Arch. Rational Mech. Anal. 152 (2000) 81–92. © Springer-Verlag 2000

# *On the First Boundary Value Problem for Quasilinear Parabolic Equations with Two Independent Variables*

ALKIS S. TERSENOV

*Communicated by* J. SERRIN

**Abstract**

This paper is concerned with the global solvability of the first initial boundary value problem for the quasilinear parabolic equations with two independent variables: $a(t, x, u, u_x)u_{xx} - u_t = f(t, x, u, u_x)$. We investigate the case when the growth of $\frac{|f(t,x,u,p)|}{a(t,x,u,p)}$ with respect to $p$ is faster than $p^2$ when $|p| \to \infty$. Conditions which guarantee the global classical solvability of the problem are formulated.

## 0. Introduction

In the present paper we investigate the global solvability of the following problem

$$a(t, x, u, u_x)u_{xx} - u_t = f(t, x, u, u_x) \text{ in } Q_T = (-l, l) \times (0, T), \qquad (0.1)$$

$$u(0, x) = u_0(x) \text{ for } |x| < l, \quad u(t, \pm l) = 0, \qquad (0.2)$$

where for $(t, x) \in \bar{Q}_T$, $|u| \leqq M$ ($M > 0$ is some constant) and for any $p$ the functions $a(t, x, u, p)$, $f(t, x, u, p)$ are Hölder continuous and

$$a(t, x, u, p) > 0, \qquad (0.3)$$

$$|f(t, x, u, p)| \leqq a(t, x, u, p)\psi(|p|). \qquad (0.4)$$

In addition we suppose that $u_0(x)$ is a Lipschitz continuous function and $u_0(\pm l) = 0$. S. N. KRUZHKOV in [1] shows that if the $C^1$ function $\psi(\rho)$ is such that $\psi(\rho) > 0$ for $\rho > 0$ and

$$\int_1^{+\infty} \frac{\rho \, d\rho}{\psi(\rho)} = +\infty \quad (\text{or } \psi(\rho) = \text{Const}(1 + \rho^2)), \qquad (0.5)$$

then under the above mentioned assumptions on $a$, $f$ and $u_0$ there exists a global (i.e., for any $T > 0$) solution of problem (0.1), (0.2) from the class $C_{t,x}^{1+\beta/2,2+\beta}(Q_T) \cap C^0(\bar{Q}_T)$ for some $\beta \in (0,1)$. It is well known that assumption (0.5) (or Bernstein-Nagumo-Tonelli condition (see [2]–[4]) on no more than quadratic growth of the function $\frac{|f(t,x,u,p)|}{a(t,x,u,p)}$ with respect to $p$ when $|p| \to +\infty$) is generally speaking necessary for the global solvability of problem (0.1), (0.2). Examples show that in the case of violation of condition (0.5) the gradient of the bounded solution may blow up on the boundary of the domain (see [5]–[11]) as well as in the interior of the domain (see [12]–[14]); i.e., there exists a $t^*$ such that $|u_x(t,x_0)| \to +\infty$ when $t \to t^*$ for some $x_0 \in [-l,l]$.

The present paper is devoted to the generalization of condition (0.5). In the first two sections we obtain *a priori* estimates of the gradient of the solution. Note that when obtaining these estimates we do not need any assumption on the smoothness of the functions $a$, $f$. We use the Hölder continuity of these functions only in the third section where we prove the existence theorem.

The first section deals with the boundary gradient estimate. We obtain here the following result. Let $\psi(0) \geqq 0$, $\psi(\rho) > 0$ for $\rho > 0$ and suppose that there exist $p_0$ and $p_1$ such that

$$\int_{p_0}^{p_1} \frac{\rho\, d\rho}{\psi(\rho)} \geqq \mu \equiv \max\{M, \operatorname{osc}(u), K_0 l\} \tag{0.6}$$

and

$$\int_{p_0}^{p_1} \frac{d\rho}{\psi(\rho)} \leqq l, \tag{0.7}$$

where $0 < p_0 < p_1 < +\infty$, $M = \sup_{Q_T} |u|$, $|u_0(x)| \leqq K_0(l - |x|)$, $K_0 > 0$ is some constant and $\operatorname{osc}(u) = \sup u - \inf u$. Then the gradient of the solution is bounded on the boundary of the domain for any $T > 0$. If condition (0.7) is not fulfilled then in order to obtain the same result we should impose an additional condition on $u_0(x)$:

$$|u_0(x)| \leqq h_i(x), \quad i = 1, 2, \tag{0.8}$$

where the functions $h_i(x)$ are defined in Section 1. Note that the fulfilment of condition (0.5) implies the fulfilment of conditions (0.6), (0.7). In fact, let us take $p_0 = \mu/l$ and select $p_1$ such that

$$\int_{p_0}^{p_1} \frac{\rho d\rho}{\psi(\rho)} = \mu$$

(that is possible due to the divergence of the integral on $+\infty$). We have

$$\int_{p_0}^{p_1} \frac{d\rho}{\psi(\rho)} \leqq \frac{1}{p_0} \int_{p_0}^{p_1} \frac{\rho\, d\rho}{\psi(\rho)} = \frac{\mu}{p_0} = l.$$

Let us give an example (see also the example in Section 3).

*Example.* Consider the following problem

$$u_{xx} - u_t = f(t, x, u)u_x^3 \text{ in } (-l, l) \times (0, T), \tag{0.9}$$

$$u(0, x) = u_0(x) \text{ in } (-l, l) \text{ and } u(t, \pm l) = 0 \text{ for } t \in [0, T], \tag{0.10}$$

where $T$ is an arbitrary positive, $u_0(x)$ is Lipschitz continuous function, $u_0(\pm l) = 0$ and $f(t, x, u)$ is bounded in $[-l, l] \times [0, T] \times [-M, M]$. Denote by $f_0$ the sup $|f|$. Let us write conditions (0.6), (0.7):

$$\frac{1}{p_0} - \frac{1}{p_1} \geqq \max\{M, \operatorname{osc}(u), K_0 l\} f_0,$$

$$\frac{1}{p_0^2} - \frac{1}{p_1^2} \leqq 2l f_0.$$

Select $p_0$ and $p_1$ so that

$$\frac{1}{p_0} - \frac{1}{p_1} = \max\{M, \operatorname{osc}(u), K_0 l\} f_0. \tag{0.11}$$

In the second condition we also take "=", and using (0.11) obtain

$$\frac{1}{p_0} + \frac{1}{p_1} = \frac{2l}{\max\{M, \operatorname{osc}(u), K_0 l\}}. \tag{0.12}$$

Obviously system (0.11), (0.12) is solvable with $p_1 > p_0 > 0$ if

$$2l > (\max\{M, \operatorname{osc}(u), K_0 l\})^2 f_0.$$

This inequality is equivalent to

$$2 > K_0^2 l f_0, \tag{0.13}$$

because for the solution of the problem (0.9), (0.10) we have $M = \sup |u_0| \leqq K_0 l$ and (see Remark at the end of Section 1) in order to obtain the boundary gradient estimate it is sufficient to take $\mu = \max\{M, K_0 l\}$. The fact that $\mu \geqq \operatorname{osc}(u)$ is used in the proof of the global gradient estimate. Thus if condition (0.13) is fulfilled we have the boundary gradient estimate independent of $T$. Note that we can always select $p_1 > p_0 > 0$ so that condition (0.11) is fulfilled. If it is impossible to satisfy at the same time condition (0.12), then in order to estimate the gradient on the boundary we should impose condition (0.8). If in (0.9) the function $f$ does not depend on $u$, is differentiable with respect to $x$ and $f_x \geqq 0$, then (having the boundary gradient estimates) we can easily obtain the global gradient estimate by differentiating the equation and applying the maximum principle to the function $w(t, x) \equiv u_x(t, x)$. In the general case, in order to obtain global gradient estimates we also need the fulfilment of conditions (0.4), (0.5). Otherwise we can have interior gradient blow-up (see [12]).

In the second section we show that fulfilment of conditions (0.4), (0.6), (0.7) or (0.4), (0.6), (0.8) together with the following condition on the modulus of continuity of $u_0(x)$: $|u_0(x) - u_0(y)| \leqq h_1(|x - y| - l)$, implies the global *a priori* estimate of the gradient. If we want to avoid this condition on $u_0(x)$ we need additional assumptions on the function $f(t, x, u, p)$. Namely, the function $f$ can be represented in the form $f(t,x,u,p) = f_1(t,x,u,p) + f_2(t,x,u,p)$ where the first term $f_1$ satisfies conditions (0.4), (0.5), the second $f_2$ satisfies conditions which guarantee the boundary gradient estimates and

$$f_2(t,x,u,p) - f_2(t,y,v,p) \geqq 0, \quad f_2(t,y,u,-p) - f_2(t,x,v,-p) \geqq 0 \tag{0.14}$$

when $x \geqq y$, $u \geqq v$, $p \geqq 0$. The function $f_2$ satisfies conditions (0.14) if, for example, $f_2 = f_2(t,p)$ or $f_2 = g(t,x)h_1(t,p)$, where $g$ is nondecreasing with respect to $x$ and $ph_1(t,p) \geqq 0$, or $f_2 = g(t,u)h_2(t,p)$, where $h_2(t,p) \geqq 0$ for any $p$ and $g$ is nondecreasing with respect to $u$. As was mentioned above, we do not need any assumption on the smoothness of the coefficients in order to obtain the global gradient estimate. Obviously in (0.9) $fu_x^3$ satisfies conditions (0.14) if $f$ is independent of $u$ and nondecreasing with respect to $x$.

In the last section based on the *a priori* estimates of Sections 1, 2 we prove the existence theorems.

## 1. Boundary Gradient Estimate

Consider problem (0.1), (0.2). Assume that the functions $a(t,x,u,p)$ and $f(t,x,u,p)$ are defined on the set $\bar{Q}_T \times [-M,M] \times R$ and are bounded for $(t,x) \in Q_T$, $|u| \leqq M$ and for any $p$. Suppose that conditions (0.3), (0.4),(0.6) are fulfilled with $\psi(\rho) \in C^1([0,+\infty))$, $\psi(\rho) > 0$ for $\rho > 0$ and $\psi(0) \geqq 0$. Introduce the functions $h_1(x)$ and $h_2(x)$ by the following

$$\begin{aligned} &h_1'' + \psi(|h_1'|) = 0, \quad h_1(-l) = 0, \quad h_1(-l+\tau_0) = \mu, \\ &h_2'' + \psi(|h_2'|) = 0, \quad h_2(l-\tau_0) = \mu, \quad h_2(l) = 0, \end{aligned}$$

where $\mu = \max\{M, \operatorname{osc}(u), K_0 l\}$ and

$$|u_0(x)| \leqq K_0(l - |x|). \tag{1.1}$$

The constant $\tau_0$ will be selected below. Represent the solution of the first equation in parametric form (using the substitution $q(h_1) = h_1'$, $\frac{dq}{dx} = q\frac{dq}{dh_1}$):

$$h_1 = h_1(q) = \int_q^{q_1} \frac{\rho\, d\rho}{\psi(\rho)}, \quad x = x(q) = \int_q^{q_1} \frac{d\rho}{\psi(\rho)} - l, \tag{1.2}$$

where the parameter $q$ varies in the interval $[q_0, q_1]$ and $q_0, q_1$ are chosen so as to have $q_1 > q_0 > 0$ and

$$h_1(q_0) = \int_{q_0}^{q_1} \frac{\rho\, d\rho}{\psi(\rho)} = \mu,$$

which is possible due to (0.6). We put

$$\tau_0 = \int_{q_0}^{q_1} \frac{d\rho}{\psi(\rho)}.$$

Define $D_1$ and $D_2$:

$$D_1 = \{(t, x) : 0 < t \leqq T, x \in (-l, -l + \tau_0) \cap (-l, l)\},$$
$$D_2 = \{(t, x) : 0 < t \leqq T, x \in (l - \tau_0, l) \cap (-l, l)\}.$$

**Lemma 1.1.** *Let $u(t, x)$ be a classical solution ($u(t, x) \in C^{1,2}_{t,x}(Q_T) \cap C^0(\bar{Q}_T)$) of problem* (0.1), (0.2)*; assume that conditions* (0.3), (0.4), (0.6), (1.1) *are fulfilled. If $\tau_0 \leqq l$ or $|u_0(x)| \leqq h_i(x)$, then*

$$|u(t, x)| \leqq h_i(x) \quad in \ \ \bar{D}_i, \quad i = 1, 2.$$

**Proof.** Denote by $\Gamma_i$ the parabolic boundary of $D_i$; i.e., if $\tau_0 \leqq l$ then

$$\Gamma_1 = \{t = 0, x \in [-l, -l + \tau_0]\} \cup \{t \in [0, T], x = -l\} \cup \{t \in [0, T], x = -l + \tau_0\},$$
$$\Gamma_2 = \{t = 0, x \in [l - \tau_0, l]\} \cup \{t \in [0, T], x = l - \tau_0\} \cup \{t \in [0, T], x = l\}$$

and if $\tau_0 > l$ then

$$\Gamma_1 = \Gamma_2 = \{t = 0, x \in [-l, l]\} \cup \{t \in [0, T], x = -l\} \cup \{t \in [0, T], x = l\}.$$

Let us show that $|u(t, x)| \leqq h_i(x)$ on $\Gamma_i$, $i = 1, 2$. Due to (1.1), if $\tau_0 \leqq l$ we have

$$|u_0(-l + \tau_0)| \leqq K_0 l \leqq h_1(-l + \tau_0), \quad |u_0(l - \tau_0)| \leqq K_0 l \leqq h_2(l - \tau_0).$$

Taking into account that $h_1'' \leqq 0$ and $h_2'' \leqq 0$, we conclude that $|u_0(x)| \leqq h_i(x)$, $i = 1, 2$. If $\tau_0 > l$, then $|u_0(x)| \leqq h_i(x)$, $\ i = 1, 2$ from the conditions of the lemma.

Further, for $t \in [0, T]$ we have

$$u(t, -l) = h_1(-l) = u(t, l) = h_2(l) = 0,$$
$$|u(t, -l + \tau_0)| \leqq M \leqq h_1(-l + \tau_0), \ \ |u(t, l - \tau_0)| \leqq M \leqq h_2(l - \tau_0),$$

and if $\tau_0 > l$, then for $t \in [0, T]$ we have $u(t, l) = 0 < h_1(l), u(t, -l) = 0 < h_2(l)$.

Let

$$L_0 u \equiv A(t, x)(u_{xx} + \psi(|u_x|)) - u_t,$$

where $A(t, x) = a(t, x, u, u_x)$. It is clear that $L_0 u \geqq 0$, $L_0 h_i = 0$, $i = 1, 2$ and for $v_i \equiv u - h_i$ the following is valid:

$$L_0 u - L_0 h_i \equiv \tilde{L}_0 v_i \equiv A(t, x)(v_{ixx} + \beta_i v_{ix}) - v_{it} \geqq 0 \ \text{ in } \ D_i, i = 1, 2,$$

where $\beta_i$ are bounded in $\bar{D}_i \setminus \Gamma_i$ due to the fact that $u$ is a classical solution and $\psi$ is $C^1$ function. By means of the standard arguments based on the maximum principle, we can show that the function $v$ cannot achieve a positive maximum in $\bar{D}_i \setminus \Gamma_i$. Hence $u - h_i \leqq 0$ in $D_i$.

Replacing $u$ by $-u$ we obtain the inequality $u + h_i \geqq 0$ (note that $L_0(-u) \geqq 0$). Lemma 1.1 is proved.

**Remark.** One can easily see that in order to prove Lemma 1.1 it is sufficient to take $\mu = \max\{M, K_0 l\}$, the fact that $\mu \geqq \operatorname{osc}(u)$ will be used in the proof of Lemma 2.1 and of Lemma 2.2.

## 2. Global gradient estimate

In this section we obtain the global gradient estimate without assumptions on the smoothness of the coefficients based on S. N. KRUZHKOV's idea of introducing a new spatial variable [1] (see also [15]–[19]). We consider two cases:

1. The assumptions on the function $f(t,x,u,p)$ are the same as in Section 1, but the function $u_0(x)$ satisfies a certain additional condition (Lemma 2.1).
2. The function $u_0(x)$ is the same as in Section 1 and $f(t,x,u,p)$ can be represented in the form

$$f(t,x,u,p) = f_1(t,x,u,p) + f_2(t,x,u,p), \tag{2.1}$$

where the first term $f_1$ satisfies conditions (0.4), (0.5), the second $f_2$ satisfies conditions (0.4), (0.6) and

$$f_2(t,x,u,p) - f_2(t,y,v,p) \geqq 0, \quad f_2(t,y,u,-p) - f_2(t,x,v,-p) \geqq 0 \tag{2.2}$$

when $x \geqq y, \;\; u \geqq v, \;\; p \geqq 0$ (Lemma 2.2).

**Lemma 2.1.** *Assume that the conditions of Lemma 1.1 are fulfilled and the function* $u_0(x)$ $(u_0(\pm l) = 0)$ *satisfies the inequality*

$$|u_0(x) - u_0(y)| \leqq h_1(|x-y| - l) \;\; \textit{for} \;\; |x-y| \leqq \tau_0. \tag{2.3}$$

*Then for any classical solution of problem* (0.1), (0.2) *we have*

$$|u_x(t,x)| \leqq C_0,$$

*where the constant* $C_0$ *depends only on* $\psi$, $l$ *and* $M$.

(The function $h_1$ and the constant $\tau_0$ are defined in Section 1.)

**Proof.** Consider equation (0.1) at points $(t,x), (t,y) \in Q_T$ where $x \neq y$ :

$$a(t,x,u(t,x),u_x(t,x))u_{xx} - u_t(t,x) = f(t,x,u(t,x),u_x(t,x)), \tag{2.4}$$

$$a(t,y,u(t,y),u_y(t,y))u_{yy} - u_t(t,y) = f(t,y,u(t,y),u_y(t,y)). \tag{2.5}$$

Subtracting (2.5) from (2.4) for the function $v(t,x,y) \equiv u(t,x) - u(t,y)$, we obtain

$$\begin{aligned} &a(t,x,u(t,x),v_x)v_{xx} + a(t,y,u(t,y),-v_y)v_{yy} - v_t \\ &\qquad\qquad = f(t,x,u(t,x),v_x) - f(t,y,u(t,y),-v_y). \end{aligned}$$

Taking into account (0.4), we have

$$Lv \equiv A(t,x)[v_{xx} + \psi(|v_x|)] + A(t,y)[v_{yy} + \psi(|v_y|)] - v_t \geqq 0,$$

where $A(t,z) \equiv a(t,z,u(t,z),u_z(t,z))$. Let us now define the function $h(\tau)$ by the following:

$$h''(\tau) + \psi(|h'(\tau)|) = 0, \quad h(0) = 0, \quad h(\tau_0) = \mu = \max\{M, \operatorname{osc}(u), K_0 l\}. \tag{2.6}$$

Obviously $Lh(x-y)=0$. Compare the functions $v$ and $h$ in the prism

$$P=\{(t,x,y):0<t\leqq T,|x|<l,|y|<l,0<x-y<\tau_0\}.$$

For $w\equiv v(t,x,y)-h(x-y)$ we have

$$0\leqq Lv-Lh\equiv\tilde{L}w\equiv A(t,x)(w_{xx}+\beta_1 w_x))+A(t,y)(w_{yy}+\beta_2 w_y))-w_t,$$

where $|\beta_i|<+\infty$, $i=1,2$ due to the smoothness of $\psi$ and to the fact that $u(t,x)$ is the classical solution. By means of the standard arguments based on the maximum principle, we can show that the function $w$ cannot achieve a positive maximum in $\bar{P}\setminus\Gamma$ where $\Gamma$ is a parabolic boundary of $P$. Consider $\Gamma$:

(1) For $x=y$ we have $w=0$.

(2) For $x-y=\tau_0$ we have $u(t,x)-u(t,y)-\mu\leqq \operatorname{osc}(u)-\mu\leqq 0$.

(3) For $y=-l$, $x\in(-l,-l+\tau_0)$, $t\in(0,T]$ we have $w=u(t,x)-h(x+l)$. Let us show that $u(t,x)\leqq h(x+l)$. To this end it is sufficient to show that $h_1(x)\leqq h(x+l)$ (recall that from Lemma 1.1 we have $u(t,x)\leqq h_1(x)$). The latter inequality follows directly from the fact that

$$h_1''(x)+\psi(|h_1'(x)|)=0,\quad h_1(-l)=0,\quad h_1(-l+\tau_0)=\mu,$$
$$\tilde{h}''(x)+\psi(|\tilde{h}'(x)|)=0,\quad \tilde{h}(-l)=0,\quad \tilde{h}(-l+\tau_0)=\mu,$$

where $\tilde{h}(x)\equiv h(x+l)$.

(4) For $x=l$, $y\in(l-\tau_0,l)$, $t\in(0,T]$ we have $w=-u(t,y)-h(l-y)$. Let us show that $u(t,y)\geqq -h(l-y)$. It is sufficient to show that $h(l-y)\geqq h_2(y)$ (from Lemma 1.1 we have $u(t,y)\geqq -h_2(y)$). The latter inequality follows from

$$h_2''(y)+\psi(|h_2'(y)|)=0,\quad h_2(l-\tau_0)=\mu,\quad h_2(l)=0,$$
$$\bar{h}''(y)+\psi(|\bar{h}'(y)|)=0,\quad \bar{h}(l-\tau_0)=\mu,\quad \bar{h}(l)=0,$$

where $\bar{h}(y)\equiv h(l-y)$.

(5) Finally, from the conditions of Lemma 2.1, we have for $t=0$

$$u_0(x)-u_0(y)-h_1(x-y-l)\leqq 0.$$

Thus we have proved that

$$u(t,x)-u(t,y)\leqq h(x-y)\quad\text{in } \bar{P}.$$

By analogy, taking the function $\tilde{v}\equiv u(t,y)-u(t,x)$ in the place of $v$, we obtain

$$u(t,x)-u(t,y)\geqq -h(x-y)\quad\text{in } \bar{P}.$$

In view of the symmetry of the variables $x$ and $y$, we examine the case $y>x$ in the same way. As a result we have that for

$$0\leqq t\leqq T,\quad |x|<l,\quad |y|<l,\quad 0<|x-y|<\tau_0$$

the inequality

$$\frac{|u(t,x)-u(t,y)|}{|x-y|} \leqq \frac{h(|x-y|)-h(0)}{|x-y|}$$

holds, implying that $|u_x(t,x)| \leqq h'(0)$. Lemma 2.1 is proved.

**Lemma 2.2.** *Suppose that the conditions of Lemma 1.1 and conditions* (2.1), (2.2) *hold. Then for any classical solution of problem* (0.1), (0.2) *we have*

$$|u_x(t,x)| \leqq C_1,$$

*where the constant* $C_1$ *depends only on* $\psi, l, M, K_0 \equiv \sup \frac{|u_0(x)-u_0(y)|}{|x-y|}$.

**Proof.** Similarly to the proof of Lemma 2.1 for $v(t,x,y) \equiv u(t,x)-u(t,y)$ we obtain

$$\begin{aligned} Lv \equiv A(t,x)(v_{xx}+\psi_0(|v_x|)) + A(t,y)(v_{yy}+\psi_0(|v_y|)) - v_t \\ \geqq f_2(t,x,u(t,x),u_x(t,x)) - f_2(t,y,u(t,y),u_y(t,y)), \end{aligned}$$

where $\psi_0(\rho) \geqq 1$ is $C^1([0,\infty))$ function and

$$\int_1^{+\infty} \frac{\rho\, d\rho}{\psi_0(\rho)} = +\infty. \tag{2.7}$$

Define the function $h_0(\tau)$ by the following

$$h_0''(\tau) + \psi_0(|h_0'(\tau)|) = 0, \quad h_0(0) = 0, \quad h_0(\tau_1) = \mu = \max\{M, \operatorname{osc}(u), K_0 l\}.$$

In order to select $\tau_1$ represent the solution of (2.7) in parametrical form

$$h_0(q) = \int_q^{q_1} \frac{\rho\, d\rho}{\psi_0(\rho)}, \quad \tau(q) = \int_q^{q_1} \frac{d\,\rho}{\psi_0(\rho)},$$

where the parameter $q$ varies in the interval $[q_0, q_1]$ and $q_0, q_1$ are chosen so that $q_1 > q_0 \geqq \max\{K_0, h_1'(-l)\}$ (the function $h_1$ was defined in Section 2) and

$$h_0(q_0) = \int_{q_0}^{q_1} \frac{\rho\, d\rho}{\psi_0(\rho)} = \mu,$$

which is possible due to (2.7). Define $\tau_1$ by

$$\tau_1 = \int_{q_0}^{q_1} \frac{d\,\rho}{\psi_0(\rho)}.$$

Consider the function $v(t,x,y)$ and $h_0(x-y)$ in the domain

$$P = \{(t,x,y) : 0 < t < T, |x| < l, |y| < l, 0 < x-y < \tau_1\}.$$

Obviously for $w \equiv v(t,x,y) - h(x-y)$ we have the inequality

$$\begin{aligned} A(t,x)(w_{xx}+\beta_1 w_x) + A(t,y)(w_{yy}+\beta_2 w_y) - w_t \\ \geqq f_2(t,x,u(t,x),u_x(t,x)) - f_2(t,y,u(t,y),u_y(t,y)) \end{aligned}$$

and for $\tilde{w} = we^{-t}$ the inequality

$$A(t,x)(\tilde{w}_{xx} + \beta_1\tilde{w}_x) + A(t,y)(\tilde{w}_{yy} + \beta_2\tilde{w}_y) - \tilde{w} - \tilde{w}_t \geqq e^{-t}(f_2(t,x,u(t,x),u_x(t,x)) - f_2(t,y,u(t,y),u_y(t,y))). \quad (2.8)$$

Suppose that $\tilde{w}$ achieves its positive maximum at the point $N \in \bar{P} \setminus \Gamma$ ($\Gamma$ is the parabolic boundary of $P$), then at this point we have $\tilde{w} > 0$, $\tilde{w}_x = \tilde{w}_y = 0$, i.e., $u(t,x) > u(t,y), u_x = h' > 0, u_y = h' > 0$. Thus, due to (2.2) the right side of (2.8) at $N$ is positive. This contradicts the fact that $\tilde{w}$ attains a positive maximum at $N$. Consider the parabolic boundary of $P$:

(1) For $x = y$ we have $\tilde{w} = 0$.
(2) For $x - y = \tau_1$ we have $u(t,x) - u(t,y) - \mu \leqq \operatorname{osc}(u) - \mu \leqq 0$ hence $\tilde{w} \leqq 0$.
(3) For $t = 0$ we have $u_0(x) - u_0(y) - h_0(x-y) \leqq 0$ because $u_0(x) - u_0(y) \leqq K_0(x-y)$ and $h_0(x-y) = h_0(x-y) - h_0(0) = h_0'(\tau^*)(x-y) \geqq K_0(x-y)$.

It remains to consider the following two parts of $\Gamma$:

(4) $y = -l, \quad x \in (-l, -l+\tau_1), \quad t \in (0,T]$ and
(5) $x = l, \quad y \in (l-\tau_1, l), \quad t \in (0,T]$.

On part (4) we have $\tilde{w} = e^{-t}(u(t,x) - h_0(x+l))$ and on (5) $\tilde{w} = e^{-t}(-u(t,y) - h_0(l-y))$. In order to prove that on (4) and (5) $\tilde{w}$ is non-positive it is sufficient to show that

$$h_1(x) \leqq h_0(x+l) \quad \text{for } x \in (-l, -l+\tau_1),$$
$$h_2(y) \leqq h_0(l-y) \quad \text{for } y \in (l-\tau_1, l),$$

and then to apply Lemma 1.1. These inequalities follow immediately from the fact that $h_0' \geqq h_1'(-l)$. Note that $\tau_1 \leqq \tau_0$ because $h_0(x+l) = h_1(x)$ for $x = -l$, $h_0'(x+l) \geqq h_1'(-l) = \sup h_1'(x)$ and $h_0(\tau_1) = h_1(-l+\tau_0)$.

Thus we have proved that

$$u(t,x) - u(t,y) \leqq h_0(x-y) \text{ in } \bar{P}.$$

By analogy, taking the function $\tilde{v} \equiv u(t,y) - u(t,x)$ in the place of $v$ we obtain

$$u(t,x) - u(t,y) \geqq -h_0(x-y) \text{ in } \bar{P}$$

(here we use the second inequality (2.2)).

Similarly to the proof of Lemma 2.1 we conclude that

$$|u_x(t,x)| \leqq h_0'(0) = p_1.$$

Lemma 2.2 is proved.

## 3. Existence and Uniqueness Theorem

Let us now formulate the existence and uniqueness theorem for problem (0.1), (0.2). Denote by $D_i$ the set $\bar{Q}_T \times [-M, M] \times [-C_i, C_i]$, $i = 0, 1$.

**Theorem 3.1.** *Suppose that conditions* (0.3), (0.4), (0.6), (0.7) (*or* (0.8)), (2.3) *are fulfilled. In addition suppose that some condition which guarantee the boundedness of* $|u|$ *is fulfilled* (*see* [18], [20]). *If* $a(t, x, u, p)$, $f(t, x, u, p) \in C^{\alpha}(D_0)$, $\alpha \in (0, 1)$ *then for any* $T > 0$ *problem* (0.1),(0.2) *has at least one solution* $u(t, x)$ *from* $C^{1+\beta/2,2+\beta}_{t,x}(Q_T) \cap C^0(\bar{Q}_T)$ *for some* $\beta \in (0, 1)$.

**Theorem 3.2.** *Suppose that conditions* (0.3), (0.4), (0.6), (0.7) (*or* (0.8)), (2.1), (2.2) *are fulfilled. In addition suppose that some condition which guarantee the boundedness of* $|u|$ *is fulfilled. If* $a(t, x, u, p)$, $f(t, x, u, p) \in C^{\alpha}(D_1)$, $\alpha \in (0, 1)$ *and* $u_0(x)$ *is Lipschitz continuous function such that* $u_0(\pm l) = 0$ *then for any* $T > 0$ *problem* (0.1), (0.2) *has at least one solution* $u(t, x) \in C^{1+\beta/2,2+\beta}_{t,x}(Q_T) \cap C^0(\bar{Q}_T)$ *for some* $\beta \in (0, 1)$.

Note that (see [1]) the boundedness of $|u_x|$ implies Hölder continuity of the solution with respect to $t$ with Hölder exponent 1/2 and Hölder constant depending only on $\sup |u_x|$ and on the maximum of functions $a(t, x, u, p)$, $|f(t, x, u, p)|$ on the set $D_i$. The boundedness of $|u_x|$ implies also the Hölder continuity of $u_x$ (see[1]) with Hölder constant and Hölder exponent depending also on $\sup |u_x|$ and on the maximum of $a$ and $|f|$ on $D_i$. These estimates imply the existence of the required solution (see for example [20]).

**Remark on Uniqueness.** If the functions $a(t, x, u, p)$, $f(t, x, u, p)$ are differentiable with respect to $u$, then the solution in Theorems 3.1, 3.2 is unique (see [18]).

*Example.* In [10] it was shown that the solution of the problem

$$u_{xx} - u_t = (x + 1/2)u_x^3 \quad \text{in } (-1/2, 1/2) \times (0, +\infty),$$
$$u(0, x) = u_0(x), \quad \text{for } |x| < 1/2, \quad u_0(-1/2) = 0, \quad u_0(1/2) = \pi/2,$$

where $u_0(x)$ is smooth compatible with boundary conditions, remains bounded but it cannot possess a bounded derivative. It means that $\sup |u_x| \to +\infty$ as $t$ goes to the proper value $t^*$ (finite or infinite). Note that the only steady-state solution of this problem is the function $\arcsin(x + 1/2)$. Let us examine the following problem

$$u_{xx} - u_t = (x + l)u_x^3 \quad \text{in } (-l, l) \times (0, +\infty),$$
$$u(0, x) = u_0(x) \quad \text{for } -l < x < l, \quad u_0(-l) = 0, \quad u_0(l) = U,$$

where $U$ is some constant and the function $u_0(x)$ is smooth compatible with boundary conditions. For the function $v(t, x) = u(t, x) - U(x + l)/2l$ we obtain

$$v_{xx} - v_t = (x + l)(v_x + U/2l)^3, \quad v(t, -l) = v(t, l) = 0,$$
$$v(0, x) = u_0(x) - U(x + l)/2l.$$

We take $\psi(\rho) \equiv 2l(\rho + |U|/2l)^3$ and obtain

$$\int_{p_0}^{p_1} \frac{\rho\, d\rho}{2l(\rho + |U|/2l)^3}$$
$$= \frac{1}{2lp_0 + |U|} - \frac{1}{2lp_1 + |U|} + \frac{|U|}{2}\left[\frac{1}{(2lp_1 + |U|)^2} - \frac{1}{(2lp_0 + |U|)^2}\right],$$
$$\cdot \int_{p_0}^{p_1} \frac{d\rho}{2l(\rho + |U|/2l)^3} = \frac{l}{(2lp_0 + |U|)^2} - \frac{l}{(2lp_1 + |U|)^2}.$$

Thus, if there exist $p_1 > p_0 > 0$ such that

$$\int_{p_0}^{p_1} \frac{\rho\, d\rho}{2l(\rho + |U|/2l)^3} = \max\{\sup|v|, \operatorname{osc}(v), lK_0\}$$

and

$$(2lp_0 + |U|)^{-2} - (2lp_1 + |U|)^{-2} = 1 \text{ or condition (0.8) is fulfilled,}$$

then we have the global gradient estimate independent of $T$. The estimate of $|v(t, x)|$ can be easily obtained from the fact that $u(x, t)$ attains maximum and minimum on the parabolic boundary of the domain (i.e., $\{t = 0,\ \ |x| < 1/2\} \cup \{t \in (0, +\infty),\ \ x = \pm 1/2\}$).

**Remark.** The case $U = \pi/2,\ \ l = 1/2$ does not satisfy condition (0.6) for any initial data. In fact, for $U = \pi/2$ and $l = 1/2$ we have

$$\int_{0}^{+\infty} \frac{\rho\, d\rho}{2l(\rho + |U|/2l)^3} = \frac{1}{\pi}$$

and $\sup|v| \to V_0 > 1/\pi$ when $t \to t^*$ for some $t^*$ (see [10]).

## References

1. S. N. Kruzhkov, Quasilinear parabolic equations and systems with two independent variables, *Trudy Sem. Petrovsk.* **5** (1979) 217–272 (Russian). English transl. in *Topics in Modern Math.*, Consultants Bureau, New York, 1985.
2. S. N. Bernstein, On the equations of calculus of variations, *Sobr. soch. M.: Izd-vo AN SSSR* **3** (1960) 191–242 (Russian).
3. M. Nagumo, Über die gleichmässige Summierbarkeit und ihre Anwendung auf ein Variationsproblem, *Japan J. Math.* **6** (1929) 173–182.
4. L. Tonelli, *Fondamenti di calcolo delle variazioni*. Bologna: Zanichelli, 1923.
5. A. F. Filippov, On condition for the existence of a solution of a quasilinear parabolic equation, *Dokl.Akad.Nauk SSSR* **141** (1961) 568-570 (Russian) English transl. *Soviet Math. Dokl.* **2** (1961) 1517–1519.
6. M. Fila & G. Lieberman, Derivative blow-up and beyond for quasilinear parabolic equations, *Diff. and Integral Equations*, **7** (1994) 811–822.
7. G. Lieberman, The first initial-boundary value problem for quasilinear second order parabolic equations, *Ann. Scuola Norm. Sup. Pisa* **13** (1986), 347–387.
8. T. Dlotko, Examples of parabolic problems with blowing-up derivatives, *J. Math. Anal. Appl.* **154** (1991) 226–237.

9. N. KUTEV, Global solvability and boundary gradient blow up for one dimensional parabolic equations, *Progress in PDE: Elliptic and Parabolic Problems* (C. BANDLE, et al., eds.), Longman, 1992, pp. 176–181.
10. M. P. VISHNEVSKII, T. I. ZELENYAK & M. M. LAVRENTIEV JR., Behavior of solutions to parabolic equations for large time, *Sib. Mat. Zhurn.* **36** (1995) 510–530 (Russian). English translation in *Sib. Math. J.* **36** (1995), 435–453.
11. M. LAVRENTIEV JR., P. BROADBRIDGE & V. BELOV, Boundary value problems for strongly degenerate parabolic equations, *Communications in PDE*, **22** (1997), 17–38.
12. S .ANGENENT & M. FILA, Interior gradient blow-up in a semilinear parabolic equation, *Diff. and Integral Equations* **9** (1996) 865–877.
13. PH. BLANC, Existence de solutions discontinues pour des equations paraboliques *C. R. Acad. Sci. Paris Serie 1 Math.* **31**0 (1990) 53–56.
14. Y. GIGA, Interior blow up for quasilinear parabolic equations, *Discrete and Continuous Dynamical Systems* **1** (1995) 449–461
15. S. N. KRUZHKOV, Nonlinear parabolic equations in two independent variables, *Trudy Mosk. Mat. Ob.* **16** (1967) 329–346 (Russian). English transl. in *Transact. Moscow Math. Soc.* **16**, (1968) 355–373.
16. N. V. KHUSNUTDINOVA, On conditions for the boundedness of the gradient of the solutions of degenerated parabolic equations, *Dinamika Sposhnoi Sredy* **72** (1985) 120–129 (Russian).
17. A. S. TERSENOV, On a certain class of degenerated non-uniformly parabolic equations, *Vestn. Mosk. Univ. Ser. 1* **6** (1996) 94–97 (Russian). English translation in *Moscow Univ. Math. Bull.* **1** (1996) 63–65.
18. A. S. TERSENOV, On quasilinear non-uniformly parabolic equations in general form, *J. of Diff. Equations* **142** (1998) 263–276.
19. A. S. TERSENOV, On quasilinear non-uniformly elliptic equations in some nonconvex domains, *Communications in PDE* **23** (1998) 2165–2186.
20. O. A. LADYZHENSKAJA, V. A. SOLONNIKOV & N. N. URAL'CEVA, *Linear and Quasilinear Equations of Parabolic Type*. Moscow: Izdat. "Nauka" 1967 (Russian). English Translation: *Amer. Math. Soc. Transl.* **23** (1968).

Department of Mathematics, University of Crete
71409 Heraklion - Crete, Greece

and

Lavrentyev Institute of Hydrodynamics
Novosibirsk, Russia

(*Accepted June 9, 1999*)