Symmedian

by Paris Pamfilos

> The reason is not to glorify *"bit chasing"*; a more fundamental issue is at stake here: Numerical subroutines should deliver results that satisfy simple, useful mathematical laws whenever possible. . . . Without any underlying symmetry properties, the job of proving interesting results becomes extremely unpleasant. The enjoyment of one's tools is an essential ingredient of successful work.
>
> *Donald Knuth, Seminumerical Algorithms, section 4.2.2*

Contents

1 Symmedian

Here we discuss two related concepts: the *"symmedian line"* and the *"symmedian point"* or *"Lemoine point"* of a triangle $\tau = ABC$. The symmedian *line* of τ from vertex A is the symmetric of its median from A w.r.t. to the bisector from A. The symmedian *point* K is

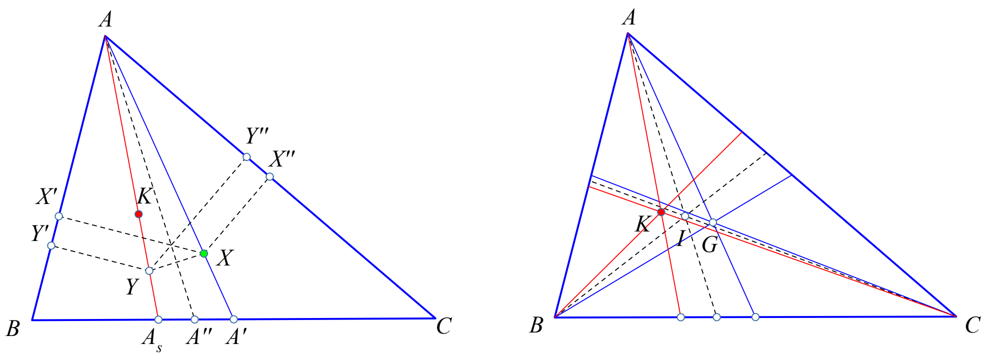

Figure 1: Symmedian line from A and symmedian point K

the intersection point of the symmedian lines from the three vertices (See Figure 1). Next theorem expresses a *"characteristic property"* of the points Y of a symmedian line.

Theorem 1. *The distances of points Y of the symmedian line AA_s from A, from the adjacent sides $\{AB, AC\}$, are analogous to these sides*:

$$\frac{YY'}{YY''} = \frac{AB}{AC}. \tag{1}$$

This follows from the symmetry w.r.t. to the bisector AA'' and the corresponding property of the points X of the median AA', by which, due to the equality of the areas of triangles $\{ABA', AA'C\}$, the analogous ratio *is inversely proportional to the adjacent sides*:

$$\frac{XX'}{XX''} = \frac{AC}{AB}.$$

Projecting the intersection point K of two of the symmedians on the sides of the triangles and applying the theorem, we see that the third symmedian passes also through that point, thus:

Theorem 2. *The three symmedians of the triangle pass through a common point K called the symmedian point of the triangle.*

2 Antiparallels

Closely related to the concept of *"symmedian"* is the concept of *"antiparallels"* to a side of the triangle. An antiparallel $\varepsilon = B'C'$ to side BC of the triangle is the line defined by any circle λ passing through B, C and intersecting the sides $\{AB, AC\}$ a second time at the points $\{C', B'\}$ (See Figure 2). There are two prominent antiparallels to BC :
1. The tangent ε_0 to the circumcircle at A.
2. The side $B''C''$ of the *"orthic triangle"*, defined by the feet of the altitudes of the triangle.
The tangent ε_0 is the limiting position for which the points $\{B', C'\}$ coincide with A and the circle λ coincides with the circumcircle κ of the triangle. The line $B''C''$ is antiparallel because the quadrangle $BCB''C''$ is *"cyclic"*.

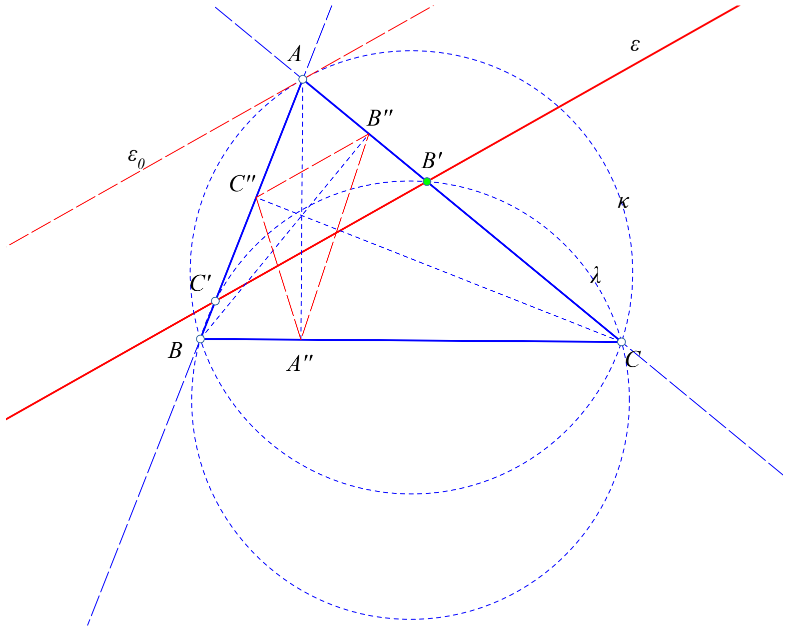

Figure 2: Antiparallels ε to the side BC

Theorem 3. *All triangles $AB'C'$ are similar to ABC.*

This results immediately from the corresponding angle equalities of the triangles:

$$\widehat{C'} = \widehat{C} \quad \text{and} \quad \widehat{B'} = \widehat{B}. \tag{2}$$

The symmetry on the bisector from A maps triangle $AB'C'$ to a triangle AB_1C_1 with side B_1C_1 parallel to BC hence the median line from A of $AB'C'$ maps to the median line of ABC. This proves the following:

Theorem 4. *The symmedian of ABC coincides with the median line of the triangles $\{AB'C'\}$.*

3 Harmonic quadrangle

Theorem 5. *The tangents at $\{B,C\}$ to the circumcircle of triangle ABC intersect at a point D on the symmedian AA_s.*

This follows from the characteristic property of the points of the symmedian line expressed by equation 1. Calculating the distances of D from the sides and using the sines-rule we get (See Figure 3):

$$\frac{DD'}{DD''} = \frac{DB\sin(\widehat{C})}{DC\sin(\widehat{B})} = \frac{\sin(\widehat{C})}{\sin(\widehat{B})} = \frac{AB}{AC}.$$

Theorem 6. *The intersection A_t of the tangent to the circumcircle at A with BC is the "harmonic conjugate" of A_s w.r. to $\{B,C\}$.*

This because the triangles $\{AA_tB, CA_tA\}$ are similar, implying (See Figure 3):

$$\frac{A_tC}{A_tB} = \frac{A_tC\cdot A_tB}{A_tB^2} = \frac{A_tA^2}{A_tB^2} = \frac{\sin(\widehat{B})^2}{\sin(\widehat{C})^2} = \frac{A_sC}{A_sB}.$$

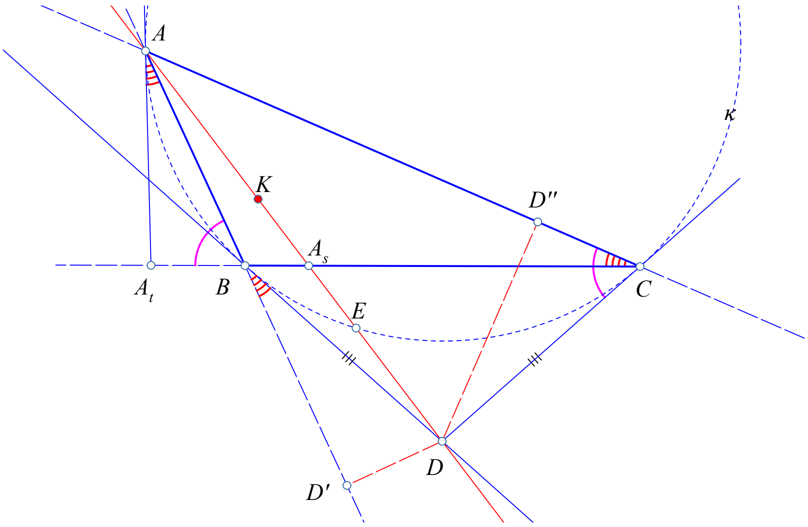

Figure 3: Tangents intersecting on the symmedian AA_s

It follows that the pencil of lines $A(B, C, A_s, A_t)$ is harmonic, hence it defines on the circumcircle a *"harmonic quadrangle"*, i.e. a circumscribed quadrangle for which the *"cross ratio"* $(BCAE) = (BCA_tA_s) = -1$. This kind of quadrangles is also characterized by the property of being cyclic and having *equal products of opposite sides.*

In the course of the previous proof we have also seen the following result:

Theorem 7. *The ratios XB'/XC' defined by the sides $\{AB, AC\}$ and the symmedian line from A on every parallel $B'XC'$ to the side BC are equal to the ratio $\sin(\widehat{C})^2/\sin(\widehat{B})^2$.*

Theorem 8. *The intersection E of the symmedian from A with the circumcircle of ABC defines a harmonic quadrangle $ABEC$.*

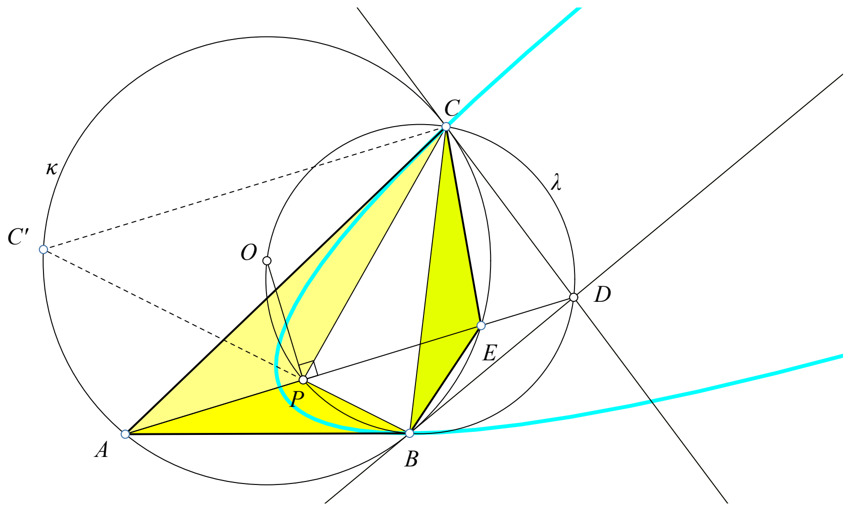

Figure 4: Property of the symmedian related to the harmonic quadrangle $ABEC$

Theorem 9. *The circle λ through $\{B, C\}$ and the intersection D of the tangents at B, C to the circumcircle $\kappa(O)$ of the triangle ABC passes through the center O and intersects the symmedian AD a second time at P, which defines triangles APC, BPC simmilar to BEC. In addition holds the relation $PA^2 = PB \cdot PC$.*

The angles $\{OPD, OCD\}$ are right showing that λ passes through O (See Figure 4). A similar angle chasing shows that the angles of the triangles at P, E are equal to the supplement of $\widehat{A}$. Note here that, since D is the middle of the arc BC on λ the symmedian AD bisects the angle $\widehat{BPC}$. Also $\widehat{ECB} = \widehat{BAP}$ completes the proof that $ABP \sim CBE$. Analogously $\widehat{EBP} = \widehat{PAC}$ completes the proof that $BEC \sim APC$.

For the proof of the last claim see that the parallel CC' to the symmedian AD and PB intersect on $C' \in \kappa$, hence $PB \cdot PC = PB \cdot PC' = PA^2$.

Figure 4 shows also a so-called *"Artzt parabola"* of the triangle ABC, which is the parabola passing through $\{B, C\}$ and being there tangent to the sides $\{AB, AC\}$. Its focus is point P.

4 Gergonne point of the triangle

The *"Gergonne point"* G_e of the triangle ABC is the intersection point of the lines $\{AA', BB', CC'\}$ joining the vertices with the contact points of the opposite sides with the

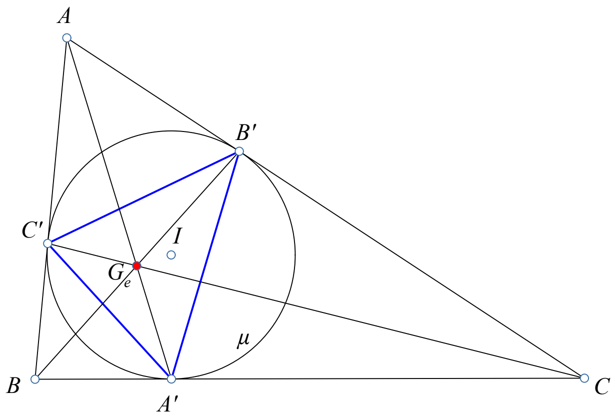

Figure 5: Gergonne point G_e of ABC = symmedian point of $A'B'C'$

"incircle" $\mu(I)$ of the triangle (See Figure 5). A direct consequence of theorem 5 is the following, which proves also the existence of this point.

Theorem 10. *The three lines $\{AA', BB', CC'\}$ are symmedians of the triangle $A'B'C'$ and the Gergonne point G_e of ABC coincides with the symmedian point of $A'B'C'$.*

The two triangles $\{ABC, A'B'C'\}$ being *"point perspective"* w.r. to point K, are also, according to *"Desargues' theorem"* (see file **Desargues' theorem**), *"line perspective"* i.e. corresponding sides meet on a line ε. This is, per definition the *"trilinear polar"* of K, called *"Lemoine line"* of the triangle (See Figure 6).
From theorem 6 follows the following:

Theorem 11. *Points $\{A'', B'', C''\}$ are the poles of lines, respectively $\{AA', BB', CC'\}$ relative to the circumcircle κ of ABC.*

By the duality of the polarity, this implies that K is the pole of the trilinear polar ε. Hence line OK is orthogonal to the trilinear polar. This is the so-called *"Brocard axis"* of the triangle (see file **Brocard points**).

By the fundamental property of *"Apollonian circles"* (see file **Apollonian circles**), by which these circles are orthogonal to κ, follows the next property:

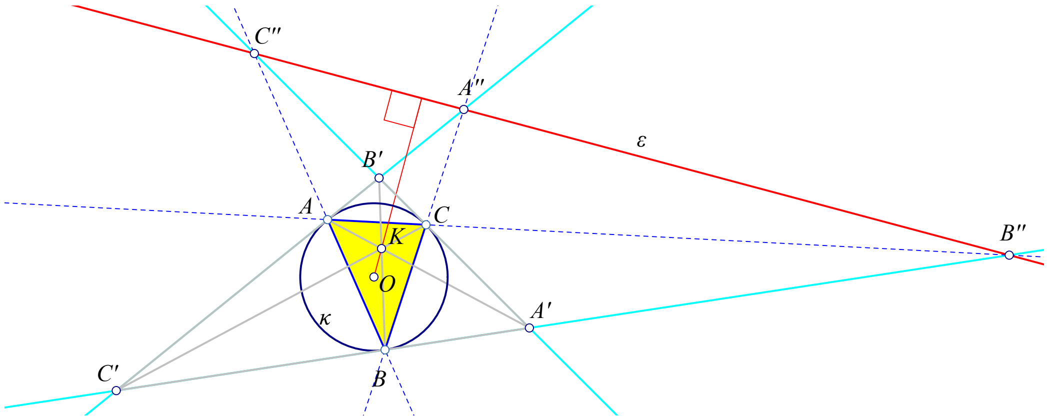

Figure 6: Lemoine line ε of triangle ABC

Theorem 12. *The points $\{A'', B'', C''\}$ are the centers of the Apollonian circles, passing correspondingly through the vertices $\{A, B, C\}$ of the triangle ABC.*

5 First Lemoine circle

The *"first Lemoine circle"* of the triangle ABC results by drawing parallels to the sides of the triangle from the symmedian point K.

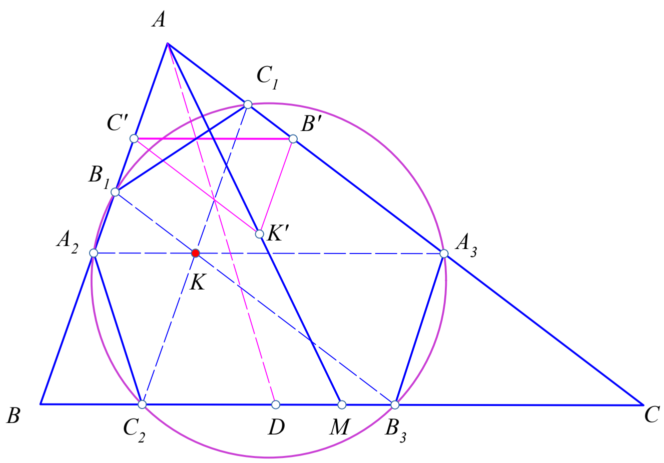

Figure 7: First Lemoine circle of the triangle ABC

Theorem 13. *The parallels to the sides of the triangle from the symmedian point intersect the sides in six concyclic points.*

The proof results from the fact that the three segments $\{B_1C_1, C_2A_2, A_3B_3\}$ joining the intersection points (See Figure 7) are antiparallel respectively to the sides $\{BC, CA, AB\}$. To see this for B_1C_1 reflect K on the bisector AD to the point K', which is on the median AM. Then, the whole parallelogram AB_1KC_1 is reflected to the congruent to it $AB'K'C'$, which has the diagonal $B'C'$ parallel to BC, thus proving that B_1C_1 is antiparallel to BC. Analogously is proved the same property for the segments $\{C_2A_2, A_3B_3\}$. This implies that the quadrangles

$$B_1C_1A_3A_2, \quad C_2A_2B_1B_3, \quad A_3C_1C_2B_3 \tag{3}$$

are cyclic. From this follows that $\widehat{C_1B_1A} = \widehat{BA_2C_2} = \widehat{C}$. This, in turn, implies that the trapezium $C_1B_1A_2C_2$ is isosceles and inscribed in the circumcircle of $C_1B_1A_2A_3$. From this follows that the three quadrangles (3) have pairwise three common points, consequently their circumcircles coincide and define the so called *"Lemoine circle"* of the triangle.

The lines $\{B_1C_1,\ C_2A_2,\ A_3B_3\}$ define also another triangle $A'B'C'$ with remarkable properties (See Figure 8).

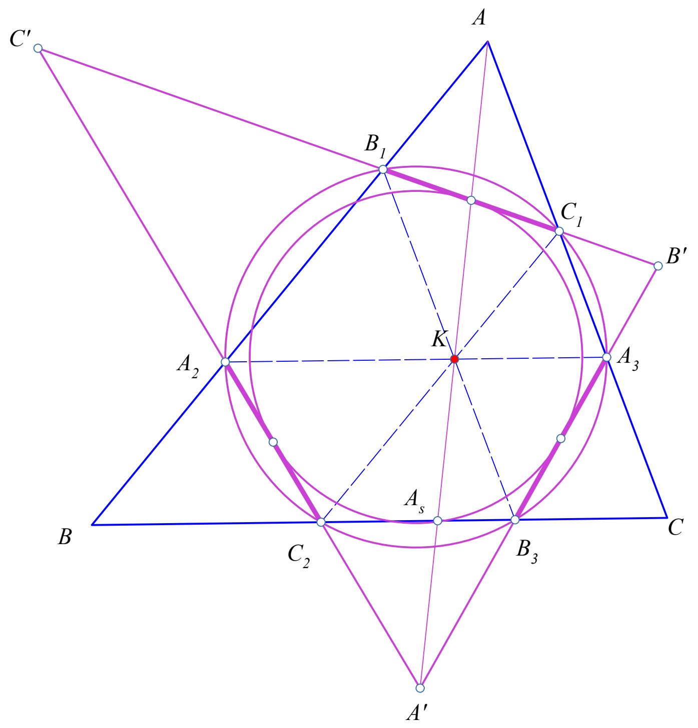

Figure 8: Additional properties related to the Lemoine circle

Theorem 14. *Related to the Lemoine circle of the triangle ABC and the triangle $A'B'C'$ are the following properties.*

1. *The segments $\{B_1C_1, C_2A_2, A_3B_3\}$ are equal.*
2. *The triangles $\{AB_1C_1, BC_2A_2, CA_3B_3\}$ are similar to ABC.*
3. *The triangles $\{A'B_3C_2, B'C_1A_3, C'A_2B_1\}$ are isosceli.*
4. *The incircle λ of $A'B'C'$ is concentric to the Lemoine circle κ of ABC .*
5. *The symmedians $\{AK, BK, CK\}$ pass respectively through $\{A', B', C'\}$.*
6. *The Gergonne point G_e of $A'B'C'$ coincides with the symmedian point K of ABC.*

nr-1, *nr-2* and *nr-3* are immediate consequences of theorem 13.
nr-4 follows from the equality of segments in *nr-1*, since these are equal chords of the Lemoine circle.
nr-5 is proved by showing that for the symmedian line AKA_s the ratios KA_2/KA_3 and A_sC_2/A_sB_3 are equal:

$$\frac{A_sC_2}{A_sB_3} = \frac{A_sB - C_2B}{A_sC - B_3C} = \frac{t \cdot KA_2 - KA_2}{t \cdot KA_3 - KA_3} = \frac{KA_2}{KA_3}.$$

Nr-6 is an immediate consequence of the previous properties.

Notice that the first Lemoine circle is a special case of the so called *"Tucker circles"* of the triangle (see file **Tucker circles**).

6 Adams' circle

The first Lemoine circle of ABC seen as a construct relative to the triangle $A'B'C'$ is the so-called *"Adams' circle"* of the triangle $A'B'C'$, reflecting a property, usually formulated as follows ([Hon95, p.63]):

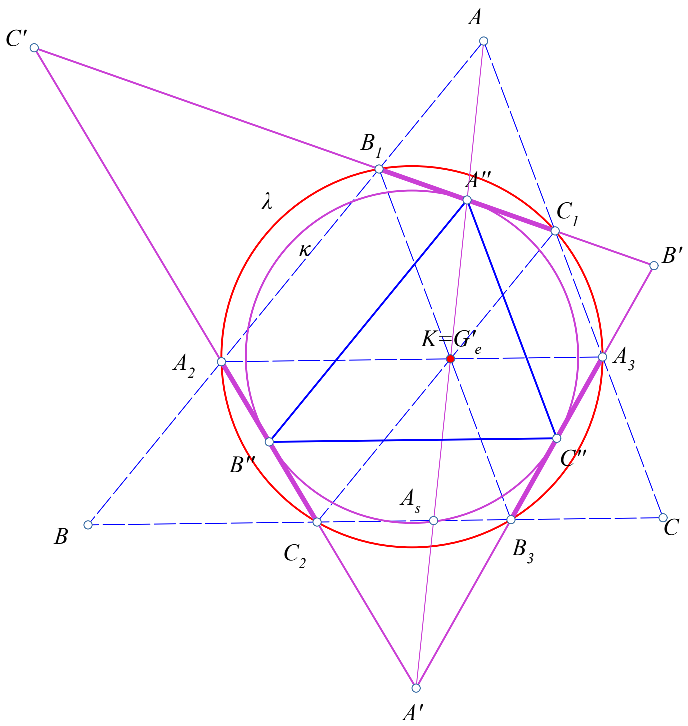

Figure 9: The Adams' circle of the triangle $A'B'C'$

Theorem 15. *The parallels to the sides of the cevian triangle $A''B''C''$ of the Gergonne point G'_e of the triangle $A'B'C'$ from point G'_e intersect the sides of $A'B'C'$ in 6 points lying on a circle λ concentric to the incircle κ of $A'B'C'$.*

The proof follows from the discussion in section 6 and the obvious fact that $A''B''C''$ is homothetic to ABC relative to the homothety with center $K = G'_e$ and ratio $1/2$.

7 Conics through parallels

This and the next section give another aspect of the first Lemoine circle. We start with a point D not lying on the side-lines of the triangle ABC. Of central importance for this aspect is the following property.

Theorem 16. *The parallels to the sides of a triangle ABC passing through a point D not lying on the side-lines of the triangle define respectively on the non-parallel sides six points lying on a conic.*

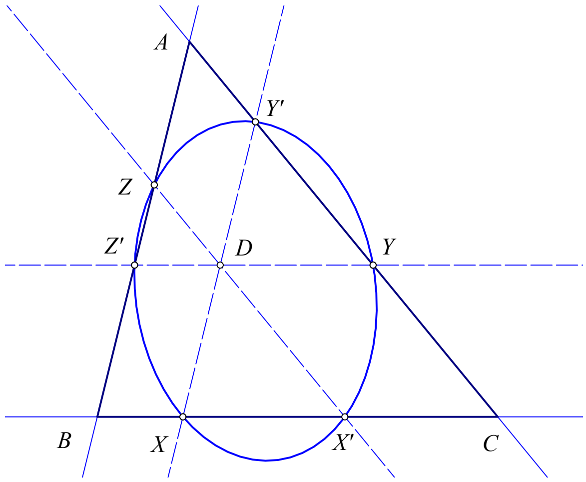

Figure 10: The conic defined by parallels to the sides from D

Proof. The proof is a trivial verification of *"Carnot's theorem"* ([Yiu13, p.117]), by which the intersection points $\{X, X', \ldots\}$ are on a conic if and only if the product of ratios

$$\frac{XB}{XC} \cdot \frac{X'B}{X'C} \cdot \frac{YC}{YA} \cdot \frac{Y'C}{Y'A} \cdot \frac{ZA}{ZB} \cdot \frac{Z'A}{Z'B} = 1.$$

In our case this product is easily seen to be equal to

$$\frac{a^2}{b^2} \cdot \frac{b^2}{c^2} \cdot \frac{c^2}{a^2} = 1,$$

where $\{a, b, c\}$ the side lengths of the triangle. Hence the points are indeed on a conic. □

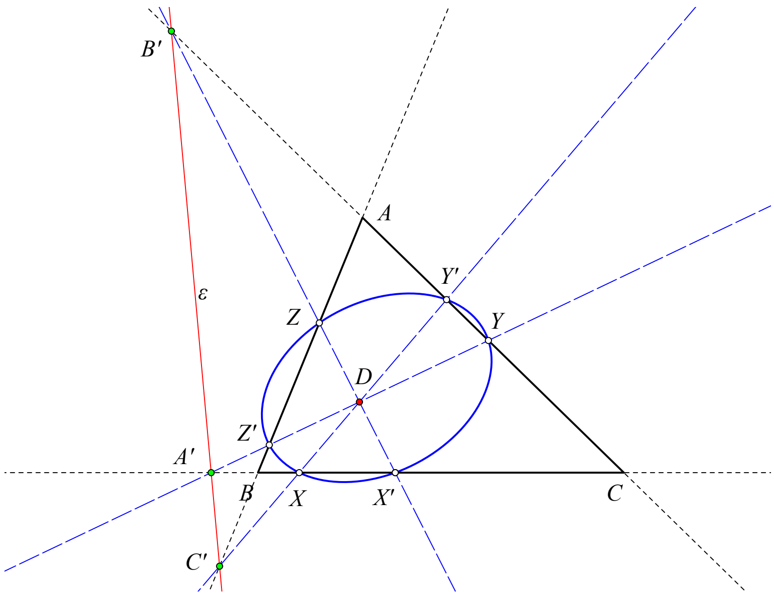

Figure 11: Conic defined by a point and a line

Notice that this property can be generalized in the following sense (See Figure 11).

Theorem 17. *Given a line ε intersecting the sides of the triangle ABC at the points $\{A', B', C'\}$, consider the intersections $\{X, X', Y, Y', Z, Z'\}$ of the sides of the triangle with the lines through $\{A', B', C'\}$ and an arbitrary point D. Then the six points lie on a conic.*

The proof of this is reduced to the one of the previous theorem by means of a *"projectivity"* leaving the points $\{A, B, C\}$ fixed and sending the line ε at the line at infinity.

8 Characterization of the first Lemoine circle

Next theorem characterizes the *"symmedian point"* of the triangle as the one for which the previous construction of conics delivers a circle. Here the symmedian point K of the triangle ABC is identified by its property to have distances from the sides proportional to these. Thus its *"trilinear coordinates"* are described by $K \cong (a : b : c)$.

Theorem 18. *With the notation and conventions of the previous section the conic is a circle if and only if the point D coincides with the symmedian point of the triangle ABC.*

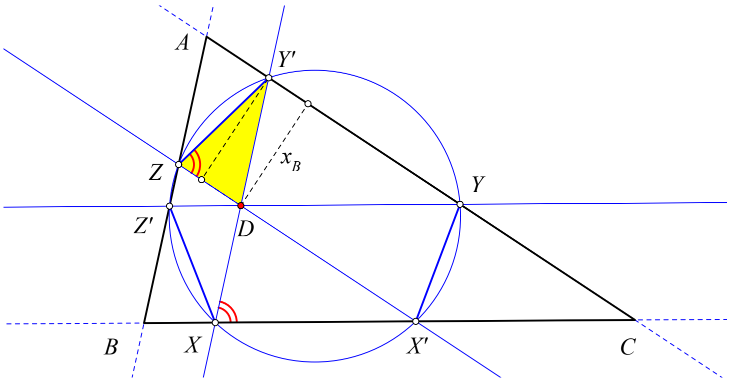

Figure 12: The conic is a circle

Proof. If the conic is a circle (See Figure 12), then the three trapezia $\{X'YY'Z, \ldots\}$ are equilateral and the three segments $\{X'Y, Y'Z, Z'X\}$ have equal lengths. By the equality of the inscribed angles seen in the figure follows that the distance x_B of D from AC is $x_B = ZY' \sin(\widehat{B})$. This implies immediately

$$(x_A : x_B : x_C) = (\sin(\widehat{A}) : \sin(\widehat{B}) : \sin(\widehat{C}) = (a : b : c),$$

thereby proving the necessity part of the theorem.

For the sufficiency consider the triangle DZY'. By assumption its altitudes satisfy

$$\frac{b}{c} = \frac{x_B}{x_C} = \frac{ZY' \sin(\widehat{Z})}{ZY' \sin(\widehat{Y'})} = \frac{\sin(\widehat{Z})}{\sin(\widehat{Y'})} = \frac{DY'}{DZ}.$$

Thus, the triangles $\{ABC, DZY'\}$, having the angles $\widehat{A} = \widehat{D}$ and the containing them sides proportional, are similar and the angles $\widehat{Z} = \widehat{B}$. Analogous argument shows that the angle $\widehat{YX'Z}$ is equal to $\widehat{B}$. This implies that $X'YY'Z$ is an isosceles trapezium inscribed in a circle passing through X. A similar argument shows the analogous property for the trapezia $Y'ZZ'X$ and $Z'XX'Y$. This implies that all six points $\{X, X', Y, Y', Z, Z'\}$ are on the same circle. □

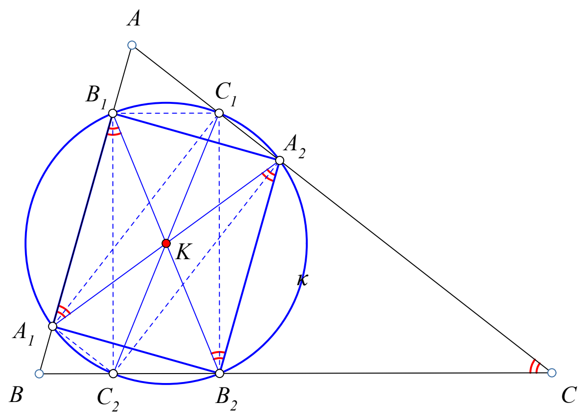

Figure 13: Second Lemoine circle of the triangle

9 Second Lemoine circle

The *"second Lemoine circle"* of the triangle ABC is a circle with center at the symmedian point K, whose existence follows from next theorem

Theorem 19. *(See Figure 13). On the antiparallels to the sides of the triangle ABC, which are drawn from the symmedian point K, the other sides define respectively equal segments $\{KA_1 = KA_2, \ldots\}$, which define diameters of a circle κ with center K.*

The quadrangles

$$A_1B_1A_2B_2, \quad B_1C_1B_2C_2, \quad C_1A_2C_2A_1 \tag{4}$$

are rectangles. For $A_1B_1A_2B_2$, using the fact that the quadrangles

$$A_1A_2CB, \quad B_1B_2CA, \quad C_1C_2BA$$

are cyclic, this follows from the equality of the angles noticed in figure 13. Analogously is proved the property for the other quadrangles in (4). The rectangles (4) have pairwise one common diagonal, and this proves the theorem.

10 Inscribed rectangles

In this section we consider rectangles inscribed in a given triangle, with one side coincident with a side of the triangle. Next theorem relates these rectangles with the symmedian point of the triangle (See Figure 14).

Theorem 20. *The centers M of the rectangles, inscribed in the triangle ABC and having one side on the line BC, are contained in a line ε, which passes through the middle N of BC, the middle L of the altitude AY and the symmedian point K of the triangle.*

In fact, the centers M of the rectangles are the middles of the segments PI joining the middles of opposite parallels $\{DH, ZE\}$. Since all triangles $\{PIN\}$ are similar, the claim that M describes a line passing through $\{N, L\}$ is clear. That this line passes also through K follows from the proof of theorem 19, where we saw that there is such a rectangle with center at K.

Theorem 21. *The symmedian point K of the triangle is the common point of the lines joining the middles $\{L_i\}$ of the altitudes with the middles $\{N_i\}$ of the opposite sides of the triangle ABC.*

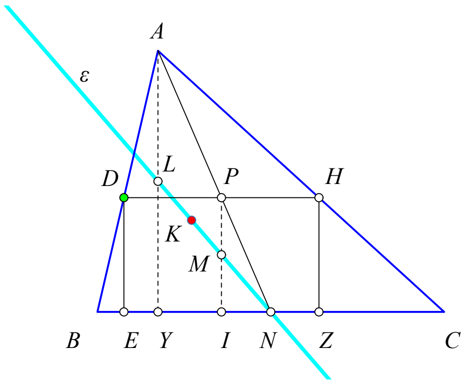

Figure 14: Inscribed rectangles

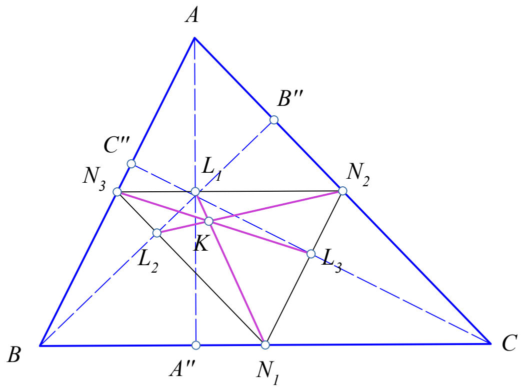

Figure 15: Alternative definition of the symmedian point

This is a direct corollary of theorem 20. Notice that this point K is the *"triangle center"* $X(69)$ w.r. to the medial triangle $N_1N_2N_3$ ([Kim18]).

11 Medial and Orthic triangle intersections

Consider the *"medial triangle"* $A''B''C''$ with vertices the middles of the sides of triangle ABC. Each symmedian line of ABC intersects the sides of the medial triangle, which are adjacent to the opposite vertex of the corresponding parallelogram at two points. For example the symmedian BK intersects the sides $\{B''A'', B''C''\}$ at the points $\{V_2, U_2\}$. Analogously are defined the points $\{V_1, U_1\}$ and $\{V_3, U_3\}$ (See Figure 15). Next theorem list some properties of these points.

Theorem 22. *Let $A'B'C'$ be the "orthic" triangle of ABC, with vertices the feet of the altitudes. Then the following are valid properties.*

1. *The sides of the orthic at $\{A', B', C'\}$ pass respectively through $\{(U_1, V_1), (U_2, V_2), (U_3, V_3)\}$.*
2. *The triangles $\{A'A''V_1, \ldots, A'A''U_1, \ldots\}$ are all similar to ABC.*

To show *nr-1* in a case, for example, to show that $A'C'$ passes through V_1 , consider for the moment V_1 as the intersection of lines $\{A'C', A''B''\}$. Measuring the angles of $A'A''V_1$ we see that this triangle is similar to ABC and the quadrangle $A'V_1CB''$ is cyclic. All the angles at $\{A', A'', B''\}$ can be measured using standard properties of the orthic triangle and show that $B''CV_1$ is similar to ABC.

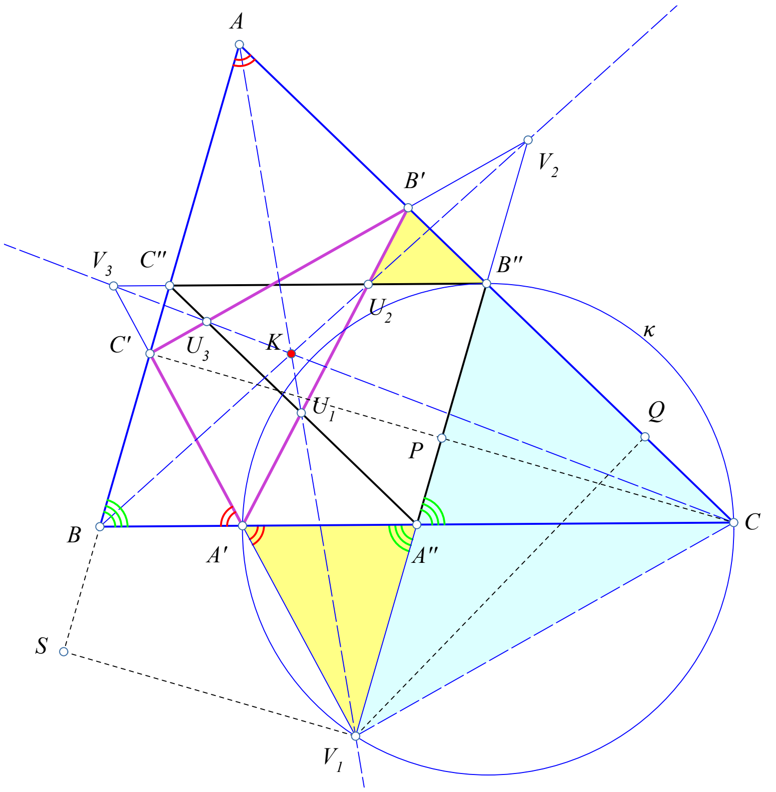

Figure 16: Intersections of sides of the medial and the orthic

Having that, project V_1 on the sides of ABC at points $\{S, Q\}$. Then we have:

$$\frac{V_1S}{V_1Q} = \frac{PC'}{V_1Q} = \frac{CP}{V_1Q} = \frac{CB''}{V_1B''} = \frac{AB}{AC},$$

proving that V_1 satisfies the characteristic property of the points of the symmedian, hence it is on the symmedian AK. Analogously are proved the other cases of this claim. *Nr-2* results, as alluded to before, by measuring the angles of these triangles. Figure 16 gives a hint for this in the case of the triangle $A'A''V_1$.

12 Vecten squares of the triangle

On the sides of triangle $\tau = ABC$ erect squares. These are the *"Vecten squares"* of the triangle. The triangle of the opposite sides of the squares $\tau' = A'B'C'$ has its sides parallel to those of τ, hence is similar to it. (See Figure 17).

Theorem 23. *The similarity center of $\{\tau, \tau'\}$ is the symmedian point K of τ, which is also the symmedian point of τ'.*

The proof is immediately seen in the figure:

$$\frac{AA'}{AA''} = \frac{AB}{AC} = \frac{A'B'}{A'C'},$$

showing that A is on the symmedian line of τ' and τ.

The same result holds for the configuration created by erecting squares on the sides, each lying on the same side with the opposite vertex (See Figure 18). The only difference in this case is that the two triangles ABC and $A'B'C'$ are related by an *"anti-homothety"* with center the symmedian point K of both triangles.

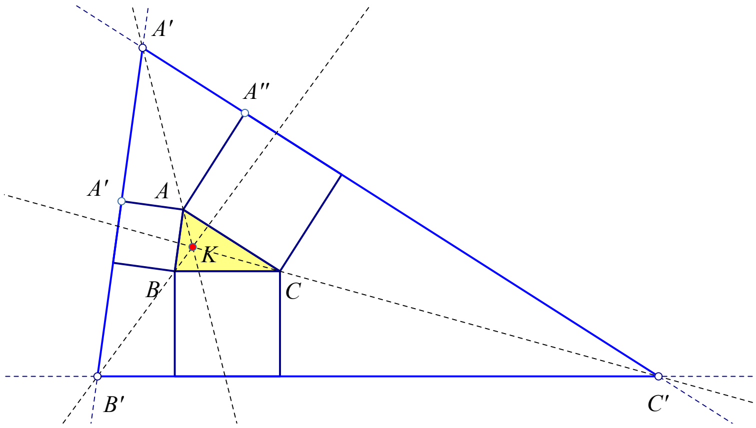

Figure 17: Vecten squares of the triangle

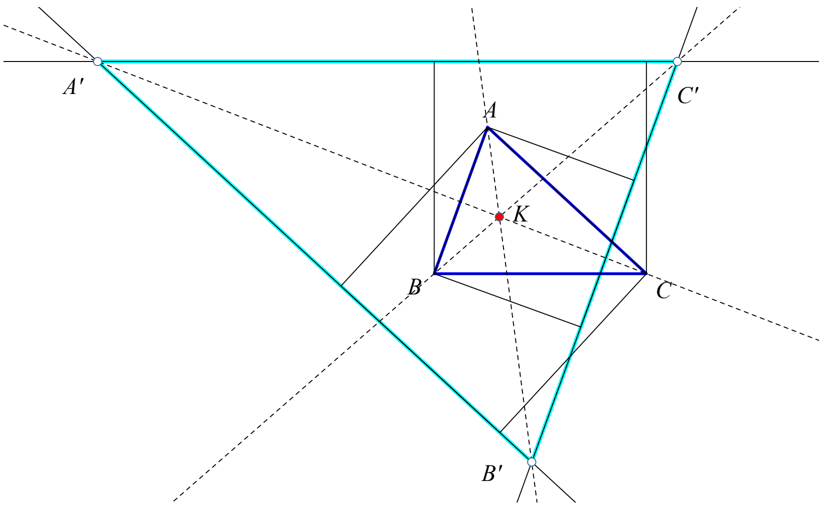

Figure 18: Inner Vecten squares of the triangle

13 A point on the symmedian line

The following properties of the symmedian line from A characterize point point E, which is the focus of the *"A-Artzt parabola"* of the triangle ABC . From its definition this parabola is tangent to the sides $\{AB, AC\}$ at points $\{B, C.\}$ To define the point, draw from C the parallel CD to the symmedian AJ and draw also the line BD intersecting the symmedian at E (See Figure 19).

Theorem 24. *Referring to figure 19, the following are valid properties:*

1. *Triangles EBA' and AED are similar to ABC.*
2. *Triangles EBA' and $EA'C$ are also similar.*
3. *Triangle DEC is isosceles and triangles ADE and $A'CE$ are equal.*
4. *E is the middle of AA' and $AE^2 = BE \cdot ED = BE \cdot EC$.*
5. *$\widehat{BEC} = 2\widehat{A}$ and AE bisects $\widehat{BEC}$.*

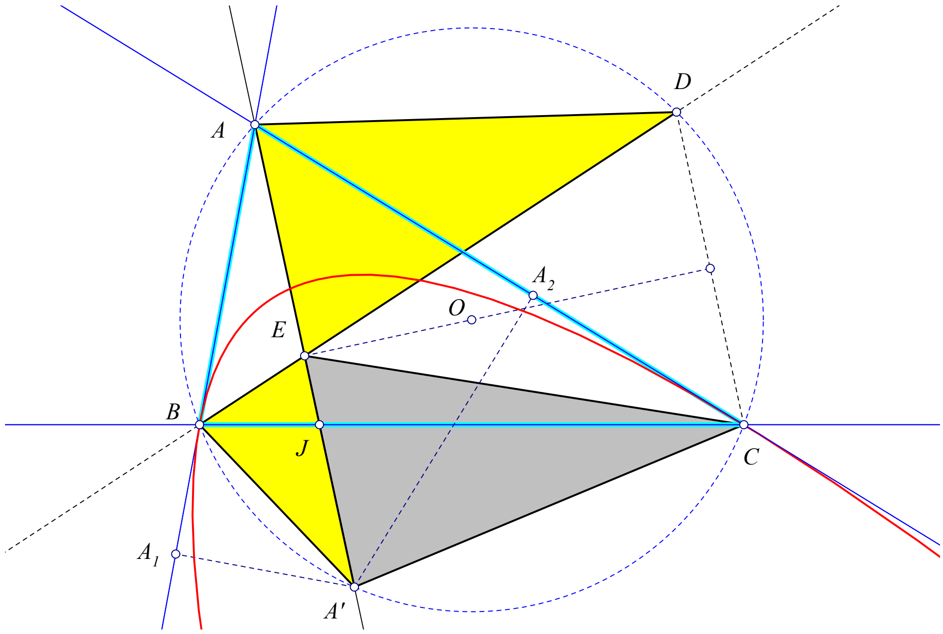

Figure 19: Focus E of *"Artzt parabola"*

6. *Triangles BEA and AEC are also similar. In particular $\widehat{ABE} = \widehat{EAC}$ and rotating triangle ABE about E so that B moves on AB and the triangle remains all the time similar to BEA, makes vertex A move then on line AC*

Nr-1. In fact, EBA' is similar to EAD. Later has $\widehat{ADE} = \widehat{ABC}$) and $\widehat{EAD} = \widehat{BAC}$), because $ADCA'$ is equilateral trapezium and $\widehat{EAD} = \widehat{EA'C} = \widehat{ABC}$. Notice that this property is valid for every cevian AA' and its parallel CD and the resulting construction. Nowhere in the proof was needed the symmedian property of AA'. But now it comes.

Nr-2. In fact they have both $\widehat{EBA'} = \widehat{EA'C}$ and corresponding sides satisfying the relation $A'B/AC = A'A_1/A'A_2 = c/b$, points A_1, A_2 being the projections of A' on the sides and using the characteristic property of the points on the symmedian.

Nrs 3-6. By *nr-2* we have also $EB/EA' = c/b$. This proves the similarity of EBA and $EA'C$, the other properties following trivially.

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