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A remark on the porous medium equation with nonlinear source

Alkis S. Tersenov

Department of Mathematics, University of Crete, 71409 Heraklion - Crete, Greece

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ABSTRACT

In the present paper, we obtain a new a priori estimate of the solution of the initial-boundary value problem for the porous medium equation with nonlinear source and formulate the conditions guaranteeing the global solvability of this problem.

1. Introduction and formulation of the result

Consider the following parabolic equation

$$u_t - \Delta u^q = k(t, \mathbf{x})u^p \quad \text{in } Q_T = (0, T) \times \Omega, \Omega \subset \mathbf{R}^n, T > 0, \tag{1.1}$$

where $q > 1$, $p > 0$, $0 \le k(t, \mathbf{x}) \le \kappa$, coupled with initial and boundary conditions

$$u\Big|_{\Gamma_T} = \phi\Big|_{\Gamma_T} \ge 0, \quad \Gamma_T = \Omega \cup [0, T] \times \partial\Omega \tag{1.2}$$

which imply that $u \ge 0$ in Q_T. This equation appears in different applications (see [1–3] and the references therein). It is well known [4–6,1,7] that solutions of this problem may blow-up in finite time. The global solvability (i.e. for arbitrary $T > 0$) was proved in [5] for $k \equiv 1$ and homogeneous boundary conditions in the following three cases (see also [1]):

if $q > p$;

if $q = p$ and the first eigenvalue of the problem $\Delta u = -\lambda u$ in Ω, $u\Big|_{\partial\Omega} = 0$ is greater than 1;

if $q < p$ and ϕ satisfies smallness type restrictions, for $n \ge 3$ the additional restriction $p \in (q, q\frac{n+2}{n-2})$ is required.

In [4], the global solvability of problem (1.1), (1.2) with homogeneous boundary conditions in the one dimensional case ($n = 1, x \in (0, 1)$) was proved under the similar assumptions, namely:

if $q > p$;

if $q = p$ and κ is sufficiently small;

if $q < p$ and ϕ satisfies the smallness type restrictions.

(Note that in [4] more general equation $u_t = [h(u)_x + \varepsilon g(u)]_x + kf(u)$ was considered.)

The goal of the present paper is to obtain a new a priori estimate of the solution and to propose slightly different conditions guaranteeing the global solvability.

For simplicity, in order to work with classical solution, we suppose that

$$u\Big|_{\Gamma_T} = \phi\Big|_{\Gamma_T} > 0 \tag{1.3}$$

E-mail address: tersenov@math.uoc.gr.

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which implies that $u > 0$ in Q_T. Assume that ϕ is continuous and k is a continuously differentiable function. The domain Ω satisfies the exterior sphere condition and

$$\Omega \subset \{\mathbf{x}: |x_i| \le l_i,\ i = 1, \dots, n\},$$

without loss of generality suppose that $l_1 = \min_i\{l_i\}$. Define the constant K by the following

$$K = \max\left\{\left(\frac{\kappa}{q} l_1^{2(p-q+1)} 2^{2-p-q} \frac{(4q-3)^p}{(q-1)^{q-1}}\right)^{\frac{1}{q-p}}, \frac{m}{2l_1^2(q-1)}\right\}, \quad \text{for } q \ne p$$

and

$$K = \frac{m}{2l_1^2(q-1)} \quad \text{for } q = p$$

where

$$m = \max \phi\Big|_{\Gamma_T}.$$

Theorem. *There exists a global classical solution of* (1.1), (1.3) *satisfying the estimate*

$$0 < u(t, \mathbf{x}) \le \frac{l_1^2}{2}(4q-3)K \quad \forall \mathbf{x} \in \Omega \text{ and } t \ge 0 \tag{1.4}$$

in the following three cases:

1. *if* $q > p$;
2. *if* $q = p$ *and*
$$l_1^2 \le \frac{q}{\kappa} \frac{(4q-4)^{q-1}}{(4q-3)^q};$$
3. *if* $q < p$ *and*
$$m \le 2l_1^2(q-1)\left(\frac{\kappa}{q} l_1^{2(p-q+1)} 2^{2-p-q} \frac{(4q-3)^p}{(q-1)^{q-1}}\right)^{\frac{1}{q-p}}.$$

Remark. In the case $q = p$ the smallness restriction on the size of the domain is only in one direction and estimate (1.4) takes the form

$$0 < u(t, \mathbf{x}) \le \frac{4q-3}{4q-4} m.$$

In the case $q < p$ we do not need any additional restrictions on p for $n \ge 3$.

2. Proof of the Theorem

Rewrite Eq. (1.1) in the following form

$$u_t = q\,u^{q-1}\Delta u + q(q-1)\,u^{q-2}|\nabla u|^2 + k(t, \mathbf{x})u^p. \tag{2.1}$$

Consider the auxiliary equation

$$u_t - q\,u^{q-1}\Delta u = q(q-1)\,u^{q-2}|\nabla u|^2 + k(t, \mathbf{x})g(u) \tag{2.2}$$

where

$$g(u) = \begin{cases} u^p, & \text{if } u \le \frac{1}{2} l_1^2(4q-3)K \\ \left(\frac{1}{2} l_1^2 (4q-3)K\right)^p, & \text{if } u > \frac{1}{2} l_1^2(4q-3)K \end{cases}.$$

The existence of a classical solution of problem (2.2), (1.3) follows from the standard theory (see, for example, [8]). Our goal is to obtain the a priori estimate $u \le \frac{l_1^2}{2}(4q-3)K$ for the solution of problem (2.2), (1.3) and by this to show that Eqs. (2.2) and (2.1) coincide.

Consider the function

$$h(x_1) = \frac{K}{2}(l_1^2 - x_1^2) + 2l_1^2(q-1)K.$$

For $v(t, \mathbf{x}) \equiv u(t, \mathbf{x}) - h(x_1)$ we have

$$\begin{aligned} v_t - q\,u^{q-1}\Delta v &= q(q-1)\,u^{q-2}|\nabla u|^2 + k(t, \mathbf{x})g(u) + qu^{q-1}h'' \\ &= q(q-1)\,u^{q-2}|\nabla u|^2 + k(t, \mathbf{x})g(u) - qu^{q-1}K. \end{aligned}$$

Assume that the function v attains its positive maximum at the point $N \in \overline{Q}_T \setminus \Gamma_T$, at this point $v > 0$ and $\nabla v = 0$, i.e.

$$u > h \geq 2l_1^2(q-1)K, \qquad u_{x_1} = h' = -Kx_1, \qquad u_{x_i} = 0 \quad \text{for } i = 2, \ldots, n.$$

Thus we have

$$\begin{aligned}
v_t - q\,u^{q-1}\Delta v\Big|_N &= q(q-1)u^{q-2}(-Kx_1)^2 + k(t,\mathbf{x})g(u) - qu^{q-1}K\Big|_N \\
&< q(q-1)u^{q-2}K^2 l_1^2 + \kappa\left(\frac{l_1^2}{2}(4q-3)K\right)^p - qu^{q-1}K\Big|_N \\
&= \left(q(q-1)u^{q-2}K^2 l_1^2 - \frac{q}{2}u^{q-1}K\right) + \left(\kappa\left[\frac{l_1^2}{2}(4q-3)K\right]^p - \frac{q}{2}u^{q-1}K\right)\Bigg|_N \\
&< \frac{q}{2}u^{q-2}K\left(2l_1^2(q-1)K - u\right) + \left(\kappa\left[\frac{l_1^2}{2}(4q-3)K\right]^p - \frac{q}{2}\left[2l_1^2(q-1)K\right]^{q-1}K\right)\Bigg|_N \\
&< K^p\left[\kappa\left(\frac{l_1^2}{2}(4q-3)\right)^p - \frac{q}{2}\left(2l_1^2(q-1)\right)^{q-1}K^{q-p}\right] \leq 0.
\end{aligned}$$

Hence we obtain that $v_t - q\,u^{q-1}\Delta\, v\Big|_N < 0$ which is impossible. Taking into account the fact that $v \leq 0$ on Γ_T we conclude that

$$u(t,\mathbf{x}) \leq h(x_1) \leq h(0) = \frac{l_1^2}{2}(4q-3)K.$$

The inequality

$$\kappa\left(\frac{l_1^2}{2}(4q-3)\right)^p - \frac{q}{2}\left(2l_1^2(q-1)\right)^{q-1}K^{q-p} \leq 0 \tag{2.3}$$

for $q > p$ follows directly from the definition of K, for $q = p$ it follows from the restriction on l_1. If $q < p$ then (2.3) takes the form

$$K^{p-q} \leq \frac{q}{2}(2l_1^2(q-1))^{q-1}\left[\kappa\left(\frac{l_1^2}{2}(4q-3)\right)^p\right]^{-1}$$

which is fulfilled if

$$\frac{m}{2l_1^2(q-1)} \leq \left(\frac{\kappa}{q}l_1^{2(p-q+1)}2^{2-p-q}\frac{(4q-3)^p}{(q-1)^{q-1}}\right)^{\frac{1}{q-p}}.$$

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