Homotheties and Similarities

A file of the Geometrikon gallery by Paris Pamfilos

We shall not cease from exploration
And the end of all our exploring
Will be to arrive where we started
And know the place for the first time.

T.S. Eliot, Little Gidding

Contents

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1 Homotheties

Homotheties of the plane and their generalization, the *"similarities"*, to be discussed below, are transformations which generalize those of *"isometries"* or *"congruences"* of the plane (see file **Isometries**).

Given a number $\kappa \neq 0$ and point O of the plane, we call *"homothety"* of *"center"* O and *"ratio"* κ the transformation which corresponds: a) to point O, itself, b) to every point $X \neq O$ the point X' on the line OX, such that the following signed ratio relation holds:

$$\frac{OX'}{OX} = \kappa.$$

A direct consequence of the definition is, that for every point O the homothety of center

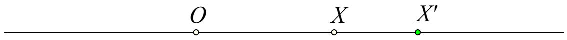

Figure 1: Homothety

O and ratio $\kappa = 1$ is the identity transformation. Often, when the ratio is $\kappa < 0$ we say that the transformation is an *"antihomothety"*. Its characteristic is that point O is between X and X'.

Theorem 1. *The composition of two homotheties with center O and ratios κ and λ is a homothety of center O and ratio $\kappa \cdot \lambda$.*

Proof. Obvious consequence of the definition. If f and g are the two homotheties with the same center O and ratios respectively κ and λ, then, for every point X, points $Y = f(X)$, $Z = g(Y)$ and O will be four points on the same line and will satisfy,

$$\frac{OY}{OX} = \kappa, \quad \frac{OZ}{OY} = \lambda \quad \Rightarrow \quad \frac{OZ}{OX} = \frac{OZ}{OY} \cdot \frac{OY}{OX} = \lambda \cdot \kappa \,.$$

□

Corollary 1. *The inverse transformation of a homothety f, of center O and ratio κ, is the homothety with the same center and ratio $\frac{1}{\kappa}$.*

Remark 1. The homothety is a special transformation closely connected with Thales' theorem and the *"similarity of triangles"*, i.e *"triangles which have equal corresponding angles" Lef trightarrow "triangles which have proportional corresponding sides"*.

Two triangles, and more general two shapes $\{\Sigma, \Sigma'\}$ are called *"homothetic"* when there exists a homothety f mapping one onto the other $f(\Sigma) = \Sigma'$.

Exercise 1. *Find all homotheties that transform a given point X to another point $X' \neq X$.*

Homothetic triangles are particular cases of *"similar triangles"* and are the key to investigate properties of more general homothetic shapes.

2 Homotheties and triangles

Theorem 2. *A homothety f with center O maps a triangle OXY to a similar triangle $OX'Y'$ with $\{X' = f(X), Y' = f(Y)\}$, the sides $\{XY, X'Y'\}$ being parallel.*

Proof. A simple application of Thales' theorem. □

Corollary 1. *A homothety maps a line ε to a parallel line ε'.*

Exercise 2. *Find all homotheties that transform a given line ε to a given parallel to it ε'. Distinguish the cases $\{\varepsilon \neq \varepsilon' \, , \, \varepsilon = \varepsilon'\}$.*

Theorem 3. *A homothety maps a triangle ABC to a similar triangle $A'B'C'$.*

Proof. Application of the previous corollary and Thales' theorem. □

Corollary 2. *A homothety f preserves the angles and multiplies the distances between points with its ratio. In other words, for every pair of points X, Y and their images $X' = f(X), Y' = f(Y)$ holds $|X'Y'| = \kappa|XY|$ and for every three points the respective angles are preserved $\widehat{Y'X'Z'} = \widehat{YXZ}$.*

Exercise 3. *Show that there is no genuine homothety with ratio $k \neq 1$ mapping a triangle to itself.*

Theorem 4. *Two triangles ABC and $A'B'C'$, which have their corresponding sides parallel are similar.*

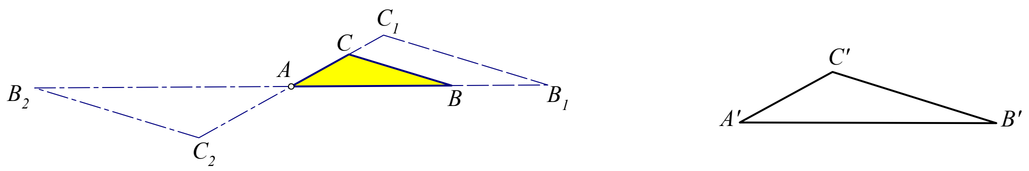

Figure 2: Triangles with corresponding sides parallel

Proof. Translate triangle $A'B'C'$ and place it in such a way, that the vertices A and A' coincide and the lines of their sides AB, AC coincide respectively with $A'B'$ and $A'C'$ (See Figure 2). The translated triangle will take the position AB_1C_1 or AB_2C_2, with its third side parallel to BC. Therefore, it will be similar to ABC, while it is also congruent to the initial $A'B'C'$. □

Theorem 5. *For two triangles ABC and $A'B'C'$, which have parallel corresponding sides, the lines AA', BB' and CC', which join the vertices with the corresponding equal angles, either pass through a common point and the triangles are homothetic, or are parallel and the triangles are congruent.*

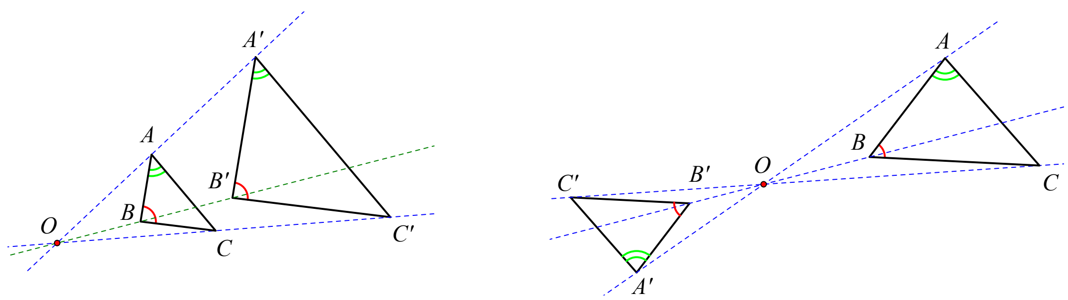

Figure 3: Homothetic triangles

Proof. Let O be the intersection point of AA' and BB'. We will show that CC' also passes through point O. According to Thales, we have equal ratios $\frac{|AB|}{|A'B'|} = \frac{|OA|}{|OA'|} = \frac{|OB|}{|OB'|} = \kappa$. Consider therefore on OC point C'' with $\frac{|OC|}{|OC''|} = \kappa$. The created triangle $A'B'C''$ has sides proportional to those of ABC, therefore it is similar to it and consequently has the same angles. It follows, that $A'B'C'$ and $A'B'C''$ have $A'B'$ in common and same angles at A' and B', therefore they coincide and $C' = C''$, in other words, OC passes through C' too.

This reasoning shows also that, if the two lines AA' and BB' do not intersect, that is if they are parallel, then the third line will also be necessarily parallel to them and $ABB'A'$, $BCC'B'$ and $ACC'A'$ will be parallelograms, therefore the triangles will have corresponding sides equal. □

3 Homotheties with different centers

Theorem 6. *The composition of two homotheties f and g with different centers O and P and ratios respectively κ and λ, with $\kappa \cdot \lambda \neq 1$, is a homothety with center T on the line OP and ratio equal to $\kappa \cdot \lambda$.*

Proof. The proof is an interesting application of Menelaus' theorem (see file **Menelaus' theorem**). Let X be an arbitrary point and $Y = f(X)$, $Z = g(Y)$. This defines the triangle OYP and the points X, Z are contained in its sides OY and YP respectively. Let T be the intersection point of ZX with OP. Applying Menelaus' theorem we have,

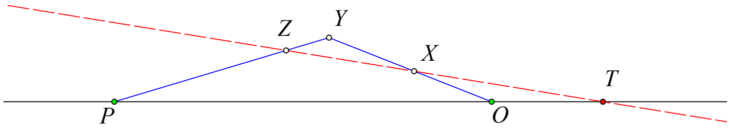

Figure 4: Composition of homotheties with $\kappa\lambda \neq 1$

$$\frac{XO}{XY} \cdot \frac{ZY}{ZP} \cdot \frac{TP}{TO} = 1 \quad \Rightarrow \quad \frac{TP}{TO} = \frac{XY}{XO} \cdot \frac{ZP}{ZY}.$$

However, for the oriented line segments holds

$$\begin{aligned}
XY &= XO + OY \;\Rightarrow \frac{XY}{XO} = \frac{XO+OY}{XO} = 1 + \frac{OY}{XO} = 1 - \kappa, \\
ZY &= ZP + PY \;\Rightarrow \frac{ZY}{ZP} = \frac{ZP+PY}{ZP} = 1 + \frac{PY}{ZP} = 1 - \frac{1}{\lambda} \quad \Rightarrow \\
\frac{TP}{TO} &= \frac{XY}{XO} \cdot \frac{ZP}{ZY} = (1-\kappa) \cdot \left(\frac{1}{1-\frac{1}{\lambda}} \right) = \frac{\lambda \cdot (1-\kappa)}{\lambda - 1}.
\end{aligned}$$

The last formula shows, that the position of T on the line OP is fixed and independent of X. In addition, the ratio $\mu = \frac{TZ}{TX}$ is calculated, by applying Menelaus' theorem to the triangle OXT, this time with PY as secant:

$$\begin{aligned}
\frac{PT}{PO} \cdot \frac{ZX}{ZT} \cdot \frac{YO}{YX} = 1 & \quad \Rightarrow \\
\frac{ZX}{ZT} = \frac{YX}{YO} \cdot \frac{PO}{PT} & \quad \Leftrightarrow \\
\frac{ZT+TX}{ZT} = \frac{YO+OX}{YO} \cdot \frac{PT+TO}{PT} & \quad \Leftrightarrow \\
1 - \frac{1}{\mu} = \left(1 + \frac{OX}{YO}\right)\left(1 + \frac{TO}{PT}\right) & \quad \Leftrightarrow \\
1 - \frac{1}{\mu} = \left(1 - \frac{1}{\kappa}\right)\left(1 - \frac{\lambda - 1}{\lambda(1-\kappa)}\right) & \quad \Leftrightarrow \\
\mu = \kappa\lambda \,. &
\end{aligned}$$

□

Theorem 7. *The composition of two homotheties f and g with different centers O and P respectively and ratios κ and λ with $\kappa \cdot \lambda = 1$ is a translation by interval parallel to OP.*

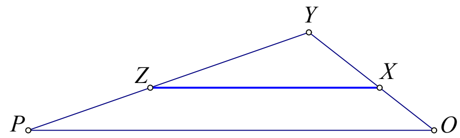

Figure 5: Composition of homotheties with $\kappa\lambda = 1$

Proof. Let X be an arbitrary point and $Y = f(X)$, $Z = g(Y)$. This defines the triangle OYP and the points X, Z are contained in its sides OY and YP respectively. According to the hypothesis

$$\frac{YX}{YO} = \frac{YO+OX}{YO} = 1 - \frac{1}{\kappa}, \qquad \frac{YZ}{YP} = \frac{YP+PZ}{YP} = 1 - \frac{PZ}{YP} = 1 - \lambda = 1 - \frac{1}{\kappa}.$$

The equality of the ratios shows, that the line segment XZ is parallel to OP. From the similarity of triangles YOP and YXZ, follows that

$$XZ = (1-\lambda)OP,$$

therefore XZ has fixed length and direction. □

4 Homotheties and translations

Theorem 8. *The composition $g \circ f$ of a homothety and a translation g is a homothety.*

Proof. Assume that the homothety has center O and ratio κ and the translation is defined by the fixed, oriented line segment AB. Let also $X \neq O$ be arbitrary and $Y = f(X)$, $Z = g(Y)$. Assume finally that P is the intersection of the line XZ and the line ε is the parallel

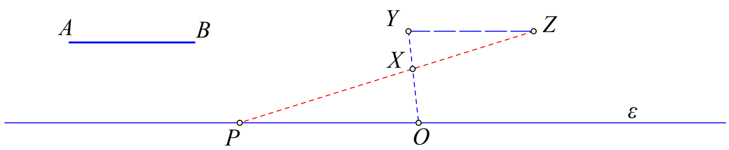

Figure 6: Compositions of homotheties and a translation

to AB from O (See Figure 6). From the similarity of the triangles XYZ and OXP follows that

$$\frac{OP}{AB} = \frac{OP}{YZ} = \frac{OX}{YX} = \frac{OX}{YO+OX} = \frac{1}{\frac{YO+OX}{OX}} = \frac{1}{1-\kappa} \quad \Rightarrow \quad OP = \frac{1}{1-\kappa}AB.$$

It follows that the position of P on ε is fixed and independent of X. Also for the ratio,

$$\frac{PZ}{PX} = \frac{OY}{OX} = \kappa.$$

Therefore the composition $g \circ f$ is a homothety of center P and ratio κ. □

Exercise 4. *Show that the composition $g \circ f$ of a translation f and a homothety g is a homothety.*

Remark 2. The last theorems and the exercise show that homotheties and translations build a closed, as we say, set of transformations with respect to composition. We saw something similar also for rotations and translations (see file **Isometries**).

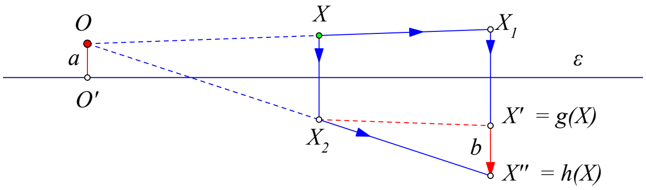

Figure 7: Composition of a homothety and a reflection

Exercise 5. *Consider a line ε and a point O at distance a from it. Let f_1 be the homothety with center at O and ratio k and f_2 the reflection in ε. Show that the compositions $g = f_2 \circ f_1$ and $h = f_1 \circ f_2$ differ by a translation. In other words, for every point X of the plane it is valid $h(X) - g(X) = b$, where b is a line segment of length $|2a(1-k)|$ and direction orthogonal to ε (See Figure 7).*

Exercise 6. *Given two circles with different radii, show that there exist homotheties which map one to the other. How many are there? What are their centers and ratios?*

Exercise 7. *Given two circles κ and λ, draw a line intersecting them, which forms chords AB, CD, having given lengths ([2, p. 21]).*

Exercise 8. *Show that a shape , with more than one points, for which there is a homothety f, different from the identity, leaving Σ invariant ($f(\Sigma) = \Sigma$), extends to infinity. Find a shape example with this property.*

Hint: If f leaves invariant, then also the inverse homothety $g = f^{-1}$ will leave it invariant. If $\{O, k\}$ is the center and the ratio of f, then $\{O, \frac{1}{k}\}$ will be respectively the center and ratio of the inverse homothety. Thus, we can assume $k > 1$. Then if X is an arbitrary point of Σ, the $X' = f(X)$ will satisfy $|OX'| = k|OX|$. Repeating this procedure we find $X'' = f(X')$, with $|OX''| = k^2|OX|$ and after n similar steps, we find points $X^{(n)} = f(X^{(n-1)})$, with $|OX^{(n)}| = k^n|OX|$.

A shape example with the aforementioned property is a set of lines through a fixed point O.

5 Representation and group properties of homotheties

Fixing a cartesian coordinate system, the homothety with center O and ratio k is represented using vectors by :

$$Y = O + k \cdot (X - O) \quad \Leftrightarrow \quad \{y_1 = o_1 + k(x_1 - o_1)\,,\ y_2 = o_2 + k(x_2 - o_2)\}. \tag{1}$$

Using matrices, this is equivalent to :

$$\begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} k & 0 & (1-k)o_1 \\ 0 & k & (1-k)o_2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}. \tag{2}$$

The product of two such matrices is of the same form :

$$\begin{pmatrix} k & 0 & (1-k)o_1 \\ 0 & k & (1-k)o_2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} k' & 0 & (1-k')o'_1 \\ 0 & k' & (1-k')o'_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} k'' & 0 & (1-k'')o''_1 \\ 0 & k'' & (1-k'')o''_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3}$$

$$\text{with} \quad k'' = k \cdot k' \quad \text{and} \quad O'' = \frac{k(1-k')}{1-kk'}O + \frac{1-k}{1-kk'}O' .$$

Equation (2) shows that the general homothetic transformation is a product of a homothety centered at the origin and a translation with corresponding matrix representations:

$$\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & (1-k)o_1 \\ 0 & 1 & (1-k)o_2 \\ 0 & 0 & 1 \end{pmatrix} .$$

In the language of *"groups"* the preceding discussion and matrix representations reflect the following properties:

1. The set of *"homotheties + translations"* constitutes a group G.
2. The homotheties with a given fixed center O constitute a subgroup G_O of G.
3. The set of translations constitute also a subgroup T of G.

6 Similarities, general definitions

The first encounter with the notion of similarity is perhaps that of *"similar triangles"*, i.e. *"triangles having equal respective angles"* ⇔ *"triangles having proportional respective sides"*. The material discussed here handles this notion and its generalizations for more general shapes, like polygons, from the viewpoint of transformations of the plane.

"Similarity" is called a transformation f of the plane, which multiplies the distances of points with a constant $\kappa > 0$, which is called *"ratio"* or *"scale"* of the similarity. By definition then, for every pair of points $\{X, Y\}$ a similarity corresponds points

$$X' = f(X) , \ Y' = f(Y) , \quad \text{which satisfy} \quad |X'Y'| = \kappa \cdot |XY| .$$

This general definition includes the *"isometries"* or *"congruences"*, for which $\kappa = 1$, and the *homotheties*. Simillarities not coincident with isometries, in other words, similarities for which $\kappa \neq 1$ are called *"proper"* similarities. As we will see further down (Theorem 15), proper similarities are divided into two categories: *"direct similarities"* or *"rotational similarities"* and *"antisimilarities"* or *"reflective similarities"* [3, *p.* 217].

A *direct similarity* or *rotational similarity* is defined as a composition $g \circ f$ of a rotation f and a homothety g, which shares the same center with f. The rotation angle of f is called *"angle of similarity"*. An *antisimilarity* is defined as a composition $g \circ f$ of a reflection f and a homothety g with center on the axis of the reflection f. The axis of f is called *"axis of antisimilarity"*.

Remark 3. In both categories therefore there exists a point, the center O of the homothety g which is fixed under the transformation. Obviously, proper similarities cannot have also a second fixed point T different from O. For if they had, then for the two points and their images $O' = f(O) = O$, $T' = f(T) = T$ would hold $|OT| = |O'T'|$, while a proper similarity requires $|O'T'| = \kappa|OT|$ with $\kappa \neq 1$. This unique fixed point is called *"center"* of the proper similarity.

The order of the transformations, which participate in the definition of a proper similarity, is irrelevant because of the following theorem.

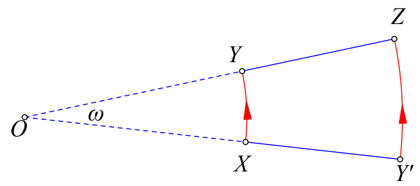

Figure 8: Commutativity of rotation and concentric homothety

Theorem 9. *The two transformations, which participate in the definition of a proper similarity, commute ($g \circ f = f \circ g$).*

Proof. Let us see the proof for the direct similarities, which are compositions $g \circ f$ of rotations f and homotheties g (See Figure 8). The proof for antisimilarities is similar. For the proof then, it suffices to observe the orbit of an arbitrary point X under the application of the two transformations. According to $g \circ f$, we first rotate X, about the center O of the rotation, to Y and next we take the homothetic Z of Y. It holds therefore $(XOY) = \omega$ and $\frac{OZ}{OY} = \kappa$, where ω is the angle of rotation of f and κ the homothety ratio of g. According to $f \circ g$, we first take the homothetic Y' of X and next we rotate Y' by ω. It is obvious that the two processes give the same final result, which is the point Z. □

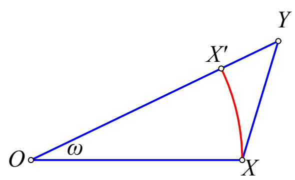

Figure 9: Triangles OXY for direct similarities

Theorem 10. *For every direct similarity f with center O and rotation angle, which is not a multiple of π, the triangles OXY with $Y = f(X)$, which result for the different positions of X on the plane, are similar.*

Proof. Direct consequence of the definition, according to which $\widehat{XOY}$ is the rotation angle ω and the ratio $\frac{|OY|}{|OX|}$ is the ratio κ of the similarity, □

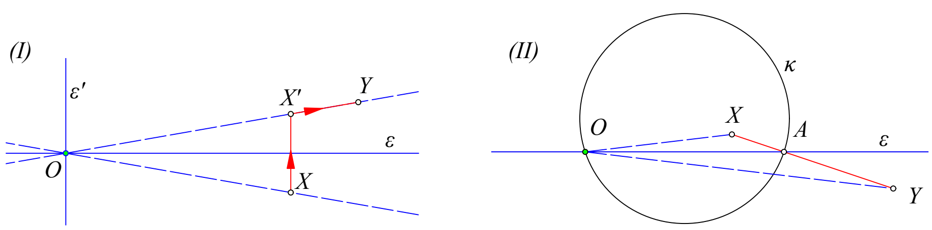

Figure 10: Antisimilarity ... and Apollonian circles

Theorem 11. *For every antisimilarity f with center O and axis ε and every point X of the plane, for which points O, X, $Y = f(X)$ are not collinear, the angles $\widehat{XOY}$ have the same bisectors, which coincide with ε and its orthogonal ε' at O. Points X of ε and ε' are the only points for which O, X, Y are collinear.*

Proof. Direct consequence of the definition, according to which the lines OX, OY are always symmetric relative to ε (See Figure 10-I). □

Exercise 9. *Show that, if f is an anitsimilarity with center at O and axis ε, then for every point X different from O and its image $Y = f(X)$, holds $|k| = \frac{|OY|}{|OX|} = \frac{|AY|}{|AX|}$, where k is the similarity ratio and A is the intersection point of line XY with line ε. Conclude that the Apollonian circle κ of the segment XY for the ratio $|k|$ (see file* **Apollonian circles***), passes through points $\{O, A\}$ and points $\{X, Y\}$ are inverse with respect to κ (See Figure 10-II).*

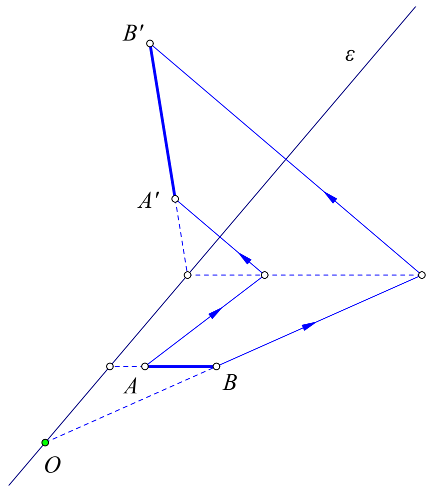

Figure 11: $\{AB, A'B'\}$ equally inclined to the axis ε

Exercise 10. *Show that, if f is an anitsimilarity with center at O and axis ε, then for any two points $\{A, B\}$ and their images $\{A', B'\}$ the segments $\{AB, A'B'\}$ are equally inclined to the axis, i.e. a bisector of their angle is parallel to ε (see figure 11).*

Exercise 11. *Show that for every triple of non collinear points X, Y, Z and their images X', Y', Z' through a similarity, triangles XYZ and $X'Y'Z'$ are similar.*

Exercise 12. *Show that a direct similarity maps a triangle ABC to a similar triangle $A'B'C'$, which is also similarly oriented to ABC. An antisimilarity reverses the orientation of the triangles.*

Exercise 13. *Show that a similarity maps a line ε to a line ε' and a circle κ to a circle κ'.*

Exercise 14. *Show that two similarities f, g, which are coincident at two different points A and B, they are coincident at every point of the line AB. Conclude then, that the composition of the transformations $g^{-1} \circ f$ is either the identity transformation or a reflection.*

Exercise 15. *Show that two similarities f, g, which are coincident at three non collinear points, they are coincident at every point of the plane.*

7 Similarities defined by two segments

A basic property of similarities is expressed with the following theorem:

Theorem 12. *For two line segments AB and $A'B'$ of the plane, of different length, there exists a unique direct similarity which maps A to A' and B to B', consequently mapping AB to $A'B'$.*

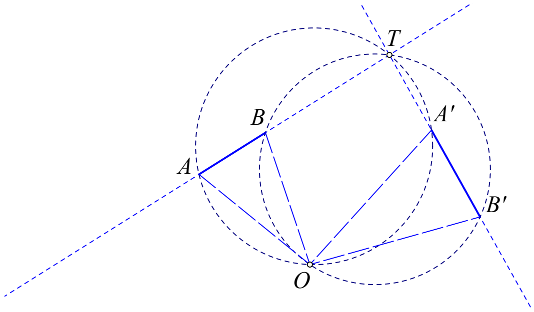

Figure 12: Similarity from two line segments

Proof. Leaving the special cases for the end, let us assume that the two segments are in general position and the lines they define intersect at a point T. This defines two circles $(AA'T)$ and $(BB'T)$ which intersect not only at T but also at a second point O. The quadrilaterals $TBOB'$ and $TAOA'$ are inscriptible in circle, therefore their angles at O are equal as supplementary to the angle at T. This shows that $(AOA') = (BOB')$ and defines the rotation f of the similarity. From this property follows that the angles of triangles $A'B'O$ and ABO at O are equal as are their angles at A' and A (as internal and opposite external in quadrilateral $AOA'T$). It follows that the two triangles are similar and the similarity ratio is $\kappa = \frac{|A'B'|}{|AB|}$. The wanted similarity then is the composition of the rotation f and the similarity with ratio κ and center O.

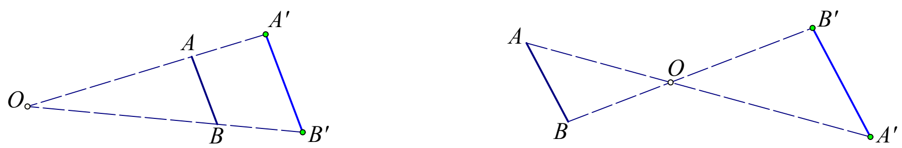

Figure 13: Similarity from two parallel line segments

In the special case where T does not exist, that is when AB and $A'B'$ are parallel and not collinear, then O is the intersection point of AA' and BB'. If AB and $A'B'$ are equally oriented, then the wanted similarity is the homothety with center O and ratio $\kappa = \frac{|A'B'|}{|AB|}$. If AB and $A'B'$ are inversely oriented, then the wanted similarity is the composition of the rotation f by π about O (which coincides with point symmetry relative to O) and the homothety with ratio $\kappa = \frac{|A'B'|}{|AB|}$ relative to O. The reasoning for collinear AB and $A'B'$ is similar, but I leave this case as an exercise.

The uniqueness of this similarity follows from the fact that the arguments can be reversed. If O is the center of a similarity, which maps AB to $A'B'$, then for the angles, $(AOA') = (BOB')$ and further the triangles AOB and $A'OB'$ will be similar. This however means that the quadrilaterals $AOA'T$ and $BOB'T$ are inscriptible in circles and O is the intersection point of the circles (ATA') and (BTB'), as in the previous case. □

Theorem 13. *For two line segments AB and $A'B'$ of the plane, of different length, there exists a unique antisimilarity, which maps A to A' and B to B', consequently mapping AB to $A'B'$.*

Proof. We want an antisimilarity whose ratio $\kappa = \frac{|A'B'|}{|AB|}$ we know. Therefore it suffices to find its center O'. This point will be on the bisectors of the angles $\widehat{AO'A'}$ and $\widehat{BOB'}$ (theorem 11). These bisectors will intersect the respective sides AA' and BB' of the triangles

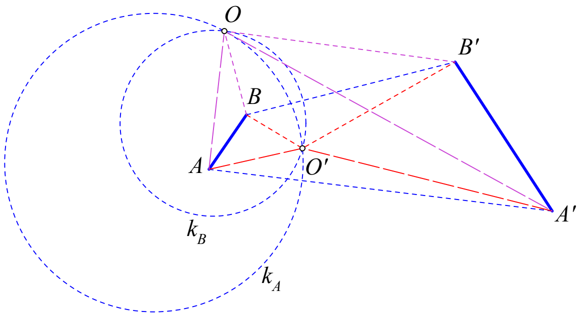

Figure 14: Antisimilarity from two line segments

AOA' and BOB' at points which divide them in ratio κ. Therefore O' will be contained in the two Apollonian circles k_A and k_B, which are respectively the loci of the points which divide segments AA' and BB' in ratio κ. Consequently it will coincide with an intersection point of these circles. A similar property will be valid also for the center O of the direct similarity, which is guaranteed by the previous theorem. Therefore this, too will be contained in the intersection of k_A and k_B. Consequently the two circles will intersect. From the equality of ratios

$$\frac{|OA|}{|OA'|} = \frac{|OB|}{|OB'|} = \frac{|O'A|}{|O'A'|} = \frac{|O'B|}{|O'B'|} = \frac{|AB|}{|A'B'|},$$

it follows that triangles $O'AB$ and $O'A'B'$ are similar and OAB, $OA'B'$ are also equal.

In the case where the two circles intersect at exactly two points (See Figure 14), it is impossible for both pairs of similar triangles to consist of similarly oriented triangles. This, because otherwise we would have two direct similarities with centers at O and O', something which is excluded by the previous theorem. Therefore one of the two pairs will consist of reversely oriented triangles and consequently one of the two will be an antisimilarity and the other a direct similarity (see remark 4).

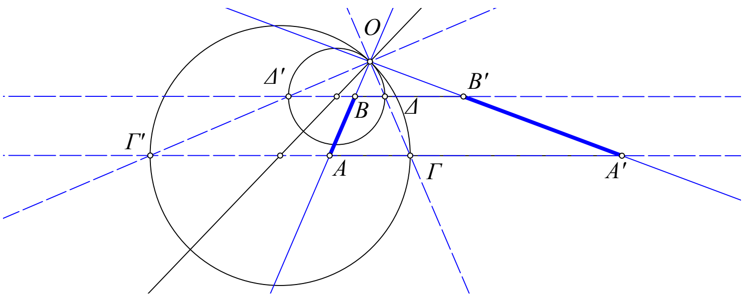

Figure 15: Coincidence of centers of similarity and antisimilarity

If the two points O and O' coincide then AA' and BB' must be parallel (See Figure 15). Indeed, then, the bisectors of the angles AOA' and BOB' will coincide and the two circles k_A and k_B will be tangent at O. However the lines AA' and BB' contain the diametrically opposite pairs of points C, C' and D, D' respectively, which are defined by the mutually orthogonal bisectors which pass through O. Because of the circle tangency at O, the diameters CC' and DD', which are excised by the two orthogonal lines on the circles are parallel, something which proves the claim. In this case the direct similarity has rotation angle $\widehat{AOA'}$ and the antisimilarity has axis line CD.

In the special case, in which the lines AB and $A'B'$ are parallel, the two Apollonian circles pass through the intersection point O of AA' and BB', which is a homothety center

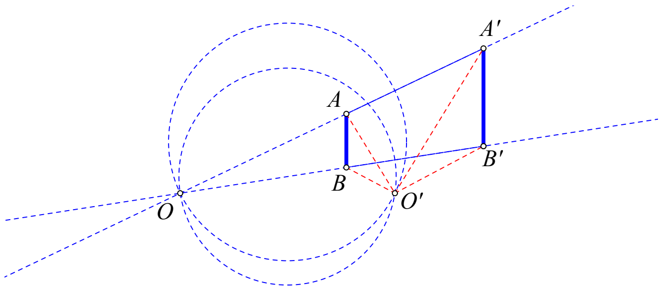

Figure 16: Centers of similarity and antisimilarity when $AB||A'B'$

and, consequently, the center of a direct similarity between AA' and BB'. The antisimilarity center coincides in this case also with the other intersection point O' of the two circles (See Figure 16). □

Exercise 16. *Complete the proof of the last two theorems, by examining the case where AB and A'B' are on the same line.*

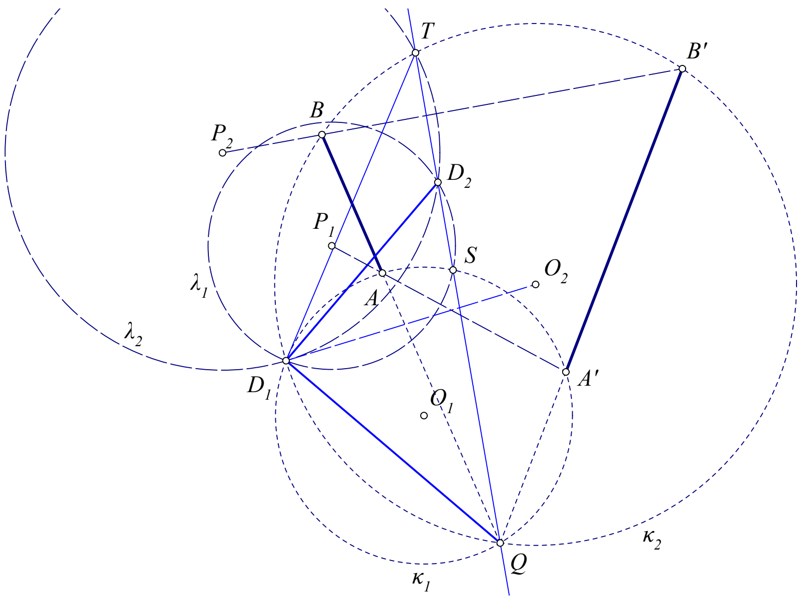

Figure 17: The line QT and the right angle $Q\widehat{D_1}D_2$

Theorem 14. *Let $\{D_1, D_2\}$ be the similarity centers of the direct and indirect(antisimilarity) mapping the segment AB onto $A'B'$ (see figure 17). Let also Q denote the intersection $AB \cap A'B'$ and consider the circles $\{\kappa_1 = (BQB'), \kappa_2 = (AQQ')\}$, whose second intersection defines D_1. The second intersection point $D_2 \neq D_1$ of the Apollonian circles $\{\lambda_1, \lambda_2\}$ of the segments $\{AA', BB'\}$ w.r.t. the ratio $r = AB/A'B'$ defines the center of the antisimilarity mapping AB onto $A'B'$. Let also $\{S = \kappa_1 \cap \lambda_1, T = \kappa_2 \cap \lambda_2\}$. The following are valid properties.*

1. *Triangles $\{ASA', BTB'\}$ are similar.*
2. *Points $\{T, S, Q\}$ are collinear.*
3. *D_2 is collinear with $\{T, S, Q\}$.*
4. *The angle $D_2\widehat{D_1}Q$ is right.*
5. *The lines $\{QD_1, QD_2\}$ are harmonic conjugate w.r.t. $\{AB, A'B'\}$.*

Proof. *Nr-1* is valid because the ratios $SA/SA' = TB/TB' = r$ and the angles are equal: $\hat{S} = \widehat{T} = \pi - \widehat{Q}$.

Nr-2 is valid because $\widehat{AQS} = \widehat{AA'S} = \widehat{BB'T} = \widehat{BQT}$.

Nr-3 is valid because the line TS is characterized by the ratio of distances of its points from the segments $d(X, AB)/d(X, A'B') = r$, satisfied by $\{S, T\}$. But D_2 satisfies also this condition, hence belongs to that line.

Nr-4 follows by an angle chasing argument. $D_2\widehat{D_1}O_2 = D_1\widehat{T}D_2$ because D_1O_2 is tangent to λ_2 at D_1. This follows from the fact that κ_2 passing through $\{B, B'\}$, which are inverse relative to λ_2 is orthogonal to λ_2. Also $O_2\widehat{D_1}Q = \frac{1}{2}(\pi - D_1\widehat{O_2}Q) = \pi/2 - D_1\widehat{T}Q$.

Nr-5 is a consequence of the characterization of their points to have ratio of distances from $\{AB, A'B'\}$: $d(X, AB)/d(X, A'B') = r$. □

Remark 4. The preceding theorem makes more precise the distinction between the centers $\{D_1, D_2\}$ respectively of the direct similarity and the antisimilarity mapping the segment AB onto $A'B'$: In the right angled triangle QD_1D_2 the right angle is at D_1.

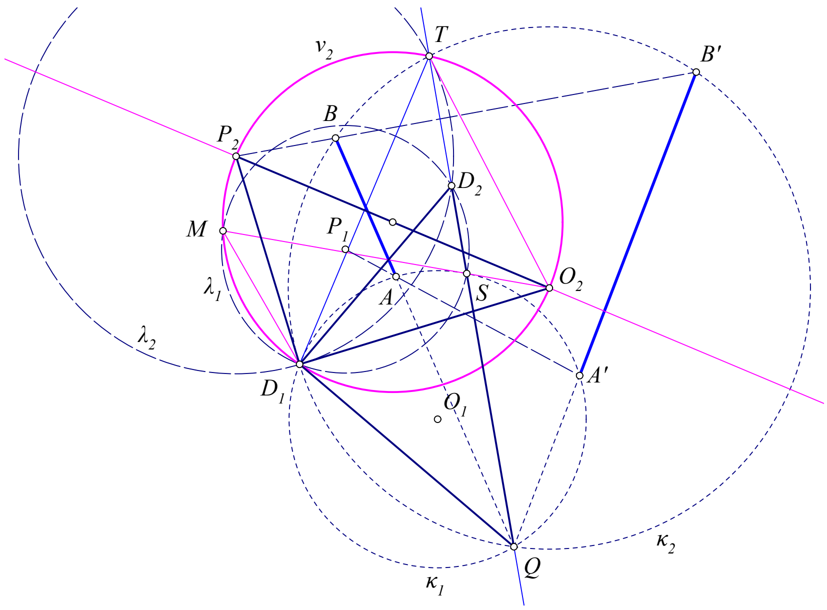

Figure 18: Collinear points $\{O_2, S, M\}$

Exercise 17. *With the notation and conventions of the preceding theorem show that:*

1. *The triangle $D_1O_2P_2$ is similar to $\triangle D_2D_1Q$ (see figure 18).*
2. *The line O_2S passes through the second intersection point M of the two circles λ_1 and $P_2D_1O_2$.*

Hint: For *nr-2* consider M as second intersection of the circle $\nu_2 = (P_2D_1O_2)$ with line O_2S and show that $M \in \lambda_2$. For this it suffices to show that $D_1\widehat{M}S = D_1\widehat{D_2}S$.

8 Similarities and orientation

Theorem 15. *Every proper similarity is a direct similarity, if it preserves the orientation of triangles and an antisimilarity if it reverses the orientation of triangles.*

Proof. Indeed, let X, Y be two different points and $X' = f(X)$, $Y' = f(Y)$ their images by the similarity. Assume also that f preserves the orientation of triangles and g is the direct similarity, which maps X to X' and Y to Y' (Theorem 12). Then the two similarities f and g coincide on the entire line XY (Exercise 12). Let Z be a point not on the line XY. The triangle XYZ maps by f to the similar and similarly oriented (to XYZ) triangle $X'Y'Z'$. The same happens with g. It also maps XYZ to a similar and similarly oriented triangle $X'Y'Z''$. Triangles XYZ, $X'Y'Z'$, $X'Y'Z''$ are similar and similarly oriented, and the last two have $X'Y'$ in common. Therefore they either coincide or one is the mirror image of the other. The latter however cannot happen, because then the two triangles would have reverse orientation. Therefore the triangles coincide and consequently $Z' = Z''$, in other words f and g are coincident on three non collinear points, therefore they are coincident everywhere and holds $f = g$.

The case where the transformation f reverses the orientation of the triangles is proved similarly. □

Corollary 2. *Every proper similarity has exactly one fixed point.*

Exercise 18. *Determine the fixed point of a given proper similarity f.*

Hint: Use Theorem 12 for direct similarities and Theorem 13 for antisimilarities ([4, *p.* 74]).

9 Similarities and triangles

Exercise 19. *Show that, for two similar but not congruent triangles ABC and $A'B'C'$, there exists a unique proper similarity which maps ABC to $A'B'C'$.*

Hint: Use the similarity (direct or antisimilarity) which maps AB to $A'B'$.

The next two exercises show, that in the definition of the proper similarity it is not necessary to restrict ourselves to homotheties and rotations (resp. reflections) with coincident centers (resp. with homothety center on the the axis of the reflection). Even if the centers are different (resp. the center is not on the axis of reflection), the composition of a homothety and a rotation (resp. reflection) is a proper similarity.

Exercise 20. *Show that the composition $g \circ f$ of a homothety f with center O and a rotation g with center $P \neq O$ is a direct similarity with rotation angle that of g, ratio that of f and center which is determined by f and g. Show that the same happens also for the composition $f \circ g$.*

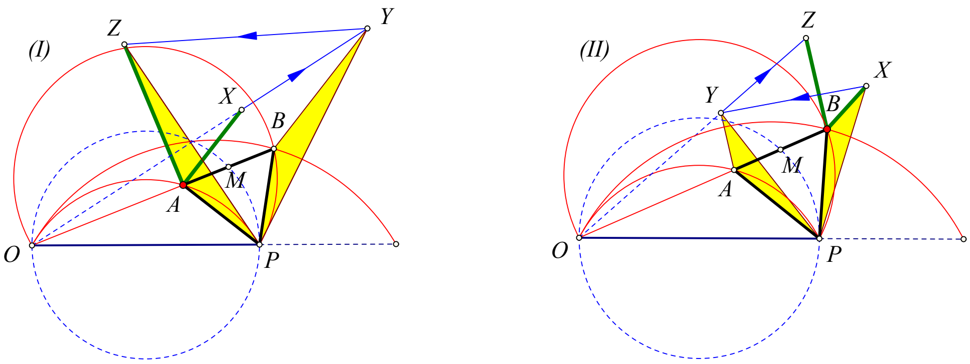

Figure 19: Composition of homothety and rotation

Hint: Let κ be the ratio of the homothety f and ω be the angle of the rotation g. There exists an isosceles triangle PAB, with vertex at the center P of the rotation, whose two other vertices A, B are centers of the similarities $g \circ f$ and $f \circ g$ ((I) and (II) in figure 19, respectively). This triangle can be constructed using two characteristic properties it has: a) an apical angle equal to ω and b) $B = f(A)$.

Indeed, if such a triangle exists, then B will see the line segment OP under the angle $\frac{\pi-\omega}{2}$ and A will see OP under the angle $\frac{\pi+\omega}{2}$. Both of the latter if $\kappa > 1$. If $\kappa < 1$ the roles of A and B must be reversed. Let us then assume that $\kappa > 1$ and that $f(A) = B$. Point B is on the intersection of the arc of the points which see OP under angle $\frac{\pi-\omega}{2}$ and of the arc which results through the homothety f from the arc of points which see OP under angle $\frac{\pi+\omega}{2}$. Consequently point B is constructible and from it the isosceles PAB with angle ω at P is also constructible. Then $g(f(A)) = g(B) = A$, therefore point A is a fixed point of $h = g \circ f$.

Let X an arbitrary point, $Y = f(X)$ and $Z = g(Y)$. The angle $(XAZ) = \omega$. Indeed, the triangles PAZ and PBY are congruent, because they have $|PA| = |PB|$ by hypothesis, $|PY| = |PZ|$, since point Z results from Y through a rotation about P and the angles $\widehat{APZ}$, $\widehat{BPY}$ are equal since both added to $\widehat{ZPB}$ give ω. Also, because of the similarity, $\kappa = \frac{|OB|}{|OA|} = \frac{|OY|}{|OX|}$, therefore AX and BY are parallel and $|BY| = |AZ|$. Therefore $\frac{|AZ|}{|AX|} = \kappa$ and the angle between the lines AZ and AX is equal to the angle between AZ and BY, which is ω. Consequently, the correspondence $Z = g(f(X))$ coincides with the composition $g' \circ f'$, where f' is the rotation about A by ω and g' is the homothety relative to A with ratio κ. We have then $g \circ f = g' \circ f'$ and the second composition satisfies the definition of the direct rotation.

The proof of the claim for the other ordering of the composition, that is $f \circ g$ (corresponding to case (II) in figure 19) is similar.

Exercise 21. *Show that the composition $g \circ f$ of a homothety f with center O and a reflection g with axis ε, which does not contain O, is an antisimilarity with axis a line ε', parallel to ε and center the projection P of O on ε'. Show that the same happens also for the composition $f \circ g$.*

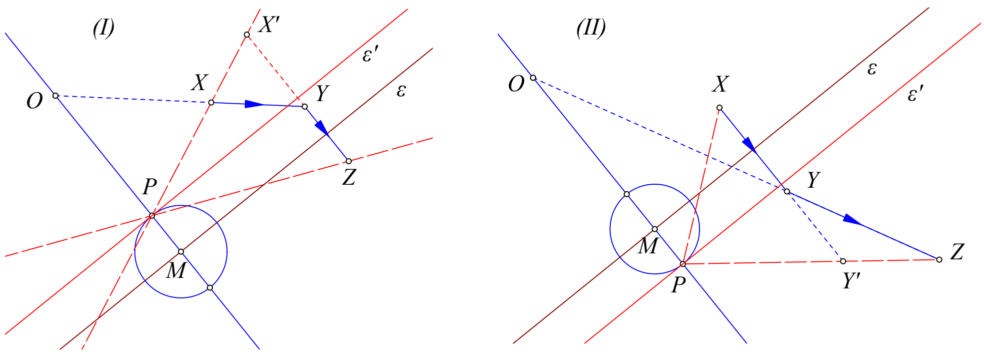

Figure 20: Composition of homothety and reflection

Hint: The key role here is played by the circle with center the projection M of O on the axis of g and radius $r = \frac{\kappa-1}{\kappa+1}|OM|$. The intersection point P of this circle with OM, which is contained between points O and M is proven to be a fixed point of $g \circ f$. Its diametrically opposite is proven to be a fixed point of $f \circ g$ (cases (I) and (II) respectively in figure 20). The rest follows easily from the figures, in which X is an arbitrary point of the plane, $Y = f(X)$ (resp. $Y = g(X)$) and $Z = g(Y)$ (resp. $Z = f(Y)$). In the first case $g \circ f = g' \circ f'$, where f' is the homothety with center P and ratio equal to the ratio κ of f and g' is the reflection relative to the line ε', which is parallel to ε and passes through point P. In the second case $f \circ g = f'' \circ g''$, where g'' is the reflection relative to ε', which is parallel to ε and passes through point P and f'' is the homothety with center P and ratio equal to κ. The figures show the trajectory of the arbitrary point X under the application of these new transformations. In the first case point X maps by f' to X', which next, by g' maps to Z. In the second case point X maps by g'' to Y', which, by f'' maps to Z'.

Exercise 22. *Show that the composition of two direct similarities is a direct similarity with rotation the sum of the rotations and ratio the product of the ratios if their angles sum up to $\omega + \omega' \neq 2k\pi$ and the ratios κ and κ' satisfy $\kappa \cdot \kappa' \neq 1$, otherwise it is a translation.*

Hint: Combination of the two previous exercises and of the fact, that the product of rotations is a rotation (see file **Isometries**). Write the two similarities in their *"normal"* form, as compositions of homotheties and rotations with the same center: $g \circ f$ and $g' \circ f'$. Then their composition would be $(g' \circ f') \circ (g \circ f) = (g' \circ f') \circ (f \circ g) = g' \circ (f' \circ f) \circ g$. Consider next the aforementioned fact about the composition of two rotations, which is also a rotation, applied to $f' \circ f$ and subsequently to the resulting rotation or translation h apply Exercise 20 or Theorem 8, etc ([9, *p.* 42, II]).

Exercise 23. *State and prove an exercise similar to the previous one for the composition of two antisimilarities and the composition of an antismilarity and a direct similarity.*

As it happens with isometries and congruence, so it happens also with similarity transformations, which are at the root of a general definition of the similarity for plane shapes: Two shapes S, S' of the plane are called *"similar"*, when there exists a similarity f which maps one to the other ($f(S) = S'$).

10 Triangles varying by similarity

Theorem 16. *The triangle ABC varies in such a way, that the measure of its angles remain fixed, its vertex at A also remains fixed and its vertex B moves along a fixed line ε. Then, its third vertex C moves along another fixed line ζ, which forms with ε an angle equal to $\widehat{BAC}$.*

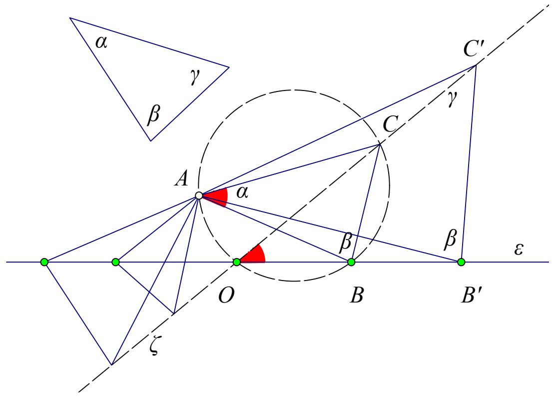

Figure 21: Variable triangle with fixed angles

Proof. Let ABC be one triangle with the aforementioned properties (See Figure 21). Consider the circumscribed circle of this triangle and the second intersection point O of this circle (different from B) with line ε. Because O sees BC under the same angle as A, the angle at O will be fixed and equal to the triangle angle α (the angles remain fixed, only the dimensions and the position of the triangle change). Because the quadrilateral $AOBC$ is inscribed in the circle, the angle $\widehat{AOB}$, which is opposite to γ, will be its supplementary, something which completely determines the position of point O. Consequently, C is contained in the line, which passes through point O and forms angle α with ε. □

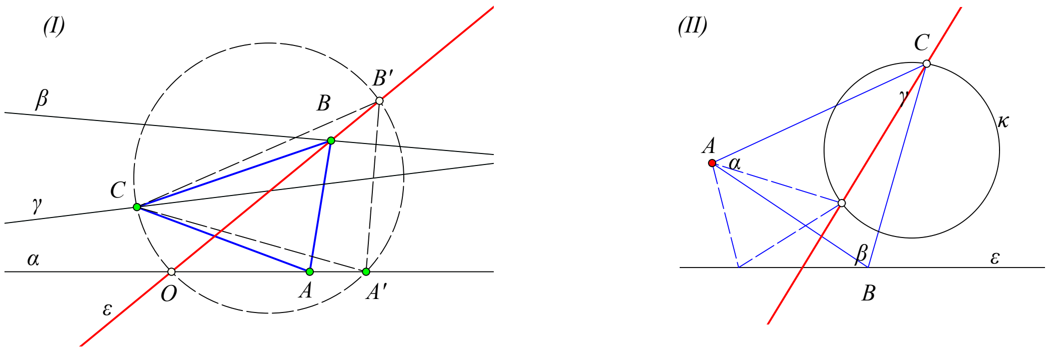

Figure 22: Vertices on lines Vertices on line and circle

Exercise 24. *Construct a triangle ABC with given angles, whose vertices $\{A, B, C\}$ are contained respectively in three lines $\{\alpha\ ,\ \beta\ ,\ \gamma\}$.*

Hint: Consider an arbitrary point C of γ and triangles $CA'B'$ with angles equal respectively to the given and the vertex A' on line α (See Figure 22-I). Then (Theorem 16) vertex B' is contained always in a fixed line ε. Consider the intersection point B of ε and of β and define the triangle ABC.

Exercise 25. *Construct a triangle ABC with given angles, whose vertex at A is a fixed point, the vertex at B is contained in a line ε and the vertex C is contained in a circle κ (See Figure 22-II).*

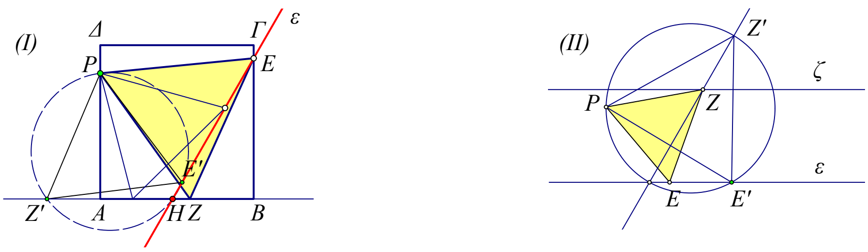

Figure 23: Equilateral inscribed in square Equilateral between parallels

Hint: Consider all equilateral triangles $PZ'E'$, with Z' on line AB. According to Theorem 16, the other vertex E' of all these triangles varies on a definite line ε (See Figure 23-I). One intersection point E of this line and the square, different from the intersection point H of ε with AB, defines the base of the wanted triangle PZE.

Exercise 26. *Construct an equilateral triangle, which has one vertex at a given point and the other two vertices on given parallel lines (See Figure 23-II). Also construct an equilateral triangle which has its three vertices on three given parallel lines.*

Figure 24: M and M' have the same relative position with respect to the polygon

11 Relative position in similar figures

Given two similar polygons $p = ABCD...$ and $p' = A'B'C'D'...$, we say that the points M and M' have respectively the same *"relative position"* with respect to the polygons, when all the triangles which are formed by connecting M and M' with corresponding vertices of the polygons are respectively similar. In figure 24 points M and M' have the same relative position with respect to the two similar polygons. The ratios $\frac{|MA|}{|M'A'|} = \frac{|MB|}{|M'B'|} = ...$ are all equal to the similarity ratio of the two polygons. It follows directly from the definition, that the distances between two points with the same relative positions have themselves ratio $\frac{|MN|}{|M'N'|}$ equal to the ratio of similarity of the two polygons. More generally, it can be proved easily, that if the points X, Y, Z, ... and X', Y', Z', ... have respectively the same relative position with respect to the similar polygons p and p', then the polygons $XYZ...$ and $X'Y'Z'...$ are similar and their similarity ratio is equal to the similarity ratio of p, p'. Some special points in p, p' which have the same relative position, are corresponding vertices, the middles of corresponding sides, etc.

Remark 5. Two similar polygons $\{p, p'\}$ define a similarity f which maps one to the other: $f(p) = p'$. Thus, to say that $\{X, X'\}$ have the same relative position with respect to $\{p, p'\}$ is the same with saying that they correspond under f : $f(X) = X'$.

Often in applications we consider polygons $p = ABCD...$ which vary, remain however similar to a fixed polygon. We say then that the polygon varies *"by similarity"*. Points which are co-varied with such a polygon, retaining however their relative position with respect to the similar polygons, we say that they remain *"similarly invariant"*. In figure 24 polygon $p' = A'B'C'...$ results by similarity from $p = ABC...$ and, following this change, points M, N remain similarly invariant relative to the changing polygon, taking the same relative positions M', N' with respect to the similar polygon p'. Such a variation by similarity we met already in Theorem 16, which can be re-expressed as follows:

If the triangle ABC varies by similarity, so that one of its vertices remains fixed, while another moves on a line, then the third vertex as well moves on a line.

The next two theorems generalize this property.

Theorem 17. *Suppose that the polygon $p = ABCD...$ varies by similarity and in such a way, that the point M of its plane retains its position fixed, not only its relative position with respect to p, but also its absolute position on the plane. Suppose further that, under this variation, the similarly invariant point X of the polygon moves on a line ε_X, then every other similarly invariant point Y of p will move on a line ε_Y.*

Proof. This follows directly by applying Theorem 16 to triangle MXY, which, according to the previous comments, varies but remains similar to itself. Let us note that the polygon's vertices are special points which remain similarly invariant, therefore they too will draw

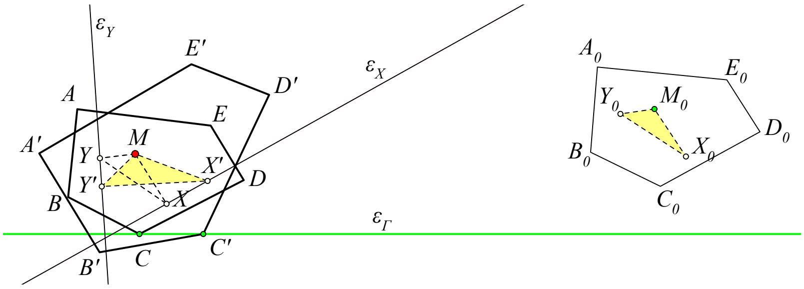

Figure 25: Variation by similarity with M's position relatively and absolutely fixed

lines. In figure 25 the line ε_C is shown, which the vertex C of the polygon slides on. In the same figure can also be seen a similar polygon $p_0 = A_0B_0C_0...$ to p, on which the corresponding points M_0, X_0, Y_0 can be distinguished. These points have in p_0 the same relative position with that of M, X, Y in p. □

Theorem 18. *Suppose that the polygon $p = ABCD...$ varies by similarity and in such a way, that the point of its plane M retains its position fixed, not only its relative position with respect to p, but also its absolute position on the plane. Suppose also that, during this variation, the similarly invariant point X of the polygon moves on a circle κ_X, then every other similarly invariant point Y of p will be moving on a circle κ_Y.*

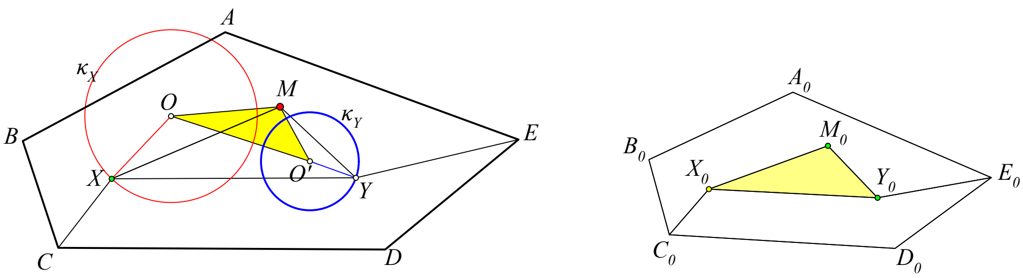

Figure 26: Variation by similarity with the position of M relatively and absolutely fixed II

Proof. During the variation of the polygon p, the triangle MXY, whose vertices remain similarly invariant, will remain similar to itself. The angles of this triangle will remain fixed, point M will remain fixed and X will move on a fixed circle $\kappa_X(O,\rho)$ (See Figure 26). We consider the triangle MOX and we construct its similar $MO'Y$, such that the angle $\widehat{YMO'}$ is equal to $\widehat{XMO}$ and $\widehat{O'YM}$ is equal to $\widehat{OXM}$. Triangles MOO' and MXY have then their sides at M proportional and their angles at M equal. Consequently the triangles are similar. Because OM is fixed, it follows that MO' is also fixed, consequently point O' will be fixed. From the similarity of triangles OXM, $O'YM$ it follows that $\rho' = |O'Y|$ will also be fixed, therefore Y will be moving on the circle $\kappa_Y(O',\rho')$. Figure 26 also shows a polygon $p_0 = A_0B_0C_0...$, similar to p, on which the corresponding points M_0, X_0, Y_0 are shown. These points have on p_0 the same relative position with that of M, X, Y on p. □

12 Polygons on the sides of a triangle

Theorem 19. *The ratio of areas of two similar polygons Π and Π' is*

$$\frac{\epsilon(\Pi)}{\epsilon(\Pi')} = \kappa^2,$$

where κ is the similarity ratio of the two polygons.

Proof. From one vertex in Π and its corresponding in Π' draw the diagonals and divide Π and respectively Π' into triangles respectively similar to the previous. The areas of the polygons are written as a sum of the areas of these triangles $\epsilon(\Pi) = \epsilon(t_1) + \epsilon(t_2) + ...$ and respectively $\epsilon(\Pi') = \epsilon(t'_1) + \epsilon(t'_2) + ...$ The corresponding triangles are similar therefore$\epsilon(t'_1) = \kappa^2\epsilon(t_1)$, $\epsilon(t'_2) = \kappa^2\epsilon(t_2), ...$ and the assertion follows using these relations in the previous equations for areas:

$$\begin{aligned}\epsilon(\Pi') &= \epsilon(t'_1) + \epsilon(t'_2) + ...\\ &= \kappa^2\epsilon(t_1) + \kappa^2\epsilon(t_2) + ...\\ &= \kappa^2 \cdot (\epsilon(t_1) + \epsilon(t_2) + ...)\\ &= \kappa^2\epsilon(\Pi),\end{aligned}$$

□

Remark 6. The preceding relation, between areas of similar polygons, leads to another form of the Pythagorean theorem in which, instead of squares on the sides of the right triangle, we construct on the sides polygons similar to a given polygon.

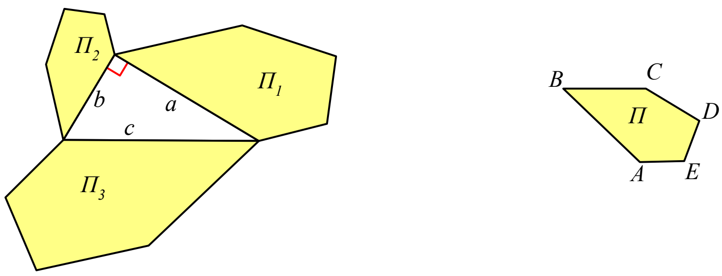

Figure 27: Generalized theorem of Pythagoras

Theorem 20. *Given is a polygon $\Pi = ABCD...$ and a right triangle. On the sides of the right triangle are constructed polygons Π_1, Π_2, Π_3 similar to Π, such that the sides of the right triangle a, b, c are homologous to the side AB of Π (See Figure 27). Then the sum of the areas of the polygons on the orthogonal sides is equal to the area of the polygon on the hypotenuse*

$$\epsilon(\Pi_1) + \epsilon(\Pi_2) = \epsilon(\Pi_3).$$

Proof. Let $|AB| = d$ be the length of the side AB of Π. According to Theorem 19, the ratios of areas will be

$$\frac{\epsilon(\Pi_1)}{\epsilon(\Pi)} = \frac{a^2}{d^2}, \quad \frac{\epsilon(\Pi_2)}{\epsilon(\Pi)} = \frac{b^2}{d^2}, \quad \frac{\epsilon(\Pi_3)}{\epsilon(\Pi)} = \frac{c^2}{d^2}.$$

The claim follows directly by solving for a^2, b^2, c^2 the previous relations and replacing in the theorem of Pythagoras: $a^2 + b^2 = c^2$. □

Exercise 27. *Construct a square DEZH, inscribed in the triangle ABC with its side DE on BC.*

Hint: Let us suppose that the square has been constructed (See Figure 28-I). We extend AD and AE until they meet the orthogonals of BC at B and C, respectively, at points K and I. (AHD, ABK) and (AZE, ACI) are pairs of similar triangles, consequently

$$\frac{|HD|}{|BK|} = \frac{|AH|}{|AB|}, \quad \frac{|ZE|}{|CI|} = \frac{|AZ|}{|AC|}.$$

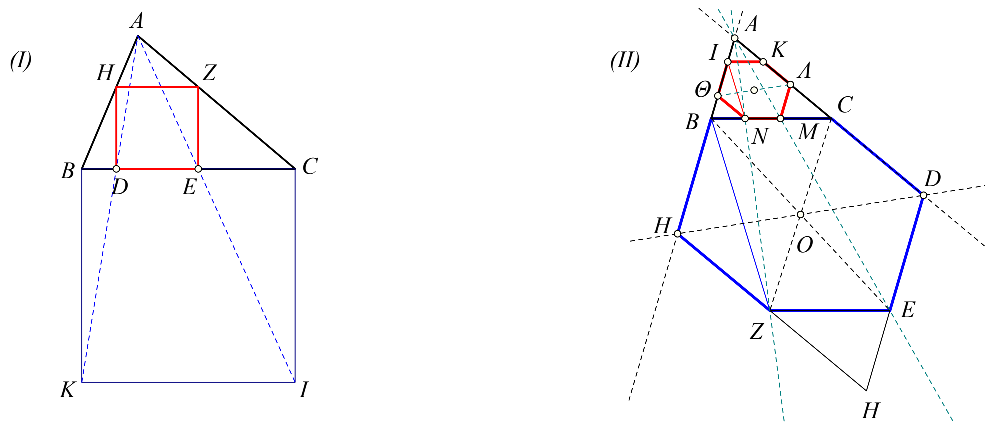

Figure 28: Square inscribed in triangle   Hexagon inscribed in triangle

Because HZ and BC are parallel, the ratios on the right sides of the equalities are equal, therefore the ratios on the left sides will be equal. This implies that $|BK| = |CI|$, therefore $BCIK$ is a rectangle and consequently the triangles ADE and AKI are similar. Because $\frac{|DE|}{|KI|} = \frac{|AD|}{|AK|} = \frac{|AH|}{|AB|}$, it follows that KI and BK are equal, therefore $BCIK$ is a square. This square can be constructed directly and by drawing AK and AI we define DE on BC and from this $DEZH$, which is proved being a square with similar reasoning.

Exercise 28. *Construct an equilateral hexagon with three sides on the sides of triangle ABC and three parallel to them.*

Hint: Let $\Theta IK\Lambda MN$ be the wanted hexagon (See Figure 28-II). Construct the similar to it on side BC. For this, define equal segments on the extensions of AB and AC respectively: $|BH| = |BC| = |CD|$ and consider the middle O of HD. Next consider the symmetric E, Z relative to O of B, C respectively. The first hexagon is equilateral by assumption and the second by construction. Also the two hexagons have corresponding sides parallel, hence they are also homothetic. The three points A, N and Z are collinear, as well as the three points A, M and E. Triangles $I\Theta N$ and BHZ are isosceli with equal apical angles Θ and H. Therefore IN and BZ are parallel and their ratio is

$$\frac{|IN|}{|BZ|} = \frac{|\Theta I|}{|HB|} = \frac{|IK|}{|BC|}.$$

However AIK and ABC are similar, consequently

$$\frac{|IK|}{|BC|} = \frac{|AI|}{|AB|} = \kappa.$$

The two hexagons, $\Theta IK\Lambda MN$ and $HBCDEZ$ are homothetic relative to A with homothetic ratio κ. The equilateral hexagon $HBCDEZ$ can be constructed directly and, by drawing AZ and AE, we define the points N and M on BC. From there on the construction of $\Theta IK\Lambda MN$ is easy, through parallels to the sides of the triangle ABC.

Exercise 29. *Show that the equilateral hexagon of the preceding exercise is unique, consequently the three hexagons which result from the previous construction, but starting from a different triangle side, coincide.*

Hint: From the previous exercise, it follows that for every hexagon like $\Theta IK\Lambda MN$ (See Figure 28-II), its vertex N is contained in a line AZ which is dependent only from the triangle ABC. This line intersects BC at exactly one point, N, and consequently there is only one triangle $I\Theta N$ defined and similar to BHZ with its vertex N on BC. From the uniqueness of N follows that of Θ, next of I etc. The second part of the exercise is a direct logical consequence of its first part.

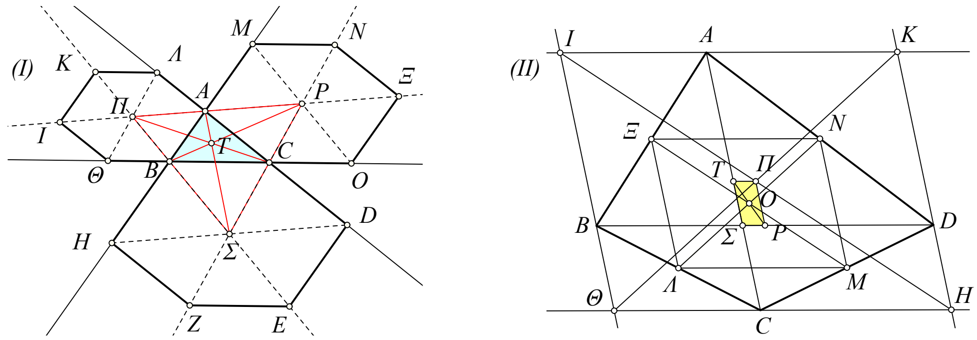

Figure 29: Escribed equilateral hexagons Circumscribed parallelogram

Exercise 30. *Construct the three escribed equilateral hexagons on the sides of triangle ABC (See Figure 29-I). Show that the lines which join the centers of symmetry Π, P and Σ of these hexagons pass through corresponding vertices of the triangle ABC.*

Hint: Show first that the halves of two such polygons, like for example the quadrilaterals *CDHB* and *CAMO* are similar.

Exercise 31. *From the vertices of the convex quadrilateral ABCD draw parallels to the diagonals, which do not contain them. This forms a parallelogram HΘIK (See Figure 29-II). Show that the point Π of the intersection of its diagonals, the intersection point Σ of the diagonals of ABCD as well as the middles of its diagonals, define the vertices of another parallelogram. Also show that the parallelogram ΛMNΞ of the middles of the sides of ABCD is homothetic to HΘIK and find the ratio and the center of the homothety.*

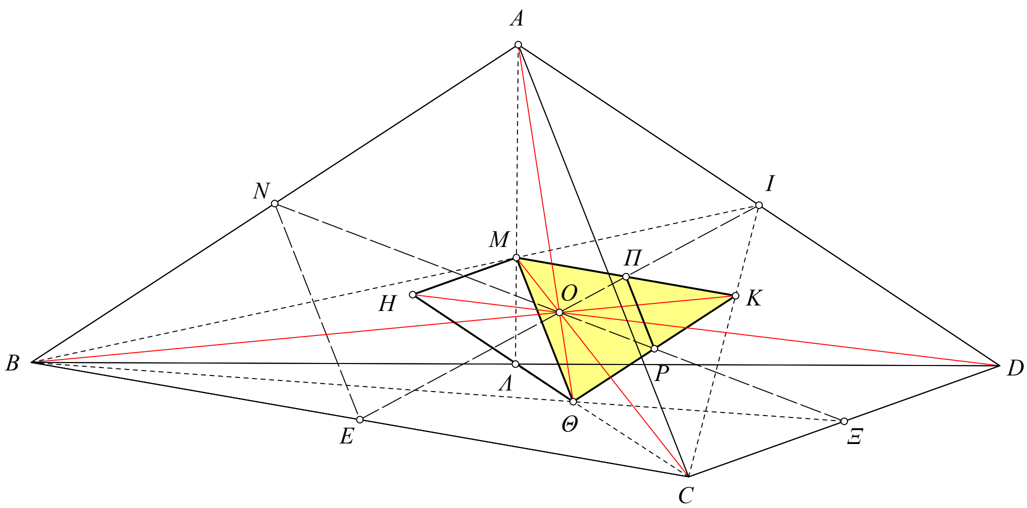

Figure 30: Homothetic to quadrilateral

Exercise 32. *In the convex quadrilateral ABCD we define the centroids H, Θ, K, M of the triangles ABC, BCD, CDA, DAB, respectively. Show that HΘKM is homothetic to ABCD with homothety* *ratio $\frac{1}{3}$ and homothety center the common middle O of the line segments which join the middles of its opposite sides (See Figure* 30*).*

Hint: First show that the triangles like BNE and $K P \Pi$ are homothetic relative to the homothety with center O and ratio 3. Points P, Π are respectively the middles of $K\Theta$ and KM.

Next exercise gives an application of similarity, which produces a relatively simple solution to a complex problem. The problem is related to the construction of triangles on the sides of a given triangle. Given a triangle ABC and three other triangles τ_1, τ_2, τ_3, we choose a side on each one of the three last triangles. Next we construct on the sides of triangle ABC externally lying triangles BDC, CEA, AZB respectively similar to τ_1, τ_2, τ_3, on ABC, so that the respectively similar to the selected sides of the three triangles coincide with the sides of the triangle ([5, p. 141]). I call such a construction briefly: a *welding by similarity* of τ_1, τ_2, τ_3 onto ABC. The resulting points $\{D, E, Z\}$ I call *vertices of the welding* (See Figure 31).

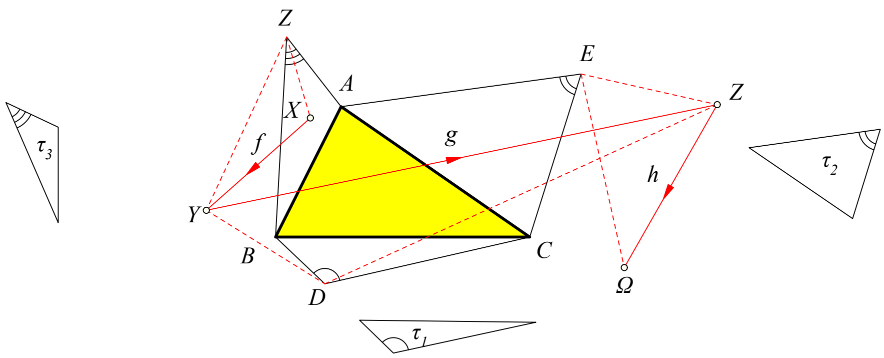

Figure 31: Triangles on the sides of ABC

Exercise 33. *Construct the triangle ABC from the triangles τ_1, τ_2, τ_3 and the respective vertices D, E, Z of a welding by similarity on ABC.*

Hint: From the given triangles and points D, E, Z there are defined respectively three similarities f, g, h. Similrarity f has center Z, angle $\omega_1 = (AZB)$ and ratio $\kappa_1 = \frac{|ZB|}{|ZA|}$. Similarity g has center D, angle $\omega_2 = (BDC)$ and ratio $\kappa_2 = \frac{|DC|}{|DB|}$. Finally, the similarity h has center E, angle $\omega_3 = (CEA)$ and ratio $\kappa_3 = \frac{|EA|}{|EC|}$. The angles and the ratios of the similarities are determined completely from the given triangles τ_1, τ_2, τ_3. We then observe that point A satisfies

$$f(A) = B, \;\; g(B) = C, \;\; h(C) = A \quad \Rightarrow \quad (h \circ g \circ f)(A) = A.$$

Point A therefore coincides with the unique fixed point of the composed similarity $h \circ g \circ f$. Consequently point A is determined from the givens (even if its actual construction is somewhat involved). As soon as A is determined, the rest of the vertices of the wanted triangle are constructed by applying the similarities: $B = f(A)$ and $C = g(B)$.

The preceding exercise includes many interesting special cases, which offer themselves for further study, for example, when the three triangles τ_1, τ_2, τ_3 coincide or have a more special form (isosceles, equilateral) or when for the corresponding ratios holds $k_1 k_2 k_3 = 1$.

13 Representation and group properties of similarities

Fixing a cartesian coordinate system, a rotation about the origin and a reflection on a line through the origin is described respectively by the matrices R_ϕ of the form:

$$\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ \sin(\phi) & -\cos(\phi) \end{pmatrix} .$$

The reflective behavior of the second matrix is seen by applying it to the vectors

$$a = (\cos(\phi/2), \sin(\phi/2))^t \quad \text{and} \quad b = (-\sin(\phi/2), \cos(\phi/2))^t ,$$

of which, the first maps to itself and the second, which is orthogonal to the first, maps to its negative. This identifies the line of the first vector $\{ta, t \in \mathbb{R}\}$ with the axis ε of the reflection. The general similarity, according to the definition, is described by the vector equation:

$$X' = O + R_\phi(X - O) \quad \text{multiplied by} \quad X'' = O + k(X' - O) .$$

This in matrix notation is expressed through the product of matrices

$$H \cdot R \quad \text{with} \quad H = \begin{pmatrix} k & 0 & (1-k)o_1 \\ 0 & k & (1-k)o_2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} u & v & o_1 - (uo_1 + vo_2) \\ w & z & o_2 - (wo_1 + zo_2) \\ 0 & 0 & 1 \end{pmatrix} ,$$

where the constants represent the matrix:

$$R_\phi = \begin{pmatrix} u & v \\ w & z \end{pmatrix} \quad \Rightarrow \quad H \cdot R = \begin{pmatrix} ku & kv & (1-ku)o_1 - kvo_2 \\ kw & kz & -kwo_1 + (1-kz)o_2 \\ 0 & 0 & 1 \end{pmatrix} .$$

In the language of *"groups"* the preceding discussion and matrix representations reflect the following properties:

1. The set of *"similarities"* constitutes a group G.
2. The *"direct"* similarities constitute a subgroup G^+ of G. They are characterized by the sign of the determinant of the matrix representing the similarity, which is *positive*.
3. The *"antisimilarities"* constitute a *"coset"* G^- in G, so that $G = G^+ \cup G^-$. The antisimilarities are characterized by the sign of the determinant of their matrix representation, which is negative.
4. The homotheties constitute a subgroup H of G^+.

14 Logarithmic spiral and pursuit curves

Exercise 34. *Starting from a "golden" rectangle $KP\Theta I$ and successively subtracting the squares of its small sides, we construct a sequence of other, pairwise similar rectangles (See Figure 32). Show that each one of them, for example $M\Theta H\Lambda$, results from its previous $ZI\Theta H$ through a similarity f with center the intersection point O of $K\Theta$, HI, rotation angle $\frac{\pi}{2}$ and ratio $x = \frac{\sqrt{5}-1}{2}$.*

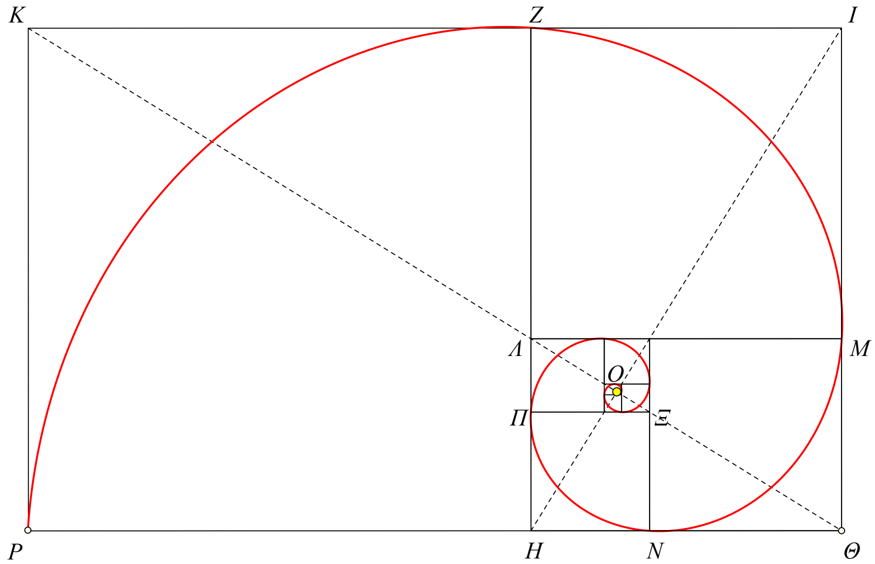

Figure 32: Golden section rectangles and logarithmic spiral

Hint: Simple use of the definitions and the properties of the golden section .

Figure 32, shows a curve, called a **logarithmic spiral**, which passes through a vertex of the initial rectangle P, as well as its successive positions $Z = f(P)$, $M = f(Z)$, $N = f(M)$, $= f(N)$, ..., which result by applying repeatedly the similarity f ([1, $p.$227]). Similar sequences of points and logarithmic spirals containing them result by starting from any point $P \neq O$ and taking the successive $f(P), f^2(P), f^3(P)$, ..., etc.

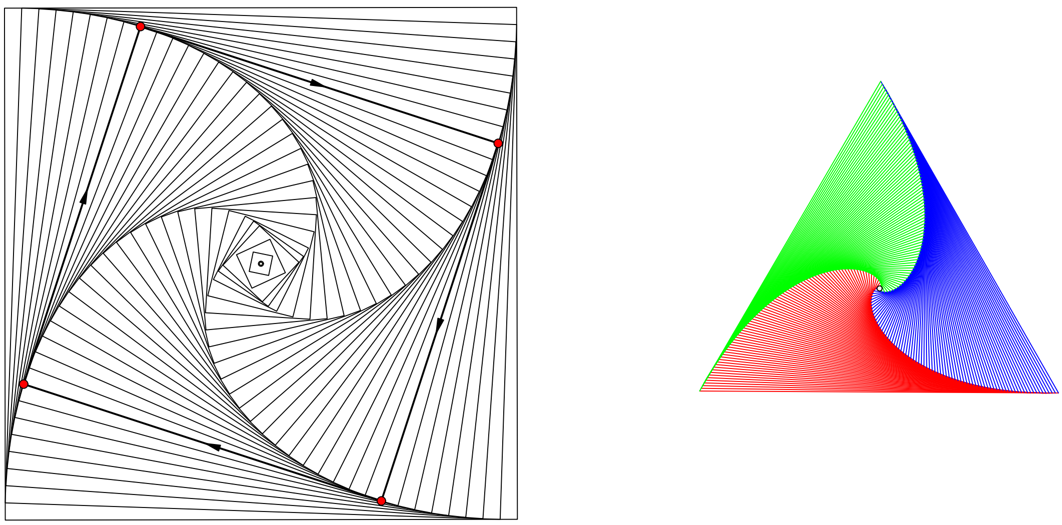

Figure 33: Logarithmic spirals as pursuit curves

This kind of curve shows up also as a *"pursuit trajectory"* in *"pursuit problems"* , like that with 4 bugs initially placed at the vertices of a square. The bugs start pursuing each other, moving at a constant speed. Each time their positions are at the vertices of a square, which gradually shrinks and simultaneously rotates, until they all meet at the center of the square. Figure 33, on the left, shows the positions of the bugs at different moments in time and the corresponding square defined by their positions. The same figure, on the right, shows the corresponding curves for three bugs, which start at the vertices of an equilateral triangle ([8, *p.* 136], [6, *p.* 203], [7, *p.* 109]).

Exercise 35. *On the sides* $\{AB, CA\}$ *of the triangle* ABC *we take respectively points* $\{E, D\}$ *such that* $\{AE = x \cdot AB, CD = x \cdot CA\}$*. Show that the segment* DE *becomes minimal, when it is orthogonal to the median* AM *of the triangle. Compute the minimal length of* DE *and the value of* x *for which this is obtained.*

Hint: Draw CD' parallel, equal and equal oriented to DE. Show that D' lies on the median AM of the triangle. The minimal DE is the altitude from C of ACA', where A' the symmetric of A relative to M.

Bibliography

[1] J. M. Aarts. *Plane and Solid Geometry*. Springer Verlag, Berlin, 2008.

[2] August Adler. *Theorie der Geometrischen Konstruktionen*. Goeschensche Verlagshandlung, Leipzig, 1906.

[3] William Barker and Roger Howe. *Continuous Symmetry, From Euclid to Klein*. American Mathematical Society, 2007.

[4] H Coxeter. *Introduction to Geometry*. John Wiley and Sons Inc., New York, 1961.

[5] Roland Deaux. *Introduction to the Geometry of Complex Numbers*. Dover, New York, 1956.

[6] L. Graham. *Ingenious Mathematical Problems and Methods*. Dover Publications, New York, 1959.

[7] Paul J. Nahin. *Chases and Escapes, The Mathematics of Pursuit and Evasion*. Princeton University Press, Princeton, 2007.

[8] Hugo Steinhaus. *Mathematical Snapshots*. Oxford University Press, 3rd Edition, Oxford, 1983.

[9] I. M. Yaglom. *Geometric Transformations I, II, III, IV*. The Mathematical Association of America, 1962.

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