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Smoothing effect of absorption for degenerate parabolic equations in 1-d

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Abstract In the present paper we consider the Cauchy and the Dirichlet problems for the equation

$$u_t = \big(\kappa(t,u)u_x\big)_x - f(t,u),$$

where $\kappa(t,u) \geq 0$ for $u \geq 0$. We formulate conditions on $\kappa(t,u)$ and $f(t,u)$ guaranteeing the existence of a global and local solutions such that $u_x \in L^\infty$ and $u_t \in L^2$.

Keywords Porous medium equation · Gradient estimates · Quasilinear equations

Mathematics Subject Classification 35K65 · 35K59 · 35K20

1 Introduction and formulation of the results

Consider the equation

$$u_t = \big(\kappa(t,u)u_x\big)_x - f(t,u), \quad (t,x) \in \Pi_T = (0,T) \times \mathbf{R}, \tag{1.1}$$

coupled with initial condition

$$u(0,x) = u_0(x) \quad \text{for } x \in \mathbf{R}, \tag{1.2}$$

where $T > 0$ is an arbitrary constant. We assume that

$$0 \leq u_0(x) \leq M, \quad |u_0(x) - u_0(y)| \leq K|x-y|, \quad \forall x, y \in \mathbf{R} \quad \text{and} \quad \|u_{0x}\|_{L^2(\mathbf{R})} < \infty, \tag{1.3}$$

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where M and K are some positive constants. Concerning the smoothness of coefficients we suppose that

$$\kappa(t,u) \in C^{1,1+\alpha}_{t,u}([0,T]\times(0,+\infty)), \quad f(t,u) \in C^{\alpha}([0,T]\times(0,+\infty)) \tag{1.4}$$

for some $\alpha \in (0,1)$. We impose the following structure restrictions

$$\kappa(t,0) = f(t,0) = 0, \quad \kappa(t,u) \geq 0 \quad \text{and} \quad f(t,u) \geq 0 \quad \text{for } u \geq 0 \tag{1.5}$$

and

$$K^2\big[\kappa_u(t,u_2) - \kappa_u(t,u_1)\big] \leq f(t,u_2) - f(t,u_1) \quad \text{for } u_2 > u_1 > 0, \quad t \in [0,T]. \tag{1.6}$$

Definition 1 We say that a bounded nonnegative Hölder continuous function $u(t,x)$ is a strong solution of problem (1.1), (1.2) if $u_x \in L^{\infty}(\Pi_T)$, $u_t \in L^2(\Pi_T)$, $\sqrt{\kappa(t,u)}u_x \in L^2(\Pi_T)$, and for an arbitrary smooth $\phi(t,x)$ vanishing for large $|x|$ the following identity is satisfied

$$\int_{\Pi_T} u_t\phi \mathrm{d}t\mathrm{d}x + \int_{\Pi_T} \kappa(t,u)u_x\phi_x \mathrm{d}t\mathrm{d}x + \int_{\Pi_T} f(t,u)\phi \mathrm{d}t\mathrm{d}x = 0.$$

Theorem 1 (global existence) *Assume that conditions* (1.3)–(1.6) *are fulfilled, then for any* $T > 0$ *there exists a strong solution* $u(t,x)$ *of problem* (1.1), (1.2). *Moreover*

$$0 \leq u \leq M \ \ \forall (t,x) \in \Pi_T, \quad \|u_x\|_{L^{\infty}(\Pi_T)} \leq K,$$

and

$$\|u\|^2_{L_2(\mathbf{R})} + 2\|\sqrt{\kappa}u_x\|^2_{L_2(\Pi_T)} \leq \|u_0\|^2_{L_2(\mathbf{R})} \quad \forall t \in (0,T).$$

Let us apply this Theorem to the porous media equation with absorption:

$$u_t = \big(mu^{m-1}u_x\big)_x - f(t,u). \tag{1.7}$$

Condition (1.6) takes the form

$$m(m-1)K^2\big[u_2^{m-2} - u_1^{m-2}\big] \leq f(t,u_2) - f(t,u_1), u_2 > u_1 > 0. \tag{1.8}$$

Example 1 Obviously this condition is fulfilled for an arbitrary nondecreasing Hölder continuous f if $m \in [1,2]$ and for such f, m above Theorem 1 guarantees the existence of a global strong solution of problem (1.7), (1.2). Note that we can take here $f \equiv 0$ as well.

Example 2 Consider equations

$$u_t = \big(u^{5/2}\big)_{xx} - \lambda u^{1/2}, \quad \lambda \geq 15/4\,K^2, \tag{1.9}$$

$$u_t = \big(u^{5/2}\big)_{xx} - \lambda u^{1/4}, \quad \lambda \geq 15/2\,K^2\sqrt{M}, \tag{1.10}$$

where $M = \max_{\Pi_T} u = \max_{\mathbf{R}} u_0$. One can easily see that (1.4)–(1.6) are fulfilled in both cases, and thus Theorem 1 guarantees the global solvability of problem (1.9), (1.2) and (1.10), (1.2). See also Example 3 in Sect. 2.4.

What happens if no absorption term is present? We will give an answer on this question for the porous medium equation:

$$u_t = \big(u^m\big)_{xx} \ \text{ in } \ \Pi_T. \tag{1.11}$$

Let us remind (see [2,6]) that in the one-dimensional case the optimal regularity of a weak solution of the Cauchy problem (1.11), (1.2) (with $m > 1$) is the following one: if u_0^{m-1} is

Lipschitz continuous then $u \in \mathcal{C}^\beta$ where $\beta = \min\{1, (m-1)^{-1}\}$. In particular $\beta = 1$ for $m \in (1, 2]$ which is consistent with the above Theorem 1 (see Example 1 with $f \equiv 0$). In the multidimensional case the situation is more complicated and is not completely set yet (see, for example [3,16] and the references therein).

Recall that by weak solution of problem (1.11), (1.2) we understand (see for example [1, 16]) a nonnegative bounded function u, with bounded generalized derivative $\left(u^m\right)_x$, satisfying the identity

$$\int_{\Pi_T} -u\psi_t \mathrm{d}t\mathrm{d}x + \int_{\Pi_T} \left(u^m\right)_x \psi_x \mathrm{d}t\mathrm{d}x = \int_R u_0(x)\psi(0,x)\mathrm{d}x,$$

for all smooth $\psi(t, x)$ vanishing for large $|x|$ and for $t = T$. Such solution exists for arbitrary $T > 0$, see for example [1,8,11,16].

Theorem 2 (local existence) *Assume that condition* (1.3) *is fulfilled and* $m \geq 3$*, then for some* $T^* > 0$ *there exists a strong solution* $u(t, x)$ *of problem* (1.11)*,* (1.2)*. Moreover*

$$0 \leq u \leq M \ \ \forall (t,x) \in \Pi_{T^*}, \quad \|u_x(t,x)\|_{L^\infty(\Pi_{T^*})} \leq K^*,$$

and

$$\|u\|^2_{L_2(\mathbf{R})} + 2\|\sqrt{\kappa}u_x\|^2_{L_2(\Pi_{T^*})} \leq \|u_0\|^2_{L_2(\mathbf{R})} \quad \forall t \in (0, T^*),$$

where $\Pi_{T^*} = (0, T^*) \times \mathbf{R}$*,* T^* *depends only on* m*,* K^2*,* M^{m-3} *and* K^* *depends only on* K*,* T^**.*

Actually Theorem 2 states that a weak solution (which as it was mentioned above exists for any $T > 0$) has better differential properties for $t \in (0, T^*)$. Note that T^* and K^* can be find explicitly.

Thus we give an answer to the question formulated above for $m \in (1, 2] \cup [3, \infty)$. For $m \in (2, 3)$ our approach seems not working.

Let us turn now to the Dirichlet problem. Consider equation

$$u_t = \left(\kappa(t,u)u_x\right)_x - f(t,u), \quad (t,x) \in Q_T = (0,T) \times (-l, l), \tag{1.12}$$

coupled with initial and boundary conditions

$$u(0,x) = u_0(x) \quad \text{for } |x| < l, \ u_0(\pm l) = 0 \quad \text{and} \quad u(t, \pm l) = 0 \quad \text{for} \ \ t \in [0,T]. \tag{1.13}$$

We suppose that

$$0 \leq u_0(x) \leq M, \quad |u_0(x) - u_0(y)| \leq K|x - y|, \quad \forall x, y \in [-l, l]. \tag{1.14}$$

Assume that conditions (1.4), (1.5) are fulfilled and instead of (1.6) we impose the more restrictive one

$$K^2\left[\kappa_u(t,u_2) - \kappa_u(t,u_1)\right] \leq f(t,u_2) - f(t,u_1) \quad \text{for} \ \ u_2 > u_1 \geq 0, \quad t \in [0,T] \tag{1.15}$$

Note that (1.15) unlike to (1.6) implies that

$$K^2\kappa_u(t,u) \leq f(t,u) \quad \text{for} \ \ u \geq 0, \quad t \in [0,T]$$

which will be used below in the boundary gradient estimates (see Lemma 2.2).

Definition 2 We say that a nonnegative Hölder continuous function $u(t,x)$ is a strong solution of problem (1.12), (1.13) if $u_x \in L^\infty(Q_T)$, $u_t \in L^2(Q_T)$ and for an arbitrary smooth $\phi(t,x)$ the following identity is satisfied

$$\int_{Q_T} u_t \phi \mathrm{d}t\mathrm{d}x + \int_{Q_T} \kappa(t,u) u_x \phi_x \mathrm{d}t\mathrm{d}x + \int_{Q_T} f(t,u)\phi \mathrm{d}t\mathrm{d}x = 0.$$

Theorem 3 (global existence) *Assume that conditions* (1.4), (1.5), (1.14), (1.15) *are fulfilled, then for any* $T > 0$ *there exists a strong solution* $u(t,x)$ *of problem* (1.12), (1.13). *Moreover*

$$0 \le u \le M \quad \forall (t,x) \in Q_T, \quad \|u_x(t,x)\|_{L^\infty(Q_T)} \le K.$$

Example 2 (but not Example 1) can be extended to the Dirichlet problem by obvious way (see also Example 3 in Sect. 2.4).

Let us consider the Dirichlet problem for the standard porous medium equation with $m \ge 3$

$$u_t = \left(u^m\right)_{xx}, \quad m \ge 3 \text{ in } Q_T, \tag{1.16}$$

coupled with conditions (1.13).

Theorem 4 (local existence) *Assume that condition* (1.14) *is fulfilled, then for some* $T^* > 0$ *there exists a strong solution* $u(t,x)$ *of problem* (1.16), (1.13). *Moreover*

$$0 \le u \le M \;\; \forall (t,x) \in Q_{T^*}, \quad \|u_x(t,x)\|_{L^\infty(Q_{T^*})} \le K^*$$

where $Q_{T^*} = (0, T^*) \times (-l, l)$, T^* *depends only on* m, K^2, M^{m-3} *and* K^* *depends only on* K, T^*.

As in Theorem 2, constants T^* and K^* will be find explicitly (see proof of this theorem in Sect. 2).

Note that the regularized effect of the low order term for the Burgers equation was demonstrated in [14] and was based on the modification of conditions (0.10)–(0.11) from [13]. In the present paper we actually apply similar approach giving another modification of conditions (0.10)–(0.11) from [13].

Different aspects of degenerate equations with absorption were investigated in [4,7,12,15].

The paper is organized as follows. We start with the Dirichlet problem in Sect. 2 and in Sect. 3 we consider the Cauchy problem.

2 The Dirichlet problem

2.1 Regularization

Rewrite Eq. (1.12) in the nondivergent form and consider the regularized equation

$$u_t = \big(\kappa(t,u) + \varepsilon\big) u_{xx} + \kappa_u(t,u) u_x^2 - f(t,u), \tag{2.1}$$

where ε is an arbitrary positive constant.

First consider the auxiliary equation

$$u_t = \big(\overline{\kappa}(t,u) + \varepsilon\big) u_{xx} + \kappa_u(t,u) u_x^2 - \overline{f}(t,u) \tag{2.2}$$

coupled with initial and boundary conditions (1.13), here

$$\overline{\kappa}(t,z)=\begin{cases}\kappa(t,z), & \text{for } z\ge 0\\ 0, & \text{for } z<0\end{cases}\quad \overline{f}(t,z)=\begin{cases} f(t,z), & \text{for } z\ge 0\\ 0, & \text{for } z<0\end{cases}.$$

The existence of classical solution u_ε of problem (2.2), (1.13) under the smoothness assumptions (1.4) follows from [13].

2.2 A priori estimates

For simplicity in the proofs of lemmas below we omit the subindex ε.

Lemma 2.1 *For a classical solution $u_\varepsilon(t,x)$ of problem* (2.2), (1.11) *the estimate*

$$0\le u_\varepsilon(t,x)\le M,$$

holds in Q_T.

Proof For $v=u\,e^{-\delta t}$, $\delta>0$ constant, we have

$$v_t-\big(\overline{\kappa}(t,u)+\varepsilon\big)v_{xx}-\kappa_u(t,u)v_x^2e^{\delta t}+\delta v=-\overline{f}(t,u)e^{-\delta t}. \tag{2.3}$$

Denote by Γ_T the parabolic boundary of Q_T, i.e.

$$\Gamma_T=\partial Q_T\setminus\{t=T,|x|<l\}.$$

Suppose that the function v attains its negative minimum at the point $N\in\overline{Q_T}\setminus\Gamma_T$, then at this point the left-hand side of (2.3) is strictly negative and the right-hand side is zero (since $v<0$ implies $u<0$ implies $\overline{f}(t,u)=0$). From this contradiction we conclude that v cannot attain its negative minimum at the internal points and hence, taking into account the initial and the boundary conditions, we conclude that $v(t,x)\ge 0$.

Assume now that the function v attains its positive maximum at the point $N_1\in\overline{Q}_T\setminus\Gamma_T$, then at this point the left-hand side of (2.3) is strictly positive and the right-hand side is nonpositive (since $v>0$ implies $u>0$ implies $\overline{f}(t,u)\ge 0$). Hence v cannot attain its positive maximum at the internal points and taking into account the boundary conditions we conclude that $v(t,x)\le M$ and thus

$$0\le u(t,x)\le e^{\delta t}M.$$

Passing to the limit as $\delta\to 0$ we obtain the needed estimate. □

From the above estimate it immediately follows that Eq. (2.2) coincides with Eq. (2.1), so, instead of problem (2.2), (1.13) we will consider problem (2.1), (1.13).

Lemma 2.2 *For a classical solution $u_\varepsilon(t,x)$ of problem* (2.1), (1.13) *the estimates*

$$u_\varepsilon(t,x)\le K(x+l),\quad u_\varepsilon(t,x)\le K(l-x)$$

hold in Q_T.

Proof Let

$$v(t,x)=\big(u(t,x)-K(x+l)\big)e^{-\delta t}.$$

By direct calculations we obtain that

$$v_t+\delta v-(\kappa(t,u)+\varepsilon)v_{xx}=e^{-\delta t}[\kappa_u(t,u)u_x^2-f(t,u)]. \tag{2.4}$$

Suppose that at the point $N \in \overline{Q}_T \setminus \Gamma_T$ the function v attains its positive maximum, we have

$$v_t + \delta v - (\kappa(t,u) + \varepsilon)v_{xx}\Big|_N > 0, \tag{2.5}$$

on the other hand at this point

$$v > 0, \quad v_x = 0 \;\Leftrightarrow\; u > 0, \quad u_x = K$$

and hence, due to (1.5), (1.15)

$$e^{-\delta t}[\kappa_u(t,u)u_x^2 - f(t,u)]\Big|_N = e^{-\delta t}[\kappa_u(t,u)K^2 - f(t,u)]\Big|_N \le 0.$$

This contradicts (2.5); thus, v cannot attain its positive maximum at the internal points of Q_T. Consider v on the parabolic boundary of Q_T :

for $t = 0$ we have

$$v(0,x) = \big(u_0(x) - K(x+l)\big)e^{-\delta t} \le 0,$$

for $x = \pm l$ we have

$$v(t,\pm l) = \big(-K(\pm l + l)\big)e^{-\delta t} \le 0.$$

Hence

$$v(t,x) \le 0 \;\text{ for }\; (t,x) \in \overline{Q}_T$$

and consequently

$$u(t,x) \le K(x+l) \;\text{ for }\; (t,x) \in \overline{Q}_T.$$

Similarly, for

$$w(t,x) = \big(u(t,x) - K(l-x)\big)e^{-\delta t}$$

we obtain that

$$w_t + \delta w - (\kappa(t,u) + \varepsilon)w_{xx} = e^{-\delta t}[\kappa_u(t,u)u_x^2 - f(t,u)] \tag{2.6}$$

Suppose that at the point $N_1 \in \overline{Q}_T \setminus \Gamma_T$ the function w attains its positive maximum, then at this point the left-hand side of (2.6) is strictly positive, on the other hand at this point

$$u > 0, \;\; u_x = -K$$

and hence, due to (1.5), (1.15) the right-hand side of (2.6) is less or equal zero. From this contradiction we conclude that w cannot attain its positive maximum at the internal points of Q_T. Consider w on the parabolic boundary of Q_T :

$$w(0,x) = \big(u_0(x) - K(l-x)\big)e^{-\delta t} \le 0, \;\; w(t,\pm l) = \big(-K(l \mp l)\big)e^{-\delta t} \le 0.$$

Hence

$$w(t,x) \le 0 \;\text{ for }\; (t,x) \in \overline{Q}_T$$

and consequently

$$u(t,x) \le K(l-x) \;\text{ for }\; (t,x) \in \overline{Q}_T.$$

□

Lemma 2.3 *For a classical solution of problem* (2.1), (1.13) *the estimate*

$$|u_{\varepsilon x}(t,x)| \le K$$

holds in Q_T.

Proof Consider Eq. (2.1) at two different points (t,x), (t,y), $x>y$:

$$u_t = (\kappa(t,u)+\varepsilon)u_{xx} + \kappa_u(t,u)u_x^2 - f(t,u), \quad u = u(t,x), \tag{2.7}$$

$$u_t = (\kappa(t,u)+\varepsilon)u_{yy} + \kappa_u(t,u)u_y^2 - f(t,u), \quad u = u(t,y). \tag{2.8}$$

Subtracting (2.8) from (2.7) for

$$\bar{w}(t,x,y) = u(t,x) - u(t,y) - K(x-y)$$

we obtain

$$\begin{aligned} &\bar{w}_t - (\kappa(t,u(t,x))+\varepsilon)\bar{w}_{xx} - (\kappa(t,u(t,y))+\varepsilon)\bar{w}_{yy} \\ &\quad = \kappa_u(t,u(t,x))u_x^2(t,x) - f(t,u(t,x)) - \kappa_u(t,u(t,y))u_y^2(t,y) + f(t,u(t,y)) \end{aligned}$$

and for $w = \bar{w}e^{-t}$ we have

$$\begin{aligned} L\,w &\equiv w_t + w - (\kappa(t,u(t,x))+\varepsilon)w_{xx} - (\kappa(t,u(t,y))+\varepsilon)w_{yy} \\ &= \Big[\kappa_u(t,u(t,x))u_x^2(t,x) - f(t,u(t,x)) \\ &\quad -\kappa_u(t,u(t,y))u_y^2(t,y) + f(t,u(t,y))\Big]e^{-t}. \end{aligned} \tag{2.9}$$

Consider Eq. (2.9) in the domain

$$P = \{(t,x,y) : t\in(0,T), x\in(-l,l), y\in(-l,l), x>y\}.$$

Denote by Γ the parabolic boundary of P, i.e.

$$\Gamma = \partial P \setminus \{t=T,\ x\in(-l,l), y\in(-l,l), x>y\}.$$

Suppose that the function w attains its positive maximum at the point $N \in \overline{P}\setminus\Gamma$, then at this point we have

$$u(t,x) > u(t,y) \quad \text{and} \quad u_x(t,x) = u_y(t,y) = K,$$

hence due to condition (1.5)

$$L\,w\Big|_N < 0$$

which is impossible. Consequently w cannot attain its positive maximum at the internal points of P. Consider the parabolic boundary of the domain P:

1. for $t=0$ we have $w(0,x,y) = u_0(x) - u_0(y) - K(x-y) \le 0$;
2. for $x=y$ we have $w=0$;
3. for $x=l$, $y\in[-l,l]$, since $-u(t,y)\le 0$, we have

$$w(t,l,y) = \big(-u(t,y) - K(l-y)\big)e^{-t} \le 0;$$

   and finally
4. for $y=-l$, $x\in[-l,l]$, due to Lemma 2.2, we have

$$w(t,x,-l) = \big(u(t,x) - K(x+l)\big)e^{-t} \le 0.$$

Thus we obtain that for $x > y$

$$u(t,x) - u(t,y) \le K(x-y).$$

Similarly, subtracting (2.7) from (2.8) for

$$v(t,x,y) = \Big(u(t,y) - u(t,x) - K(x-y)\Big)e^{-t}$$

we obtain

$$\begin{aligned} L\,v = \Big[&\kappa_u(t,u(t,y))u_y^2(t,y) - f(t,u(t,y)) \\ &-\kappa_u(t,u(t,x))u_x^2(t,x) + f(t,u(t,x))\Big]e^{-t}. \end{aligned} \tag{2.10}$$

Consider Eq. (2.10) in the domain P. Suppose that the function v attains its positive maximum at the point $N \in \overline{P} \setminus \Gamma$, then at this point we have

$$u(t,y) > u(t,x) \ \text{ and } \ u_y(t,y) = u_x(t,x) = -K,$$

hence due to condition (1.5)

$$L\,v\Big|_N < 0$$

which is impossible. Thus v cannot attain its positive maximum at the internal points of P. Consider the parabolic boundary of the domain P:

1. for $t = 0$ we have $v(0,x,y) = u_0(y) - u_0(x) - K(x-y) \le 0$;
2. for $x = y$ we have $v = 0$;
3. for $x = l$, $y \in [-l, l]$ due to Lemma 2.2, we have

$$v(t,l,y) = \big(u(t,y) - K(l-y)\big)e^{-t} \le 0,$$

   and finally
4. for $y = -l$, $x \in [-l, l]$ we have

$$v(t,x,-l) = \big(-u(t,x) - K(x+l)\big)e^{-t} \le 0$$

   since $-u(t,y) \le 0$.

Thus we obtain that for $x > y$

$$|u(t,x) - u(t,y)| \le K(x-y).$$

In view of the symmetry of the variables x, y we conclude that for $t \in [0,T]$, $|x| \le l$, $|y| \le l$ we have

$$|u(t,x) - u(t,y)| \le K|x-y|$$

which implies the required estimate. □

Lemma 2.4 *For a classical solution of problem* (2.1), (1.13) *the estimate*

$$\int_{Q_T} u_{\varepsilon t}^2(t,x)\mathrm{d}t\mathrm{d}x \le C$$

holds, where the constant C do not depend on ε.

Proof Rewrite Eq. (2.1) in divergent form

$$u_{\varepsilon t} = \big(\kappa(t, u_\varepsilon) + \varepsilon) u_{\varepsilon x}\big)_x - f(t, u_\varepsilon). \tag{2.11}$$

Multiplying (2.11) by u_t and integrating by parts with respect to x we obtain

$$\int_{-l}^{l} u_t^2 \mathrm{d}x = -\frac{1}{2}\int_{-l}^{l} (\kappa + \varepsilon)\big(u_x^2\big)_t \mathrm{d}x - \int_{-l}^{l} f u_t \mathrm{d}x.$$

Now integrating by parts with respect to t we obtain

$$\int_{Q_T} u_t^2 \mathrm{d}t\mathrm{d}x = -\frac{1}{2}\int_{-l}^{l} (\kappa + \varepsilon) u_x^2\Big|_{t=0}^{t=T} \mathrm{d}x + \frac{1}{2}\int_{Q_T} \kappa_t u_x^2 \mathrm{d}t\mathrm{d}x$$
$$+\frac{1}{2}\int_{Q_T} \kappa_u u_t u_x^2 \mathrm{d}t\mathrm{d}x - \int_{Q_T} f u_t \mathrm{d}t\mathrm{d}x.$$

Applying the Hölder and Young inequalities we conclude

$$\int_{Q_T} u_t^2 \mathrm{d}t\mathrm{d}x + 2\int_{-l}^{l} (\kappa(T, u(T, x)) + \varepsilon) u_x^2(T, x)\mathrm{d}x)$$
$$\leq 2\int_{-l}^{l} (\kappa(0, u_0(x)) + \varepsilon) u_0^2(x)\mathrm{d}x + 2\int_{Q_T} |\kappa_t| u_x^2 \mathrm{d}t\mathrm{d}x$$
$$+ \int_{Q_T} \kappa_u u_x^4 \mathrm{d}t\mathrm{d}x + 2\int_{Q_T} f^2 \mathrm{d}t\mathrm{d}x,$$

from where the assertion of the Lemma follows. □

2.3 Proof of Theorem 3

We have constructed functions u_ε-classical solutions of (2.1), (1.2). Multiplying (2.11) by an arbitrary smooth ϕ and integrating by part we obtain

$$\int_{Q_T} u_{\varepsilon t}\phi \mathrm{d}t\mathrm{d}x + \int_{Q_T} \big(\kappa(t, u_\varepsilon) + \varepsilon\big) u_{\varepsilon x}\phi_x \mathrm{d}t\mathrm{d}x + \int_{Q_T} f(t, u_\varepsilon)\phi \mathrm{d}t\mathrm{d}x$$
$$= \varepsilon \int_0^T u_{\varepsilon x}\phi\Big|_{x=-l}^{x=l} \mathrm{d}t. \tag{2.12}$$

Note that the for u_ε the following estimate (see [5,9]) takes place:

$$|u_\varepsilon(t_1, x) - u_\varepsilon(t_2, x)| \leq C|t_1 - t_2|^{1/2} \tag{2.13}$$

with constant C depending only on M, K and max $(\kappa + \varepsilon)$, max κ_u, max f (maximum is taken over the set $[0, T] \times [0, M]$). The above estimate and Lemmas 2.1, 2.3, 2.4 guarantee the existence of subsequence ε_n such that

$$u_{\varepsilon_n} \to u \text{ uniformly},$$
$$u_{\varepsilon_n x} \to u_x \text{ *weakly in } L_\infty(Q_T),$$
$$u_{\varepsilon_n t} \to u_t \text{ *weakly in } L_2(Q_T)$$

as $n \to \infty$ ($\varepsilon_n \to 0$). Based on these we can pass to the limit in the identity (2.12) and by this prove the existence.

2.4 Proof of Theorem 4

For the function $v = ue^{-\mu t}$, $\mu > 0$ the problem (1.16), (1.13) takes the form

$$v_t = (\kappa(t, v)v_x)_x - \mu v, \quad m \geq 3, \tag{2.14}$$

$$v(0, x) = u_0(x) \geq 0, \quad v(t, \pm l) = 0, \tag{2.15}$$

with

$$\kappa(t, v) = me^{\mu(m-1)t} v^{m-1}.$$

Condition (1.15) in this case can be written as follows

$$m(m-1)K^2\left[v_2^{m-2} - v_1^{m-2}\right] \leq \mu e^{\mu(1-m)t}[v_2 - v_1], \quad v_2 > v_1 \geq 0 \tag{2.16}$$

or

$$t \leq \frac{1}{\mu(m-1)} \ln\left(\frac{\mu}{m(m-1)K^2} \frac{v_2 - v_1}{v_2^{m-2} - v_1^{m-2}}\right).$$

The last is fulfilled if

$$t \leq \frac{1}{\mu(m-1)} \ln\left(\frac{\mu}{m(m-1)K^2} \frac{1}{(m-2)M^{m-3}}\right),$$

where $M = \max v = \max u$. Define $T^* > 0$ by the following

$$T^* = \sup_{\mu} \frac{1}{\mu(m-1)} \ln\left(\frac{\mu}{m(m-1)(m-2)K^2 M^{m-3}}\right),$$

here supremum is taking for $\mu > m(m-1)(m-2)K^2M^{m-3}$. Thus Theorem 3 guarantees the existence of a strong solution of problem (2.14), (2.15) on the interval $(0, T^*)$ and the estimate

$$\|v_x\|_{L^\infty(0,T^*)} \leq K = \max |v_x(0, x)| = \max |u_{0x}(x)|.$$

Returning to problem (1.16), (1.13) we finish the proof of the existence. Obviously

$$\|u_x\|_{L^\infty(0,T^*)} \leq Ke^{\mu t} \leq K^* = Ke^{\mu T^*}.$$

Example 3 Consider equation

$$u_t = \left(u^3\right)_{xx} - \lambda u, \quad \lambda \geq 6K^2 \text{ in } Q_T, \tag{2.17}$$

One can easily see that (1.15) is fulfilled and thus Theorem 3 guarantees the global solvability of problem (2.17), (1.13).

Now consider equation

$$u_t = \left(u^3\right)_{xx} \text{ in } Q_T, \tag{2.18}$$

For $v = ue^{-\mu t}$ we obtain

$$v_t = \left(3e^{2\mu t} v^2 v_x\right)_x - \mu t.$$

Condition (2.16) takes the form

$$6K^2 \leq \mu e^{-2\mu t} \quad \Leftrightarrow \quad t \leq \frac{1}{2\mu} \ln \frac{\mu}{6K^2}.$$

Supremum of $1/2\mu\ \ln(\mu/6K^2)$ (with respect to μ under restriction $\mu > 6K^2$) is obtained for $\mu = e\,6K^2$, hence for

$$t \in \left(0, \frac{1}{12eK^2}\right)$$

we have

$$|v_x(t,x)| \le K \quad \Leftrightarrow \quad |u_x(t,x)| \le Ke^{6eK^2t} \le K^* = K\sqrt{e}.$$

Thus we conclude that Theorem 4 guarantees the existence of a strong solution to the problem (2.18), (1.13) on the interval

$$t \in (0, T^*), \quad T^* = \frac{1}{12eK^2}.$$

3 The Cauchy problem

3.1 Regularized problem

Consider the following regularized problem

$$u_t = \big(\kappa(t,u) + \varepsilon\big)u_{xx} + \kappa_u(t,u)u_x^2 - f(t,u) \ \text{ in } \ Q_T^l = (0,T) \times (-l,l) \tag{3.1}$$

coupled with initial and boundary conditions

$$u(0,x) = h_l(x), \quad u_x(t,\pm l) \pm \delta u(t,\pm l) = 0, \tag{3.2}$$

where

$$0 \le h_l(x) \le M, \quad |h_l'(x)| \le K, \quad \forall x \in [-l,l]$$

and $h_{lx}(\pm l) \pm \delta h_0(\pm l) = 0$, here ε and l are arbitrary positive constants and positive constant δ is such that

$$M\delta < K$$

(we will use the last inequality in the proof of Lemma 3.2).

First consider the following auxiliary equation

$$u_t = \big(\overline{\kappa}(t,u) + \varepsilon\big)u_{xx} + \kappa_u(t,u)u_x^2 - \overline{f}(t,u) \ \text{ in } \ Q_T^l \tag{3.3}$$

where functions $\overline{\kappa}(t,u)$, $\overline{f}(t,u)$ where defined in the previous section.

The existence of classical solution of problem (3.3), (3.2) under the smoothness assumptions (1.4) follows from [13].

3.2 A priori estimates

Lemma 3.1 *For a classical solution $u_\varepsilon(t,x)$ of problem (3.3), (3.2) the estimate*

$$0 \le u_\varepsilon(t,x) \le M,$$

holds in Q_T.

Proof The proof is similar to the proof of Lemma 2.1. The only difference is on the lateral boundary of the domain, i.e. $x = \pm l$, $t \in [0, T]$ where for $v = ue^{-\delta t}$ we have

$$v_x(t, \pm l) \pm \delta v(t, \pm l) = 0.$$

Hence on the lateral boundary the function v cannot attain positive maximum or negative minimum. Thus, similarly to the proof of Lemma 2.1 we conclude that the function v attains its positive maximum and negative minimum at the initial moment and consequently

$$0 \le u \le M.$$

□

From the above estimate it immediately follows that equation (3.3) coincides with equation (3.1), so, instead of problem (3.3), (3.2) we will consider problem (3.1), (3.2).

Lemma 3.2 *For a classical solution of problem* (3.1), (3.2) *the estimate*

$$|u_{\varepsilon x}(t, x)| \le K$$

holds in Q_T.

Proof The proof is similar to the proof of Lemma 2.3. The only difference is on the lateral boundary of the domain P.

Consider the function $w(t, x, y)$ (see the proof of Lemma 2.3) on the lateral boundary of the domain P:

1. for $x = y$ we have $w = 0$;
2. for $x = l$, $y \in [-l, l]$ we have

$$w_x(t, l, y) = \big(u_x(t, l) - K\big)e^{-t} = \big(-\delta u(t, l) - K\big)e^{-t} < 0$$

   and thus the function w cannot attain its positive maximum on this part of the lateral boundary;
3. for $y = -l$, $x \in [-l, l]$ we have

$$w_y(t, x, -l) = \big(-u_y(t, -l) + K\big)e^{-t} = \big(-\delta u(t, -l) + K\big)e^{-t} > 0$$

   and the function w cannot attain its positive maximum on this part of the lateral boundary as well (recall that $K > \delta M$ and note that on this part of the boundary the derivative with respect to y is inward derivative).

Thus similarly to Lemma 2.3 we conclude that for $x > y$

$$u(t, x) - u(t, y) \le K(x - y).$$

Now consider the function $v(t, x, y)$ on the lateral boundary of the domain P:

1. for $x = y$ we have $v = 0$;
2. for $x = l$, $y \in [-l, l]$ we have

$$v_x(t, l, y) = (-u_x(t, l) - K)e^{-t} = (\delta u(t, l) - K) < 0,$$

3. for $y = -l$, $x \in [-l, l]$ we have

$$v_y(t, x, -l) = (u_y(t, -l) + K)e^{-t} = \big(\delta u(t, -l) + K\big)e^{-t} > 0$$

   and hence the function v cannot attain its positive maximum on part 2. and 3. of the lateral boundary.

Thus we obtain that for $x > y$

$$|u(t,x) - u(t,y)| \le K(x-y).$$

In view of the symmetry of the variables x y we conclude that for $t \in [0,T]$, $|x| \le l$, $|y| \le l$ we have

$$|u(t,x) - u(t,y)| \le K|x-y|$$

which implies the required estimate. □

3.3 Proof of Theorems 1 and 2

We have constructed functions u_ε-classical solutions of (3.1), (3.2). Note that the estimates of u_ε, $u_{\varepsilon x}$ as well as the estimate

$$|u_\varepsilon(t_1,x) - u_\varepsilon(t_2,x)| \le C|t_1 - t_2|^{1/2} \tag{3.4}$$

are independent of δ (and of ε). From Schauder estimates we conclude that for some $\gamma \in (0,1)$ the $\mathcal{C}^{1+\gamma/2,2+\gamma}_{t,x}$ norm of u_ε is bounded by a constant independent of δ (but of course depending on ε). Thus we can pass to the limit as $\delta \to 0$ and obtain the classical solution of the homogeneous Neumann problem:

$$u_{\varepsilon t} = \big(\kappa(t,u_\varepsilon) + \varepsilon\big)u_{\varepsilon xx} + \kappa_u(t,u_\varepsilon)u^2_{\varepsilon x} - f(t,u_\varepsilon) \ \text{ in } \ Q^l_T, \tag{3.5}$$

$$u(0,x) = h_l(x), \quad u_x(t,\pm l) = 0, \tag{3.6}$$

where $h_{lx}(\pm l) = 0$.

Obviously the estimates of Lemma 3.1 and 3.2 as well as the estimate (3.4) hold for the solution of problem (3.5), (3.6). Let us obtain the needed integral estimates.

Lemma 3.3 *For a classical solution of problem* (3.5), (3.6) *the estimate*

$$\int_{Q_T} u^2_{\varepsilon t}\mathrm{d}t\mathrm{d}x \le C$$

holds, where the constant C *do not depend on* ε.

The proof is similar to the proof of Lemma 2.4.

Lemma 3.4 *For a classical solution of problem* (3.5), (3.6) *the estimate*

$$\int_{-l}^{l} u^2_\varepsilon \mathrm{d}x + 2\int_{Q_T} \kappa(t,u_\varepsilon)u^2_{\varepsilon x}\mathrm{d}x\mathrm{d}t \le \int_{-l}^{l} u^2_0(x)\mathrm{d}x$$

holds $\forall t \in (0,T)$, *where the constant* C *do not depend on* ε.

Proof Multiplying (3.5) by u and integrating by parts we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-l}^{l} u^2\mathrm{d}x + \int_{-l}^{l} \big(\kappa(t,u) + \varepsilon\big)u^2_x\mathrm{d}x = -\int_{-l}^{l} f\, u\mathrm{d}x \le 0.$$

Integrating with respect to t we obtain the required estimate. □

Let us extend $h_l(x)$ by zero for $|x| > l$. Now we approach the initial function u_0 by (extended) functions h_l and obtain the (classical) solution of the Cauchy problem

$$u_{\varepsilon t} = \big(\kappa(t,u_\varepsilon) + \varepsilon\big)u_{\varepsilon xx} + \kappa_u(t,u_\varepsilon)u^2_{\varepsilon x} - f(t,u_\varepsilon) \ \text{ in } \ \Pi_T, \tag{3.7}$$

$$u(0,x) = u_0(x), \tag{3.8}$$

passing to the limit as $l \to \infty$ and applying the standard diagonal process ([10]). Note that the a priori estimates obtained above are independent of l and hence are valid for the problem (3.7), (3.8) as well.

Thus there exists a subsequence ε_n such that

$$u_{\varepsilon_n} \to u \ \text{uniformly},$$
$$u_{\varepsilon_n x} \to u_x \ \text{*weakly in} \ L_\infty(\Pi_T),$$
$$u_{\varepsilon_n t} \to u_t \ \text{*weakly in} \ L_2(\Pi_T)$$

as $n \to \infty$ ($\varepsilon_n \to 0$). Based on these we can pass to the limit in the identity

$$\int_{\Pi_T} u_{\varepsilon t}\phi \mathrm{d}t\mathrm{d}x + \int_{\Pi_T} \big(\kappa(t,u_\varepsilon) + \varepsilon\big)u_{\varepsilon x}\phi_x \mathrm{d}t\mathrm{d}x + \int_{\Pi_T} f(t,u_\varepsilon)\phi \mathrm{d}t\mathrm{d}x = 0$$

with an arbitrary smooth $\phi(t,x)$ vanishing for large $|x|$. By this we finish the proof of Theorem 1.

The proof of Theorem 2 is similar to the proof of Theorem 4.

Note that Example 3 by obvious way can be extended to the Cauchy problem.

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