The reason is not to glorify “bit chasing”; a more fundamental issue is at stake here: Numerical subroutines should deliver results that satisfy simple, useful mathematical laws whenever possible. ... Without any underlying symmetry properties, the job of proving interesting results becomes extremely unpleasant. The enjoyment of one’s tools is an essential ingredient of successful work.

*Donald Knuth, Seminumerical Algorithms, section 4.2.2*
1 Symmedians and symmedian point

Here we discuss two related concepts: the “symmedian line” and the “symmedian point” or “Lemoine point” of a triangle \( \tau = ABC \). The symmedian line of \( \tau \) from vertex \( A \) is the symmetric of its median from \( A \) w.r. to the bisector from \( A \). The symmedian point \( K \) is the intersection point of the symmedian lines from the three vertices (See Figure 1). Next theorem expresses a “characteristic property” of the points \( Y \) of a symmedian line.

**Theorem 1.** The distances of points \( Y \) of the symmedian line \( AA_s \) from \( A \), from the adjacent sides \( \{AB, AC\} \), are proportional to these sides:

\[
\frac{YY'}{YY''} = \frac{AB}{AC}.
\]

**Proof.** This follows from the symmetry w.r.t. the bisector \( AA'' \) and the corresponding property of the points \( X \) of the median \( AA' \), by which, due to the equality of the areas of triangles \( \{ABA', AA'C\} \), the analogous ratio is inversely proportional to the adjacent sides:

\[
\frac{XX'}{XX''} = \frac{AC}{AB}.
\]

Projecting the intersection point \( K \) of two of the symmedians on the sides of the triangles and applying the theorem, we see that the third symmedian passes also through that point, thus:

**Theorem 2.** The three symmedians of the triangle pass through a common point \( K \) called the symmedian point of the triangle.

2 Antiparallels

Closely related to the concept of “symmedian” is the concept of “antiparallels” to a side of the triangle. An antiparallel \( \varepsilon = B'C' \) to side \( BC \) of the triangle is the line defined by any circle \( \lambda \) passing through \( B, C \) and intersecting the sides \( \{AB, AC\} \) a second time at the points \( \{C', B'\} \) (See Figure 2). There are two prominent antiparallels to \( BC \):

1. The tangent \( \varepsilon_0 \) to the circumcircle at \( A \).
2. The side \( B''C'' \) of the “orthic triangle”, defined by the feet of the altitudes of the triangle.
The tangent $\epsilon_0$ is the limiting position for which the points $\{B', C'\}$ coincide with $A$ and the circle $\lambda$ coincides with the circumcircle $\kappa$ of the triangle. The line $B''C''$ is antiparallel because the quadrangle $B'C''$ is “cyclic”.

**Theorem 3.** All triangles $AB'C'$ are similar to $ABC$.

This results immediately from the corresponding angle equalities of the triangles:

$$\widehat{C'} = \widehat{C} \quad \text{and} \quad \widehat{B'} = \widehat{B}.$$  \hfill (3)

The symmetry on the bisector from $A$ maps triangle $AB'C'$ to a triangle $AB_1C_1$ with side $B_1C_1$ parallel to $BC$ hence the median line from $A$ of $AB'C'$ maps to the median line of $ABC$. This proves the following:

**Theorem 4.** The symmedian of $ABC$ coincides with the median line of the triangles $\{AB'C'\}$.

### 3 Harmonic quadrangle

**Theorem 5.** The tangents at $(B, C)$ to the circumcircle of triangle $ABC$ intersect at a point $D$ on the symmedian $AA_s$.

**Proof.** This follows from the characteristic property of the points of the symmedian line expressed by equation 1. Calculating the distances of $D$ from the sides and using the sines-rule we get (See Figure 3):

$$\frac{DD'}{DD''} = \frac{DB \sin(\widehat{C})}{DC \sin(\widehat{B})} = \frac{\sin(\widehat{C})}{\sin(\widehat{B})} = \frac{AB}{AC}.$$

\[\square\]

**Theorem 6.** The intersection $A_t$ of the tangent to the circumcircle at $A$ with $BC$ is the “harmonic conjugate” of $A_s$ w.r. to $(B, C)$ and the ratios $XB'/XC'$ defined by the sides $(AB, AC)$ and the symmedian line from $A$ on every parallel $B'XC'$ to the side $BC$ are equal to the ratio $\sin(\widehat{C})^2/\sin(\widehat{B})^2 = c^2/b^2$. 

![Figure 2: Antiparallels $\epsilon$ to the side $BC$](image-url)
Proof. This because the triangles \( \{AA_tB, CA_tA\} \) are similar, implying (See Figure 3):

\[
\frac{A_tC}{A_tB} = \frac{A_tC \cdot A_tB}{A_tB^2} = \frac{A_tA^2}{A_tB^2} = \frac{\sin(B)^2}{\sin(C)^2} = \frac{A_tB}{A_tA}.
\]

\[\square\]

It follows that the pencil of lines \( A(B, C, A_s, A_t) \) is harmonic, hence it defines on the circumcircle a “harmonic quadrangle”, i.e. a circumscribed quadrangle for which the “cross ratio” \( (BCA_tA_s) = (BCA_A) = -1 \). This kind of quadrangles is also characterized by the property of being cyclic and having equal products of opposite sides.

**Theorem 7.** The intersection \( E \) of the symmedian from \( A \) with the circumcircle of \( ABC \) defines a harmonic quadrangle \( ABEC \).

**Theorem 8.** The circle \( \lambda \) through \( \{B, C\} \) and the intersection \( D \) of the tangents at \( B, C \) to the circumcircle \( \kappa(O) \) of the triangle \( ABC \) passes through the center \( O \) and intersects the symmedian.
AD a second time at P, which defines triangles \( \triangle APC \), \( \triangle BPC \) simmilar to \( \triangle BEC \), so that the symmedian \( PE \) bisects the angle \( \angle BPC \). In addition holds the relation \( PA^2 = PB \cdot PC \).

Proof. The angles \( \angle OPD, \angle OCD \) are right showing that \( \lambda \) passes through \( O \) (see figure 4). A similar angle chasing shows that the angles of the triangles at \( P, E \) are equal to the supplement of \( A \). Note here that, since \( D \) is the middle of the arc \( BC \) on \( \lambda \) the symmedian \( AD \) bisects the angle \( \angle BPC \). Also \( ECB = \angle BAP \) completes the proof that \( \triangle ABP \sim \triangle CBE \).

Analogously \( \angle EBP = \angle PAF \) completes the proof that \( \triangle BEC \sim \triangle APC \) and shows that \( PE \) is a bisector of the angle \( \angle BPC \).

For the proof of the last claim see that the parallel \( CC' \) to the symmedian \( AD \) is parallel to the bisector of \( BPC \) hence \( PC = PC' \) and line \( PB \) passes through \( C' \), hence \( PB \cdot PC = PB \cdot PC' = PA^2 \).

Figure 4 shows also a so-called “Artzt parabola” of the triangle \( ABC \), which is the parabola passing through \( (B, C) \) and being there tangent to the sides \( (AB, AC) \). Its focus is point \( P \) (see section 14).

4 Second Brocard triangle

The “second Brocard triangle” of the triangle \( ABC \) is defined by its vertices which are the projections \( \{A_2, B_2, C_2\} \) of the circumcenter \( O \) of \( \triangle ABC \) on respective symmedians.

![The second Brocard triangle](image)

**Theorem 9.** The circumcircle \( \lambda \) of the second Brocard triangle has the symmedian point \( K \) and the circumcenter \( O \) of \( \triangle ABC \) as diametral points.

Proof. This follows directly from the definition of the vertices \( \{A_2, B_2, C_2\} \) as projections of \( O \) on respective symmedians, implying that each of them is viewing the segment \( OK \) under a right angle.

The circle with diameter \( OK \) is called “Brocard circle” of the triangle. We’ll see in theorem 25 that \( \{A_2, O\} \) are inverses w.r.t. the Apollonian circle \( \lambda_A \) through \( A \). This implies that the Brocard circle is orthogonal to \( \lambda_A \). Analogously it is seen that this circle is orthogonal to all three Apollonian circles \( \{\lambda_A, \lambda_B, \lambda_C\} \) of \( \triangle ABC \). This implies that the circle \( \lambda \), like the circumcircle \( \kappa \) belongs to the pencil of circles which are orthogonal to the Apollonian circles, called “Schoute pencil” of the triangle \( ABC \) (see file Pedal triangles).

**Theorem 10.** The “cyclocevian” triangle \( EFG \) of the vertices of the second Brocard triangle is congruent and inversely oriented to the triangle \( ABC \).
5 Gergonne point

Proof. “Cyclocevian” of a point $P$ w.r.t. to the triangle $ABC$ is called the triangle formed by the second intersections of the lines $(PA, PB, PC)$ and the circumcircle $\kappa$. We prove the theorem for $A_2$. In the course of proof of theorem 8 we have seen that drawing parallels to the symmedian $AK$ from $(B, C)$ we get respectively their second intersections $(G, F)$ with the circumcircle $\kappa$, which are respectively collinear with $(A_2C, A_2B)$ (see figure 6). This implies easily that the two triangles $(ABC, EGF)$ are each the reflection of the other relative to the line $OA_2$. 

\section{5 Gergonne point}

The “Gergonne point” $G_e$ of $\triangle A'B'C'$ is the intersection point of the lines $(AA', BB', CC')$ joining the vertices with the contact points of the opposite sides with the “incircle” $\mu(I)$ of $\triangle A'B'C'$ (see figure 7). A direct consequence of theorem 5 is the following, which proves also the existence of this point.

\textbf{Theorem 11.} The three lines $(AA', BB', CC')$ are symmedians of the triangle $ABC$ and the Gergonne point $G_e$ of $A'B'C'$ coincides with the symmedian point $K$ of $ABC$.

The two triangles $(ABC, A'B'C')$ being “point perspective” w.r. to point $K$, are also, according to “Desargues’ theorem” (see file Desargues’ theorem), “line perspective” i.e. corresponding sides meet on a line $\epsilon$. This is, per definition the “trilinear polar” of $K$, called “Lemoine line” of the triangle (see figure 8).
From theorem 6 follows the following:

**Theorem 12.** Points \( \{A'', B'', C''\} \) are the poles of lines, respectively \( \{AA', BB', CC'\} \) relative to the circumcircle \( \kappa \) of \( ABC \).

By the duality of the polarity, this implies that \( K \) is the pole of the trilinear polar \( \varepsilon \). Hence line \( OK \) is orthogonal to the trilinear polar. This is the so-called “Brocard axis” of the triangle (see file Brocard).

By the fundamental property of “Apollonian circles” (see file Apollonian), by which these circles are orthogonal to \( \kappa \), follows the next property:

**Theorem 13.** The points \( \{A'', B'', C''\} \) are the centers of the Apollonian circles, passing correspondingly through the vertices \( \{A, B, C\} \) of the triangle \( ABC \).

### 6 First Lemoine circle

The “first Lemoine circle” of the triangle \( ABC \) results by drawing parallels to the sides of the triangle from the symmedian point \( K \).
**Theorem 14.** The parallels to the sides of the triangle from the symmedian point intersect the sides in six concyclic points.

The proof results from the fact that the three segments \( \{B_1C_1, C_2A_2, A_3B_3\} \) joining the intersection points (see figure 9) are antiparallel respectively to the sides \( \{BC, CA, AB\} \). To see this for \( B_1C_1 \) reflect \( K \) on the bisector \( AD \) to the point \( K' \), which is on the median \( AM \). Then, the whole parallelogram \( AB_1KC_1 \) is reflected to the congruent to it \( AB'K'C' \), which has the diagonal \( B'C' \) parallel to \( BC \), thus proving that \( B_1C_1 \) is antiparallel to \( BC \).

Analogously is proved the same property for the segments \( \{C_2A_2, A_3B_3\} \). This implies that the quadrangles \( B_1C_1A_3A_2, C_2A_2B_1B_3, A_3C_1C_2B_3 \) are cyclic. From this follows that \( C_1B_1A = BA_2C_2 = \widehat{C} \). This, in turn, implies that the trapezium \( C_1B_1A_2C_2 \) is isosceles and inscribed in the circumcircle of \( C_1B_1A_2A_3 \). From this follows that the three quadrangles \( (4) \) have pairwise three common points, consequently their circumcircles coincide and define the so called “first Lemoine circle” of the triangle.

The lines \( \{B_1C_1, C_2A_2, A_3B_3\} \) define also another triangle \( A'B'C' \) with remarkable properties (see figure 10).

**Theorem 15.** Related to the Lemoine circle of the triangle \( ABC \) and the triangle \( A'B'C' \) are the following properties.

1. The segments \( \{B_1C_1, C_2A_2, A_3B_3\} \) are equal.
2. The triangles \( \{AB_1C_1, BC_2A_2, CA_3B_3\} \) are similar to \( ABC \).
3. The triangles \( \{A'B_3C_2, B'C_1A_3, C'A_2B_1\} \) are isosceles.
4. The incircle \( \lambda \) of \( A'B'C' \) is concentric to the Lemoine circle \( \kappa \) of \( ABC \).
5. The symmedians \( AK, BK, CK \) pass respectively through \( A', B', C' \).

6. The Gergonne point \( G_e \) of \( A'B'C' \) coincides with the symmedian point \( K \) of \( ABC \).

nr-1, nr-2 and nr-3 are immediate consequences of theorem 14.
nr-4 follows from the equality of segments in nr-1, since these are equal chords of the Lemoine circle.
nr-5 is proved by showing that for the symmedian line \( AKA_s \) the ratios \( KA_2/KA_3 \) and \( A_sC_2/A_sB_3 \) are equal:

\[
\frac{A_sC_2}{A_sB_3} = \frac{A_sB - C_2B}{A_sC - B_3C} = \frac{t \cdot KA_2 - KA_2}{t \cdot KA_3 - KA_3} = \frac{KA_2}{KA_3}.
\]

Nr-6 is an immediate consequence of the previous properties.

Notice that the first Lemoine circle is a special case of the so called “Tucker circles” of the triangle (see file Tucker circles).

7 Adams’ circle

The first Lemoine circle of \( ABC \) seen as a construct relative to the triangle \( A'B'C' \) is the so-called “Adams’ circle” of the triangle \( A'B'C' \), reflecting a property, usually formulated as follows ([Hon95, p.63]):

![Figure 11: The Adams’ circle of the triangle \( A'B'C' \)](image)

**Theorem 16.** The parallels to the sides of the cevian triangle \( A''B''C'' \) of the Gergonne point \( G'_e \) of the triangle \( A'B'C' \) from point \( G'_e \) intersect the sides of \( A'B'C' \) in 6 points lying on a circle \( \lambda \) concentric to the incircle \( \kappa \) of \( A'B'C' \).

The proof follows from the discussion in section 7 and the obvious fact that \( A''B''C'' \) is homothetic to \( ABC \) relative to the homothety with center \( K = G'_e \) and ratio 1/2.
8 Conics through parallels

This and the next section give another aspect of the first Lemoine circle. We start with a point \( D \) not lying on the side-lines of the triangle \( ABC \). Of central importance for this aspect is the following property.

**Theorem 17.** The parallels to the sides of a triangle \( ABC \) passing through a point \( D \) not lying on the side-lines of the triangle define respectively on the non-parallel sides six points lying on a conic.

![Figure 12: The conic defined by parallels to the sides from \( D \)](image)

**Proof.** The proof is a trivial verification of “Carnot’s theorem” ([Yiu13, p.117]), by which the intersection points \( \{X, X', \ldots\} \) are on a conic if and only if the product of ratios

\[
\frac{XB}{XC} \cdot \frac{X'B}{X'C} \cdot \frac{YC}{YA} \cdot \frac{Y'C}{Y'A} \cdot \frac{ZA}{Z'B} \cdot \frac{Z'A}{Z'B} = 1.
\]

In our case this product is easily seen to be equal to

\[
\frac{a^2}{b^2} \cdot \frac{b^2}{c^2} \cdot \frac{c^2}{a^2} = 1,
\]

where \( \{a, b, c\} \) the side lengths of the triangle. Hence the points are indeed on a conic.

Notice that this property can be generalized in the following sense (See Figure 13).

**Theorem 18.** Given a line \( \varepsilon \) intersecting the sides of the triangle \( ABC \) at the points \( \{A', B', C'\} \), consider the intersections \( \{X, X', Y, Y', Z, Z'\} \) of the sides of the triangle with the lines through \( \{A', B', C'\} \) and an arbitrary point \( D \). Then the six points lie on a conic.

The proof of this is reduced to the one of the previous theorem by means of a “projectivity” leaving the points \( \{A, B, C\} \) fixed and sending the line \( \varepsilon \) at the line at infinity.

9 Characterization of the first Lemoine circle

Next theorem characterizes the “symmedian point” of the triangle as the one for which the previous construction of conics delivers a circle. Here the symmedian point \( K \) of the triangle \( ABC \) is identified by its property to have distances from the sides proportional to these. Thus its “trilinear coordinates” are described by \( K \equiv (a : b : c) \).
Theorem 19. With the notation and conventions of the preceding section the conic is a circle if and only if the point $D$ coincides with the symmedian point of the triangle $ABC$.

Proof. If the conic is a circle (See Figure 14), then the three trapezia $(X'YY'Z, ...)$ are equilateral and the three segments $(X'Y, Y'Z, Z'X)$ have equal lengths. By the equality of the inscribed angles seen in the figure follows that the distance $x_B$ of $D$ from $AC$ is $x_B = ZY' \sin(\hat{B})$. This implies immediately

$$(x_A : x_B : x_C) = (\sin(\hat{A}) : \sin(\hat{B}) : \sin(\hat{C}) = (a : b : c),$$

thereby proving the necessity part of the theorem.

For the sufficiency consider the triangle $DZY'$. By assumption its altitudes satisfy

$$\frac{b}{c} = \frac{x_B}{x_C} = \frac{ZY' \sin(\hat{Z})}{Z'Y \sin(\hat{Y}')} = \frac{\sin(\hat{Z})}{\sin(\hat{Y}')} = \frac{DY'}{DZ}.$$
10 Second Lemoine circle

The “second Lemoine circle” of the triangle $ABC$ is a circle with center at the symmedian point $K$, whose existence follows from next theorem

![Second Lemoine circle](image)

**Figure 15**: Second Lemoine circle of the triangle

**Theorem 20.** (See Figure 15). On the antiparallels to the sides of the triangle $ABC$, which are drawn from the symmedian point $K$, the other sides define respectively equal segments {${KA_1 = KA_2, \ldots}$}, which define diameters of a circle $\alpha$ with center $K$.

The quadrangles

$${A_1B_1A_2B_2, \ B_1C_1B_2C_2, \ C_1A_2C_2A_1}$$

are rectangles. For $A_1B_1A_2B_2$, using the fact that the quadrangles

$${A_1A_2CB, \ B_1B_2CA, \ C_1C_2BA}$$

are cyclic, this follows from the equality of the angles noticed in figure 15. Analogously is proved the property for the other quadrangles in (5). The rectangles (5) have pairwise one common diagonal, and this proves the theorem.

11 Inscribed rectangles

In this section we consider rectangles inscribed in a given triangle, with one side coincident with a side of the triangle. Next theorem relates these rectangles with the symmedian point of the triangle (see figure 16).

**Theorem 21.** The centers $M$ of the rectangles, inscribed in the triangle $ABC$ and having one side on the line $BC$, are contained in a line $\varepsilon$, which passes through the middle $N$ of $BC$, the middle $L$ of the altitude $AY$ and the symmedian point $K$ of the triangle.
In fact, the centers $M$ of the rectangles are the middles of the segments $PI$ joining the middles of opposite parallels $(DH, ZE)$. Since all triangles $(PIN)$ are similar, the claim that $M$ describes a line passing through $(N, L)$ is clear. That this line passes also through $K$ follows from the proof of theorem 20, where we saw that there is such a rectangle with center at $K$.

**Theorem 22.** The symmedian point $K$ of the triangle is the common point of the lines joining the middles $(L_i)$ of the altitudes with the middles $(N_i)$ of the opposite sides of the triangle $ABC$.

This is a direct corollary of theorem 21. Notice that this point $K$ is the “triangle center” $X(69)$ w.r. to the medial triangle $N_1N_2N_3$ ([Kim18]).

12 Medial and Orthic triangle intersections

Consider the “medial triangle” $A'B''C''$ with vertices the middles of the sides of triangle $ABC$. Each symmedian line of $ABC$ intersects the sides of the medial triangle, which are adjacent to the opposite vertex of the corresponding parallelogram at two points. For example the symmedian $BK$ intersects the sides $(B''A'', B''C'')$ at the points $(V_2, U_2)$. Analogously are defined the points $(V_1, U_1)$ and $(V_3, U_3)$ (see figure 18). Next theorem lists some properties of these points.

**Theorem 23.** Let $A'B'C'$ be the “orthic” triangle of $ABC$, with vertices the feet of the altitudes. Then the following are valid properties.
1. The sides of the orthic at \( \{A', B', C'\} \) pass respectively through the points \((U_1, V_1), (U_2, V_2), (U_3, V_3)\).

2. The triangles \((A'A''V_1, ..., A'A''U_1, ...)\) are all similar to \(ABC\).

To show \(nr-1\) in a case, for example, to show that \(A'C'\) passes through \(V_1\), consider for the moment \(V_1\) as the intersection of lines \((A'C', A''B'')\). Measuring the angles of \(A'A''V_1\) we see that this triangle is similar to \(ABC\) and the quadrangle \(A'V_1CB''\) is cyclic. All the angles at \((A', A'', B'')\) can be measured using standard properties of the orthic triangle and show that \(B''CV_1\) is similar to \(ABC\).

![Figure 18: Intersections of sides of the medial and the orthic](image)

Having that, project \(V_1\) on the sides of \(ABC\) at points \((S, Q)\). Then we have:

\[
\frac{V_1S}{V_1Q} = \frac{PC'}{V_1Q} = \frac{CP}{V_1Q} = \frac{CB''}{V_1B''} = \frac{AB}{AC'}
\]

proving that \(V_1\) satisfies the characteristic property of the points of the symmedian, hence it is on the symmedian \(AK\). Analogously are proved the other cases of this claim.

\(Nr-2\) results, as alluded to before, by measuring the angles of these triangles. Figure 18 gives a hint for this in the case of the triangle \(A'A''V_1\).

13 Vecten squares of the triangle

On the sides of triangle \(\tau = ABC\) erect squares. These are the "Vecten squares" of the triangle. The triangle of the opposite sides of the squares \(\tau' = A'B'C'\) has its sides parallel to those of \(\tau\), hence is similar to it. (see figure 19).
Theorem 24. The similarity center of \((\tau, \tau')\) is the symmedian point \(K\) of \(\tau\), which is also the symmedian point of \(\tau'\).

The proof is immediately seen in the figure:

\[
\frac{AA'}{AA''} = \frac{AB}{AC} = \frac{A'B'}{A'C'}
\]

showing that \(A\) is on the symmedian line of \(\tau'\) and \(\tau\).

The same result holds for the configuration created by erecting squares on the sides, each lying on the same side with the opposite vertex (see figure 20). The only difference in this case is that the two triangles \(ABC\) and \(A'B'C'\) are related by an “anti-homothety” with center the symmedian point \(K\) of both triangles.

14 Artzt parabolas

The following properties relate the symmedian \(AK\) to the Apollonian circle \(\lambda_A(O_A)\), which is the locus of the points \((X : XB/XC = b/c)\) and lead also to a characterization
of the projection $A_2$ of the circumcenter on the symmedian $AK$ as the focus of the “$A$-Artzt parabola” of the triangle $ABC$. By its definition, this parabola is tangent to the sides $(AB, AC)$ at the points $(B, C)$. Analogously are defined the $B$-Artzt parabola tangent to the sides $(BC, BA)$ and the $C$-Artzt parabola tangent to $(CA, CB)$ (see figure 21). Notice that point $A_2$ is a vertex of the so called “second Brocard triangle” ([Cou80, p.279]). Its isogonal conjugate $J$ appearing in the next theorem is a vertex of the “fourth Brocard triangle” ([Gib21]).

Figure 22: The projection $A_2$ of $O$ on the symmedian $AK$

**Theorem 25.** Referring to figure 22, we denote by $\kappa(O)$ the circumcircle of $\triangle ABC$ and consider the points: the second intersection $A'$ of the symmedian $AK$ with $\kappa$, the second intersection $F$ of the median $AM$ with $\kappa$ and the reflection $J$ of $A'$ in $BC$.

1. $BA'FC$ is a trapezium and $A'F$ is parallel to $BC$.
2. Point $A'$ is on the Apollonian circle $\lambda_A(O_A)$, satisfying $A'B/A'C = b/c$.
3. Point $A_2$ is the middle of $AA'$, line $OA_2$ passes through $O_A$.
4. Point $A_2$ is the inverse of $O$ w.r.t. to $\lambda_A$.
5. Point $J$ is the second intersection of the Apollonian circle $\lambda_A$ with the median $AM$. 

Figure 21: The Artzt parabolas of $\triangle ABC$
6. \( JBFC \) is a parallelogram and point \( J \) is the isogonal conjugate of \( A_2 \).

7. Line \( HJ \) is orthogonal to the median \( AM \), where \( H \) is the orthocenter of \( \triangle ABC \).

Proof. Nr-1 follows from the equality of the angles \( \hat{BAA'} = \hat{FAC} \).

Nr-2: By nr-1 and the similar triangles \( (MFC \sim MBA, MFB \sim MCA) \) we have

\[
\frac{A'B}{AC} = \frac{FC}{FB} = \frac{MB}{MC} = \frac{AB}{AM} \cdot \frac{AM}{AC} = \frac{AB}{AC}.
\]

Nrs 3-4 are a direct consequence of nr-2.

Nr-5 Follows from the fact that \( A'MF \) is isosceles and \( \overline{JA'} \) is a right angle. This implies that \( J \) is on the median \( AF \). Obviously it is also on \( \lambda_A \) since \( BC \) is a diameter of this circle.

Nr-6: By nr-4 point \( M \) is the middle of both segments \( (BC, JF) \) implying that \( JBFC \) is a parallelogram. By theorem 8 and the preceding claim these triangles are similar:

\[
BA_2A \sim AA_2C \sim BA'C \sim CFB \sim BJ'C.
\]

This implies \( A_2BA = JC = JBA' = A_2CB \). Analogously is seen that \( A_2CB = JCA \). Since we have also \( A_2AB = JA'C \) the claim is proved.

nr-7: This involves a computation in barycentrics (see file Barycentric coordinates) in which the points have corresponding coordinates

\( A(1, 0, 0), G(1, 1, 1), H(S_BS_C, S_CS_A, S_AS_B), J(a^2, 2S_A, 2S_A) \).

We test the orthogonality formula for lines \( S_{App'} + S_{Bqq'} + S_Crr' = 0 \), where \( (p, q, r) \) and \( (p', q', r') \) are the points at infinity (their directions) of the lines \( (AG, HJ) \) given by the respective triple products \( (A \times G) \times G, (H \times J) \times G \). The result is the expression \( 2(S_B + S_C - a^2)(S_B S_C + S_C S_A + S_A S_B) = 0 \), according to the identity for the “Conway symbols” \( S_B + S_C = a^2 \) (see file Conway symbols). \( \square \)

Figure 23: Generating the Artzt parabola by varying triangle \( B^*A_2A^* \sim BA_2A \)
Next theorem complements some properties discussed in theorem 8. Point $D$ is defined as the second intersection of the parallel $CD$ to the symmedian $AA'$ with the circumcircle of triangle $ABC$. In that theorem was shown that the projection $A_2$ of the circumcenter on the symmedian $AA'$, which is a vertex of the second Brocard triangle, was a point of the line $BD$ (see figure 23).

**Theorem 26.** Referring to figure 23, the following are valid properties:

1. Triangles $A_2BA'$ and $AA_2D$ are similar to $ABC$.
2. Triangle $DA_2C$ is isosceles and triangles $ADA_2$ and $A'CA_2$ are equal.
3. Triangles $A_2BA'$ and $A_2A'C$ are also similar.
4. Varying triangle $BA_2A$ to a similar $B^*A_2A^*$ so that $B^*$ moves on $AB$ makes vertex $A^*$ move on line $AC$ and the line $e = B^*A^*$ is tangent to the parabola.

Nr-1. In fact, $\triangle ABC$ is similar to $\triangle A_2AD$. Later has $\triangle ADA_2 = \triangle ABC$ and $\triangle A_2AD = \triangle ABC$, because $ADCA'$ is equilateral trapezium and $\triangle A_2AD = \triangle A_2A'C = \triangle ABC$. Obviously $\triangle A_2BA'$ is similar to $\triangle A_2AD$.

Nr-2 is obvious, since the triangles $(A_2AD, A_2A'C)$ are each the reflection of the other in line $OA_2$.

Nr-3 is a consequence of $\text{nr}-2$.

Nr-4 is a general property of parabolas related to a triangle like $ABC$, which has two sides tangent to the parabola and the third side joins the contact points. In such a case it is well known ([Cha65, p.52]) that a line $A^*B^*$ intersecting the tangents at points such that $AB^*/B^*B = CA^*/A^*A$ is tangent to the parabola (see file Triangles tangent to parabolas). The relation $AB^*/B^*B = CA^*/A^*A$ follows from the similar triangles $(AB^*A_2 \sim CA^*A_2)$.

**Bibliography**


Related topics

1. Barycentric coordinates
2. Brocard
3. Cross Ratio
4. Desargues’ theorem
5. Pedal triangles
6. Projective line
7. The quadratic equation in the plane

Any correction, suggestion or proposal from the reader, to improve/extend the exposition, is welcome and could be send by e-mail to: pamfilos@uoc.gr