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# The Cauchy problem for a class of quasilinear parabolic equations

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**Abstract.** The present paper is concerned with the Cauchy problem for the parabolic equation $u_t + H(t, \mathbf{x}, u, \nabla u) = \varepsilon \Delta u$. New conditions guaranteeing the global classical solvability are formulated. Moreover, it is shown that the same conditions guarantee the global existence of the Lipschitz continuous viscosity solution for the related Hamilton–Jacobi equation.

**Mathematics Subject Classification (2000).** 35K15, 35F25

**Key words.** quasilinear parabolic equations – Hamilton–Jacobi equation

## Introduction and main results

Consider the following Cauchy problem:

$$u_t + H(t, \mathbf{x}, u, \nabla u) = \varepsilon \Delta u \text{ in } \Pi_T = \mathbf{R}^n \times (0, T), \tag{0.1}$$

$$u(0, \mathbf{x}) = u_0(\mathbf{x}) \text{ for } \mathbf{x} \in \mathbf{R}^n, \tag{0.2}$$

where $\varepsilon$ is an arbitrary positive constant.

We suppose that function $H(t, \mathbf{x}, u, \mathbf{q})$ satisfies the following two restrictions. The first one is:

$$-uH(t, \mathbf{x}, u, \mathbf{0}) \leq \alpha_1 u^2 + \alpha_2, \tag{0.3}$$

where $\alpha_1$ and $\alpha_2$ are positive constants. In order to formulate the second restriction, introduce the following notation: for $i = 2, \dots, n-1$

$$\mathbf{x}' = (x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n), \quad \mathbf{x}'' = (x_1, \dots, x_{i-1}, x_i'', x_{i+1}, \dots, x_n),$$

$$\mathbf{q} = (q_1, \dots, q_n), \quad \mathbf{q}^- = (q_1, \dots, q_{i-1}, -q_i, q_{i+1}, \dots, q_n);$$

for $i = 1$

$$\mathbf{x}' = (x_1', x_2, \dots, x_n), \quad \mathbf{x}'' = (x_1'', x_2, \dots, x_n), \quad \mathbf{q}^- = (-q_1, q_2, \dots, q_n)$$

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and for $i = n$

$$\mathbf{x}' = (x_1, \dots, x_{n-1}, x_n'), \quad \mathbf{x}'' = (x_1, \dots, x_{n-1}, x_n''), \quad \mathbf{q}^- = (q_1, \dots, , q_{n-1}, -q_n).$$

Now let us formulate the second restriction. Suppose that

$$\begin{aligned} H(t, \mathbf{x}', u, \mathbf{q}) - H(t, \mathbf{x}'', v, \mathbf{q}) \geq 0, \\ H(t, \mathbf{x}'', u, \mathbf{q}^-) - H(t, \mathbf{x}', v, \mathbf{q}^-) \geq 0, \end{aligned} \tag{0.4}$$

when $x_i' \geq x_i''$, $u \geq v$, $q_i \geq 0$ and arbitrary $q_j$ for $i = 1, \dots, n$, $i \neq j$.

For the case of one spatial variable these conditions appear as $H(t, x', u, q) - H(t, x'', v, q) \geq 0$, $H(t, x'', u, -q) - H(t, x', v, -q) \geq 0$ for $x' \geq x''$, $u \geq v$, $q \geq 0$.

Conditions (0.4) are fulfilled if, for example:

1) $H = H(t, \nabla u)$, where $H(t, \nabla u)$ is an arbitrary function;
2) $H = f(t, u)g(t, \nabla u)$, where $f(t, u)$ is a nondecreasing (non-increasing) function with respect to $u$, $g(t, \mathbf{q})$ is non-negative (nonpositive) function.
3) $H = \sum_{i=1}^{n} f_i(t, x_i)g_i(t, u_{x_i})$, where $f_i$ are nondecreasing (non-increasing) functions with respect to $x_i$ and functions $g_i$ satisfy the conditions $q_i g(t, q_i) > 0$ ($q_i g(t, q_i) < 0$) for $|q_i| > 0$.

Concerning the initial data we suppose that

$$|u_{0x_i}(\mathbf{x})| \leq K_i, \quad |u_0(\mathbf{x})| \leq M_0 \text{ and } |u_{0x_i}(\mathbf{x})| \to 0 \text{ when } |x_i| \to +\infty, \tag{0.5}$$

where $i = 1, \dots, n$, $M_0 > 0$ and $K_i > 0$ are some constants.

Let us formulate the main result for problem (0.1), (0.2).

**Theorem 1.** *Suppose that conditions* (0.3)–(0.5) *are fulfilled and, in addition, suppose that* $u_0(\mathbf{x}) \in C^1(\mathbf{R}^n)$, $H(t, \mathbf{x}, u, \mathbf{q}) \in C^\alpha(\Pi_T \times \mathbf{R} \times \mathbf{R}^n)$ *for some* $\alpha \in (0, 1)$. *Then for any* $T > 0$ *there exists a solution of problem* (0.1), (0.2) *which belongs to* $C_{t,\mathbf{x}}^{1+\alpha/2, 2+\alpha}(\Pi_T) \cap C_{t,\mathbf{x}}^{0,1}(\bar{\Pi}_T)$.

*Besides,* $|u_{x_i}(t, \mathbf{x})| \leq K_i$ *for* $i = 1, \dots, n$. *If for some* $m$ $\max |u_{0x_m}(\mathbf{x})| = 0$ *then* $u_{x_m}(t, \mathbf{x}) \equiv 0$ *in* $\Pi_T$.

In order to prove Theorem 1 we approximate the Cauchy problem by the third initial boundary-value problem (1.1)–(1.3) (see Section 1). Condition (0.3) (see [9]) guarantee the following apriori estimate for the solution of problem (1.1)–(1.3):

$$\max_{\Pi_T^l} |u| \leq M \equiv \min_{\lambda > \alpha_1} e^{\lambda T} \max \left\{ \sqrt{\frac{\alpha_2}{\lambda - \alpha_1}}, \max_{\Omega^l} |u_0(\mathbf{x})| \right\}. \tag{0.6}$$

The main step is the establishment of the a priori estimate of $|\nabla u|$. Here we use the idea of Kruzhkov [7], [8] of introducing a new spatial variable (see also [4], [5], [14]). Next we show that

$$\frac{|u(t_1, \mathbf{x}) - u(t_2, \mathbf{x})|}{|t_1 - t_2|^{1/2}} \leq C_0, \tag{0.7}$$

where the constant $C_0$ does not depend on $\varepsilon$. The last step is the establishment of the estimate of $|\nabla u|$ in $C^\alpha$ norm (this estimate depends on $\varepsilon$). The existence of the solution of problem (1.1)–(1.3) follows from these a priori estimates and Schauder's fixed-point theorem (see, for example, [10]).

The solution of the Cauchy problem is obtained as the limit of a sequence of solutions of the third initial boundary-value problem under an unlimited dilation of the domain.

Let us mention here that the usual restriction (see [3], [9], [10]) on $H$ in order to obtain the apriori estimate of the gradient of the solution is the following one:

$$|H(t, \mathbf{x}, u, \mathbf{q})| \le Const(1 + |\mathbf{q}|)^2.$$

In the present paper the function $H(t, \mathbf{x}, u, \mathbf{q})$ may have an arbitrary growth with respect to $\mathbf{q}$ when $|\mathbf{q}| \to +\infty$. In the one-dimensional case analogous results were obtained in [14] and [16] (in [15] the radially symmetric case was investigated).

Concerning the equation

$$u_t + H(t, \mathbf{x}, u, \nabla u) = 0 \tag{0.8}$$

we are interested in the existence of the Lipschitz continuous viscosity solution for problem (0.8), (0.2). The viscosity solution [2], [11] is a uniformly continuous function. In order to prove the Lipschitz continuity of this function some additional restrictions are necessary [1], [17]. We give a new sufficient condition guaranteeing the Lipschitz continuity of the viscosity solution. Let us formulate the main result.

**Theorem 2.** *Suppose that all conditions of Theorem* 1 *are fulfilled. Then for any* $T > 0$ *there exists a viscosity solution of problem* (0.8), (0.2) *satisfying equation* (0.8) *almost everywhere in* $\Pi_T$ *such that* $u_t, \nabla u \in L_\infty(\Pi_T)$.

*Besides* $||u_{x_i}(t, \mathbf{x})||_{L_\infty} \le K_i$ *for* $i = 1, \dots, n$. *If for some* $m$ $\max |u_{0x_m}(\mathbf{x})| = 0$ *then* $u_{x_m}(t, \mathbf{x}) \equiv 0$ *in* $\Pi_T$.

In order to obtain such a solution we pass to the limit when $\varepsilon \to 0$ in equation (0.1) based on the fact that the estimate of the gradient and estimates (0.6), (0.7) are independent of $\varepsilon$. The obtained viscosity solution is a Lipschitz continuous function and satisfies (0.8) almost everywhere.

Let us mention here that if the Hamiltonian $H$ depends only on $t$ and $\nabla u$ then in Theorem 2 it is sufficient to require the function $H(t, \mathbf{q})$ to be only continuous.

## 1. A priori estimates for the auxiliary problem

In this section we consider the problem

$$u_t + H(t, \mathbf{x}, u, \nabla u) = \varepsilon \Delta u \ \text{ in } \ \Pi_T^l, \tag{1.1}$$

$$u_{x_i} - \delta u \,\big|_{x_i=-l} = u_{x_i} + \delta u \,\big|_{x_i=l} = 0 \ \text{ for } \ i = 1, \dots, n, \tag{1.2}$$

$$u(0, \mathbf{x}) = \phi^l(\mathbf{x}) \ \text{ for } \ \mathbf{x} \in \Omega^l, \tag{1.3}$$

where $\phi^l_{x_i} \pm \delta\phi\,\big|_{x_i=\pm l} = 0$, $|\phi^l| \le M_0$, $|\phi^l_{x_i}| \le K_i$, $i = 1, \ldots, n$, $\Omega^l = \{\mathbf{x} : |x_i| < l, i = 1, \ldots, n\}$ and $\Pi^l = \Omega^l \times (0, T)$. The constant $\delta$ satisfies the inequality $0 < \delta \le \min\{K_1, \ldots, K_n\}/M$. Recall that we select $K_i$ to be strictly positive.

**Lemma 1.** *Let $u(t, \mathbf{x})$ be a classical solution of problem* (1.1)–(1.3) *(i.e. $u(t, \mathbf{x}) \in C^{1,2}_{t,\mathbf{x}}(\Pi^l_T) \cap C^{0,1}_{t,\mathbf{x}}(\bar{\Pi}^l_T)$) and suppose that conditions* (0.3)–(0.5) *are fulfilled. Then in $\bar{\Pi}^l_T$ the inequality*

$$|u_{x_i}(t, \mathbf{x})| \le K_i, \quad i = 1, \ldots, n$$

*holds.*

*Proof.* Let us prove the lemma for $i = 1$. For $i = 2, \ldots, n$ the proof is similar. Consider equation (0.1) at two different points of $\Pi^l_T$, $(t, \mathbf{x})$ and $(t, \mathbf{y})$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, x_2, \ldots, x_n)$ $(x_1 \ne y_1)$:

$$u_t(t, \mathbf{x}) + H(t, \mathbf{x}, u(t, \mathbf{x}), \nabla u(t, \mathbf{x})) = \varepsilon\Delta u(t, \mathbf{x}),$$
$$u_t(t, \mathbf{y}) + H(t, \mathbf{y}, u(t, \mathbf{y}), \nabla u(t, \mathbf{y})) = \varepsilon\Delta u(t, \mathbf{y}).$$

One can easily see that the function $v(t, \mathbf{x}, y_1) \equiv u(t, \mathbf{x}) - u(t, \mathbf{y})$ satisfies the equation

$$\varepsilon\Delta v - v_t = H(t, \mathbf{x}, u(t, \mathbf{x}), \nabla u(t, \mathbf{x})) - H(t, \mathbf{y}, u(t, \mathbf{y}), \nabla u(t, \mathbf{y}))$$

in the domain $\{(t, \mathbf{x}, y_1) : t \in (0, T), |x_i| < l, i = 1, \ldots, n, |y_1| < l\}$. Here $\Delta v \equiv \sum_1^n v_{x_i x_i} + v_{y_1 y_1}$. Define the following operator:

$$Lv \equiv \varepsilon\Delta v - v_t.$$

Consider the function $w \equiv v(t, \mathbf{x}, y_1) - K_1'(x_1 - y_1)$ in the prism

$$P = \{(t, \mathbf{x}, y_1) : 0 < t \le T, |x_i| < l, i = 1, \ldots, n, |y_1| < l, 0 < x_1 - y_1\},$$

here $K_1' = K_1 + \delta_0$, $\delta_0 > 0$ is an arbitrary small constant. We have

$$Lw \equiv \varepsilon\Delta w - w_t = H(t, \mathbf{x}, u(t, \mathbf{x}), \nabla u(t, \mathbf{x})) - H(t, \mathbf{y}, u(t, \mathbf{y}), \nabla u(t, \mathbf{y})).$$

Let $\tilde{w} = we^{-t}$, then

$$\begin{aligned}\tilde{L}\tilde{w} &\equiv \varepsilon\Delta\tilde{w} - \tilde{w} - \tilde{w}_t \\ &= e^{-t}[H(t, \mathbf{x}, u(t, \mathbf{x}), \nabla u(t, \mathbf{x})) - H(t, \mathbf{y}, u(t, \mathbf{y}), \nabla u(t, \mathbf{y}))].\end{aligned} \tag{1.4}$$

Denote by $\Gamma$ the parabolic boundary of $P$ ($\Gamma = \partial P \setminus \{(t, \mathbf{x}, y_1) : \ t = T, \ |x_i| < l, |y_1| < l, \ 0 < x_1 - y_1\}$). If the function $\tilde{w}$ attains its positive maximum at the point $N^0 = (t^0, \mathbf{x}^0, y_1^0) \in \bar{P} \setminus \Gamma$ then at this point $\tilde{w}_{x_i} = \tilde{w}_{y_1} = 0$, $\tilde{w} > 0$, hence $u_{x_1}(t, \mathbf{x}) = u_{y_1}(t, \mathbf{y}) = K_1' > 0$, $u_{x_i}(t, \mathbf{x}) = u_{x_i}(t, \mathbf{y})$, $i = 2, 3, \ldots, n$, and $u(t, \mathbf{x}) > u(t, \mathbf{y})$ $(x_1^0 > y_1^0)$. Thus from (1.4), taking into account (0.4), we obtain

$$\tilde{L}\tilde{w}\,\big|_{N^0} \ge 0,$$

on the other hand, the fact that at $N^0$ the function $\tilde{w}$ attains its positive maximum implies the inequality

$$\varepsilon\Delta\tilde{w} - \tilde{w} - \tilde{w}_t \Big|_{N^0} < 0.$$

From this contradiction we conclude that $\tilde{w}$ can not attain its positive maximum in $\bar{P} \setminus \Gamma$.

Consider $\Gamma$:

1) for $x_1 = y_1$ we have $\tilde{w} = 0$;
2) for $t = 0$ we have $\tilde{w} = e^{-t}(\phi^l(\mathbf{x}) - \phi^l(\mathbf{y}) - K_1'(x_1 - y_1)) \le e^{-t}(K_1(x_1 - y_1) - K_1'(x_1 - y_1)) \le 0$.

Now consider the following parts of $\Gamma$:

3) $\big\{(t, \mathbf{x}, y_1) : 0 < t \le T,\ \ -l < x_1 \le l,\ \ y_1 = -l,\ \ |x_i| \le l, i = 2, \dots, n\big\}$;
4) $\big\{(t, \mathbf{x}, y_1) : 0 < t \le T,\ \ x_1 = l,\ \ -l \le y_1 < l,\ \ |x_i| \le l, i = 2, \dots, n\big\}$.

Taking into account condition (1.2) we conclude that

$$\begin{aligned} -\tilde{w}_{y_1}(t, \mathbf{x}, -l) &= e^{-t}(u_{y_1}(t, -l, x_2, \dots, x_n) - K_1') \\ &= e^{-t}(\delta u(t, -l, x_2, \dots, x_n) - K_1') < 0 \end{aligned} \tag{1.5}$$

$$\begin{aligned} \tilde{w}_{x_1}(t, l, x_2, \dots, x_n, y_1) &= e^{-t}(u_{x_1}(t, \mathbf{x}) - K_1') \\ &= e^{-t}(-\delta u(t, \mathbf{x}) - K_1') - K_1') < 0. \end{aligned} \tag{1.6}$$

From (1.5) and (1.6) it follows that $\tilde{w}$ cannot attain its maximum on parts 3), 4). Let us estimate the derivatives $\tilde{w}_{x_i}$, $i = 2, \dots, n$ on the following parts of $\Gamma$:

5) $\big\{(t, x, y_1) : 0 < t \le T, x_i = -l, |y_1| < l, |x_j| < l,$
$j = 1, \dots, n,\ j \ne i, x_1 - y_1 > 0\big\}$;

and,

6) $\big\{(t, x, y_1) : 0 < t \le T, x_i = l, |y_1| < l, |x_j| < l,$
$j = 1, \dots, n,\ j \ne i, x_1 - y_1 > 0\big\}$.

We have

$$-\tilde{w}_{x_i}(t, \mathbf{x}, y_1)\Big|_{x_i=-l} = -e^{-t}\delta(u(t, \mathbf{x}) - u(t, \mathbf{y}))\Big|_{x_i=-l}, \tag{1.7}$$

$$\tilde{w}_{x_i}(t, \mathbf{x}, y_1)\Big|_{x_i=l} = e^{-t}\delta(-u(t, \mathbf{x}) + u(t, \mathbf{y}))\Big|_{x_i=l}. \tag{1.8}$$

Consider part 5). If $\tilde{w}$ attains its positive maximum at some point $(t^*, \mathbf{x}^*, y_1^*)$ of the part 5), then $\tilde{w}(t^*, \mathbf{x}^*, y_1^*) = e^{-t^*}(u(t^*, \mathbf{x}^*) - u(t^*, \mathbf{y}^*) - K_1'(x_1^* - y_1^*)) > 0$ and as a consequence $u(t^*, \mathbf{x}^*) - u(t^*, \mathbf{y}^*) > 0$. Hence from (1.7) we obtain that

$$-\tilde{w}_{x_i}(t^*, \mathbf{x}^*, y_1^*) = e^{-t*}\delta(-u(t^*, \mathbf{x}^*) + u(t^*, \mathbf{y}^*)) < 0.$$

On the other hand, at the point of its maximum we have $-\tilde{w}_{x_i} \ge 0$. From this contradiction we conclude that $\tilde{w}$ cannot attain its positive maximum on part 5).

Analogously we obtain that $\tilde{w}$ cannot attain its positive maximum on part 6). Hence $\tilde{w} \le 0$ on $\bar{P}$. Thus we have proved that

$$u(t, \mathbf{x}) - u(t, \mathbf{y}) \le K_1'(x_1 - y_1) \text{ in } \bar{P}.$$

By analogy, taking the function $v_1 \equiv u(t, \mathbf{y}) - u(t, \mathbf{x})$ in the place of $v$, we obtain

$$\begin{aligned} \tilde{L}\tilde{w}_1 &\equiv \varepsilon\Delta\tilde{w}_1 - \tilde{w}_1 - \tilde{w}_{1t} \\ &= e^{-t}[H(t, \mathbf{y}, u(t, \mathbf{y}), \nabla u(t, \mathbf{y})) - H(t, \mathbf{x}, u(t, \mathbf{x}), \nabla u(t, \mathbf{x}))], \end{aligned}$$

where $\tilde{w}_1 = (v_1 - K_1'(x_1 - y_1))e^{-t}$. If the function $\tilde{w}_1$ attains its positive maximum at the point $N^1 = (t^1, \mathbf{x}^1, y_1^1) \in \bar{P} \setminus \Gamma$ then at this point $\tilde{w}_{1x_i} = \tilde{w}_{1y_1} = 0$, $\tilde{w}_1 > 0$, hence $u_{x_1}(t, \mathbf{x}) = u_{y_1}(t, \mathbf{y}) = -K_1' < 0$, $u_{x_i}(t, \mathbf{x}) = u_{x_i}(t, \mathbf{y})$, $i = 2, 3, \ldots, n$, and $u(t, \mathbf{y}) > u(t, \mathbf{x})$ $(x_1^1 > y_1^1)$. Taking into account the (0.4), we obtain

$$\tilde{L}\tilde{w}_1 \Big|_{N^1} \ge 0,$$

on the other hand, at the point $N^1$ the function $\tilde{w}$ attains its positive maximum and hence

$$\varepsilon\Delta\tilde{w}_1 - \tilde{w}_1 - \tilde{w}_{1t} \Big|_{N^1} < 0.$$

From this contradiction we conclude that $\tilde{w}_1$ cannot attain its positive maximum in $\bar{P} \setminus \Gamma$.

Consider $\Gamma$. For $x_1 = y_1$ we have $\tilde{w}_1 = 0$ and for $t = 0$ $\tilde{w}_1 = e^{-t}(u_0(\mathbf{y}) - u_0(\mathbf{x}) - K_1'(x_1 - y_1)) \le e^{-t}(K_1(x_1 - y_1) - K_1'(x_1 - y_1)) \le 0$. On parts 3) and 4) we have

$$-\tilde{w}_{1y_1} = e^{-t}(-\delta u - K_1') < 0 \text{ and } \tilde{w}_{1x_1} = e^{-t}(\delta u - K_1') < 0,$$

respectively. Hence $\tilde{w}_1$ cannot attain its maximum on 3) nor on 4). On parts 5) and 6) we have

$$\begin{aligned} -\tilde{w}_{1x_i} \Big|_{x_i=-l} &= e^{-t}\delta(u(t, \mathbf{x}) - u(t, \mathbf{y}))) \Big|_{x_i=-l}, \\ \tilde{w}_{1x_i} \Big|_{x_i=l} &= e^{-t}\delta(u(t, \mathbf{x}) - u(t, \mathbf{y})) \Big|_{x_i=l}. \end{aligned}$$

Consider part 5). If $\tilde{w}_1$ attains its positive maximum at some point $(\hat{t}, \hat{\mathbf{x}}, \hat{y}_1)$ of the boundary 5), then $\tilde{w}_1(\hat{t}, \hat{\mathbf{x}}, \hat{y}_1) = e^{-\hat{t}}(u(\hat{t}, \hat{\mathbf{y}}) - u(\hat{t}, \hat{\mathbf{x}}) - K_1'(\hat{x}_1 - \hat{y}_1)) > 0$ and as a consequence $u(\hat{t}, \hat{\mathbf{y}}) - u(\hat{t}, \hat{\mathbf{x}}) > 0$. Hence we have

$$-\tilde{w}_{1x_i}(\hat{t}, \hat{\mathbf{x}}, \hat{y}_1) = e^{-\hat{t}}\delta(u(\hat{t}, \hat{\mathbf{x}})) - u(\hat{t}, \hat{\mathbf{y}})) < 0.$$

On the other hand, at the point of maximum we have $-\tilde{w}_{1x_i} \ge 0$. Thus we conclude that $\tilde{w}_1$ cannot attain its maximum on 5). Analogously we obtain that $\tilde{w}_1$ cannot attain its maximum on 6). Hence $\tilde{w}_1 \le 0$ on $\bar{P}$. Thus we have proved that

$$u(t, \mathbf{y}) - u(t, \mathbf{x}) \le K_1'(x_1 - y_1) \text{ in } \bar{P}.$$

In view of the symmetry of the variables $x$ and $y$, in the same way we examine the case $y_1 > x_1$. As a result we have that for

$$0 \le t \le T, \quad |x_i| \le l, i = 1, \dots, n \quad |y_1| \le l, \quad 0 < |x_1 - y_1|,$$

the inequality

$$\frac{|u(t, \mathbf{x}) - u(t, \mathbf{y})|}{|x_1 - y_1|} \le \frac{K_1' |x_1 - y_1|}{|x_1 - y_1|}$$

holds, implying that $|u_{x_1}(t, \mathbf{x})| \le K_1'$. Passing to the limit when $\delta_0 \to 0$ we fulfill the proof. □

Now let us prove the Hölder continuity of $u(t, \mathbf{x})$ and of $\nabla u(t, \mathbf{x})$. In [8], for quasilinear parabolic equations with two independent variables

$$u_t = a(t, x, u, u_x)u_{xx} + b(t, x, u, u_x),$$

the following estimates were established:

a) $|u(t_1, x) - u(t_2, x)| \le C_0 |t_1 - t_2|^{1/2}$ with the constant $C_0$ depending only on $max|u_x|$ and on the maximum of the coefficients of the equation (see also Remark 3.3 in [13]);
b) $|u_x|_{C^{\alpha/2,\alpha}_{t,x}} \le C_1$, $0 < \alpha < 1$ with $C_1$ and $\alpha$ depending on the constant $a_0$, where $0 < a_0 \le a(t, x, u, u_x)$, $\max |u|$, $\max |u_x|$ and on the maximum of the coefficients of the equation.

In the following Lemma 2 and Lemma 3 we give the proof of the similar results for the multidimensional equation (0.1). The constant $C_0$ in that case depends not only on $\max |\nabla u|$ and the maximum of $H$ but also on the dimension $n$.

**Lemma 2.** *For any classical solution of problem* (1.1)–(1.3) *we have*

$$|u(t + h, \mathbf{x}) - u(t, \mathbf{x})| \le C_0 h^{\frac{1}{2}}, 0 < h < 1, \quad t, t + h \in [0, T],$$

*where the constant* $C_0$ *depends only on* $M$, $K_i$, $n$ *and* $H_0 = \max H(t, \mathbf{x}, u, \mathbf{q})$, *where the maximum is taken over the set* $\bar{\Pi}^l_T \times [-M, M] \times [-K_1, K_1] \times \ldots \times [-K_n, K_n]$.

*Proof.* Consider the function $u(t, \mathbf{x})$ as a solution of the linear equation

$$\varepsilon \Delta u - u_t = \tilde{H}(t, \mathbf{x}),$$

with bounded $\tilde{H}(t, \mathbf{x}) \equiv H(t, \mathbf{x}, u, \nabla u)$. For $t_0 \in [0, T - h]$ denote

$$s = \max_{t \in [t_0, t_0 + h]} |u(t, \mathbf{x^0}) - u(t^0, \mathbf{x^0})|.$$

Suppose that $\mathbf{x^0}$ is an interior point of the domain $\Omega$. Consider the parallelepiped

$$Q = \left\{(t, \mathbf{x}) : t \in (t^0, t^0 + h), x_i \in \left(x_i^0 - \rho, x_i^0 + \rho\right)\right\},$$

where $0 < \rho \le d \equiv dist\{\mathbf{x^0}, \partial\Omega\}$. Introduce the following function:

$$v^{\pm} = u(t^0, \mathbf{x^0}) \pm \Big[K\rho + (t - t^0)\mu(\rho, s) + \frac{s}{\rho^2}\sum_1^n \big(x_i - x_i^0\big)^2\Big],$$

where $K = \sum_1^n K_i$ and

$$\mu(\rho, s) \equiv \varepsilon n \frac{2s}{\rho^2} + H_0 + 1.$$

Obviously

$$\begin{aligned}
v^+(t^0, \mathbf{x}) &= u(t^0, \mathbf{x^0}) + K\rho + \frac{s}{\rho^2}\sum_1^n \big(x_i - x_i^0\big)^2 \\
&\ge u(t^0, \mathbf{x^0}) + K\rho \\
&= \big(u(t^0, \mathbf{x^0}) - u\big(t^0, x_1, x_2^0, \dots, x_n^0\big) + K_1\rho\big) + \\
&\quad \big(u\big(t^0, x_1, x_2^0, \dots, x_n^0\big) - u\big(t^0, x_1, x_2, x_3^0, \dots, x_n^0\big) + K_2\rho\big) + \dots \\
&\quad + \big(u\big(t^0, x_1, \dots, x_{n-1}, x_n^0\big) - u(t^0, \mathbf{x}) + K_n\rho\big) + u(t^0, \mathbf{x}) \\
&\ge u(t^0, \mathbf{x}),
\end{aligned}$$

$$\begin{aligned}
v^+(t, \mathbf{x})\Big|_{|x_1 - x_1^0| = \rho} &= u(t^0, \mathbf{x^0}) + K\rho + (t - t^0)\mu(\rho, s) + s + \frac{s}{\rho^2}\sum_2^n \big(x_i - x_i^0\big)^2 \\
&\ge u(t^0, \mathbf{x^0}) + K\rho + s \\
&= u(t, \mathbf{x})\Big|_{|x_1 - x_1^0| = \rho} + \big(u(t^0, \mathbf{x^0}) - u(t, \mathbf{x^0}) + s\big) + \\
&\quad \big(u(t, \mathbf{x^0}) - u\big(t, x_1^0, \dots, x_{n-1}^0, x_n\big) + K_n\rho\big) + \\
&\quad \big(u\big(t, x_1^0, \dots, x_{n-1}^0, x_n\big) - u\big(t, x_1^0, \dots, x_{n-2}^0, x_{n-1}, x_n\big) + K_{n-1}\rho\big) + \dots \\
&\quad + \big(u\big(t, x_1^0, x_2, \dots, x_n\big) - u(t, \mathbf{x})\Big|_{|x_1 - x_1^0| = \rho} + K_1\rho\big) \\
&\ge u(t, \mathbf{x})\Big|_{|x_1 - x_1^0| = \rho}.
\end{aligned}$$

Analogously for $i = 2, \dots, n$ we obtain

$$v^+(t, \mathbf{x})\Big|_{|x_i - x_i^0| = \rho} \ge u(t, \mathbf{x})\Big|_{|x_i - x_i^0| = \rho}.$$

Thus

$$v^+(t, \mathbf{x}) \ge u(t, \mathbf{x})$$

on the parabolic boundary of $Q$. Furthermore

$$L(v^+) \equiv \varepsilon\Delta v^+ - v_t^+ = \varepsilon n \frac{2s}{\rho^2} - \mu(\rho, s) = -H_0 - 1 < -H_0.$$

Hence from the maximum principle (taking into account that $L(v^+ - u) < -H_0 - \tilde{H} \le 0$) we conclude that

$$v^+(t, \mathbf{x}) \ge u(t, \mathbf{x}) \text{ in } Q.$$

Analogously we obtain

$$v^-(t, \mathbf{x}) \le u(t, \mathbf{x}) \text{ in } Q.$$

So we have

$$|u(t, \mathbf{x}) - u(t^0, \mathbf{x^0})| \le K\rho + (t - t_0)\mu(\rho, s) + \frac{s}{\rho^2}\sum_1^n \left(x_i - x_i^0\right)^2,$$

and

$$|u(t, \mathbf{x^0}) - u(t^0, \mathbf{x^0})| \le K\rho + h\mu(\rho, s)$$

or

$$\begin{aligned} s = \max_{t\in[t_0,t_0+h]} |u(t, \mathbf{x^0}) - u(t^0, \mathbf{x^0})| &\le K\rho + h\mu(\rho, s) \\ \le \min_{0<\rho\le d}[K\rho + h\mu(\rho, s)] &\le \tilde{K} \min_{0<\rho\le d}[\rho + h(1 + s\rho^{-2})], \end{aligned} \tag{1.9}$$

where $\tilde{K} = max\{K, 2\varepsilon n, H_0 + 1\}$.

Suppose that $2h < d^3M^{-1}$. Obviously $(hs)^{1/3} \le (h2M)^{1/3} \le d$ (recall that $s \le 2M$). Hence, from (1.7), for $\rho_* = (hs)^{1/3}$ we have

$$s \le \tilde{K}[\rho_* + h + hs\rho_*^{-2}] = \tilde{K}(2(hs)^{\frac{1}{3}} + h).$$

Consider two cases: $h \le (hs)^{\frac{1}{3}}$ and $h > (hs)^{\frac{1}{3}}$. In the first case we have $s \le 3\tilde{K}(hs)^{\frac{1}{3}}$, hence $s^3 \le 27\tilde{K}^3hs$ and finally

$$s \le \sqrt{27\tilde{K}^3}\sqrt{h}.$$

In the second case we have $s < h^2$ and $h^2 < \sqrt{h}$ for $0 < h < 1$.

Now suppose that $2h \ge d^3M^{-1}$. We have

$$|u(t + h, \mathbf{x}) - u(t, \mathbf{x})| \le 2M = \frac{2M}{\sqrt{h}}\sqrt{h} \le (2M)^{3/2}d^{-3/2}\sqrt{h}.$$

The lemma is proved. □

**Lemma 3.** *For any classical solution of* (1.1)–(1.3) *we have*

$$|\nabla u|_{C^{\alpha,\alpha/2}(Q_T)} \le C_1, \quad \alpha \in (0, 1),$$

*where the constants $C_1$ and $\alpha$ depend only on $\varepsilon$, $K_i, i = 1, \ldots, n$, $H_0$ and $M$.*

*Proof.* The function $p(t,\mathbf{x}) \equiv u_{x_1}(t,\mathbf{x})$ can be considered as a weak solution of the equation

$$\varepsilon\Delta p - p_t = \tilde{H}_{x_1},$$

where $\tilde{H}(t,\mathbf{x}) \equiv H(t,\mathbf{x},u,\nabla u)$. Then the function $w(t,\mathbf{x},z) \equiv p(t,\mathbf{x}) + z$ is the weak solution of the equation

$$\varepsilon\Delta w - (\tilde{H}w_{x_1})_z - (\tilde{H}w_z)_{x_1} + (Bw_z)_z - w_t = 0,$$

where $B \equiv 1+\frac{H_0^2}{\varepsilon}$ and $z$ is a new independent variable. This equation is a uniformly parabolic equation for strictly positive $\varepsilon$. In fact, the matrix of the coefficients has the following form: the diagonal elements are $\varepsilon,\dots,\varepsilon,B$; on positions $1,n+1$ and $n+1,1$ we have $-\tilde{H}$, all the other elements are zero. The determinant of this matrix is $\varepsilon^{n-1}(\varepsilon B - \tilde{H}^2) \geq \varepsilon^n$. Now we can apply the well-known results of Nash–De Giorgi (see, for example, [8]). Analogously we can establish the Hölder continuity of $u_{x_i}$, $i=2,\dots,n$. The lemma is proved. □

## 2. Proof of the existence theorems

The global classical solvability of the auxiliary problem (1.1)–(1.3) follows from the estimate (0.6), estimates obtained in Lemmas 1–3, linear theory [10] and [6] (or [12]) and Schauder's fixed point theorem [10]. Recall that the classical solution is a solution belonging to $C^{1,2}_{t,\mathbf{x}}(\Pi^l_T) \cap C^{0,1}_{t,\mathbf{x}}(\bar{\Pi}^l_T)$.

The estimates obtained in Lemmas 1–3, as well as the estimate (0.6), are independent of $\delta$ and $l$. Passing to the limit when $\delta \to 0$ we obtain the global classical solvability of problem (1.1), (2.1), where

$$u_{x_i}\big|_{x_i=\pm l} = 0, \quad u(0,\mathbf{x}) = \phi^l(\mathbf{x}) \quad \text{for } \mathbf{x}\in\Omega^l, \quad \phi^l_{x_i}\big|_{x_i=\pm l} = 0, \quad i=1,\dots,n. \tag{2.1}$$

In order to prove Theorem 1 we approximate problem (0.1), (0.2) by (1.1), (2.1). The initial data $u_0(\mathbf{x})$ we approximate in $C^1$ norm by functions $u^l_0(\mathbf{x}) \in C^1(R^n)$, where $u^l_0(\mathbf{x}) = \phi^l(\mathbf{x})$ for $|x_i| \leq l$, $|u^l_0(\mathbf{x})| \leq M_0$, $|u^l_{0x_i}(\mathbf{x})| \leq K_i$.

If $u_{0x_m}(\mathbf{x}) \equiv 0$ for some $m$ then $u_{x_m}(t,\mathbf{x}) \equiv 0$. In fact, in (1.3) we take $\max|u_{0x_m}| < \delta \leq \min\{K_1,\dots,K_{m-1},K_{m+1},\dots,K_n\}/M$. One can easily see that in that case from the proof of Lemma 1 follows that $\max|u_{x_m}(t,\mathbf{x})| \leq \delta$. Passing to the limit when $\delta \to 0$ we obtain the requirement.

The solution of the Cauchy problem can be obtained as a limit of a sequence of solutions $u^l(t,\mathbf{x})$ of problem (1.1), (2.1) under an unlimited dilation of the domain $\Omega_l$ when $l \to \infty$. We apply here the standard diagonal process [9].

Approximate equation (0.8) by (0.1). From the obtained a priori estimates we have that for the solutions $u^\varepsilon(t,\mathbf{x})$ of problem (0.1), (0.4) we can find a subsequence $\varepsilon_k \to 0$ such that $u^{\varepsilon_k}$ converges uniformly to $u$. Moreover $u^{\varepsilon_k}_t \to u_t$, $u^{\varepsilon_k}_{x_i} \to u_{x_i}$, $i=1,\dots,n$ *weakly in $L_\infty(Q_T)$. Thus we obtain the Lipschitz viscosity solution satisfying equation (0.2) a.e.. Note here that we do not need $H$ to be coercive.

*Remark.* If in (0.8) $H = H(t, \nabla u)$ then in Theorem 2 it is sufficient to require the function $H(t, \mathbf{q})$ to be only continuous.

In fact, approach $H(t, \mathbf{q})$ in $L_\infty$ norm by functions $H^\epsilon(t, \mathbf{q}) \in C^\alpha((0, T) \times \mathbf{R}^n)$. The equation

$$u_t^\epsilon + H^\epsilon(t, \nabla u^\epsilon) = 0$$

satisfies the conditions of Theorem 2. We pass to the limit when $\epsilon \to 0$ using the fact that the estimate of the gradient depends only on the max $H^\epsilon$.

Let us give several evident examples of equations (0.1) and (0.8) with $H$ satisfying conditions (0.4) and (0.3) (see also the introduction):

(a) $$u_t + h(t, \nabla u) = \varepsilon \Delta u, \quad u_t + h(t, \nabla u) = 0,$$

where $h(t, \mathbf{q})$ is an arbitrary function (defined for the finite values of its arguments) Hölder continuous for the parabolic case and continuous for the hyperbolic case;

(b) $$u_t + u^3 e^{|\nabla u|} |\nabla u|^\beta = \varepsilon \Delta u, \quad u_t + u^3 e^{|\nabla u|} |\nabla u|^\beta = 0, \quad \beta \in (0, 1);$$

(c) $$u_t + u^{1/3} \sum_{i=1}^n b_i(t) |u_{x_i}|^{\beta_i} = \varepsilon \Delta u, \quad u_t + u^{1/3} \sum_{i=1}^n b_i(t) |u_{x_i}|^{\beta_i} = 0,$$

where $\beta_i \geq 0$ are constants, $b_i$ are Hölder continuous functions;

(d) let $n = 2$,

$$u_t + \phi(t) x^{1/3} u_x^{2k+1} + \psi(t) y^3 u_y^{2l+1} = \varepsilon \Delta u, \quad u_t + \phi(t) x^{1/3} u_x^{2k+1} + \psi(t) y^3 u_y^{2l+1},$$

where $\phi(t) \geq 0$, $\psi(t) \geq 0$ are Hölder continuous functions and $k, l$ are positive integers.

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