On Convergence of Sequences of Measurable Functions

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Abstract

In order to study the three basic kinds of convergence (in measure, almost every where, almost uniformly) of a sequence (f_n) of measurable functions, we define new conditions with respect to the sequence (f_n) under which one kind of the above convergence implies another one. Also we study in which measure spaces almost uniform convergence coincide with almost everywhere convergence and in which measure spaces convergence in measure implies almost everywhere convergence.

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1 Introduction

Throughout this paper (Γ, Σ, μ) will denote a measure space, $f_n (n \in \mathbb{N})$, $f : \Gamma \to \mathbb{R}$ measurable functions. Also we adopt the notations $f_n \xrightarrow{\mu} f$, $f_n \xrightarrow{\mu - ae} f$, $f_n \xrightarrow{a\ell - u} f$ to mean respectively that the sequence (f_n) converges in measure, almost everywhere, almost uniform to f.

Each pair $((f_n), f)$ induces a double sequence of measurable sets $A_n^j((f_n), f)$ or simply A_n^j, $n, j \in \mathbb{N}$, where $A_n^j = \left\{\gamma \in \Gamma | \ |f_n(\gamma) - f(\gamma)| \geq \dfrac{1}{j}\right\}$, determining the behavior of the pair $((f_n), f)$ with respect to converges. More precisely we have the following well known results:

(a) $f_n \xrightarrow{\mu} f \Leftrightarrow$ For each $j \in \mathbb{N}$ $\quad \lim\limits_{n \to \infty} \mu(A_n^j) = 0$;

(b) $\{\gamma \in \Gamma | f_n(\gamma) \nrightarrow f(\gamma)\} = \bigcup\limits_{j=1}^{\infty} \Big(\bigcap\limits_{n=1}^{\infty} E_n^j \Big)$, where

$$E_n^j = \bigcup_{k=n}^{\infty} A_k^j = \left\{\gamma \in \Gamma | \text{ there exists } k \geq n : |f_k(\gamma) - f(\gamma)| \geq \frac{1}{j}\right\};$$

(c) $f_n \xrightarrow{\mu-ae} f \Leftrightarrow$ For each $j \in \mathbb{N}$ $\quad \mu\left(\bigcap_{n=1}^{\infty} E_n^j\right) = 0;$

(d) $f_n \xrightarrow{a\ell-u} f \Leftrightarrow$ For each $j \in \mathbb{N}$ $\quad \lim_{n\to\infty} \mu(E_n^j) = 0.$

(For the proof of (d) see [?]).

The above results will be used in §2, where we provide new conditions for the pairs $((f_n), f)$ under which one kind of the above convergence, of (f_n) to f, implies another of those. Also we prove that in every measure space convergence of a sequence (f_n) to f in measure is equivalent to the condition: For each increasing sequence (k_n) in $\mathbb{N}$ there exists a subsequence (m_{k_n}) of (k_n) such that $f_{m_{k_n}} \xrightarrow{a\ell-u} f$.

In the sequel we will need also the following obvious properties of (A_n^j), (E_n^j):

(e) (A_n^j) is nondecreasing with respect to j for all $n \in \mathbb{N}$.

In §2 we will see that each double sequence (A_n^j) of measurable sets nondecreasing with respect to j is induced by some pair $((f_n), f)$.

(f) (E_n^j) is nondecreasing with respect to j for all $n \in \mathbb{N}$ and nonincreasing with respect to n for all $j \in \mathbb{N}$.

(c) and (d) imply that the only case for which $\mu - ae$ convergence does not imply $a\ell - u$ convergence is the following: For all $j \in \mathbb{N}$, $\mu\left(\bigcap_{n=1}^{\infty} E_n^j\right) = 0$ and there exists $j \in \mathbb{N}$ such that $\mu(E_n^j) = \infty$ for all $n \in \mathbb{N}$.
(Since, if for all $j \in \mathbb{N}$ there exists $n \in \mathbb{N}$ with $\mu(E_n^j) < \infty$, then by (f), we have $\lim_{n\to\infty} \mu(E_n^j) = \mu\left(\bigcap_{n=1}^{\infty} E_n^j\right) = 0$, and this implies, by (d), that $f_n \xrightarrow{a\ell-u} f$). We thus arrive naturally to the following definitions (see [?]).

Definitions.

I. The pair $((f_n), f)$ satisfies the finite restriction property (f.r.p) if for each $j \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ with $\mu(E_{n_0}^j) < \infty$; and

II. The sequence (f_n) has the finite restriction property (f.r.p) if for each $j \in \mathbb{N}$, there exist $n_0 \in \mathbb{N}$ such that $\mu\left(\left\{\gamma \in \Gamma | \exists\, k, m \geq n_0 : |f_k(\gamma) - f_m(\gamma)| \geq \frac{1}{j}\right\}\right) < \infty$.

It is easy to see that if $f_n \xrightarrow{\mu-ae} f$ and (f_n) has the (f.r.p), then the pair $((f_n), f)$ satisfies the (f.r.p) (see [?]).

These definitions, together with (c) and (d) above, help us to state, in (g) below a generalization of Egoroff's theorem; this will be used in §3, where we examine conditions on the whole space (Γ, Σ, μ), such that one kind of convergence implies another.

(g) The following are equivalent:

(i) $f_n \xrightarrow{a\ell-u} f$

(ii) $f_n \xrightarrow{\mu-a\ell} f$ and either $((f_n), f)$ or (f_n) has the (f.r.p).

We note that (g) provides a gneralization of Lebesgue's theorem: if $f_n \xrightarrow{\mu-ae} f$ and either $((f_n), f)$ or (f_n) has the (f.r.p) then $f_n \xrightarrow{\mu} f$.

We recall the following standard definitions and results (h)-(j):

Definitions.

1. A set $A \in \Sigma$ is called an atom if $\mu(A) > 0$, and for $B \in \Sigma$ with $B \subseteq A$ we have that either $\mu(B) = 0$ or $\mu(A - B) = 0$.

2. The space (Γ, Σ, μ) is purely atomic if for each $B \in \Sigma$ with $\mu(B) > 0$, there exists an atom A such that $A \subseteq B$.

3. A set $E \in \Sigma$ is atomless if $\mu(E) > 0$ and E contains no atoms.

(h) If A is an atom and $f : \Gamma \to \mathbb{R}$ measurable, then f is constant $\mu - ae$ on A.

(i) If E is atomless and $\mu(E) < +\infty$ then for each $\alpha \in \mathbb{R}$, with $0 \leq \alpha \leq \mu(E)$, there exists $F \subseteq E$, $F \in \Sigma$ with $\mu(F) = \alpha$.

(j) If (Γ, Σ, μ) is a finite measure space, the following are equivalent:

(i) μ-convergence coincides with $\mu - ae$ convergence;

(ii) Γ is the countable union of pairwise disjoint sets which are either atoms with respect to μ or empty.

(For the proof of (j) see [**?**] and for related topics see [**?**], [**?**] and [**?**]).

We will formulate in §3 a generalization of (j) for arbitrary measure space.

We note that in [**?**] condition $[M]$ is defined to mean that $(A_n^j)_n$ is nonincreasing with respect to n for all $j \in \mathbb{N}$. This is a very strong condition, which, in view of (a), (b) and (d), guarantees at once that μ-convergence implies $a\ell - u$ convergence. In §2 we introduce a weaker condition, which implies the same conclusion.

2 New conditions for convergence

We begin with a lemma, which will be useful in constructing counterexamples regarding convergence.

2.1 Lemma. *If (A_n^j) is a double sequence of measurable sets, nondecreasing with respect to j for all $n \in \mathbb{N}$, then there exists a pair $((f_n), f)$ such that*

$$A_n^j = A_n^j((f_n, f) \quad \textit{for all} \quad n, j \in \mathbb{N}.$$

Proof. Let (A_n^j) be a double sequence of measurable sets such that $A_n^1 \subseteq A_n^2 \subseteq \cdots$ for all $n \in \mathbb{N}$. We set $f_n = \chi_{A_n^1} + \frac{1}{2}\chi_{A_n^2 - A_n^1} + \frac{1}{3}\chi_{A_n^3 - A_n^2} + \cdots$ for $n \in \mathbb{N}$.

Then f_n is well defined, since for each $\gamma \in \Gamma$ only one term of the above series is different from 0 and clearly $A_n^j = A_n^j((f_n, f)$ for all $n, j \in \mathbb{N}$, where $f = 0$. □

Now, paying attention on individual pairs $((f_n), f)$, we have the following

2.2 Definition. The pair $((f_n), f)$ satisfies the condition (M') if for each $j \in \mathbb{N}$ there exists $j' > j$, $j' \in \mathbb{N}$, with the j' line $(A_1^{j'}, A_2^{j'}, \dots)$ of the double sequence (A_n^j) satisfying the following condition:
For each $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$, with $m_n \geq n$, such that

$$A_k^{j'} \subseteq A_{m_n}^{j'} \quad \text{for all} \quad k \geq m_n.$$

2.3 Proposition. *If $f_n \xrightarrow{\mu} f$ and the pair $((f_n), f)$ satisfies the condition (M'), then*

$$f_n \xrightarrow{a\ell - u} f.$$

Proof. Let $\varepsilon > 0$ and $j \in \mathbb{N}$. According to (d) of §1, it is enough to find n_0 such that $\mu(E_n^j) < \varepsilon$ for $n \geq n_0$. By hypothesis, there exists $j' > j$, $j' \in \mathbb{N}$ with the j' line of the double sequence (A_n^j) satisfying the Definition **??**. Also there exists $N = N(j', \varepsilon) \in \mathbb{N}$, such that

$$\mu(A_n^{j'}) < \varepsilon \quad \text{for all} \quad n \geq N.$$

We set $n_0 = m_N \geq N$ and we have that

$$\mu(E_n^j) \leq \mu(E_n^{j'}) \leq \mu(A_{n_0}^{j'}) < \varepsilon \quad \text{for} \quad n \geq n_0,$$

since $E_n^{j'} = \bigcup_{k=n}^{\infty} A_k^{j'} \subseteq A_{n_0}^{j'}$ and $E_n^j \subseteq E_n^{j'}$. □

Example. Condition (M') is strictly weaker than condition (M).

By Lemma **??** it is enough to construct a double sequence (A_n^j) of measurable sets satisfying the following:

(1) (A_n^j) is nondecreasing with respect to j for all $n \in \mathbb{N}$;

(2) $\lim_{n\to\infty} \mu(A_n^j) = 0$ for all $j \in \mathbb{N}$; and

(3) (A_n^j) satisfies (M') and not (M).

We consider the measure space (Γ, Σ, μ) where $\Gamma = [0,\infty)$, Σ: the Lebesgue measurable sets and μ: the Lebesgue measure. Let (a_n), (b_n) be sequences in Γ such that

$a_1 < b_1 < a_2 < b_2 < \cdots$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1$.

Define

$$
\begin{aligned}
&A_n^1 = [a_n, b_n] \quad \text{for} \quad n = 1, 2, \dots \quad \text{(the first line of the double sequence } (A_n^j)),\\
&A_n^2 = [a_{n,1}] \quad \text{for} \quad n = 2, 3, \dots,\\
&A_n^3 = A_n^2 \cup [1 + a_n, 1 + b_n] \quad \text{for} \quad n = 3, 4, \dots,\\
&A_n^4 = A_n^2 \cup [1 + a_n, 2] \quad \text{for} \quad n = 4, 5, \dots,\\
&\vdots\\
&A_n^{2k+1} = A_n^{2k} \cup [k + a_n, k + b_n] \quad \text{for} \quad n = 2k + 1, 2k + 2, \dots,\\
&A_n^{2k+2} = A_n^{2k} \cup [k + a_n, k + 1] \quad \text{for} \quad n = 2k + 2, 2k + 3, \dots,\\
&\vdots
\end{aligned}
$$

Also we define the sets below the diagonal of the double sequence (A_n^j) to be equal with the corresponding set on the diagonal, namely

$$A_n^j = A_n^n \quad \text{for} \quad j \geq n.$$

Conditions (1) and (2) are easily verified. Also we observe that condition M does not hold for j odd, while Definition **??** is satisfied for j' even, so that (M') holds.

We next give a sufficient condition under which μ-convergence implies $\mu - ae$ convergence.

2.4 Definition. The pair $((f_n), f)$ is *μ-locally finite* (μ-l.f.) if for each $j, n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$, such that

$$\sum_{\ell=n_0}^{\infty} \mu(A_n^j \cap A_\ell^j) < 0.$$

2.5 Proposition. *If $f_n \xrightarrow{\mu} f$ and $((f_n), f)$ is (μ-l.f), then*

$$f_n \xrightarrow{\mu-ae} f.$$

Proof. Suppose the hypothesis is satisfied but $f_n \overset{\mu-ae}{\nrightarrow} f$. By (c) of §1 we have that there exists $j \in \mathbb{N}$ and $E \subset \bigcap\limits_{n=1}^{\infty} E_n^j$ with

$$\mu(E) = a > 0. \tag{1}$$

For the sequel we fix a $j \in \mathbb{N}$ satisfying (1). We observe that the set $K = \{n \in \mathbb{N} | \mu(A_n^j \cap E) > 0\}$ is infinite (otherwise, if n_0 is the largest element of K, then, since $E \subset E_{n_0+1}^j = \bigcup\limits_{\ell=n_0+1}^{\infty} A_\ell^j$, we have $\mu(E) \leq \sum\limits_{\ell=n_0+1}^{\infty} \mu(A_\ell^j \cap E) = 0$, a contradiction). Since $f_n \xrightarrow{\mu} f$ and K is infinite the following holds:
there exists $n_0 \in K$ such that

$$\mu(A_{n_0}^j) < \frac{\alpha}{2} \quad \text{and} \quad \mu(A_{n_0}^j \cap E) = \beta < \frac{\alpha}{2}. \tag{2}$$

As $((f_n), f)$ is (μ-l.f), we have:
there exists $n_1 \in \mathbb{N}$, such that

$$\sum_{\ell=n_1}^{\infty} \mu(A_{n_0}^j \cap A_\ell^j) < \frac{\beta}{2}. \tag{3}$$

Since, $E \subset \bigcap\limits_{n=1}^{\infty} E_n^j \subset E_{n_1}^j = \bigcup\limits_{\ell=n_1}^{\infty} A_\ell^j$, it follows that

$$E \cap A_{n_0}^j \subset A_{n_0}^j \cap \left(\bigcup_{\ell=n_1}^{\infty} A_\ell^j \right) = \bigcup_{\ell=n_1}^{\infty} (A_{n_0}^j \cap A_\ell^j). \tag{4}$$

But, from (2), we have that $\mu(E \cap A_{n_0}^j) = \beta$, while, from (4), (3), we take that

$$\mu(E \cap A_{n_0}^j) \leq \mu \left(\bigcup_{\ell=n_1}^{\infty} (A_{n_0}^j \cap A_\ell^j) \right) < \frac{\beta}{2},$$

a contradiction. □

We note that the above proposition has an application in the Laws' of Large Numbers (see [**?**]).

2.6 Remark. The conditions (M), (M'), and (μ-l.f) are sufficient but not necessary, and the conditions (M') and (μ-l.f) are independent from each other.

In fact, let $\Gamma = \mathbb{R}$, with Lebesgue measure.

(i) If we set $f_n = \chi_{[n, n+\frac{1}{2^n}]}$, then $f_n \xrightarrow{a\ell-u} 0$ (since $E_n^j = \bigcup\limits_{k>n}^{\infty} A_k^j = \bigcup_{k=n}^{\infty} \left[k, k+\frac{1}{2^k}\right]$ and $\mu(E_n^j) = \sum\limits_{k=n}^{\infty} \frac{1}{2^k} \to 0$, $n \to \infty$), but (M'), (M) are not satisfied.

(ii) If we set $g_n = \chi_{[0,\frac{1}{n}]}$, $h_n = \chi_{[n,n+\frac{1}{n}]}$, then

(1) The pair $((g_n), 0)$ satisfies (M') but not (μ-l.f);

(2) The pair $((h_n), 0)$ satisfies (μ-l.f) but not (M'); and

(3) The pair $((g_n + h_n), 0)$ does not satisfy (M'), (μ-l.f), but $g_n + h_n \xrightarrow{\mu-ae} 0$.

The next proposition is a generalization of the known characterization, for finite measures, of convergence in measure.

2.7 Proposition. *The following are equivalent*

(i) $f_n \xrightarrow{\mu} f$; *and*

(ii) For each (k_n) *increasing in* $\mathbb{N}$ *there exists subsequence* (m_{k_n}) *of* (k_n)*, such that*

$$f_{m_{k_n}} \xrightarrow{a\ell-u} f.$$

Proof. Suppose $f_n \xrightarrow{\mu} f$, then (ii) follows from Riesz theorem; however we give a direct proof, based on the study of convergence via the double sequence (A_n^j).

Let $(k_n) = (k_n^0)$ be an increasing sequence in $\mathbb{N}$. We observe that there exist increasing sequences $(k_n^{(j)})_n$ for $j = 1, 2, 3, \dots$, such that

$$(k_n^{(j)})_n \text{ is a subsequence of } (k_n^{(j-1)})_n \text{ and } \mu(A^j_{k_n^{(j)}}) < \frac{1}{2^n} \text{ for all } n \in \mathbb{N}. \tag{1}$$

(The construction of the above sequence follows easily by induction, since $\lim_{n\to\infty} \mu(A_n^j) = 0$ for all $j = 1, 2, \dots$).

Set $m_{k_n} = k_n^{(n)}$ for $n = 1, 2, \dots$; then it follows, by (1), that (m_{k_n}) is a subsequence of (k_n). Also for $n \geq j$ we have

$$\mu(E^j_{m_{k_n}}) = \mu\left(\bigcup_{\ell=n}^{\infty} A^j_{m_{k_\ell}}\right) \leq \sum_{\ell=n}^{\infty} \mu(A^j_{k_\ell^{(\ell)}}) \leq \sum_{\ell=n}^{\infty} \mu(A^\ell_{k_\ell^{(\ell)}}),$$

where $(A^j_{m_{k_n}})$, $(E^j_{m_{k_n}})$ are the corresponding double sequences of the pair $((f_{m_{k_n}}), f)$. (The last inequality above follows since $\ell \geq j$ and hence $A^\ell_{k_\ell^{(\ell)}} \supseteq A^j_{k_\ell^{(\ell)}}$).

Hence by (1) we obtain that

$$\mu(E^j_{m_{k_n}}) \leq \sum_{\ell=n}^{\infty} \frac{1}{2^\ell} \to 0, \quad n \to \infty.$$

So, by (d) of §1 we conclude that $f_{m_{k_n}} \xrightarrow{a\ell-u} f$.

For the converse direction, suppose $f_n \overset{\mu}{\nrightarrow} f$, so that there exists $j \in \mathbb{N}$, such that

$$m(A_n^j) \nrightarrow 0, \quad n \to \infty.$$

Hence, there exists $j \in \mathbb{N}$, $\varepsilon > 0$ and (k_n) increasing in $\mathbb{N}$, with

$$\mu(A_{k_n}^j) \geq \varepsilon \ \text{ for } \ n = 1, 2, \dots \ .$$

So $\mu(E_{k_n}^j) \geq \varepsilon$ for $n = 1, 2, \dots$, which implies that for each subsequence (m_{k_n}) of (k_n) we have that

$$\mu(E_{m_{k_n}}^j) \nrightarrow 0, \quad n \to \infty;$$

or equivalently that

$$f_{m_{k_n}} \overset{a\ell - u}{\nrightarrow} f.$$

$\square$

3 Spaces on which convergence coincide

In this paragraph we focus on properties of the measure space, under which one kind of convergence coincides with another.

3.1 Definition. We say that the space (Γ, Σ, μ) is *finite restrict* (f.r) if for each non-increasing sequence (E_n) in Σ, with $\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(E_{n_0}) < \infty$.

3.2 Proposition. *The following are equivalent:*

(a) $a\ell - u$ *convergence coincides with* $\mu - ae$ *convergence;*

(b) the space is finite restrict (f.r);

(c) each pair $((f_n), f)$ *with* $f_n \xrightarrow{\mu - ae} f$ *has the finite restriction property (f.r.p);*

(d) $\mu - ae$ *convergence implies* μ*-convergence;*

(e) $f_n \xrightarrow{\mu} f$ *if and only if*
for each increasing sequence (k_n) *in* $\mathbb{N}$*, there exists subsequence* (m_{k_n}) *of* (k_n) *such that* $f_{m_{k_n}} \xrightarrow{\mu - ae} f$*; and*

(f) (Γ, Σ, μ) *is finite or* Γ *is the union of a set of finite measure and a finite number of atoms with infinite measure.*

Proof.

- (a) is equivalent (c) by (g) of §1 and the comment following definition II of §1.

- (b)⇔(a)

  Let (Γ, Σ, μ) be finite restrict. If $f_n \xrightarrow{\mu-ae} f$ then, by (c) of §1, $\mu\Big(\bigcap_{n=1}^{\infty} E_n^j\Big) = 0$ for all $j \in \mathbb{N}$.

  Since the space is (f.r.) and $(E_n^j)_n$ is nonincreasing with respect to n for all $j \in \mathbb{N}$, we have that, for each $j \in \mathbb{N}$, there exists $n_0 = n_0(j)$ with $\mu(E_{n_0}^j) < \infty$.

  Hence, $\lim_{n\to\infty} \mu(E_n^j) = 0$ for all $j \in \mathbb{N}$ which implies that $f_n \xrightarrow{a\ell-u} f$.

  For the converse direction suppose that the space is not finite restrict. Then, there exists (E_n) nonincreasing in Σ with $\mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = 0$ and $\mu(E_n) = \infty$ for all $n \in \mathbb{N}$. Set $f_n = \mathcal{X}_{E_n}$ (the characteristic function of E_n) for $n = 1, 2, \ldots$. It is easy to see that

  $$f_n \xrightarrow{\mu-ae} 0 \quad \text{and} \quad f_n \overset{a\ell-u}{\nrightarrow} 0.$$

- (b)⇔(d)

  If the space is (f.r.), then $a\ell - u$ convergence coincides with $\mu - ae$ convergence, and so (d) follows

  For the converse direction suppose that the space is not (f.r). If $f_n = \mathcal{X}_{E_n}$ for $n = 1, 2, \ldots$, the above defined sequence, then we have $f_n \xrightarrow{\mu-ae} 0$, but $f_n \overset{\mu}{\nrightarrow} 0$ (since $f_n \overset{a\ell-u}{\nrightarrow} 0$).

- (b)⇔(e)

  If (b) holds, then $a\ell - u$ convergence coincides with $\mu - ae$ convergence and so (e) follows from **??**.

  Conversely, if (e) holds, then $\mu - ae$ convergence implies μ-convergence, so (d) holds, and hence also (b) holds.

- (b)⇔(f)

  Suppose (f). If (Γ, Σ, μ) is finite we have nothing to prove. Let $A_1, A_2, \ldots, A_n$ be the atoms of the space. If $(E_n)_n$ is a nonincreasing sequence in Σ with $\mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = 0$, then E_1 contains some or none of the atoms, E_2 contain fewer or the same atoms as E_1, and so on. Since $\mu\Big(\bigcap_n E_n\Big) = 0$, it is impossible that some atom A_i be contained in all E_n for $n = 1, 2, \ldots$; hence there exists n_0 such that $\mu(E_{n_0}) < \infty$, so (b) holds.

  Conversely, if (f) does not hold, then we distinguish two cases: either the space has an infinite number of atoms, each with infinite measure, or the space has a finite number of atoms, with infinite measure and the complement of the union of these atoms has infinite measure.

Case 1. There exists a sequence of atoms (A_n), with $\mu(A_n) = \infty$ for $n = 1, 2, \ldots$. Without loss of generality we may assume that the above atoms are mutually disjoint. If we set $E_n = \bigcup_{i=n}^{\infty} A_i$, then (E_n) is nonincreasing, $\mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = 0$, and $\mu(E_n) = \infty$ for all $n \in \mathbb{N}$. Hence (b) does not hold.

Case 2. $\Gamma = \Delta \cup A_1 \cup \cdots \cup A_n$, where $A_1, A_2, \ldots, A_n$ are atoms with infinite measure, $\mu(\Delta) = \infty$ and Δ contains no atoms with infinite measure.
There are two subcases to consider.

Subcase 2a. There exists $Z \subset \Delta$, $Z \in \Sigma$, such that Z does not contain sets of positive finite measure. Since Z is not an atom, we have that there exists $Z_1 \subset Z$, $Z_1 \in \Sigma$ with $\mu(Z_1) = \infty$ and $\mu(Z - Z_1) = \infty$.
Similarly, there exists $Z_2 \subset Z - Z_1$, $Z_2 \in \Sigma$ with $\mu(Z_2) = \infty$ and $\mu((Z - Z_1) - Z_2) = \infty$ and so on.

Hence we take a disjoint sequence (Z_n) in Σ with $\mu(Z_n) = \infty$ for $n = 1, 2, \ldots$. As in Case 1 it follows that (b) does not hold.

Subcase 2b. For each $Z \subset \Delta$, $Z \in \Sigma$ with $\mu(Z) > 0$ there exists $B \subset Z$, $B \in \Sigma$ such that $0 < \mu(B) < \infty$.

In this subcase we assert that there exists a disjoint sequence (B_n) in Σ such that $0 < \mu(B_n) < \infty$ for all $n \in \mathbb{N}$ and $\mu\Big(\bigcup_n B_n\Big) = \infty$.
(Set $s = \sup\{\mu(A) | A \in \Sigma, \mu(A) < \infty\}$. If $s < \infty$, it is easy to see that $\mu(A) = s < \infty$ for some $A \in \Sigma$. Since $\mu(\Delta - A) = \infty$, we can find $B \subset \Delta - A$ with $0 < \mu(B) < \infty$. Hence, $\mu(A \cup B) > s$, which is a contradiction. So $s = \infty$ and the above assertion follows easily).

Setting $E_n = \bigcup_{i=n}^{\infty} B_i$, we conclude at once that (b) does not hold. □

3.3 Proposition. *The following are equivalent:*

(a) μ convergence implies $\mu - ae$ convergence, and

(b) either the space is purely atomic, or the space has measurable atomless sets and in this case these sets have infinite measure and do not contain subsets of positive finite measure.

Proof. Suppose (a) holds and (b) does not hold. Thus there exists $E \in \Sigma$, such that E is atomless, and there exists $F \subseteq E$, $F \in \Sigma$ with $0 < \mu(F) = \alpha < \infty$. Since F is atomless, by (i) of §1, we have: for each $n \in \mathbb{N}$, there exists partition $C_n = \{F_1^{(n)}, \ldots, F_n^{(n)}\}$ of F, such that $\mu(F_i^{(n)}) = \dfrac{\alpha}{n}$ for $i = 1, 2, \ldots, n$.

Let (I_n) be an enumeration of $\bigcup\limits_{n=1}^{\infty} C_n$. It is easy to see that $\chi_{I_n} \xrightarrow{\mu} 0$ and $\chi_{I_n} \overset{\mu-ae}{\nrightarrow} 0$ (since for each $x \in F$, we have $\{n : \chi_{I_n}(x) = 0\}$ and $\{n : \chi_{I_n}(x) = 1\}$ are both infinite subsets of $\mathbb{N}$). Hence we are led to a contradiction, and (a) implies (b).

Conversely, assume (b) and let $f_n \xrightarrow{\mu} f$. We distinguish two cases.

Case 1. The space is purely atomic. Let $\{A_i\}_{i\in I}$ be the family of atoms. Then, $\Gamma = \Gamma_0 \cup \Big(\bigcup\limits_{i\in I} A_i\Big)$ and $\mu_*(\Gamma_0) = 0$ (where μ_* denotes inner measure). We have that $f_n|_{A_i} = C_{i,n}$, $f|_{A_i} = C_i$, $\mu - ae$, where $C_{i,n}$, C_i are constants for all $i \in I$.

Without loss of generality we may assume that $f_n(x) = C_{i,n}$ and $f(x) = C_i$ for all $x \in A_i$, $i \in I$. (Otherwise, instead of A_i we consider the equivalent atoms $A'_i = \{x | f_n(x) = C_{i,n}$ and $f(x) = C_i\}$, $i \in I$. Then $\Gamma = \Gamma'_0 \cup \Big(\bigcup\limits_{i} A'_i\Big)$ and $\mu_*(\Gamma'_0) = 0$).

Hence we have the following implications:

$$\begin{aligned} f_n \xrightarrow{\mu} f &\Longrightarrow f_n|_{A_i} \xrightarrow{\mu} f|_{A_i} \ \text{ for all } \ i \in I \\ &\Longrightarrow C_{i,n} \longrightarrow C_i,\ n \to \infty \ \text{ for all } \ i \in I \\ &\Longrightarrow f_n(x) \longrightarrow f(x) \ \text{ for } \ x \in A_i,\ i \in I. \end{aligned}$$

Hence $A = \{x : f_n(x) \nrightarrow f(x)\} \subseteq \Gamma_0$. Since $A = \bigcup\limits_{j=1}^{\infty} \Big(\bigcap\limits_{n=1}^{\infty} E_n^j\Big)$, we have also $A \in \Sigma$ and hence $\mu(A) = 0$. So, $f_n \xrightarrow{\mu-ae} f$.

Case 2. The space has a measurable set E, such that

$$E \text{ is atomless, } \mu(E) = \infty \text{, and } F \subseteq E,\ F \in \Sigma \text{ with } \mu(F) \neq 0 \Rightarrow \mu(F) = \infty. \tag{1}$$

In this case we have $\Gamma = \Gamma_0 \cup \Big(\bigcup\limits_{i\in I} A_i\Big)$, $f_n|_{A_i} = C_{i,n}$, $f|_{A_i} = C_i$ (as in Case 1) and Γ_0 contains null sets and sets with property (1).

As in Case 1, it holds $A = \{x | f_n(x) \nrightarrow f(x)\} \in \Sigma$ and $A \subseteq \Gamma_0$.

Hence we have the following implications

$$\begin{aligned} f_n \xrightarrow{\mu} f \Longrightarrow\ & f_n|_A \xrightarrow{\mu} f|_A \\ \Longrightarrow\ & \text{for each } j \in \mathbb{N} \text{ we have } \mu(A_n^i) \to 0,\ n \to \infty, \\ & \text{where } A_n^j = A_n^j(f_n|_A, f|_A) \\ \Longrightarrow\ & \text{for each } j \in \mathbb{N} \text{ there exists } n_0 \in \mathbb{N} \text{ such that } \mu(A_n^j) = 0 \text{ for } n \geq n_0. \end{aligned}$$

(Otherwise, if $\mu(A_n^j) \neq 0$ for infinitely many n's then by (1), $\mu(A_n^j) = \infty$ for infinitely many n's and hence $f_n|_A \overset{\mu}{\nrightarrow} f|A$, which is a contradictions).

Since $A = \bigcup\limits_j \left(\bigcap_n E_n^j \right)$, $E_n^j = \bigcup\limits_{k=n}^{\infty} A_k^i$, it follows that $\mu(A) = 0$. Hence $f_n \xrightarrow{\mu-ae} f$. $\square$

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