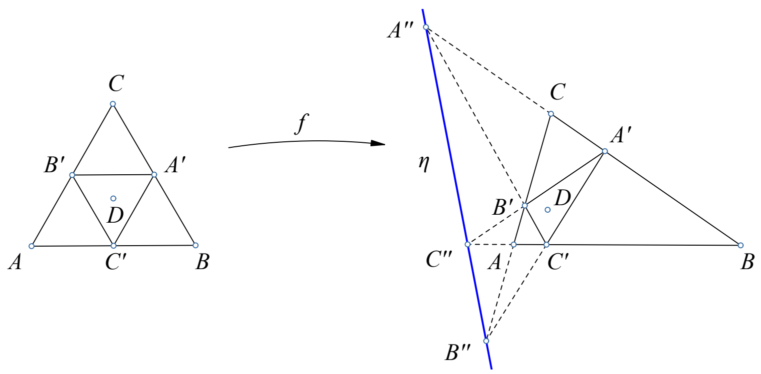

Figure 7: Mapping the line at infinity to the trilinear polar

the opposite sides $\{(AB, A'B'), \ldots\}$ to corresponding side-lines of opposite sides of the arbitrary triangle. Since the three pairs of opposite sides have intersections lying on the line at infinity, the corresponding pairs of opposite sides will intersect at the image line

$\eta = f(\eta_\infty)$ of the line at infinity via f. This proves that in an arbitrary triangle the pairs of opposite sides of the triangles $\{ABC, A'B'C'\}$ intersect on a line η, called the *"trilinear polar"* of D w.r. to ABC. Further, as in the equilateral point C' is harmonic conjugate to the point at infinity of AB, the same property holds for C' and C'' in the arbitrary triangle. This means that (C', C'') are harmonic conjugate to (A, B). This property follows from the *"preservation of the cross ratio"* by projectivities, which will be discussed below.

Remark 8. The projectivities of the plane build a group, denoted by $PGL(2,\mathbb{R})$, and identified, as we saw, with the group of real 3×3 invertible matrices, two matrices differing by a multiplicative non-zero constant being considered the same. This group contains the group $Aff(2,\mathbb{R})$ of *"affinities"* as a subgroup of projectivities *fixing the line at infinity*, which in turn contains the group $Iso(2,\mathbb{R})$ of euclidean *"isometries"* as a subgroup of affinities that preserve the distances between points. This inclusion relation

$$PGL(2,\mathbb{R}) \supset Aff(2,\mathbb{R}) \supset Iso(2,\mathbb{R}),$$

reflects the successive enrichment of structure of the plane. Starting with the projective plane $\bar{\varepsilon}$, all lines are *"equivalent"* and every pair of different lines has an intersection point. Next, deleting a line η_0 of the projective plane, we obtain the *"affine plane"* $\bar{\varepsilon} - \eta_0$ containing two kinds of pairs of lines: those that intersect, and those that do not intersect. Latter, from the viewpoint of the projective plane, are the pairs whose intersection point was contained in the deleted line (at infinity) η_0. This creates the structure of *"parallel"* lines of the affine plane. Finally we add a notion of *"distance"* between points of the affine plane and we obtain the *"Euclidean"* plane with parallel and non-parallel lines, but also with *"distances"* between points. The *"Affinities"* are *projectivities* that respect parallels, i.e. map parallel lines to parallel lines. The *"Isometries"* not only respect parallelity but also respect distances between points.

Remark 9. Besides the general properties characterizing all projectivities there are others differentiating them in distinct classes with geometric properties mainly related to the number and location of their fixed points on the plane.

14 A proof of Pappus' theorem

Pappus special

Here is another example application of the *"fundamental theorem"* of projective geometry discussed in section 13. It is based on the following exercise.

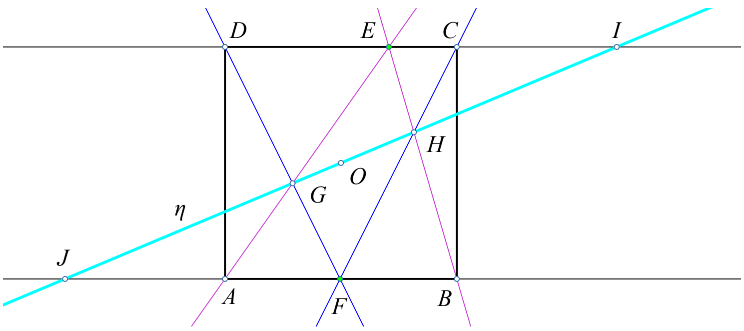

Figure 8: A case of Pappus' theorem

Exercise 2. *From the points $\{E, F\}$ on opposite sides of the square $ABCD$ we draw lines to opposite corners $\{EA, EB, FC, FD\}$ intersecting at points $\{G = EA\cap FD, H = EB\cap FC\}$. Show that line $\eta = GH$ passes through the center O of the square.*

Hint: Apply Menelaus' theorem (see file **Menelaus' theorem**) to triangles $\{EAB, FDH\}$ with secant GH (See Figure 8).

$$EAB: \ \frac{GE}{GA}\cdot\frac{JA}{JB}\cdot\frac{HB}{HE} = 1, \qquad FDH: \ \frac{GD}{GF}\cdot\frac{HF}{HC}\cdot\frac{IC}{ID} = 1.$$

From these equalities and the similar triangles $\{EDG \sim EFG, \dots\}$ follows

$$\frac{GE}{GA} = \frac{GD}{GF} \quad \text{and} \quad \frac{HB}{HE} = \frac{HF}{HC} \quad \Rightarrow \quad \frac{JA}{JB} = \frac{IC}{ID}.$$

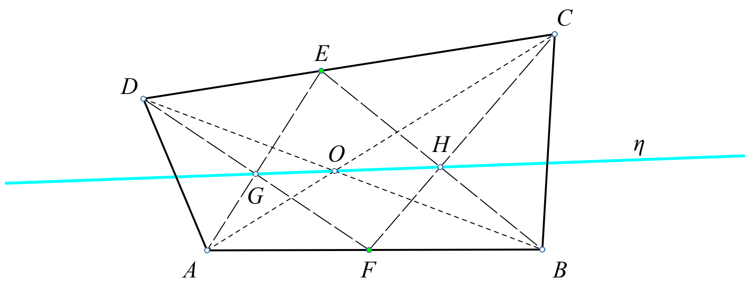

Figure 9: Pappus' theorem, the general case

Pappus general

Theorem 14. *From the points $\{E, F\}$ on opposite sides of a quadrangle $ABCD$ we draw lines to opposite corners $\{EA, EB, FC, FD\}$ intersecting at points $\{G = EA \cap FD, H = EB \cap FC\}$. Then, the line $\eta = GH$ passes through the intersection point O of the diagonals of $ABCD$.*

Proof. Use a projectivity f to map $ABCD$ onto a square. Then the claimed property transfers via f to that of the previous exercise. □

Notice that the exercise and its proof could be carried out also for a parallelogram.

15 Presevation of the cross ratio

The projectivity f maps a line $\eta = AB$, defined by two points, onto a line $\eta' = f(\eta) = A'B'$, where $\{A' = f(A), B' = f(B)\}$. Every other point $X = \mu A + \nu B$ of the line maps correspondingly to

$$X' = f(X) = f(\mu A + \nu B) = \mu f(A) + \nu f(B).$$

This rule of *"linearity"* follows from the representation of the points by projective coordinates and the fact that f, per definition, acts on the triples of these coordinates as a linear transformation. Considering a second point $Y = \chi A + \psi B$ on the line η and taking the images via f

$$Y' = f(Y) = \chi f(A) + \psi f(B) = \chi A' + \psi B',$$

we have for the cross ratios, using equation 8:

$$(AB, XY) = \frac{\chi}{\psi} : \frac{\mu}{\nu} = (A'B', X'Y'),$$

which proves a main property of projectivities:

preserves cross ratio

Theorem 15. *Projectivities preserve the cross ratio of four points on a line.*

16 Projective conics

We consider here a non-degenerated conic described in projective coordinates (u, v, w) relative to a base $\{A, B, C, D\}$ by an invertible symmetric matrix, as noticed in sections 1 and 10 with the notation slightly changed:

$$p(u, v, w) \;=\; (u, v, w)\begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} \;=\; 0.$$

Changing the projective coordinates to another system (u', v', w') we know (section 9) that the two coordinate systems are related by an invertible matrix M :

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \;=\; M \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Thus, in the new system the previous conic is described by a matrix of the form

$$(u', v', w')M^t \cdot \begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix} \cdot M \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = 0.$$

Thus the same conic in various systems of projective coordinates is described by matrices of the form:

$$P \;=\; M^t \cdot \begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix} \cdot M \quad \text{with invertible matrices} \quad M.$$

By a standard theorem ([Kap74, p.8]) of linear algebra on *"diagonalization"* of symmetric matrices, we know that there is an invertible matrix M such that P is a diagonal matrix. This, multiplying further with matrices $P' \;=\; D^t \cdot P \cdot D$ leads to a multiple of one of the matrices

$$Q' \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad Q \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

the first matrix not being admissible for non-degenerate conics, since, in the corresponding projective coordinates system, it leads to the equation

$$u^2 + v^2 + w^2 \;=\; 0.$$

The second matrix leads correspondingly to

$$u^2 + v^2 - w^2 \;=\; O,$$

and proves the theorem, showing that in the projective plane there is essentially one conic, the circle, all other being projectively equivalent to it.

all conics equivalent

Theorem 16. *All non-degenerate conics of the projective plane are pairwise projectively equivalent.*

17 Pole and polar

Of great importance in the geometry of conics is the notion of *"pole"* and *"polar of a point w.r.t. a conic"*. Functionally, the two notions are described by a linear transformation defined by the matrix of the conic w.r.t. some system of projective coordinates. If the conic is defined by the equation

$$p(u,v,w) = (u,v,w)\begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0\,, \tag{22}$$

polar Then the polar of the point $P(u,v,w)$ is a line $pu+qv+rw=0$ whose coefficients are related to the coordinates of P by the equation:

$$(p\,,\,q\,,\,r) = (u\,,\,v\,,\,w)\cdot\begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix}\,. \tag{23}$$

pole The *"pole"* of this line is by definition the p, relation between points and lines established by this equation is also invertible. In other words, the non-degenerate conic establishes a 1-1 transformation of points to lines and vice-versa.

A point $Q(u',v',w')$ lies on the polar precisely when it is satisfied the equation:

$$pu'+qv'+rw' = (u\,,\,v\,,\,w)\cdot\begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix}\cdot\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = 0\,. \tag{24}$$

This is a symmetric relation between the coordinates of the points $\{P,Q\}$ and proves the following theorem.

Theorem 17. *The point Q lies on the polar of the point P if and only if P lies on the polar of the point Q.*

The geometric content of the polar is expressed by means of the cross ratio of four points and the following theorem (see figure 10).

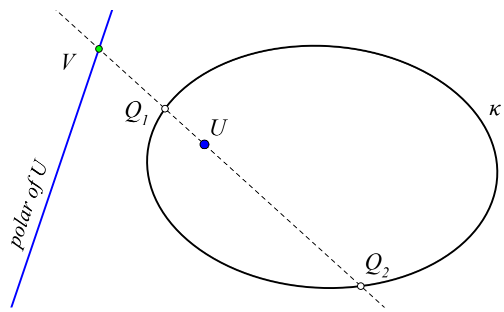

Figure 10: Pole and polar

Theorem 18. *The polar of the point U w.r.t. to the conic κ is the geometric locus of points V, such that the line UV intersecting the conic at the points $\{Q_1,Q_2\}$ satisfies $(Q_1Q_2,UV)=-1$, i.e. V is the harmonic conjugate of U w.r.t. $\{Q_1,Q_2\}$ for every line through U.*

Proof. Points $\{Q_1, Q_2\}$ can be expressed in terms of $\{U(u_0, v_0, w_0), V(u_t, v_t, w_t)\}$ as combinations (section 3).

$$Q_1 = s_1 U + t_1 V , \qquad Q_1 = s_2 U + t_2 V .$$

Using the equation 22, the points $\{Q_1, Q_2\}$ correspond to the roots $\{q_1, q_2\}$ of the equation

$$p(U + qV) = 0,$$

where we have set $q = t/s$. This is a quadratic equation in q with coefficients directly computable from the matrix, its quadratic form $p(V)$ and the corresponding bilinear form $p'(U, V)$ it defines:

$$p(U + qV) = q^2 \cdot p(V) + 2q \cdot p'(U, V) + p(U) = 0 .$$

The cross ratio $(UV, Q_1 Q_2)$ is expressed through the quotient $q2/q1$ and the harmonicity condition $(UV, Q_1 Q_2) = q_2/q_1 = -1$ is equivalent with

$$q_1 + q_2 = 0 \quad \Leftrightarrow \quad p'(U, V) = (u_0 , v_0 , w_0) \cdot \begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix} \cdot \begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix} = 0 ,$$

which shows that $V(u_t, v_t, w_t)$ belongs to the polar of U. □

We conclude this exposition with a section containing two examples of projectivities mapping respectively the circle to a parabola, and the circle to a hyperbola.

18 Projectivities mapping a circle to a parabola/hyperbola

Here we use the recipe of section 13 to create a projectivity mapping the unit circle to a parabola. As an exercise in application of the same recipe, we determine also a projectivity mapping the same circle to a hyperbola. We work in the *homogeneous system* of coordinates of section 7 and construct the projectivity f satisfying the two requirements:

1. It fixes the three points $\{A = (1, 0, 1)^t, B = (-1, 0, 1)^t, C = (0, -1, 1)^t\}$.
2. It maps the fourth point $D = (0, 1, 1)^t$ to the point at infinity in the direction of $(0, 1)$, which is $D' = (0, 1, 0)^t$.

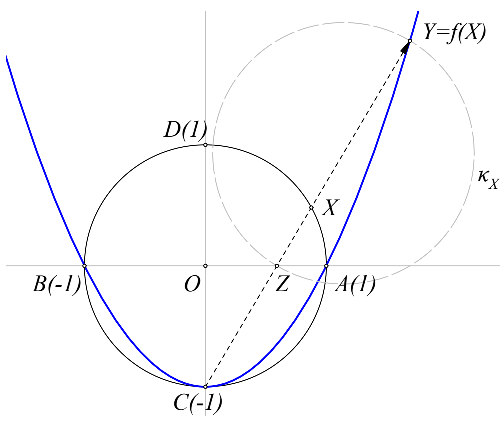

Figure 11: Projectivity (perspectivity) mapping the circle to the parabola

The matrices with columns the coordinates of the points are identical:

$$P := (A,B,C) = Q := (A',B',C') = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{with} \quad P^{-1} = \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 0 & -1 & 0 \end{pmatrix}.$$

The equation $QKP^{-1}D = k_4 D'$ takes the form:

$$\begin{pmatrix} \frac{k_1+k_2}{2} & \frac{k_1-k_2}{2} & \frac{k_1-k_2}{2} \\ 0 & k_3 & 0 \\ \frac{k_1-k_2}{2} & \frac{k_1+k_2}{2} - k_3 & \frac{k_1+k_2}{2} \end{pmatrix} = k_4 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

whose solutions are of the form $\{(k_1,k_2,k_3,k_4) = t(1,1,2,2).\}$ The matrix $M = QKP^{-1}$, representing the projectivity, takes then the form

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{mapping} \quad \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = M \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x/(1-y) \\ 2y/(1-y) \\ 1 \end{pmatrix},$$

points $\{(x',y')\}$ satisfying the equation $y' = x'^2 - 1$, representing the parabola shown in figure 11. The easily seen vanishing of the determinant

circle to parabola

$$\begin{vmatrix} x & x & 0 \\ y & 2y & -1 \\ 1 & 1-y & 1 \end{vmatrix} = 0,$$

proves that this projectivity f maps the point $X(x,y)$, to $Y = f(X) = (x',y')$ the two points being collinear with point $C(0,-1)$. Further, by inspecting the matrix M, we see that, the x-axis and point $C(0,-1)$ are the only fixed points of f, whereas the whole line $y = 1$ maps to the line at infinity. A further computation of the cross-ratio (CX,ZY), where Z is the intersection point of line CX with the x-axis, shows that this is -1 (harmonic) and the circle κ_X with diameter ZY is orthogonal to the unit circle. Actually, in the terminology used in the classification of the various kinds of projectivities f is a *"perspectivity"* with center C, axis identical to the x-axis and *"homology coefficient"*, defined by the cross ratio (CZ,XY), equal to 2.

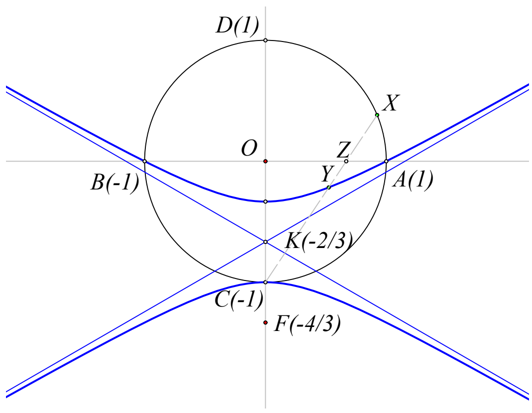

Figure 12: Projectivity (harmonic homology) mapping the circle to the hyperbola

Exercise 3. *Repeat the preceding procedure, this time to define a projectivity fixing the same points* $\{A, B, C\}$ *and mapping the point* $D = (0, 1/2, 1)^t$ *to the point at infinity* $D' = (0, 1, 0)^t$. *Show that in this case the projectivity is represented by the matrix*

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix},$$

circle to hyperbola

mapping the unit circle $x^2 + y^2 = 1$ *to the hyperbola* $x^2 - 3y^2 - 4y - 1 = 0$.

By the way, this hyperbola (See Figure 12) has its center at $K(0, -2/3)$ and its asymptotes make an angle of $60°$. An aid for the corresponding calculations can be found in the file **The quadratic equation in the plane**.

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Related material

1. **Barycentric coordinates**
2. **Ceva's theorem**
3. **Cross Ratio**
4. **Menelaus' theorem**
5. **Projective line**
6. **The quadratic equation in the plane**