A Characterization of the Focals of Hyperbolas

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Abstract
In this article we prove a property characterizing the focal points of hyperbolas.

1 Chords through a point

The property which we discuss here relates to the tangents of a hyperbola at the end points of a chord and their intersections with the asymptotes of the hyperbola. It is formulated by the following lemma.

\[ \text{Lemma 1. If the tangents to the hyperbola at the end points of a chord } AB \text{ intersect the asymptotes respectively at points } \{A_1, A_2\} \text{ and points } \{B_1, B_2\}. \text{ Then } \{A_1B_1, A_2B_2\} \text{ are parallels and } AB \text{ is their middle-parallel.} \]

Proof. The proof of the lemma, in the case \(AB\) runs in the inner domain of the hyperbola (See Figure 1), derives from the equality of the areas of the triangles \( \{A_1A_2B_1, A_1B_2B_1\} \), which have in common the area of the triangle \(A_1P'B_1\), and are complemented by the equal areas of the triangles \( \{P'A_2B_1, P'B_2A_1\}\) ([3, III.43, p. 112], [5, p.192]), point \(P'\) being the intersection of the tangents. The claim about the middle-parallel follows from the equally well known property ([3, II.3, p. 56], [4, fig. 10.18, p. 315], [5, p.191]), that \(\{A, B\}\) are respectively the middles of \(\{A_1A_2, B_1B_2\}\). The proof, when \(AB\) runs in the outer domain of the hyperbola is completely analogous\(^1\).

\(^1\)At this point I would like to express my gratitude to the referee, who kindly suggested not only the references to the classical literature, but also a complete alternative proof to the main theorem. I hope to see this proof, as well as some other, possibly better or simpler proofs, from interested readers, published in this journal.
2 The property of focal points

Next theorem, characterizes the focal points \( \{F, F'\} \) of the hyperbola by measuring the distance of the parallels \( \{A_1B_1, A_2B_2\} \), as the chord \( AB \) turns about a fixed point \( P \).

**Theorem 1.** Under the notation and conventions made above, for chords passing through a fixed point \( P \), the distance between the parallels \( \{AB, A_1B_1\} \) is variable, depending on their direction, except when \( P \) is a focal point. In the case \( P \) is a focal point, this distance is independent of the direction and equal to the conjugate axis \( b \) of the hyperbola.

![Diagram](https://via.placeholder.com/150)

Figure 2: Triangle formed by the segments cut on the asymptotes

**Proof.** To prove this, we represent the hyperbola with its canonical coordinates in the form

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.
\]

We consider also the quadratic equation, giving the product of the tangents from the point \( P'(x_1, y_1) \). This can be seen to be ([2, p.251, I])

\[
(xy_1 - x_1y)^2 = a^2(y - y_1)^2 - b^2(x - x_1)^2.
\]

The intersection points \( \{A_2, B_1\} \) and \( \{B_2, A_1\} \) of these lines with the asymptotes are found by solving the systems consisting of the previous equation and the equation of each asymptote \( x/a - y/b = 0 \) and \( x/a + y/b = 0 \) (See Figure 2). These are found to be

\[
A_2, B_1 = \frac{-ab + g}{ay_1 - bx_1} (a, b) \quad \text{and} \quad B_2, A_1 = \frac{ab + g}{ay_1 + bx_1} (a, -b),
\]

where, \( g = g(x_1, y_1) = \sqrt{a^2y_1^2 - b^2x_1^2 + a^2b^2} \). This implies that

\[
|A_2B_1|^2 = \frac{4g^2(a^2 + b^2)}{(ay_1 - bx_1)^2} \quad \text{and} \quad |B_2A_1|^2 = \frac{4g^2(a^2 + b^2)}{(ay_1 + bx_1)^2}.
\]

The required distance \( h \) of the parallels can be measured from the altitude of the triangle \( P_2A'B' \), resulting by parallel translating at an arbitrary point \( P_2 \) the segments
\{A_2 B_1, B_2 A_1\}. Since the property under consideration is invariant by similarities, we can assume that \(a^2 + b^2 = 1\). Thus, using the well known formula, deriving from the area of a triangle, \(h = \frac{b'e'\sin(\omega)}{a'}\), we find that
\[
h^2 = \frac{b'^2 e'^2 \sin(\omega)^2}{a'^2} = \frac{2(a^2 y_1^2 - b^2 x_1^2 + a^2 b^2) \sin(\omega)^2}{a^2 y_1^2 + b^2 x_1^2 + (a^2 y_1^2 - b^2 x_1^2) \cos(\omega)},
\]
where \(\omega\) is the angle of the asymptotes. Taking into account that \(\sin(\omega) = 2ab\), and \(\cos(\omega) = a^2 - b^2\), we obtain the simplified expression
\[
h^2 = \frac{4a^2 b^2 a^2 y_1^2 - b^2 x_1^2 + a^2 b^2}{a^4 y_1^2 + b^4 x_1^2}.
\]
Letting the chord \(AB\) revolve about \(P(x_0, y_0)\), the corresponding point \(P'(x_1, y_1)\) moves on the polar line \(\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1\) of \(P\) ([1, p.192]), a particular point of which is \(K_2(x_2, y_2) = (a^2/x_0, 0)\). A parametric form of the polar is consequently given by
\[
x_1 = \frac{a^2}{x_0} + t\frac{y_0}{b^2}, \quad y_1 = t\frac{x_0}{a^2}.
\]
Introducing this into equation-(5) and simplifying, we obtain
\[
h^2 = 4 \frac{p(t)}{q(t)}, \quad \text{with}
\]
\[
p(t) = t^2[-x_0^2(a^2 y_0^2 - b^2 x_0^2)] + t[-2a^4 b^2 x_0 y_0] + [a^4 b^4(x_0^2 - a^2)],
\]
\[
q(t) = t^2[x_0^2(y_0^2 + a^2)] + t[2a^2 b^2 x_0 y_0] + [a^4 b^4];
\]
The condition of constancy of \(h^2\) is equivalent with the vanishing of coefficients of the quadratic equation \(p(t) - kq(t)\), for a constant \(k\), which implies the equations
\[
x_0^2(y_0^2(a^2 + k) - x_0^2(b^2 - k)) = 0
\]
\[
(a^2 + k)x_0 y_0 = 0,
\]
\[
(x_0^2 - a^2) - k = 0.
\]
the two last equations, for \(x_0 y_0 \neq 0\) lead to contradiction. The condition \(x_0 = 0\), leads also to the contradiction \(h^2 = -4a^2\). Thus, if a point \((x_0, y_0)\) has the stated property, it must satisfy \(y_0 = 0, x_0 \neq 0\), implying \(k = b^2 = (x_0^2 - a^2)\), hence \(x_0^2 = 1\), which determines the position of a focal point \(F(\pm 1, 0)\) and the value for \(h^2 = 4b^2\), which proves the theorem.

\[\square\]

References


