Symmedian

A file of the Geometrikon gallery by Paris Pamfilos

The reason is not to glorify *"bit chasing"*; a more fundamental issue is at stake here: Numerical subroutines should deliver results that satisfy simple, useful mathematical laws whenever possible. ... Without any underlying symmetry properties, the job of proving interesting results becomes extremely unpleasant. The enjoyment of one's tools is an essential ingredient of successful work.

Donald Knuth, Seminumerical Algorithms, section 4.2.2

Contents

(Last update: 15-05-2021)

1 Symmedians and symmedian point

Here we discuss two related concepts: the *"symmedian line"* and the *"symmedian point"* or *"Lemoine point"* of a triangle $\tau = ABC$. The symmedian *line* of τ from vertex A is the symmetric of its median from A w.r.t. to the bisector from A. The symmedian *point* K is

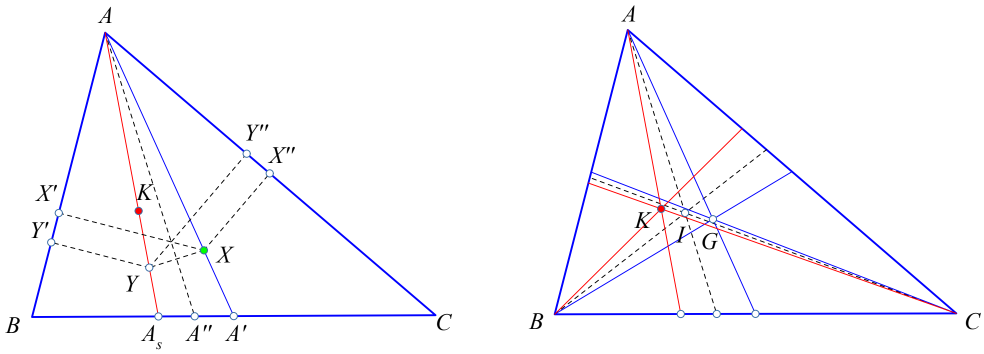

Figure 1: Symmedian line from A and symmedian point K

the intersection point of the symmedian lines from the three vertices (See Figure 1). Next theorem expresses a *"characteristic property"* of the points Y of a symmedian line.

Theorem 1. *The distances of points Y of the symmedian line AA_s from A, from the adjacent sides $\{AB, AC\}$, are proportional to these sides*:

$$\frac{YY'}{YY''} = \frac{AB}{AC}. \tag{1}$$

Proof. This follows from the symmetry w.r.t. the bisector AA'' and the corresponding property of the points X of the median AA', by which, due to the equality of the areas of triangles $\{ABA', AA'C\}$, the analogous ratio *is inversely proportional to the adjacent sides*:

$$\frac{XX'}{XX''} = \frac{AC}{AB}. \tag{2}$$

□

Projecting the intersection point K of two of the symmedians on the sides of the triangles and applying the theorem, we see that the third symmedian passes also through that point, thus:

Theorem 2. *The three symmedians of the triangle pass through a common point K called the symmedian point of the triangle.*

2 Antiparallels

Closely related to the concept of *"symmedian"* is the concept of *"antiparallels"* to a side of the triangle. An antiparallel $\varepsilon = B'C'$ to side BC of the triangle is the line defined by any circle λ passing through B, C and intersecting the sides $\{AB, AC\}$ a second time at the points $\{C', B'\}$ (See Figure 2). There are two prominent antiparallels to BC :
1. The tangent ε_0 to the circumcircle at A.
2. The side $B''C''$ of the *"orthic triangle"*, defined by the feet of the altitudes of the triangle.

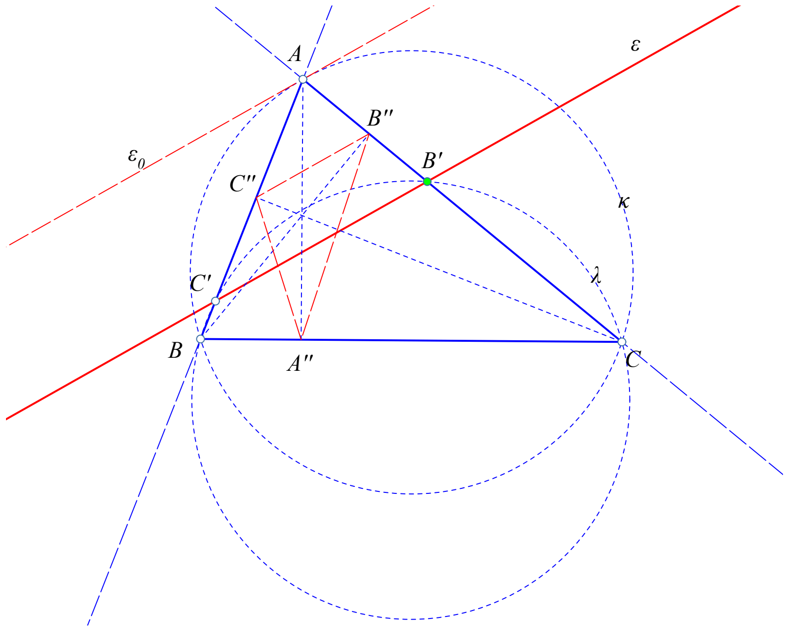

Figure 2: Antiparallels ε to the side BC

The tangent ε_0 is the limiting position for which the points $\{B', C'\}$ coincide with A and the circle λ coincides with the circumcircle κ of the triangle. The line $B''C''$ is antiparallel because the quadrangle $BCB''C''$ is *"cyclic"*.

Theorem 3. *All triangles $AB'C'$ are similar to ABC.*

This results immediately from the corresponding angle equalities of the triangles:

$$\widehat{C'} = \widehat{C} \quad \text{and} \quad \widehat{B'} = \hat{B}. \tag{3}$$

The symmetry on the bisector from A maps triangle $AB'C'$ to a triangle AB_1C_1 with side B_1C_1 parallel to BC hence the median line from A of $AB'C'$ maps to the median line of ABC. This proves the following:

Theorem 4. *The symmedian of ABC coincides with the median line of the triangles $\{AB'C'\}$.*

3 Harmonic quadrangle

Theorem 5. *The tangents at $\{B, C\}$ to the circumcircle of triangle ABC intersect at a point D on the symmedian AA_s.*

Proof. This follows from the characteristic property of the points of the symmedian line expressed by equation 1. Calculating the distances of D from the sides and using the sines-rule we get (See Figure 3):

$$\frac{DD'}{DD''} = \frac{DB\sin(\widehat{C})}{DC\sin(\hat{B})} = \frac{\sin(\widehat{C})}{\sin(\hat{B})} = \frac{AB}{AC}.$$

□

Theorem 6. *The intersection A_t of the tangent to the circumcircle at A with BC is the "harmonic conjugate" of A_s w.r.t. to $\{B, C\}$ and the ratios XB'/XC' defined by the sides $\{AB, AC\}$ and the symmedian line from A on every parallel $B'XC'$ to the side BC are equal to the ratio $\sin(\widehat{C})^2/\sin(\hat{B})^2 = c^2/b^2$.*

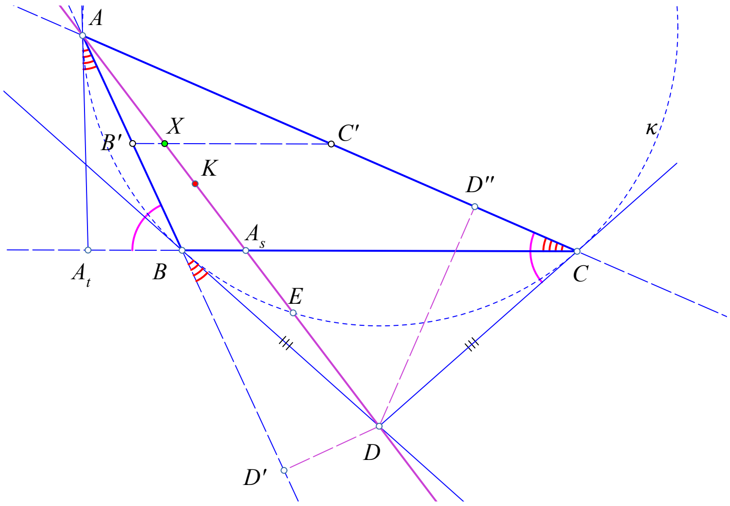

Figure 3: Tangents intersecting on the symmedian AA_s

Proof. This because the triangles $\{AA_tB, CA_tA\}$ are similar, implying (See Figure 3):

$$\frac{A_tC}{A_tB} = \frac{A_tC \cdot A_tB}{A_tB^2} = \frac{A_tA^2}{A_tB^2} = \frac{\sin(\widehat{B})^2}{\sin(\widehat{C})^2} = \frac{A_sC}{A_sB}.$$

□

It follows that the pencil of lines $A(B, C, A_s, A_t)$ is harmonic, hence it defines on the circumcircle a *"harmonic quadrangle"*, i.e. a circumscribed quadrangle for which the *"cross ratio"* $(BCAE) = (BCA_tA_s) = -1$. This kind of quadrangles is also characterized by the property of being cyclic and having *equal products of opposite sides.*

Theorem 7. *The intersection E of the symmedian from A with the circumcircle of ABC defines a harmonic quadrangle $ABEC$.*

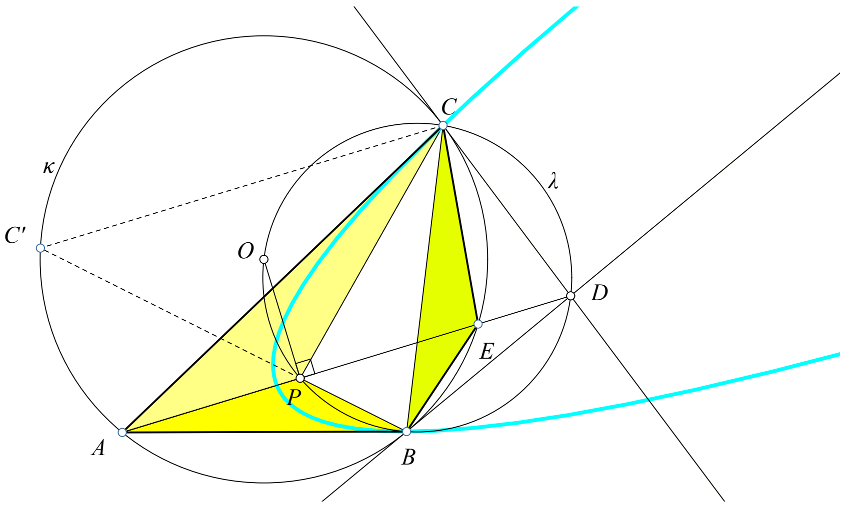

Figure 4: Property of the symmedian related to the harmonic quadrangle $ABEC$

Theorem 8. *The circle λ through $\{B, C\}$ and the intersection D of the tangents at B, C to the circumcircle $\kappa(O)$ of the triangle ABC passes through the center O and intersects the symmedian*

AD a second time at P, which defines triangles $\{APC\ ,\ BPC\}$ simmilar to BEC, so that the symmedian PE bisects the angle $\widehat{BPC}$. In addition holds the relation $PA^2 = PB \cdot PC$.

Proof. The angles $\{OPD, OCD\}$ are right showing that λ passes through O (see figure 4). A similar angle chasing shows that the angles of the triangles at P, E are equal to the supplement of $\widehat{A}$. Note here that, since D is the middle of the arc BC on λ the symmedian AD bisects the angle $\widehat{BPC}$. Also $\widehat{ECB} = \widehat{BAP}$ completes the proof that $\triangle ABP \sim \triangle CBE$. Analogously $\widehat{EBP} = \widehat{PAC}$ completes the proof that $\triangle BEC \sim \triangle APC$ and shows that PE is a bisector of the angle $\widehat{BPC}$.

For the proof of the last claim see that the parallel CC' to the symmedian AD is parallel to the bisector of BPC hence $PC = PC'$ and line PB passes through C', hence $PB \cdot PC = PB \cdot PC' = PA^2$.

Figure 4 shows also a so-called *"Artzt parabola"* of the triangle ABC, which is the parabola passing through $\{B, C\}$ and being there tangent to the sides $\{AB, AC\}$. Its focus is point P (see section 14). □

4 Second Brocard triangle

The *"second Brocard triangle"* of the triangle ABC is defined by its vertices which are the projections $\{A_2, B_2, C_2\}$ of the circumcenter O of $\triangle ABC$ on respective symmedians.

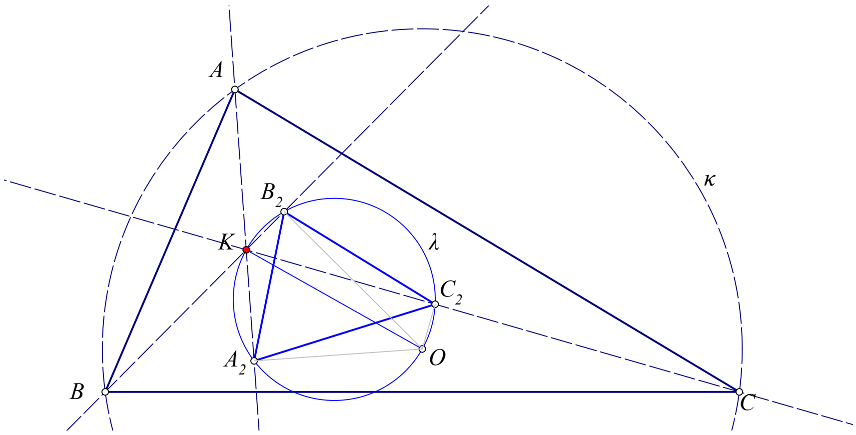

Figure 5: The second Brocard triangle $A_2B_2C_2$ of $\triangle ABC$

Theorem 9. *The circumcircle λ of the second Brocard triangle has the symmedian point K and the circumcenter O of $\triangle ABC$ as diametral points.*

Proof. This follows directly from the definition of the vertices $\{A_2, B_2, C_2\}$ as projections of O on respective symmedians, implying that each of them is viewing the segment OK under a right angle. □

The circle with diameter OK is called *"Brocard circle"* of the triangle. We'll see in theorem 25 that $\{A_2, O\}$ are inverses w.r.t. the Apollonian circle λ_A through A. This implies that the Brocard circle is orthogonal to λ_A. Analogously it is seen that this circle is orthogonal to all three Apollonian circles $\{\lambda_A, \lambda_B, \lambda_C\}$ of $\triangle ABC$. This implies that the circle λ, like the circumcircle κ belongs to the pencil of circles which are orthogonal to the Apollonian circles, called *"Schoute pencil"* of the triangle ABC (see file **Pedal triangles**).

Theorem 10. *The "cyclocevian" triangle EFG of the vertices of the second Brocard triangle is congruent and inversely oriented to the triangle ABC.*

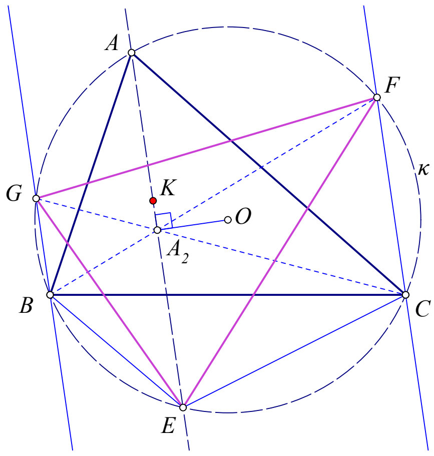

Figure 6: The cyclocevian of A_2

Proof. *"Cyclocevian"* of a point P w.r.t. to the triangle ABC is called the triangle formed by the second intersections of the lines $\{PA, PB, PC\}$ and the circumcircle κ. We prove the theorem for A_2. In the course of proof of theorem 8 we have seen that drawing parallels to the symmedian AK from $\{B, C\}$ we get respectively their second intersections $\{G, F\}$ with the circumcircle κ, which are respectively collinear with $\{A_2C, A_2B\}$ (see figure 6). This implies easily that the two triangles $\{ABC, EGF\}$ are each the reflection of the other relative to the line OA_2. □

5 Gergonne point

The *"Gergonne point"* G_e of $\triangle A'B'C'$ is the intersection point of the lines $\{AA', BB', CC'\}$ joining the vertices with the contact points of the opposite sides with the *"incircle"* $\mu(I)$ of

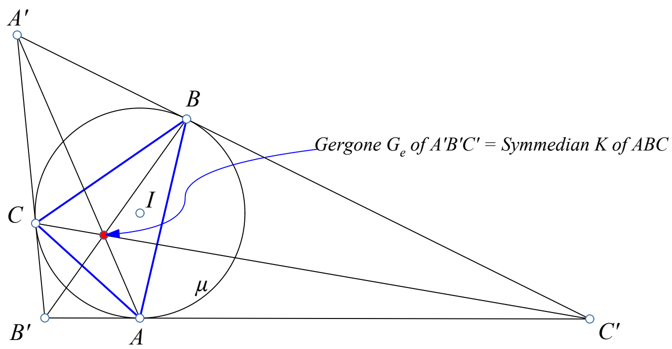

Figure 7: Gergonne point G_e of $A'B'C'$ = symmedian point K of ABC

$\triangle A'B'C'$ (see figure 7). A direct consequence of theorem 5 is the following, which proves also the existence of this point.

Theorem 11. *The three lines $\{AA', BB', CC'\}$ are symmedians of the triangle ABC and the Gergonne point G_e of $A'B'C'$ coincides with the symmedian point K of ABC.*

The two triangles $\{ABC, A'B'C'\}$ being *"point perspective"* w.r.t. to point K, are also, according to *"Desargues' theorem"* (see file **Desargues' theorem**), *"line perspective"* i.e. corresponding sides meet on a line ε. This is, per definition the *"trilinear polar"* of K, called *"Lemoine line"* of the triangle (see figure 8).

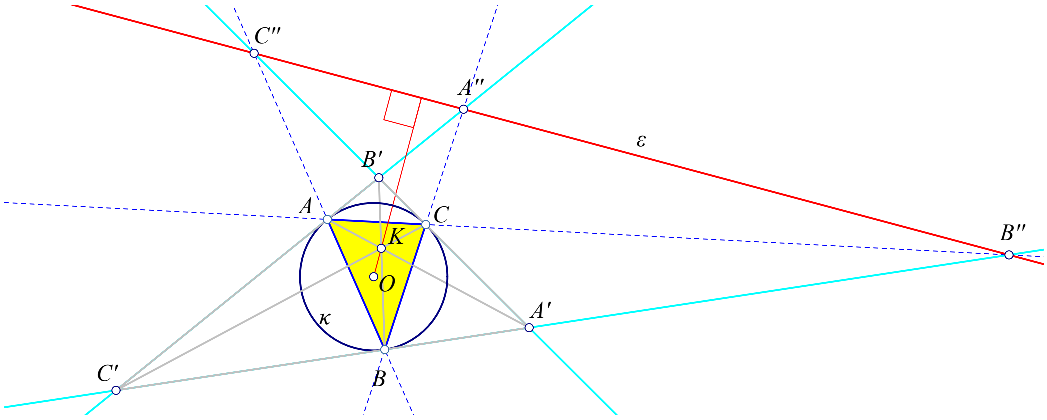

Figure 8: Lemoine line ε of triangle ABC

From theorem 6 follows the following:

Theorem 12. *Points $\{A'', B'', C''\}$ are the poles of lines, respectively $\{AA', BB', CC'\}$ relative to the circumcircle κ of ABC.*

By the duality of the polarity, this implies that K is the pole of the trilinear polar ε. Hence line OK is orthogonal to the trilinear polar. This is the so-called *"Brocard axis"* of the triangle (see file **Brocard**).

By the fundamental property of *"Apollonian circles"* (see file **Apollonian**), by which these circles are orthogonal to κ, follows the next property:

Theorem 13. *The points $\{A'', B'', C''\}$ are the centers of the Apollonian circles, passing correspondingly through the vertices $\{A, B, C\}$ of the triangle ABC.*

6 First Lemoine circle

The *"first Lemoine circle"* of the triangle ABC results by drawing parallels to the sides of the triangle from the symmedian point K.

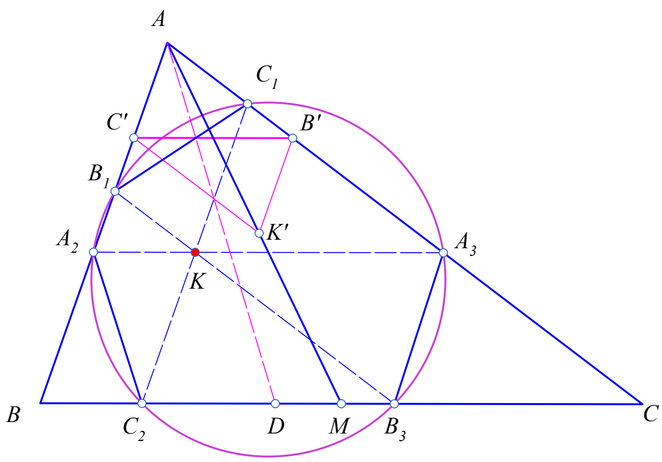

Figure 9: First Lemoine circle of the triangle ABC

Theorem 14. *The parallels to the sides of the triangle from the symmedian point intersect the sides in six concyclic points.*

The proof results from the fact that the three segments $\{B_1C_1, C_2A_2, A_3B_3\}$ joining the intersection points (see figure 9) are antiparallel respectively to the sides $\{BC, CA, AB\}$. To see this for B_1C_1 reflect K on the bisector AD to the point K', which is on the median AM. Then, the whole parallelogram AB_1KC_1 is reflected to the congruent to it $AB'K'C'$, which has the diagonal $B'C'$ parallel to BC, thus proving that B_1C_1 is antiparallel to BC. Analogously is proved the same property for the segments $\{C_2A_2, A_3B_3\}$. This implies that the quadrangles

$$B_1C_1A_3A_2, \quad C_2A_2B_1B_3, \quad A_3C_1C_2B_3 \tag{4}$$

are cyclic. From this follows that $\widehat{C_1B_1A} = \widehat{BA_2C_2} = \widehat{C}$. This, in turn, implies that the trapezium $C_1B_1A_2C_2$ is isosceles and inscribed in the circumcircle of $C_1B_1A_2A_3$. From this follows that the three quadrangles (4) have pairwise three common points, consequently their circumcircles coincide and define the so called *"first Lemoine circle"* of the triangle.

The lines $\{B_1C_1,\ C_2A_2,\ A_3B_3\}$ define also another triangle $A'B'C'$ with remarkable prop-

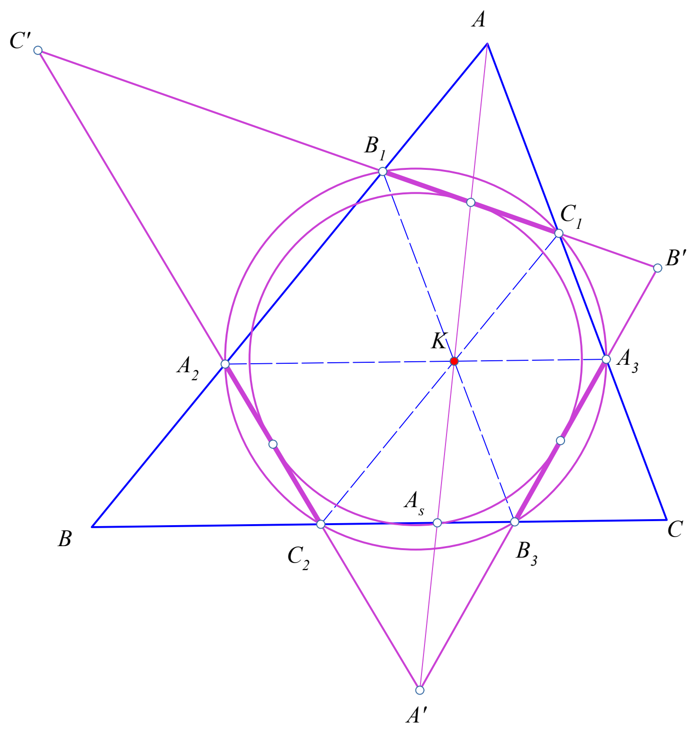

Figure 10: Additional properties related to the Lemoine circle

erties (see figure 10).

Theorem 15. *Related to the Lemoine circle of the triangle ABC and the triangle $A'B'C'$ are the following properties.*

1. *The segments $\{B_1C_1, C_2A_2, A_3B_3\}$ are equal.*
2. *The triangles $\{AB_1C_1, BC_2A_2, CA_3B_3\}$ are similar to ABC.*
3. *The triangles $\{A'B_3C_2, B'C_1A_3, C'A_2B_1\}$ are isosceli.*
4. *The incircle λ of $A'B'C'$ is concentric to the Lemoine circle κ of ABC.*

5. *The symmedians* $\{AK, BK, CK\}$ *pass respectively through* $\{A', B', C'\}$.
6. *The Gergonne point* G_e *of* $A'B'C'$ *coincides with the symmedian point* K *of* ABC.

nr-1, *nr-2* and *nr-3* are immediate consequences of theorem 14.
nr-4 follows from the equality of segments in *nr-1*, since these are equal chords of the Lemoine circle.
nr-5 is proved by showing that for the symmedian line AKA_s the ratios KA_2/KA_3 and A_sC_2/A_sB_3 are equal:

$$\frac{A_sC_2}{A_sB_3} = \frac{A_sB - C_2B}{A_sC - B_3C} = \frac{t \cdot KA_2 - KA_2}{t \cdot KA_3 - KA_3} = \frac{KA_2}{KA_3}.$$

Nr-6 is an immediate consequence of the previous properties.

Notice that the first Lemoine circle is a special case of the so called *"Tucker circles"* of the triangle (see file **Tucker circles**).

7 Adams' circle

The first Lemoine circle of ABC seen as a construct relative to the triangle $A'B'C'$ is the so-called *"Adams' circle"* of the triangle $A'B'C'$, reflecting a property, usually formulated as follows ([Hon95, p.63]):

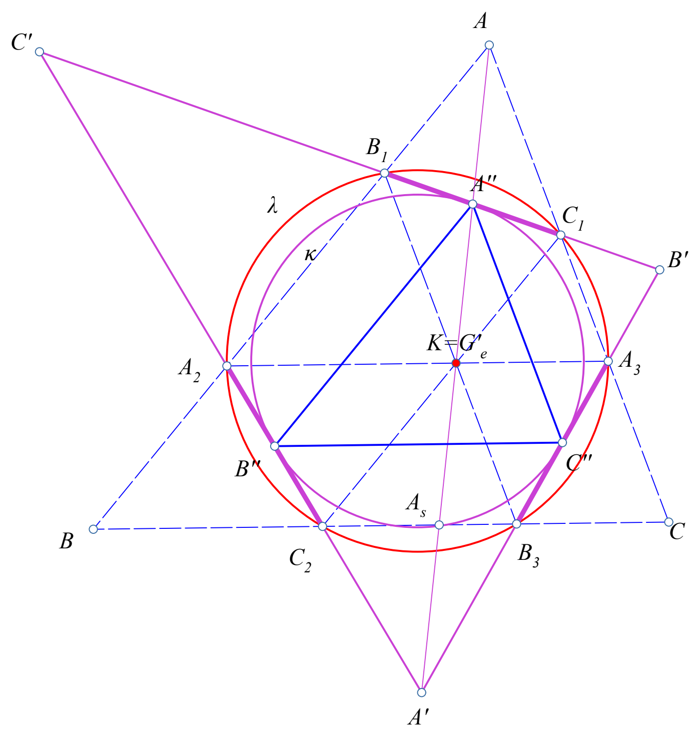

Figure 11: The Adams' circle of the triangle $A'B'C'$

Theorem 16. *The parallels to the sides of the cevian triangle* $A''B''C''$ *of the Gergonne point* G'_e *of the triangle* $A'B'C'$ *from point* G'_e *intersect the sides of* $A'B'C'$ *in 6 points lying on a circle* λ *concentric to the incircle* κ *of* $A'B'C'$.

The proof follows from the discussion in section 7 and the obvious fact that $A''B''C''$ is homothetic to ABC relative to the homothety with center $K = G'_e$ and ratio 1/2.

8 Conics through parallels

This and the next section give another aspect of the first Lemoine circle. We start with a point D not lying on the side-lines of the triangle ABC. Of central importance for this aspect is the following property.

Theorem 17. *The parallels to the sides of a triangle ABC passing through a point D not lying on the side-lines of the triangle define respectively on the non-parallel sides six points lying on a conic.*

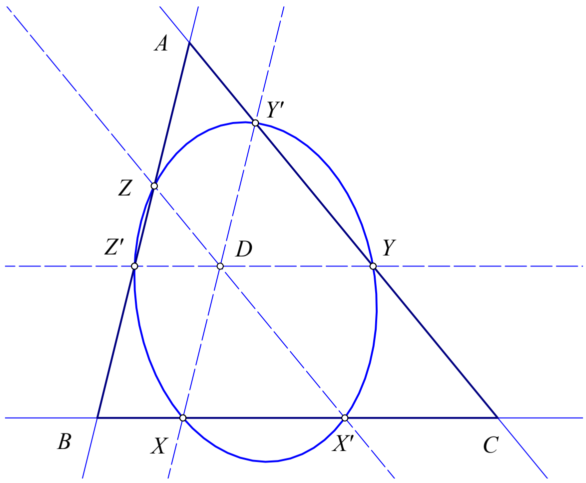

Figure 12: The conic defined by parallels to the sides from D

Proof. The proof is a trivial verification of *"Carnot's theorem"* ([Yiu13, p.117]), by which the intersection points $\{X, X', \ldots\}$ are on a conic if and only if the product of ratios

$$\frac{XB}{XC} \cdot \frac{X'B}{X'C} \cdot \frac{YC}{YA} \cdot \frac{Y'C}{Y'A} \cdot \frac{ZA}{ZB} \cdot \frac{Z'A}{Z'B} = 1.$$

In our case this product is easily seen to be equal to

$$\frac{a^2}{b^2} \cdot \frac{b^2}{c^2} \cdot \frac{c^2}{a^2} = 1,$$

where $\{a, b, c\}$ the side lengths of the triangle. Hence the points are indeed on a conic. □

Notice that this property can be generalized in the following sense (See Figure 13).

Theorem 18. *Given a line ε intersecting the sides of the triangle ABC at the points $\{A', B', C'\}$, consider the intersections $\{X, X', Y, Y', Z, Z'\}$ of the sides of the triangle with the lines through $\{A', B', C'\}$ and an arbitrary point D. Then the six points lie on a conic.*

The proof of this is reduced to the one of the previous theorem by means of a *"projectivity"* leaving the points $\{A, B, C\}$ fixed and sending the line ε at the line at infinity.

9 Characterization of the first Lemoine circle

Next theorem characterizes the *"symmedian point"* of the triangle as the one for which the previous construction of conics delivers a circle. Here the symmedian point K of the triangle ABC is identified by its property to have distances from the sides proportional to these. Thus its *"trilinear coordinates"* are described by $K \cong (a : b : c)$.

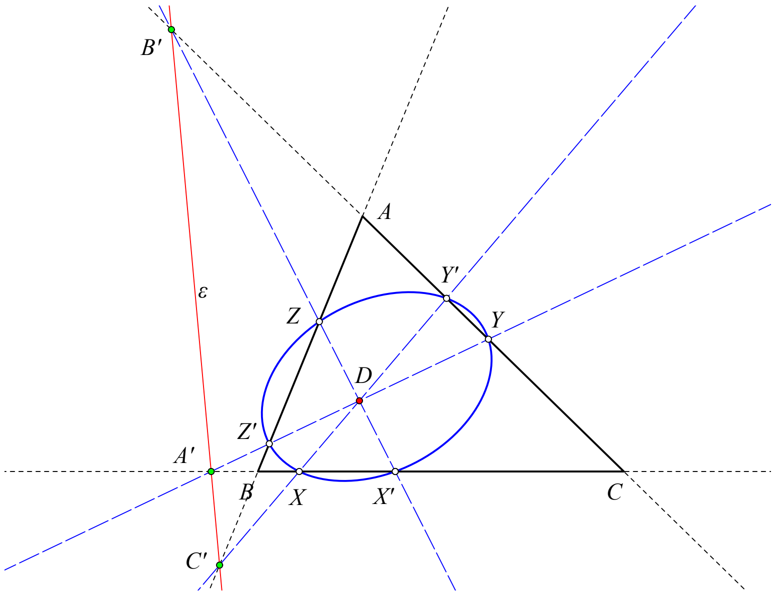

Figure 13: Conic defined by a point and a line

Theorem 19. *With the notation and conventions of the preceding section the conic is a circle if and only if the point D coincides with the symmedian point of the triangle ABC.*

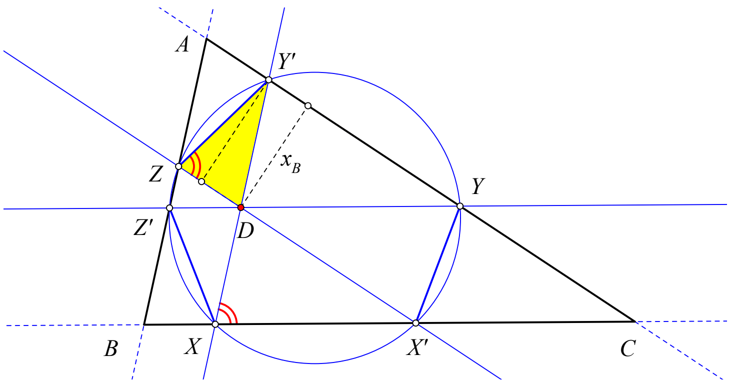

Figure 14: The conic is a circle

Proof. If the conic is a circle (See Figure 14), then the three trapezia $\{X'YY'Z, ...\}$ are equilateral and the three segments $\{X'Y, Y'Z, Z'X\}$ have equal lengths. By the equality of the inscribed angles seen in the figure follows that the distance x_B of D from AC is $x_B = ZY' \sin(\hat{B})$. This implies immediately

$$(x_A \ : \ x_B \ : \ x_C) \ = \ (\sin(\widehat{A}) \ : \ \sin(\hat{B}) \ : \ \sin(\widehat{C}) \ = \ (a \ : \ b \ : \ c),$$

thereby proving the necessity part of the theorem.

For the sufficiency consider the triangle DZY'. By assumption its altitudes satisfy

$$\frac{b}{c} = \frac{x_B}{x_C} = \frac{ZY' \sin(\widehat{Z})}{ZY' \sin(\widehat{Y'})} = \frac{\sin(\widehat{Z})}{\sin(\widehat{Y'})} = \frac{DY'}{DZ}.$$

Thus, the triangles $\{ABC, DZY'\}$, having the angles $\widehat{A} = \widehat{D}$ and the containing them sides proportional, are similar and the angles $\widehat{Z} = \hat{B}$. Analogous argument shows that the angle

$\widehat{YX'Z}$ is equal to $\hat{B}$. This implies that $X'YY'Z$ is an isosceles trapezium inscribed in a circle passing through X. A similar argument shows the analogous property for the trapezia $Y'ZZ'X$ and $Z'XX'Y$. This implies that all six points $\{X, X', Y, Y', Z, Z'\}$ are on the same circle. □

10 Second Lemoine circle

The *"second Lemoine circle"* of the triangle ABC is a circle with center at the symmedian point K, whose existence follows from next theorem

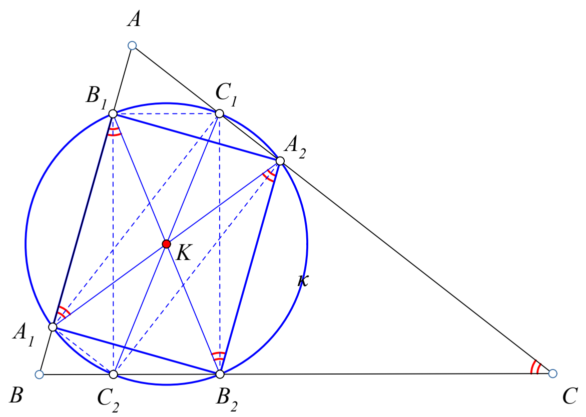

Figure 15: Second Lemoine circle of the triangle

Theorem 20. *(See Figure 15). On the antiparallels to the sides of the triangle ABC, which are drawn from the symmedian point K, the other sides define respectively equal segments $\{KA_1 = KA_2, \ldots\}$, which define diameters of a circle κ with center K.*

The quadrangles

$$A_1B_1A_2B_2, \quad B_1C_1B_2C_2, \quad C_1A_2C_2A_1 \tag{5}$$

are rectangles. For $A_1B_1A_2B_2$, using the fact that the quadrangles

$$A_1A_2CB, \quad B_1B_2CA, \quad C_1C_2BA$$

are cyclic, this follows from the equality of the angles noticed in figure 15. Analogously is proved the property for the other quadrangles in (5). The rectangles (5) have pairwise one common diagonal, and this proves the theorem.

11 Inscribed rectangles

In this section we consider rectangles inscribed in a given triangle, with one side coincident with a side of the triangle. Next theorem relates these rectangles with the symmedian point of the triangle (see figure 16).

Theorem 21. *The centers M of the rectangles, inscribed in the triangle ABC and having one side on the line BC, are contained in a line ε, which passes through the middle N of BC, the middle L of the altitude AY and the symmedian point K of the triangle.*

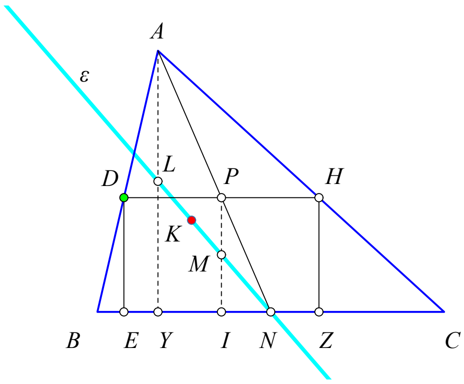

Figure 16: Inscribed rectangles

In fact, the centers M of the rectangles are the middles of the segments PI joining the middles of opposite parallels $\{DH, ZE\}$. Since all triangles $\{PIN\}$ are similar, the claim that M describes a line passing through $\{N, L\}$ is clear. That this line passes also through K follows from the proof of theorem 20, where we saw that there is such a rectangle with center at K.

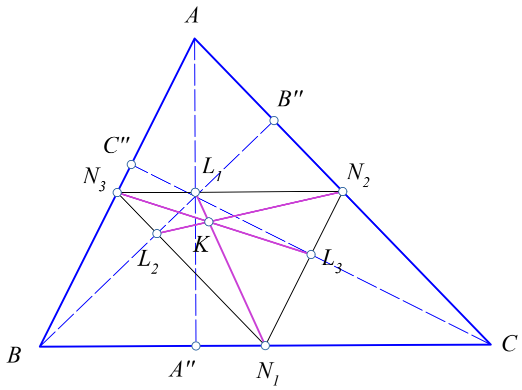

Figure 17: Alternative definition of the symmedian point

Theorem 22. *The symmedian point K of the triangle is the common point of the lines joining the middles $\{L_i\}$ of the altitudes with the middles $\{N_i\}$ of the opposite sides of the triangle ABC.*

This is a direct corollary of theorem 21. Notice that this point K is the *"triangle center"* $X(69)$ w.r. to the medial triangle $N_1N_2N_3$ ([Kim18]).

12 Medial and Orthic triangle intersections

Consider the *"medial triangle"* $A''B''C''$ with vertices the middles of the sides of triangle ABC. Each symmedian line of ABC intersects the sides of the medial triangle, which are adjacent to the opposite vertex of the corresponding parallelogram at two points. For example the symmedian BK intersects the sides $\{B''A'', B''C''\}$ at the points $\{V_2, U_2\}$. Analogously are defined the points $\{V_1, U_1\}$ and $\{V_3, U_3\}$ (see figure 18). Next theorem lists some properties of these points.

Theorem 23. *Let $A'B'C'$ be the "orthic" triangle of ABC, with vertices the feet of the altitudes. Then the following are valid properties.*

1. *The sides of the orthic at* $\{A', B', C'\}$ *pass respectively through the points* $\{(U_1, V_1), (U_2, V_2), (U_3, V_3)\}$.
2. *The triangles* $\{A'A''V_1, \ldots, A'A''U_1, \ldots\}$ *are all similar to* ABC.

To show *nr-1* in a case, for example, to show that $A'C'$ passes through V_1, consider for the moment V_1 as the intersection of lines $\{A'C', A''B''\}$. Measuring the angles of $A'A''V_1$ we see that this triangle is similar to ABC and the quadrangle $A'V_1CB''$ is cyclic. All the angles at $\{A', A'', B''\}$ can be measured using standard properties of the orthic triangle and show that $B''CV_1$ is similar to ABC.

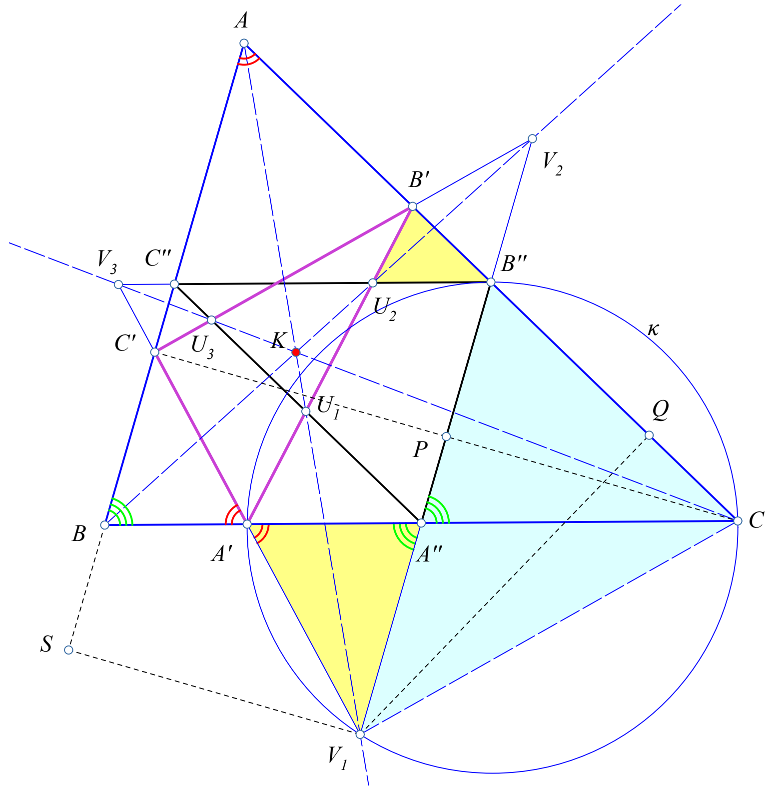

Figure 18: Intersections of sides of the medial and the orthic

Having that, project V_1 on the sides of ABC at points $\{S, Q\}$. Then we have:

$$\frac{V_1S}{V_1Q} = \frac{PC'}{V_1Q} = \frac{CP}{V_1Q} = \frac{CB''}{V_1B''} = \frac{AB}{AC},$$

proving that V_1 satisfies the characteristic property of the points of the symmedian, hence it is on the symmedian AK. Analogously are proved the other cases of this claim.
Nr-2 results, as alluded to before, by measuring the angles of these triangles. Figure 18 gives a hint for this in the case of the triangle $A'A''V_1$.

13 Vecten squares of the triangle

On the sides of triangle $\tau = ABC$ erect squares. These are the *"Vecten squares"* of the triangle. The triangle of the opposite sides of the squares $\tau' = A'B'C'$ has its sides parallel to those of τ, hence is similar to it. (see figure 19).

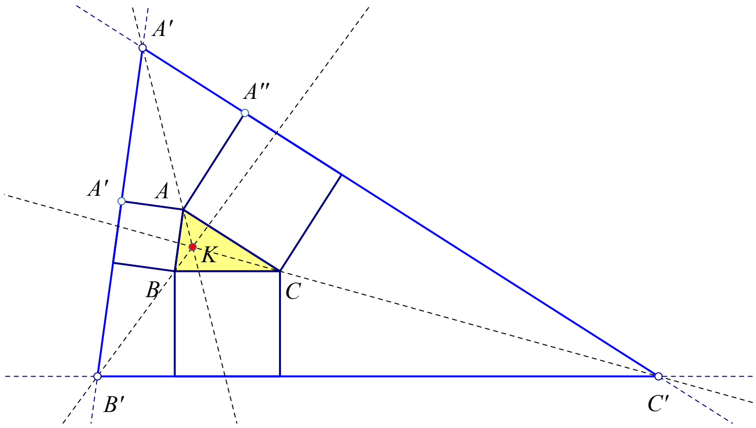

Figure 19: Vecten squares of the triangle

Theorem 24. *The similarity center of* $\{\tau, \tau'\}$ *is the symmedian point* K *of* τ, *which is also the symmedian point of* τ'.

The proof is immediately seen in the figure:

$$\frac{AA'}{AA''} = \frac{AB}{AC} = \frac{A'B'}{A'C'},$$

showing that A is on the symmedian line of τ' and τ.

The same result holds for the configuration created by erecting squares on the sides, each lying on the same side with the opposite vertex (see figure 20). The only difference in this case is that the two triangles ABC and $A'B'C'$ are related by an *"anti-homothety"* with center the symmedian point K of both triangles.

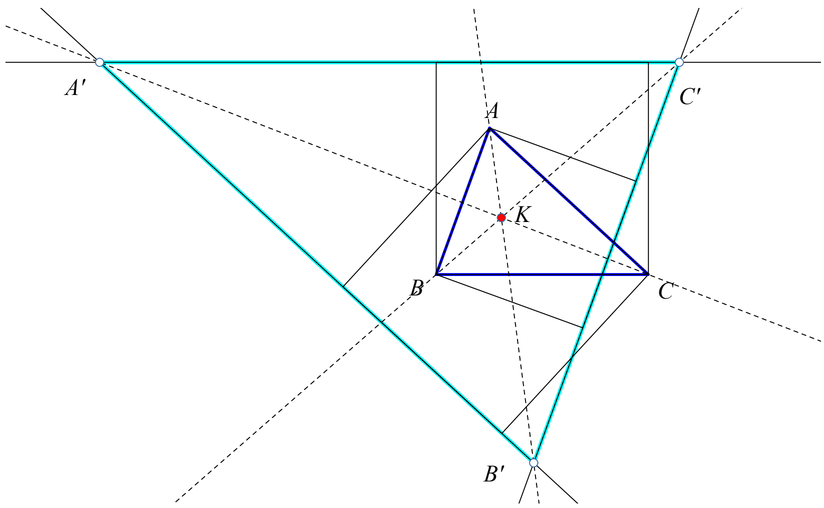

Figure 20: Inner Vecten squares of the triangle

14 Artzt parabolas

The following properties relate the symmedian AK to the Apollonian circle $\lambda_A(O_A)$, which is the locus of the points $\{X : XB/XC = b/c\}$ and lead also to a characterization

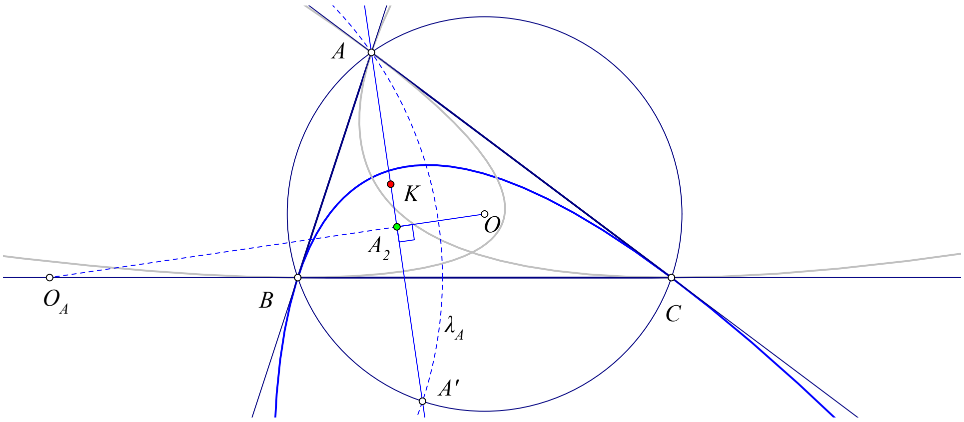

Figure 21: The Artzt parabolas of $\triangle ABC$

of the projection A_2 of the circumcenter on the symmedian AK as the focus of the *"A-Artzt parabola"* of the triangle ABC. By its definition, this parabola is tangent to the sides $\{AB, AC\}$ at the points $\{B, C.\}$ Analogously are defined the *B-Artzt parabola* tangent to the sides $\{BC, BA\}$ and the *C-Artzt parabola* tangent to $\{CA, CB\}$ (see figure 21). Notice that point A_2 is a vertex of the so called *"second Brocard triangle"* ([Cou80, p.279]). Its isogonal conjugate J appearing in the next theorem is a vertex of the *"fourth Brocard triangle"* [Gib21]).

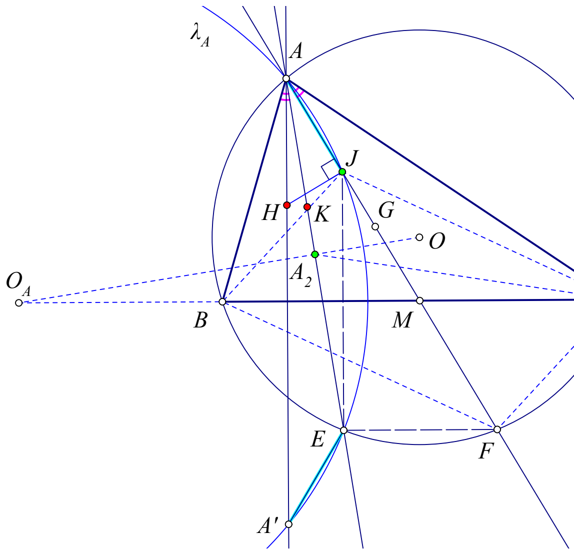

Figure 22: The projection A_2 of O on the symmedian AK

Theorem 25. *Referring to figure 22, we denote by $\kappa(O)$ the circumcircle of $\triangle ABC$ and consider the points: the second intersection A' of the symmedian AK with κ, the second intersection F of the median AM with κ and the reflection J of A' in BC.*

1. *$BA'FC$ is a trapezium and $A'F$ is parallel to BC.*
2. *Point A' is on the Apollonian circle $\lambda_A(O_A)$, satisfying $A'B/A'C = b/c$.*
3. *Point A_2 is the middle of AA', line OA_2 passes through O_A.*
4. *Point A_2 is the inverse of O w.r.t. to λ_A.*
5. *Point J is the second intersection of the Apollonian circle λ_A with the median AM.*

6. *$JBFC$ is a parallelogram and point J is the isogonal conjugate of A_2.*
7. *Line HJ is orthogonal to the median AM, where H is the orthocenter of $\triangle ABC$.*

Proof. *Nr-1* follows from the equality of the angles $\widehat{BAA'} = \widehat{FAC}$.
Nr-2: By *nr-1* and the similar triangles $\{MFC \sim MBA\ ,\ MFB \sim MCA\}$ we have

$$\frac{A'B}{A'C} = \frac{FC}{FB} = \frac{FC}{MC} \cdot \frac{MB}{FB} = \frac{AB}{AM} \cdot \frac{AM}{AC} = \frac{AB}{AC}\,.$$

Nrs 3-4 are a direct consequence of *nr-2*.
Nr-5 Follows from the fact that $A'MF$ is isosceles and $\widehat{JA'F}$ is a right angle. This implies that J is on the median AF. Obviously it is also on λ_A since BC is a diameter of this circle.
Nr-6: By *nr-4* point M is the middle of both segments $\{BC, JF\}$ implying that $JBFC$ is a parallelogram. By theorem 8 and the preceding claim these triangles are similar:

$$BA_2A\ \sim\ AA_2C\ \sim\ BA'C\ \sim\ CFB\ \sim BJC.$$

This implies $\widehat{A_2BA} = \widehat{JBC}\ \Rightarrow\ \widehat{JBA} = \widehat{A_2BC}$. Analogously is seen that $\widehat{A_2CB} = \widehat{JCA}$. Since we have also $\widehat{A_2AB} = \widehat{JAC}$ the claim is proved.
nr-7: This involves a computation in barycentrics (see file **Barycentric coordinates**) in which the points have corresponding coordinates

$$A(1,0,0)\ ,\ G(1,1,1)\ ,\ H(S_BS_C, S_CS_A, S_AS_B)\ ,\ J(a^2, 2S_A, 2S_A)\,.$$

We test the orthogonality formula for lines $S_App' + S_Bqq' + S_Crr' = 0$, where (p,q,r) and (p',q',r') are the points at infinity (their directions) of the lines $\{AG, HJ\}$ given by the respective triple products $\{(A \times G) \times G\ ,\ (H \times J) \times G\}$. The result is the expression $2(S_B + S_C - a^2)(S_BS_C + S_CS_A + S_AS_B) = 0$, according to the identity for the *"Conway symbols"* $S_B + S_C = a^2$ (see file **Conway symbols**). □

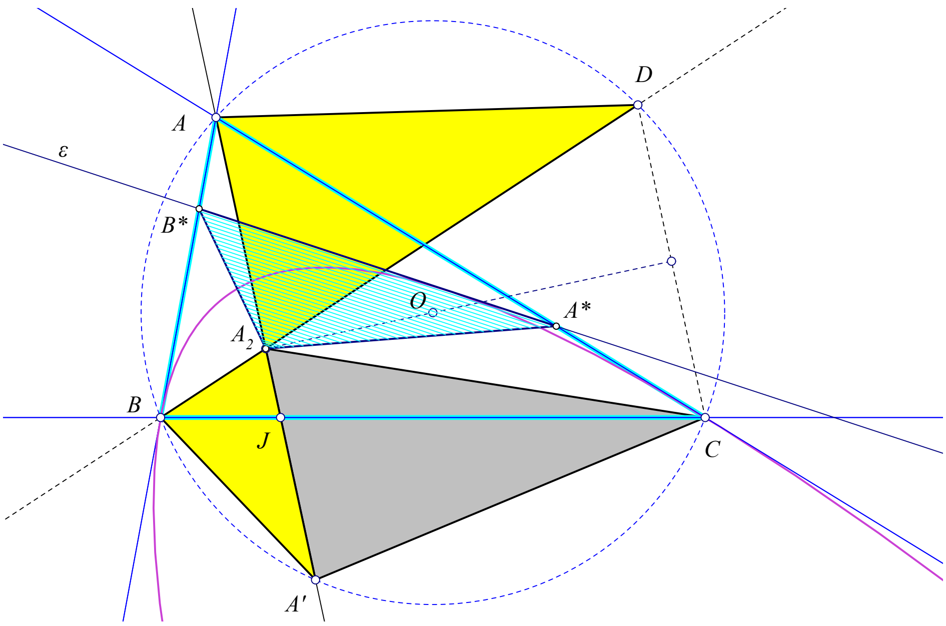

Figure 23: Generating the Artzt parabola by varying triangle $B^*A_2A^* \sim BA_2A$

Next theorem complements some properties discussed in theorem 8. Point D is defined as the second intersection of the parallel CD to the symmedian AA' with the circumcircle of triangle ABC. In that theorem was shown that the projection A_2 of the circumcenter on the symmedian AA', which is a vertex of the second Brocard triangle, was a point of the line BD (see figure 23).

Theorem 26. *Referring to figure 23, the following are valid properties:*

1. *Triangles A_2BA' and AA_2D are similar to ABC.*
2. *Triangle DA_2C is isosceles and triangles ADA_2 and $A'CA_2$ are equal.*
3. *Triangles A_2BA' and $A_2A'C$ are also similar.*
4. *Varying triangle BA_2A to a similar $B^*A_2A^*$ so that B^* moves on AB makes vertex A^* move on line AC and the line $\varepsilon = B^*A^*$ is tangent to the parabola.*

Nr-1. In fact, $\triangle ABC$ is similar to $\triangle A_2AD$. Later has $\widehat{ADA_2} = \widehat{ACB}$ and $\widehat{A_2AD} = \widehat{ABC}$, because $ADCA'$ is equilateral trapezium and $\widehat{A_2AD} = \widehat{A_2A'C} = \widehat{ABC}$. Obviously $\triangle A_2BA'$ is similar to A_2AD.

Nr-2 is obvious, since the triangles $\{A_2AD\ ,\ A_2A'C\}$ are each the reflection of the other in line OA_2.

Nr-3 is a consequence of *nr-2*.

Nr-4 is a general property of parabolas related to a triangle like ABC, which has two sides tangent to the parabola and the third side joins the contact points. In such a case it is well known ([Cha65, p.52]) that a line A^*B^* intersecting the tangents at points such that $AB^*/B^*B = CA^*/A^*A$ is tangent to the parabola (see file **Triangles tangent to parabolas**). The relation $AB^*/B^*B = CA^*/A^*A$ follows from the similar triangles $\{AB^*A_2 \sim CA^*A_2\}$.

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Related topics

1. **Barycentric coordinates**
2. **Brocard**
3. **Cross Ratio**
4. **Desargues' theorem**
5. **Pedal triangles**
6. **Projective line**
7. **The quadratic equation in the plane**

Any correction, suggestion or proposal from the reader, to improve/extend the exposition, is welcome and could be send by e-mail to: pamfilos@uoc.gr