The quadratic equation in the plane

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Abstract

This is a short review of the corresponding chapter in an analytic geometry lesson.

1 Introduction

A quadratic equation in two variables (in the plane) has the form

$$f(x,y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0.$$

It represents a curve in the plane referred to a coordinate system (x, y), which in the sequel we assume to be an **orthonormal coordinate system**. The curve itself and its geometric properties is the object of the main interest. It is the building, whereas the specific coordinate system, in which the curve is described by the above equation, is the scaffold used to make the work on the building. Changing the scaffold may influence the appearence of the equation and make it simpler or more complicated. This depends on the scaffold and how well it adapts to the geometric object. The curves though, which are represented by such equations existed also before the invention of this kind of description. They are called **conic sections** or simply **conics**. They are intersections with planes of cones or cylinders. In some cases, depending on the position of the plane, we get as intersections, points and lines. These are called **degenerate conics**. All others are called **proper conics**.

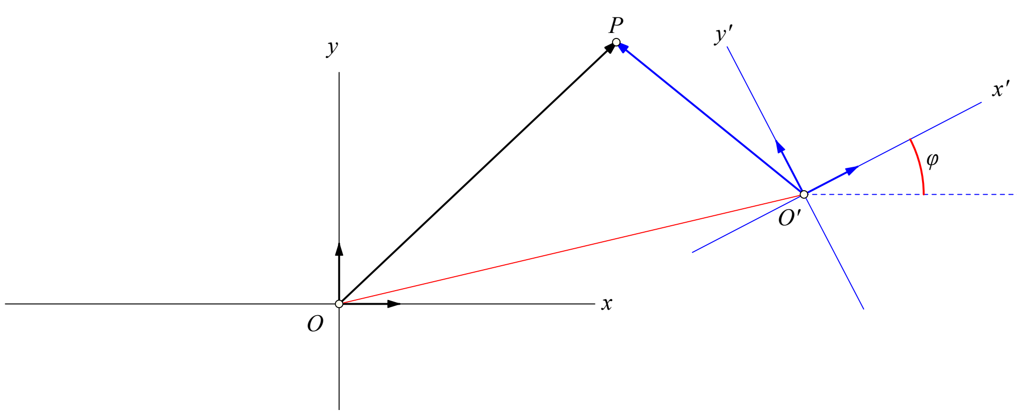

Figure 1: (x', y') system determined through $O'(x_0, y_0)$ and ϕ

2 The allowed coordinate systems

There are infinte many possible coordinate systems in which the same geometric object can be described. The acceptable coordinate systems are determined one from the other through three constants: A point $O'(x_0, y_0)$ and an angle $\frac{\pi}{4} < \phi \leq \frac{\pi}{4}$. The same point P has coordinates with respect to the two systems: (x, y) and (x', y'). These are related through the **coordinate change equations**:

$$\left.\begin{array}{l} x = x_0 + c \cdot x' - s \cdot y', \\ y = y_0 + s \cdot x' + c \cdot y'. \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{rl} x' = & c \cdot (x - x_0) + s \cdot (y - y_0), \\ y' = & -s \cdot (x - x_0) + c \cdot (y - y_0). \end{array}\right.$$

Here $c = \cos(\phi), s = \sin(\phi)$. Though there are infinite many other coordinate systems, we restrict to those that are **orthonormal** and change according to the previous relations.

3 The transformation of the coefficients

The same curve is represented by two similar equations in the two systems of coordinates:

$$\begin{array}{l} Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0, \\ A'x'^2 + 2B'x'y' + C'y'^2 + 2D'x' + 2E'y' + F' = 0. \end{array}$$

The coefficients of the second equation result by replacing (x, y) through the formulas of the preceding Nr i.e. making the substitution

$$\begin{aligned} A(x_0 + cx' - sy')^2 + 2B(x_0 + cx' - sy')(y_0 + sx' + cy') + C(y_0 + sx' + cy')^2 \\ + 2D(x_0 + cx' - sy') + 2E(y_0 + sx' + cy') + F = 0 \end{aligned}$$

and writting the resulting equation in the (x', y') system:

$$A'x'^2 + 2B'x'y' + C'y'^2 + 2D'x' + 2E'y' + F' = 0.$$

The relations between the coefficients are:

$$\begin{array}{rcl} A' & = & Ac^2 + 2Bcs + Cs^2, \\ B' & = & (C - A)cs + B(c^2 - s^2), \\ C' & = & As^2 - 2Bcs + Cc^2, \\ D' & = & c(Ax_0 + By_0 + D) + s(Bx_0 + Cy_0 + E), \\ E' & = & -s(Ax_0 + By_0 + D) + c(Bx_0 + Cy_0 + E), \\ F' & = & f(x_0, y_0) = Ax_0^2 + 2Bx_0y_0 + Cy_0^2 + 2Dx_0 + 2Ey_0 + F. \end{array}$$

These equations imply, among other things, also the following simple consequences:
1) If the transformation is a pure **translation** i.e. c=1, s=0 (the angle $\phi = 0$), then the coefficients of the quadratic terms *do not change* i.e. $A' = A, B' = B, C' = C$.
2) If the transformation ia a pure **rotation** i.e. $x_0 = y_0 = 0$, then the constant term *does not change* i.e. $F' = F$.

4 The invariants

By inspecting the above transformation relations one sees easily that the expressions

$$\begin{array}{ll} J_1 = A + C & = A' + C', \\ J_2 = AC - B^2 & = A'C' - B'^2, \\ J_3 = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} & = \begin{vmatrix} A' & B' & D' \\ B' & C' & E' \\ D' & E' & F' \end{vmatrix}. \end{array}$$

Thus, the three expressions, build from the coefficients of the equation with respect to a particular coordinate system, define three numbers, which are independent of the particular system used. These numbers are called **Invariants** of the quadratic equation. Their independence from the particular coordinate system suggests that their meaning is related to the geometric object, the curve, represented by the scaffold of the equation and carry important information concerning this geometric object. To prove this, we start with J_3 and then proceed to J_2 and J_1.

5 Product of lines $\Rightarrow J_3 = 0$

The simplest quadratic equations result by multiplying two line equations:

$$(ax + by + c)(a'x + b'y + c') = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F.$$

In these the coefficients are given by:

$$\begin{aligned}
A &= a \cdot a', \\
B &= \frac{1}{2}(a \cdot b' + b \cdot a'), \\
C &= b \cdot b', \\
D &= \frac{1}{2}(c \cdot a' + a \cdot c'), \\
E &= \frac{1}{2}(c \cdot b' + b \cdot c'), \\
F &= c \cdot c'.
\end{aligned}$$

Assume that we are in the coordinate system in which the first line ($ax + by + c = 0$) is the x-axis ($y = 0$), thus $a = 0, b = 1, c = 0$. It follows, that in this system of coordinates $A = 0, B = a'/2, C = b', D = 0, E = c'/2, F = 0$. Thus

$$J_3 = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = \begin{vmatrix} 0 & a'/2 & 0 \\ a'/2 & b' & c'/2 \\ 0 & c'/2 & 0 \end{vmatrix} = 0.$$

The inverse is also true but somewhat more difficult to prove. In fact we have proved half of the following valid theorem:

Theorem 1 *The quadratic expression $Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F$ decomposes to a product of linear factors*

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = (ax + by + c)(a'x + b'y + c'),$$

if and only if the corresponding determinant $J_3 = 0$.

6 $J_3 = 0 \Rightarrow$ a product of lines

To prove the other half of the theorem, we assume that $J_3 = 0$ and try to factor the equation $f(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$ into a product of two lines. The most complicated case is the one for which some of the coefficients A, C say A is different from zero. Thus, leaving the easiest cases for the end, we assume $A \neq 0$ and have:

$$\begin{aligned}
A \cdot f(x, y) = A^2x^2 + 2ABxy + ACy^2 + 2ADx + 2AEy + AF \quad &\Leftrightarrow \\
A^2x^2 + A(By + D) = -ACy^2 - 2AEy - AF \quad &\Leftrightarrow \\
(Ax + (By + D))^2 = (By + D)^2 - ACy^2 - 2AEy - AF \quad &\Leftrightarrow \\
(Ax + (By + D))^2 = (B^2 - AC)y^2 + 2(BD - AE)y + D^2 - AF. &
\end{aligned}$$

On the right side of the last equation stands a quadratic function in y:

$$g(y) = (B^2 - AC)y^2 + 2(BD - AE)y + D^2 - AF = Uy^2 + 2Vy + W.$$

Putting also

$$h(x, y) = Ax + (By + D),$$

we obtain the relation

$$af(x, y) = h(x, y)^2 - g(y) \quad \Leftrightarrow \quad af(x, y) = [h(x, y) - \sqrt{g(y)}] \cdot [h(x, y) + \sqrt{g(y)}].$$

In order for this to factor into two linear terms, $g(y)$ has to be a complete square i.e. its discriminant must vanish:

$$0 = UW - V^2 = (B^2 - AC)(D^2 - AF) - (BD - AE)^2 \quad \Leftrightarrow A \cdot \quad J_3 = 0.$$

In the case $A = 0$ but $B \neq 0$ we do the previous work but using this time y instead of x. In the case $A = C = 0$, we see that

$$0 = J_3 = \begin{vmatrix} 0 & B & D \\ B & 0 & E \\ D & E & F \end{vmatrix} = B \cdot (2ED - BF).$$

In this case, if also $B = 0$, then all quadratic terms vanish and the curve is a line. If $B \neq 0$, then $2ED - BF = 0$ must be valid i.e. there is a constant k such that $2E = kB, F = kD$. This implies again that the equation decomposes to a product of lines:

$$0 = 2Bxy + 2Dx + 2Ey + F = 2x(By + D) + (2Ey + F) = (2x + k)(By + D).$$

7 $J_3 = 0$: Degenerate conics

A quadratic function which decomposes to two linear factors represents a degenerate conic. A (double) line:

$$f(x, y) = (ax + by + c)^2 = a^2x^2 + 2abxy + b^2y^2 + 2acx + 2acy + c^2 = 0,$$

or a pair of lines:

$$\begin{aligned} f(x, y) &= (ax + by + c)(a'x + b'y + c') \\ &= (aa')x^2 + (ab' + ba')xy + (bb')y^2 + (ac' + ca')x + (bc' + cb')y + (cc') = 0, \end{aligned}$$

or a point

$$(x - a)^2 + (y - b)^2 = 0.$$

Last equation is satisfied only by the point with coordinates (a, b). Using complex numbers the last equation can be written:

$$(x - a)^2 + (y - b)^2 = [(x - a) + i(y - b)][(x - a) - i(y - b)] = 0.$$

Thus, in this case the quadratic equation decomposes to two linear factors too:

$$x + iy - (a + ib) = 0 \qquad \text{and} \quad x - iy - (a - ib) = 0.$$

The lines, though, are complex.

8 Example of degenerate conic

The following equation represents two intersecting lines:

$$f(x,y) = x^2 + 3xy + 2y^2 + 5x + 6y + 4 = 0.$$

One sees easily that for this equation $J_3 = 0$. To decompose into factors, order it in powers of x and complete the square:

$$\begin{aligned} f(x,y) &= x^2 + (3y+5)x + (2y^2 + 6y + 4) \\ &= \left(x + \frac{3y+5}{2}\right)^2 - \left(\frac{3y+5}{2}\right)^2 + (2y^2 + 6y + 4) \\ &= \left(x + \frac{3y+5}{2}\right)^2 - \left(\frac{y+3}{2}\right)^2 \\ &= (x+y+1)(x+2y+4). \end{aligned}$$

Exercise 1 *Show that $x^2 - xy - y^2 + 2x + 1 = 0$ decomposes in a product of two intersecting lines. Determine these lines.*

9 $J_3 \neq 0$: proper conics

If the quadratic equation

$$f(x,y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

has its invariant

$$J_3 = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} \neq 0,$$

then the corresponding curve, represented by this equation, is a proper conic i.e. either an ellipse or a hyperbola or a parabola. The ellipse and the hyperbola are **point-symmetric**.

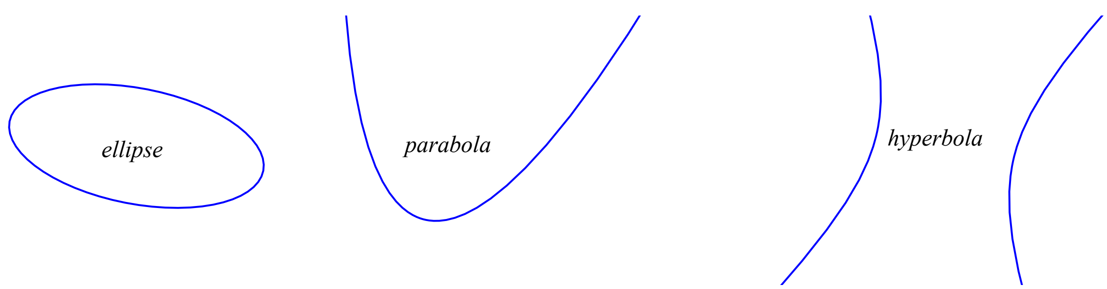

Figure 2: Proper conics

There exists a point P_0 such that for every point P on the conic, line PP_0 meets the conic in a second point P' such that $|PP_0| = |P'P_0|$. This point is called the **center** of the conic. Ellipses and hyperbolas are called collectively **central conics**. The parabola has no center of symmetry. It has though an **axis** of symmetry. This is called the **axis of the parabola** and is a line ε with the following property: for every point P on the parabola, point P' which is the **reflected** of P on the axis ε is also a point of the parabola.

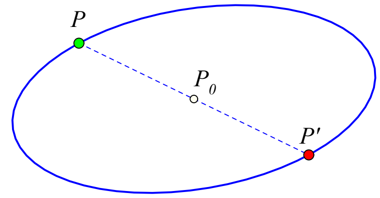

Figure 3: Ellipse's center of symmetry

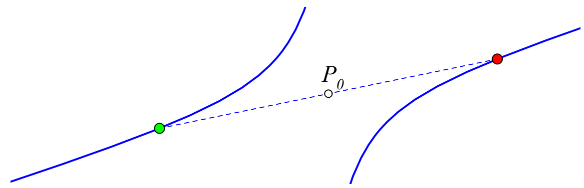

Figure 4: Hyperbola's center of symmetry

10 Central conics

The quadratic equation $f(x, y) = Ax^2+2Bxy+Cy^2+2Dx+2Ey+F = 0$, which describes a central conic, in addition to the condition $J_3 \neq 0$, is characterized by the fact that

$$J_2 = AC - B^2 \neq 0.$$

In fact, if the conic has a symmetry center and we take a coordinate system with origin at this point, then for every point $P(x, y)$ satisfying the equation the point $P'(-x, -y)$ satisfies the equation too. Thus we have:

$$\begin{aligned} f(x,y) &= Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \quad \Rightarrow \\ f(-x,-y) &= Ax^2 + 2Bxy + Cy^2 - 2Dx - 2Ey + F = 0 \quad \Rightarrow \\ f(x,y) - f(-x,-y) &= 4Dx + 4Ey = 0. \end{aligned}$$

Last equation must be true for the infinite many points $P(x, y)$ of the curve. Thus implying:

$$D = E = 0.$$

Thus, for these particular coordinate systems, which are centered at the symmetry center of the conic, the linear terms are missing and its equation obtains the simpler form:

$$f(x, y) = Ax^2 + 2Bxy + Cy^2 + F = 0.$$

It follows that in this case

$$J_3 = J_2 \cdot F \neq 0,$$

which implies that $J_2 \neq 0$.

11 Find the center

To show the inverse of the previous result, we have to prove that if both $J_3 \neq 0$ and $J_2 \neq 0$ are valid then the conic can be reduced to the above simpler form. This implies that $f(x, y) = 0 \Rightarrow f(-x, -y) = 0$ and shows that the origin of the coordinate system is the center of the conic. The problem here is to start with an equation in an arbitrary system

$$f(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

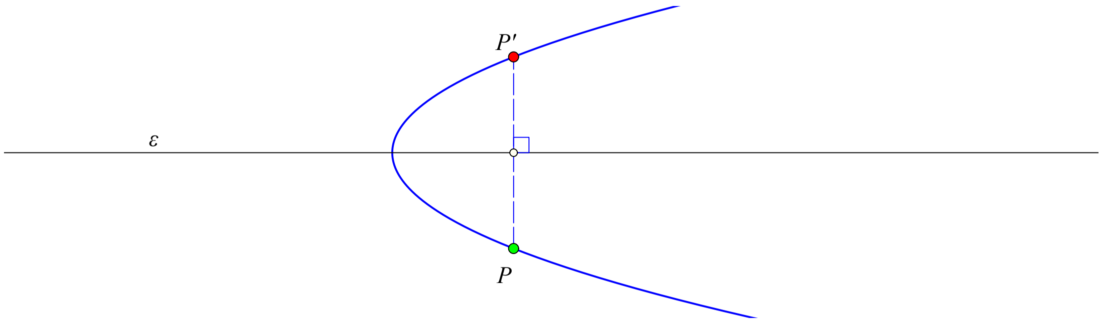

Figure 5: Parabola's axis

and find a new coordinate system centered at the center $P_0(x_0, y_0)$ of the conic. In this new system of coordinates (x', y') the quadratic equation will have the form

$$f'(x', y') = A'x'^2 + 2B'x'y' + C'y'^2 + F' = 0.$$

Hence, by the results of Nr 3, the following system of equations must hold:

$$\left.\begin{array}{l} D' = (Ax_0 + By_0 + D)c + (Bx_0 + Cy_0 + E)s = 0, \\ E' = -(Ax_0 + By_0 + D)s + (Bx_0 + Cy_0 + E)s = 0. \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{l} Ax_0 + By_0 + D = 0, \\ Bx_0 + Cy_0 + E = 0. \end{array}\right.$$

By assumption $J_2 = \begin{vmatrix} A & B \\ B & C \end{vmatrix} \neq 0$, hence the last system has a unique solution:

$$x_0 = -\frac{\begin{vmatrix} D & B \\ E & C \end{vmatrix}}{\begin{vmatrix} A & B \\ B & C \end{vmatrix}} = \frac{BE - DC}{AC - B^2},$$

$$y_0 = -\frac{\begin{vmatrix} A & D \\ B & E \end{vmatrix}}{\begin{vmatrix} A & B \\ B & C \end{vmatrix}} = \frac{DB - AE}{AC - B^2}.$$

These equations determine the location of the center $P_0(x_0, y_0)$ of the conic with respect to the original system (x, y) of coordinates. To find the new coordinates one has to make a simple translation:

$$\begin{array}{rcl} x & = & x_0 + x', \\ y & = & y_0 + y'. \end{array}$$

By Nr 3, since in this case the transformation has $c = 1, s = 0$, the coefficients of the quadratic terms are preserved and the form of the equation is

$$Ax'^2 + 2Bx'y' + Cy'^2 + F' = 0,$$

where

$$F' = f(x_0, y_0) = f(-\frac{DC - BE}{AC - B^2}, -\frac{AE - DB}{AC - B^2}) = \frac{J_3}{J_2}.$$

Last equation results by an easy calculation.

12 Find the center, examples

To find the proper conics, which are central and have coefficients $A, B, C, ...$ equal to ± 1. Locate also their centers.

$$\pm x^2 \pm 2xy \pm y^2 \pm 2x \pm 2y \pm 1 = 0.$$

Originally one thinks there are 64 such examples. But there are indeed only 8. In fact $AC - B^2 \neq 0$ means in this case $AC \neq 1$, hence A, C must have different signs. Dividing the equation by A the problem reduces to the one with coefficients

$$x^2 \pm 2xy - y^2 \pm 2x \pm 2y \pm 1 = 0.$$

Thus there are only 16 to test, and from these only 8 have $J_3 \neq 0$. These are:

$$\begin{array}{lll}
(1)\ x^2 + 2xy - y^2 + 2x + 2y - 1 = 0, & \text{center:} & (-1, 0), \\
(2)\ x^2 + 2xy - y^2 + 2x - 2y + 1 = 0, & \text{center:} & (0, -1), \\
(3)\ x^2 + 2xy - y^2 - 2x + 2y + 1 = 0, & \text{center:} & (0, 1), \\
(4)\ x^2 + 2xy - y^2 - 2x - 2y - 1 = 0, & \text{center:} & (1, 0), \\
(5)\ x^2 - 2xy - y^2 + 2x + 2y + 1 = 0, & \text{center:} & (0, 1), \\
(6)\ x^2 - 2xy - y^2 + 2x - 2y - 1 = 0, & \text{center:} & (-1, 0), \\
(7)\ x^2 - 2xy - y^2 - 2x + 2y - 1 = 0, & \text{center:} & (1, 0), \\
(8)\ x^2 - 2xy - y^2 - 2x - 2y + 1 = 0, & \text{center:} & (0, -1).
\end{array}$$

The following figure displays these 8 conics. They are all hyperbolas, even **rectangular** (we'll see what this means in a moment) and they are all **congruent** to each-other i.e. they can be set one upon the other, so that they coincide. Another way to say this is to require that *they have the same normal form* (see next Nr). In this context two normal forms are considered the same also if one of the forms results from the other by interchanging the role of x and y.

13 Axes of central conics

Central conics have also **two** axes of symmetry which are orthogonal to each other and pass through their center. Obviously it would simplify the equation if we could change to a coordinate system that has these axes as coordinate axes. In Nr 10 we saw that central conics referred to a coordinate system with origin identical with their center have a simplified corresponding equation:

$$f(x, y) = Ax^2 + 2Bxy + Cy^2 + F = 0.$$

If we assume also that the axes coincide with the symmetry axes of the curve, then with each $P(x, y)$ on the curve, points $P'(-x, y), P''(x, -y)$ must also be on the curve, so that equations

$$f(-x, y) = f(x, -y) = Ax^2 - 2Bxy + Cy^2 + F = 0,$$

must also be valid. By subtracting the two equations we see that in such a system the mixed xy-coefficient must be zero:

$$B = 0.$$

Then the quadratic equation reduces to a simpler form:

$$Ax^2 + Cy^2 + F = 0.$$

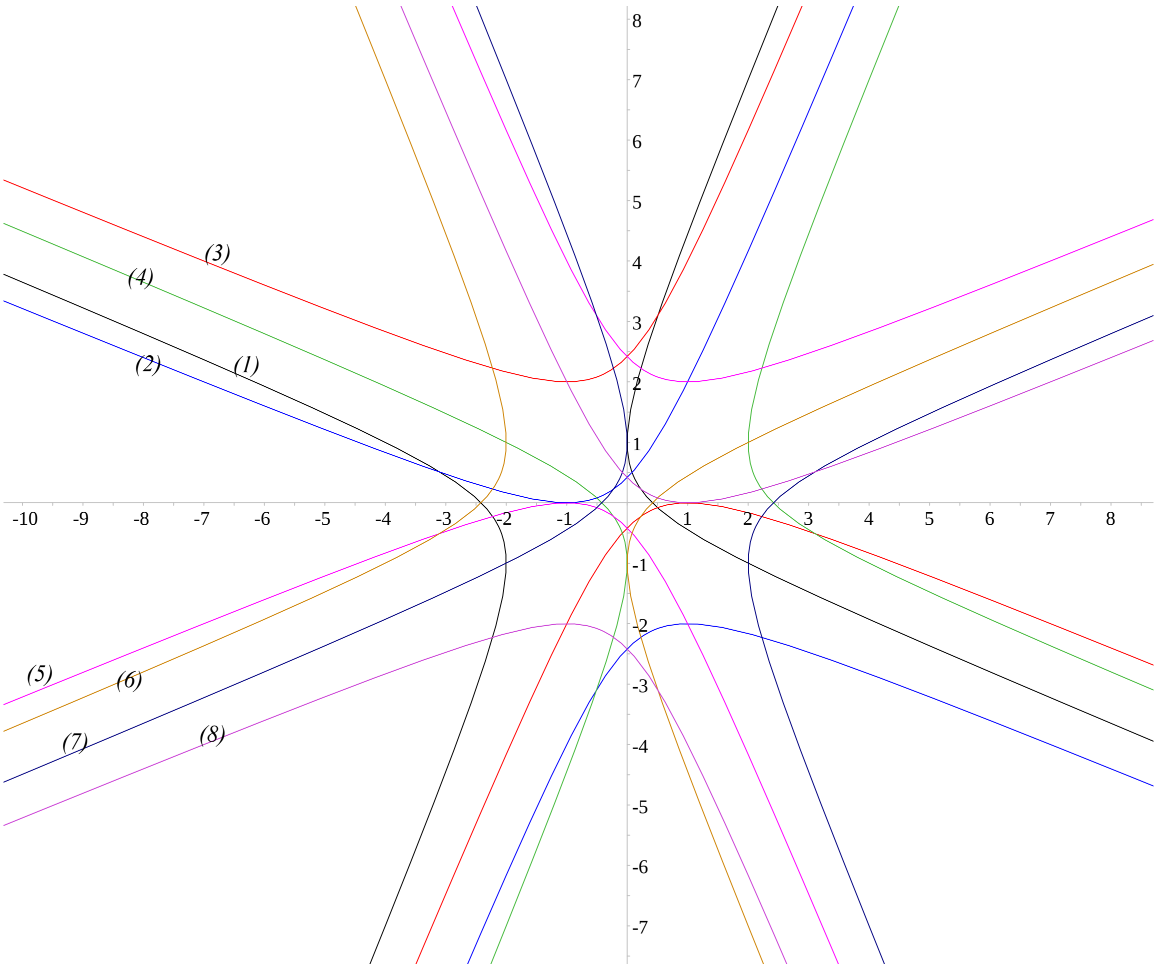

Figure 6: 8 congruent rectangular hyperbolas

Notice the difference from the equation in Nr 1. Here the equation is much simpler, the coordinates though are not the original any more. One should write the equation using something like (x', y') but I dropped the primes for aesthetic reasons. It is usual to make coordinate changes from (x, y) to (x', y'), then, possibly to others (x'', y'') and still others ... but at the final stage use the simple symbols (x, y), knowing that they are different from the original ones. The last equation is referred as the **normal form** of the equation of the conic.

14 Finding the normal form

The problem of finding the normal form of a proper central conic can be solved easily using the invariants. In fact let us now distinguish the various coordinate systems. In the first system, which is assumed to be an arbitrary one, the equation has the form:

$$f(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0.$$

If we translate to the symmetry center (x_0, y_0), then the equation becomes

$$f'(x', y') = Ax'^2 + 2Bx'y' + Cy'^2 + F' = 0, \text{ with } F' = f(x_0, y_0) = \frac{J_3}{J_2}.$$

If now, holding the origin fixed, we turn the axes so that the new axes coincide with the axes of the conic, then the equation in the new coordinates obtains the form

$$f''(x'', y'') = A'x''^2 + C'y''^2 + F' = 0.$$

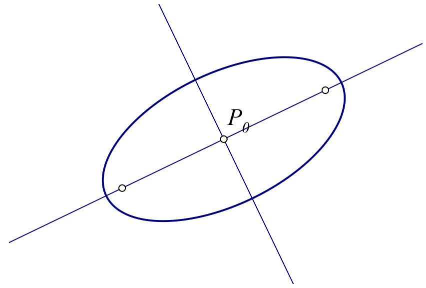

Figure 7: Ellipse with its center and its axes

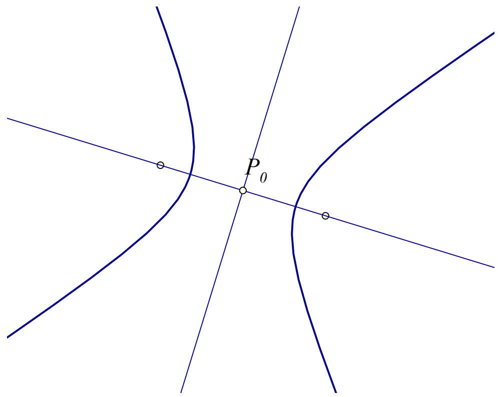

Figure 8: Hyperbola with its center and its axes

Since F' is already known, it remains to find the values of A', C'. But these constants satisfy

$$A' + C' = A + C = J_1, \quad \text{and} \quad A' \cdot C' = AC - B^2 = J_2.$$

Thus, their sum and product are obtainable from the original equation and the two constants are the roots of the equation

$$x^2 - J_1 x + J_2 = 0.$$

Even which is greater can be seen from the original equation. In fact, if $s = \sin(\phi), c = \cos(\phi)$, where ϕ is the angle by which we turn the original axes to the final axes of the conic. Then, by Nr 3, the two coefficients are related by the equation

$$A' - C' = (c^2 - s^2)(A - C) + 4Bcs.$$

But the vanishing of B' for this coordinate change implies

$$0 = B' = (C - A)cs + B(c^2 - s^2).$$

If neither B nor $C - A$ vanish, then solving the last for B and replacing in the previous equation gives:

$$A' - C' = (A - C)\left(c^2 - s^2 + 4\frac{c^2 s^2}{c^2 - s^2}\right).$$

By the conventions made in Nr 2, angle ϕ satisfies $\frac{-\pi}{4} < \phi \leq \frac{\pi}{4}$, and $c^2 - s^2 = \cos(2\phi) > 0$ under this restriction, i.e. $(A - C)$ and $(A' - C')$ have the same sign. Thus, which one from A', C' is greater from the other can be seen directly from the original equation.

In the case $B = 0$ the axes are already the symmetry axes of the conic. In the case $A - C = 0$, taking $\phi = \frac{\pi}{4}$ implies $B' = 0$.

15 Example calculation of the normal form

To see the previous procedure working, let us apply it to the example of Nr 12:

$$x^2 + 2xy - y^2 + 2x + 2y - 1 = 0.$$

Here we see easily that $J_1 = 0, J_2 = -2, J_3 = 4$. Thus,

$$F' = \frac{J_3}{J_2} = -2,$$

and A', C' are the roots of the equation

$$x^2 - 2 = 0 \quad \Rightarrow \quad A' = \sqrt{2}, C' = -\sqrt{2}.$$

In the last decision, which of the roots to equal to A' and which to C', we take into account that $A' - C'$ and $A - C = 2$ must have equal signs. Thus the normal form (dropping again the primes) is

$$(\sqrt{2})x^2 - (\sqrt{2})y^2 - 2 = 0.$$

Let us repeat the procedure for the second equation in Nr 12:

$$x^2 + 2xy - y^2 + 2x - 2y + 1 = 0, \quad \text{with} \quad J_1 = 0, J_2 = -2, J_3 = -4.$$

Thus, A', C' satisfy the equation $x^2 - 2 = 0$ and we have again

$$F' = \frac{J_3}{J_2} = 2, A' = \sqrt{2}, B' = -\sqrt{2} \quad \Rightarrow \quad (\sqrt{2})x^2 - (\sqrt{2})y^2 + 2 = 0.$$

By interchanging the roles of x and y we see that the curves corresponding to these equations are **congruent**.

16 Finding the axes of the conic

As noticed in Nr 13, the directions of the symmetry axes of a proper central conic can be determined by the condition $B' = 0$ which must be valid, when the conic is referred to its normal coordinate system with origin at its center. From Nr 3, leaving some special cases by side, this implies

$$0 = B' = (C - A)cs + B(c^2 - s^2) \quad \Rightarrow \quad \frac{2cs}{c^2 - s^2} = \frac{2B}{A - C}.$$

Since

$$\frac{2cs}{c^2 - s^2} = \frac{2\cos(\phi)\sin(\phi)}{\cos(\phi)^2 - \sin(\phi)^2} = \frac{\sin(2\phi)}{\cos(2\phi)} = \tan(2\phi),$$

the above equation becomes

$$\tan(2\phi) = \frac{2B}{A - C}.$$

This defines the angle ϕ by which the actual axes have to be rotated in order to obtain the right directions of the conic axes. The special cases left are $B = 0$ i.e. the current coordinate system is already the right one, and $A - C = 0$, in which taking $\phi = \frac{\pi}{4}$ makes $B' = 0$.

For example the first of the equations in Nr 12:

$$x^2 + 2xy - y^2 + 2x + 2y - 1 = 0,$$

has $B = 1, A - C = 2$, hence $\tan(2\phi) = 1 \;\Rightarrow\; 2\phi = \frac{\pi}{4} \;\Rightarrow\; \phi = \frac{\pi}{8}$.

17 Finding the kind of the conic

The hyperbola is unbounded and the ellipse is bounded. This is the fundamental distinction of the two kinds of proper central conics. The kind of the conic can be immediately deduced from the normal form:

$$Ax^2 + Cy^2 + F = 0.$$

If all coefficients are positive then no real conic exists. If A, C have the same sign, which is equivalent with the condition that $J_2 > 0$, then we have an ellipse in the case $F \cdot J_2 = J_3 < 0$ and no real curve if $J_3 > 0$. In fact, in this case, multiplying the whole equation by -1 if necessary, we may assume that $A, C > 0$ and $F < 0$ and the equation becomes

$$(-F) = Ax^2 + Cy^2 \geq m(x^2 + y^2), \ \Rightarrow \ , x^2 + y^2 \leq \frac{-F}{m}.$$

where m is the smaller of A, C. This shows that all points of the curve are at square-distance from the origin less than $\frac{-F}{m}$, hence the curve is bounded i.e. it is an ellipse.

If A, C have different signs, which is equivalent with $J_2 < 0$, then we have always a hyperbola. In fact, in this case, we can assume that A is positive and C, F are negative and set the equation in the form

$$Ax^2 + Cy^2 + F = 0 \ \Rightarrow \ x^2 = \frac{1}{A}(-Cy^2 - F),$$

which shows that (x, y) can obtain arbitrary big values, hence the curve is unbounded i.e. it is a hyperbola.

18 Asymptotes

The simplest way to define the asymptotes of a proper central conic is to use its normal form

$$Ax^2 + Cy^2 + F = 0.$$

The lines resulting from the equation

$$Ax^2 + Cy^2 = 0,$$

are called the asymptotes of the conic. They are two real lines only in the case of hyperbolas, i.e. when $J_2 < 0$. In this case, assuming $A > 0$ and $C < 0$, the quadratic equation decomposes to a product :

$$Ax^2 + Cy^2 = (\sqrt{A}x - \sqrt{-C}y)(\sqrt{A}x + \sqrt{-C}y) = 0.$$

The figure shows the hyperbola

$$2x^2 - y^2 - 1 = 0,$$

and its asymptotes, which are the lines

$$\sqrt{2}x - y = 0, \ \text{and} \ \ \sqrt{2}x + y = 0.$$

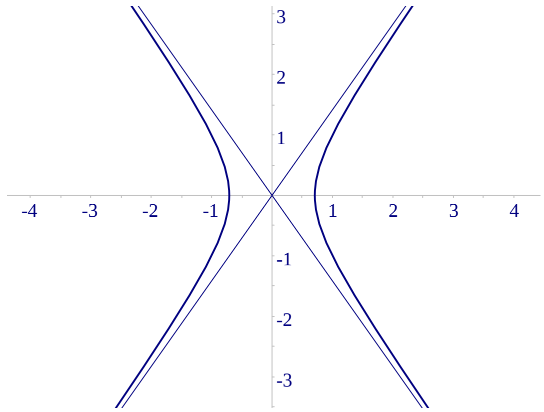

Figure 9: Hyperbola and its asymptotes

19 The angle of the asymptotes

The cosine of the angle of the asymptotes is calculated by the inner product of the unit vectors in the direction of the lines. By the preceding Nr these vectors are

$$\frac{1}{\sqrt{A-C}}(\sqrt{-C},\sqrt{A}), \text{ and } \frac{1}{\sqrt{A-C}}(-\sqrt{-C},\sqrt{A}).$$

And their inner product giving the cosine ($\cos(\theta)$) of the angle between the asymptotes is

$$\begin{aligned}
\cos(\theta) &= \frac{1}{\sqrt{A-C}}(\sqrt{-C},\sqrt{A})\cdot\frac{1}{\sqrt{A-C}}(-\sqrt{-C},\sqrt{A}) \\
&= \frac{A+C}{A-C} \\
&= \frac{A+C}{\sqrt{(A-C)^2}} \\
&= \frac{A+C}{\sqrt{(A+C)^2-4AC}} \\
&= \frac{A+C}{\sqrt{(A+C)^2-4(AC-B^2)}} \quad \text{(since } B=0\text{)}, \\
&= \frac{J_1}{\sqrt{J_1^2-4J_2}}.
\end{aligned}$$

20 Rectangular hyperbola

This kind of proper conic is characterized by the condition

$$J_1 = A + C = 0.$$

This implies

$$J_2 = AC - B^2 = -A^2 - B^2 < 0.$$

Hence, by Nr 17, this is a hyperbola. As already noticed, all examples of Nr 12 are rectangular hyperbolas. Rectangular hyperbolas have their asymptotes orthogonal (therefore the name). They are the only conics which appear as the graph of an invertible function (in its domain of definition). The most prominent example is the graph of the function $y = \frac{1}{x}$ represented by the quadratic

$$xy - 1 = 0,$$

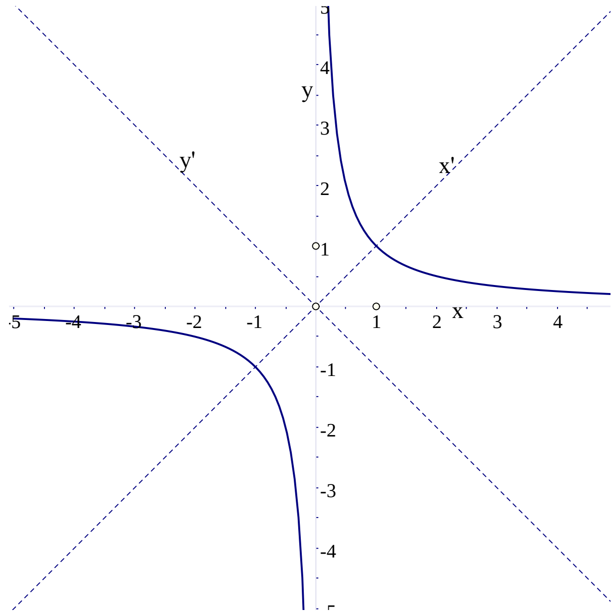

Figure 10: The graph of $y = \frac{1}{y}$ is a rectangular hyperbola

for which the corresponding invariants are $J_1 = 0, J_2 = -\frac{1}{4}, J_3 = \frac{1}{4}$, leading to the canonical form

$$A'x'^2 + C'y'^2 + F' = 0 \quad : \quad \frac{1}{2}x'^2 - \frac{1}{2}y'^2 - 1 = 0.$$

The figure 10 shows the curve and the two coordinate systems (x, y) and (x', y').

21 Asymptotes directly

Given the quadratic equation

$$f(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

the directions of asymptotes can be directly determined by dropping the linear terms and equating the remaining expression to zero:

$$Ax^2 + 2Bxy + Cy^2 = 0.$$

In fact, as noticed in Nr 3, the quadratic coefficients A, B, C do not change if we change to the coordinate system at the center, without to turn the axes. Turning now the axes to match the axes of the conic is done by a transformation of the form

$$\left.\begin{aligned} x &= c \cdot x' - s \cdot y', \\ y &= s \cdot x' + c \cdot y'. \end{aligned}\right\} \quad \Leftrightarrow \quad \left\{\begin{aligned} x' &= \quad c \cdot x + s \cdot y, \\ y' &= -s \cdot x + c \cdot y. \end{aligned}\right.$$

Taking into account the relations between A, B, C and A', B', C' given in Nr 3, we see by an easy calculation that

$$Ax^2 + 2Bxy + Cy^2 = 0 \quad \Leftrightarrow \quad A'x'^2 + 2B'x'y' + Cy'^2 = 0,$$

i.e. the last equation involving the (x', y') coordinates is valid, if and only if, the left equation for the corresponding coordinates in (x, y) is valid.

A trivial example of this fact is given by the hyperbola $xy = 1$ of the previous Nr. In the (x, y) system the asymptotes are given by equating

$$xy = 0 \text{ i.e. either } x = 0 \text{ or } y = 0.$$

In the (x', y') system the same asymptotes are given by the equation

$$\frac{1}{2}x'^2 - \frac{1}{2}y'^2 = 0 \;\Leftrightarrow\; x' - y' = 0 \text{ or } x' + y' = 0.$$

This means that in the (x, y) system the asymptotes coincide with the coordinate axes, whereas in the (x', y') system the asymptotes coincide with the bisectors of the corresponding coordinate axes.

22 Parabolas

The parabolas are the proper non-central conics. They are thus characterized by the two invariants being:

$$J_3 \neq 0 \quad \text{and} \quad J_2 = AC - B^2 = 0.$$

The expression $-J_2 = B^2 - AC$ is seen to be the discriminant of the polynomial $At^2 + 2Bt + C$. Thus its vanishing means that the polynomial has a double root ($x = -\frac{B}{A}$) and consequently can be written

$$At^2 + 2Bt + C = A(t + \frac{B}{A})^2.$$

This, setting $t = \frac{x}{y}$, implies

$$Ax^2 + 2Bxy + Cy^2 = y^2(At^2 + 2Bt + C) = y^2 A(\frac{x}{y} + \frac{B}{A})^2 = (Ax + By)^2.$$

The original equation then can be written in the form

$$(Ax + By)^2 + 2Dx + 2Ey + F = 0.$$

This suggests to make the transformation

$$\left.\begin{aligned} x &= c \cdot x' - s \cdot y', \\ y &= s \cdot x' + c \cdot y', \end{aligned}\right\} \qquad \Leftrightarrow \qquad \left\{\begin{aligned} x' &= \quad c \cdot x + s \cdot y, \\ y' &= -s \cdot x + c \cdot y, \end{aligned}\right.$$

where

$$c = \frac{A}{\sqrt{A^2 + B^2}}, \quad s = \frac{B}{\sqrt{A^2 + B^2}}.$$

By this the original equation transforms according to the rules of Nr 3 and we see that the new coefficients in the (x', y') coordinate system are

$$\begin{aligned} A' &= A + C = J_1, \\ B' &= 0, \\ C' &= 0, \\ D' &= \frac{AD + EB}{\sqrt{A^2 + B^2}}, \\ E' &= \frac{-DB + EA}{\sqrt{A^2 + B^2}}, \\ F' &= F. \end{aligned}$$

Note that $J_1 = A + C \neq 0$. This follows from the fact that $0 = J_2 = AC - B^2$ implies that A, C have the same sign. Hence $A + C = 0$ would imply $A = C = B = 0$ and the conic would be non proper, which contradicts our assumption. Thus the equation takes the form:

$$
\begin{aligned}
J_1 x'^2 + 2D'x' + 2E'y' + F = 0 \quad &\Leftrightarrow \\
x'^2 + 2\frac{D'}{J_1}x' + 2\frac{E'}{J_1}y' + \frac{F}{J_1} = 0 \quad &\Leftrightarrow \\
x'^2 + 2D''x' + 2E''y' + F'' = 0 \quad &\Leftrightarrow \\
(x' + D'')^2 + 2E''y' + (F'' - D''^2) = 0, &
\end{aligned}
$$

with the obvious substitution $D'' = \frac{D'}{J_1}, E'' = \frac{E'}{J_1}, F'' = \frac{F}{J_1}$. Again $E'' \neq 0$, since $E'' = 0$ would imply again that the conic is non-proper. Thus, by dividing through E'' and making the translation of the coordinate system:

$$
\begin{aligned}
x_1 &= x' + D'', \\
y_1 &= y' + \frac{F'' - D''^2}{E''},
\end{aligned}
$$

we obtain the equation

$$x_1^2 + 2E''y_1 = 0,$$

which is the normal form of a parabola.

Using the first form of the equation above and computing the invariant J_3 we see that $J_3 = -E'^2 J_1$ (which implies that $J_3 \cdot J_1 < 0$) and since $E' = J_1 E''$ we conclude that

$$E''^2 = -\frac{J_3}{J_1^3}.$$

Notice that the sign of E'' is not so important for the shape of the parabola, since $x^2 \pm 2Ey = 0$ represent two parabolas which are symmetric with respect to the x-axis.

23 Parabola Examples

The following cases are all possible parabolas with $A, B, C, ...$ having the values ± 1.

$$
\begin{aligned}
&(1)\ x^2 + 2xy + y^2 + 2x - 2y + 1 = 0, \\
&(2)\ x^2 + 2xy + y^2 + 2x - 2y - 1 = 0, \\
&(3)\ x^2 + 2xy + y^2 - 2x + 2y + 1 = 0, \\
&(4)\ x^2 + 2xy + y^2 - 2x + 2y - 1 = 0, \\
&(5)\ x^2 - 2xy + y^2 + 2x + 2y + 1 = 0, \\
&(6)\ x^2 - 2xy + y^2 + 2x + 2y - 1 = 0, \\
&(7)\ x^2 - 2xy + y^2 - 2x - 2y + 1 = 0, \\
&(8)\ x^2 - 2xy + y^2 - 2x - 2y - 1 = 0,
\end{aligned}
$$

They all have $J_2 = 2$ and $J_3 = -2$, so that in the canonical form appearing in the previous Nr $E'' = -\frac{1}{2}$ and all of them have the canonical form:

$$x^2 - y = 0.$$

Thus, they are all congruent. The following figure displays all of them. The numbers refer to the corresponding equation of the curve.

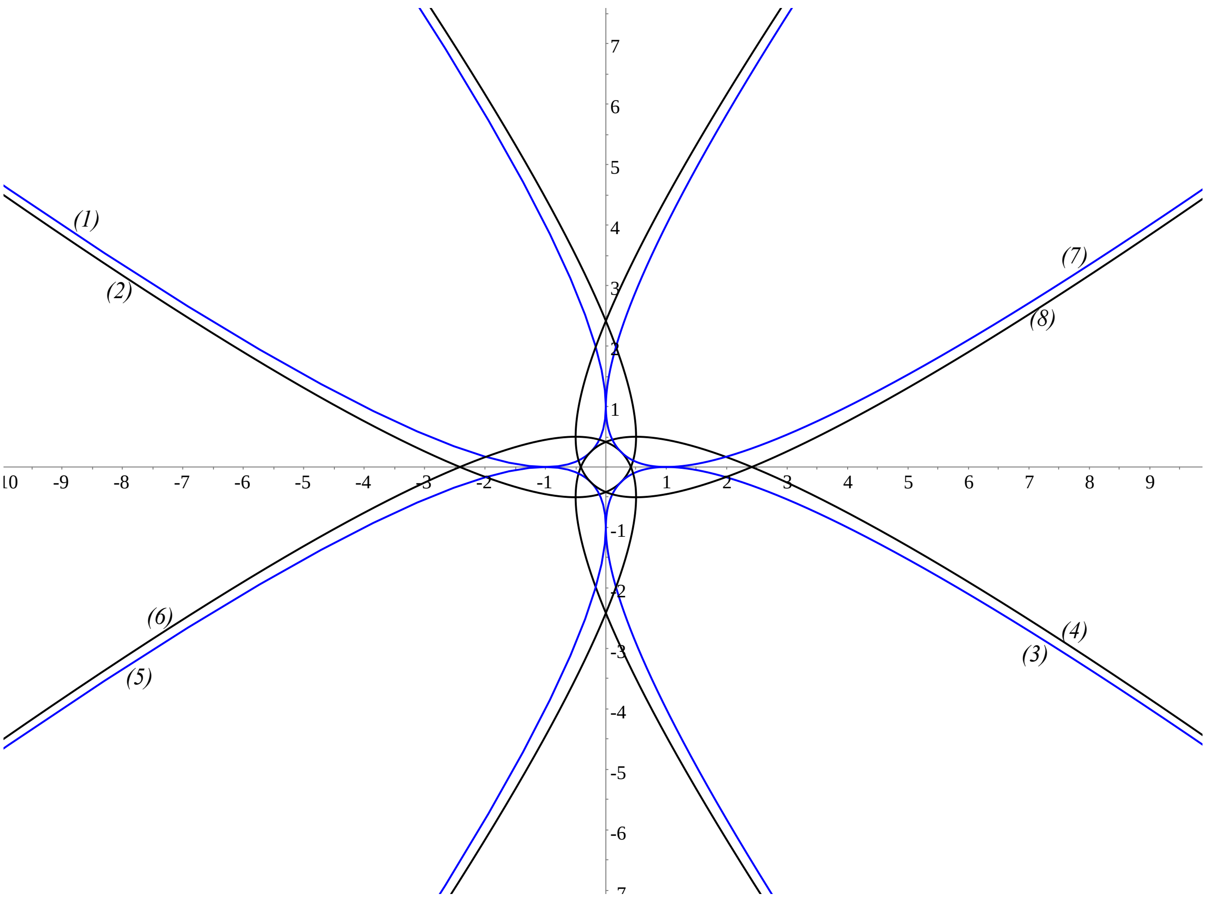

Figure 11: The eight parabolas with coefficients ± 1