Global Solvability for a Class of Quasilinear Parabolic Problems

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ABSTRACT. This paper is devoted to the existence of radially-symmetric solutions of the boundary value problems as well as of the Cauchy problem, for the equation $u_t = \varepsilon\Delta u + F(t, \mathbf{x}, u, \nabla u)$. We suppose that $F(t, \mathbf{x}, u, \mathbf{p})$ does not satisfy Bernstein's condition on no more than quadratic growth with respect to $\mathbf{p}$ when $|\mathbf{p}| \to +\infty$. Conditions which guarantee the global solvability of the problems are formulated.

1. INTRODUCTION AND MAIN RESULTS

The present paper is concerned with the global solvability of the boundary value problems in the domain K_R and the Cauchy problem for the equation

$$u_t = \varepsilon\Delta u + F(t, \mathbf{x}, u, \nabla u), \tag{1.1}$$

where $K_R = (0, T) \times B_R$, $B_R = \{\mathbf{x} : |\mathbf{x}| < R\} \subset \mathbb{R}^n$, $\mathbf{x} = (x_1, \dots, x_n)$, $\nabla u = (u_{x_1}, \dots, u_{x_n})$, and ε is a positive constant. It is well known [1]-[3] that the global solvability of the boundary value problems and of the Cauchy problem for quasilinear equations is not merely a consequence of sufficient smoothness of the coefficients. The character of nonlinearities of the coefficients plays an essential role here. The fulfillment of the condition

$$uF(t, \mathbf{x}, u, 0) \le |u|\Phi(|u|), \qquad \int^{+\infty} \frac{dz}{\Phi(z)} = +\infty, \tag{1.2}$$

where $\Phi(z)$ is a nondecreasing positive function of $z \ge 0$, guarantees the global apriori estimate of $|u|$ for the Dirichlet problem [1], [4]. The following condition (Bernstein's type condition [5])

$$|F(t, \mathbf{x}, u, \mathbf{p})| \le \nu(|u|)(1 + |\mathbf{p}|^2), \tag{1.3}$$

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where $\nu(|u|) \in \mathbb{C}^0(0,+\infty)$ is a positive monotonically increasing function, guarantees the global apriori estimate of $|\nabla u|$ [1]-[4]. Examples show that in the case of violation of assumption (1.3) the gradient of the bounded solution may blow-up on the boundary of the domain [1], [6], [7] as well as in the interior of the domain [8], [9], i.e., there exists t^* such that $|\nabla u(t,\mathbf{x})| \to +\infty$ when $t \to t^*$ at least for some $\mathbf{x}$.

Consider the following boundary conditions

$$u\,|_{S_R} = 0, \tag{1.4}$$

$$\left.\frac{\partial u}{\partial n}\right|_{S_R} = 0, \tag{1.5}$$

$$\frac{\partial u}{\partial n} + \sigma(t,u)\,|_{S_R} = 0, \tag{1.6}$$

where $S_R = (0,T) \times \partial B_R$ and by $\partial u/\partial n$ we mean the derivative in the direction of the outward normal to S_R. Let

$$u(0,\mathbf{x}) = \varphi(|\mathbf{x}|) \quad \text{for } \mathbf{x} \in B_R. \tag{1.7}$$

Suppose that the function $F(t,\mathbf{x},u,\mathbf{p})$ is defined for $(t,\mathbf{x}) \in \bar{K}_R$ and all $(u,\mathbf{p})$, and takes finite values for $(t,\mathbf{x}) \in \bar{K}_R$ and finite $(u,\mathbf{p})$. The function $\sigma(t,u)$ is defined for $t \in [0,T]$ and any u.

Suppose that $F(t,\mathbf{x},u,\nabla u)$ can be written in the (t,r) variables, where $r = |\mathbf{x}| = \sqrt{\sum_{i=1}^n x_i^2}$, in the form

$$F(t,\mathbf{x},u,\nabla u) = \widetilde{F}(t,r,u,u_r) \tag{1.8}$$

(e.g. $F = F(t,|\mathbf{x}|,u,|\nabla u|)$ or $F = F(t,|\mathbf{x}|,u,\mathbf{x}\cdot\nabla u)$, where $\mathbf{x}\cdot\nabla u = \sum_{i=1}^n x_i u_{x_i}$). We show that the global solvability of the boundary value problems for equation (1.1) can be obtained under a condition different from (1.3), which allows arbitrary growth of F with respest to $\mathbf{p}$ when $|\mathbf{p}| \to \infty$. For the sake of generality we suppose that F consists of two parts, the first one satisfies condition (1.3) and the second one satisfies another type of condition. Specifically,

$$\begin{aligned} F(t,\mathbf{x},u,\nabla u) &= g(t,\mathbf{x},u,\nabla u) + f(t,\mathbf{x},u,\nabla u) \\ &= \widetilde{g}(t,r,u,u_r) + \widetilde{f}(t,r,u,u_r), \end{aligned} \tag{1.9}$$

where

$$|\widetilde{g}(t,r,u,p)| \le \varepsilon\psi(|p|), \quad \int^{+\infty} \frac{z\,dz}{\psi(z)} = +\infty, \tag{1.10}$$

with $\psi(z) \in \mathbb{C}^1[0,+\infty)$, $\psi(z) \ge 1$ and $\widetilde{f}(t,r,u,u_r)$ satisfies the following

$$\begin{cases} \widetilde{f}(t,r_1,u_1,p) - \widetilde{f}(t,r_2,u_2,p) \ge 0, \\ \widetilde{f}(t,r_2,u_1,-p) - \widetilde{f}(t,r_1,u_2,-p) \ge 0 \end{cases} \tag{1.11}$$

for $r_2 > r_1$, $u_2 > u_1$, $p > 0$.

Let us formulate the existence theorems and then give some examples.

Theorem 1.1. *Suppose that the function* $F(t, \mathbf{x}, u, \mathbf{p})$ *is Hölder continuous in* $\mathbf{x}$*,* u*,* $\mathbf{p}$ *with exponent* α *and Hölder continuous in* t *with exponent* $\alpha/2$ *in the domain* $K_R \times \mathbb{R} \times \mathbb{R}^n$*,* $\varphi(|\mathbf{x}|) \in \mathbb{C}^{1+\alpha}(|\mathbf{x}| \leq R)$ *for some* $\alpha \in (0,1)$*,* $\sigma(t,u)$ *is a continuous function. Suppose conditions* (1.2), (1.9)-(1.11) *hold. In addition assume that* $u\sigma(t,u)\big|_{|\mathbf{x}|=R} > 0$ *for* $u \neq 0$ *and* $\varphi'(R) = -\sigma(0, \varphi(R))$*. Then for any* $T \in (0, +\infty)$ *there exists a solution of problem* (1.1), (1.6), (1.7) *which belongs to* $\mathbb{C}^{1+\beta/2, 2+\beta}_{t,\mathbf{x}}(K_R) \cap \mathbb{C}^{1/2, 1+\beta}_{t,\mathbf{x}}(\bar{K}_R)$ *for some* $\beta \in (0,1)$.

Theorem 1.2. *Suppose that the function* $F(t, \mathbf{x}, u, \mathbf{p})$ *is Hölder continuous in* $\mathbf{x}$*,* u*,* $\mathbf{p}$ *with exponent* α *and Hölder continuous in* t *with exponent* $\alpha/2$ *in the domain* $K_R \times \mathbb{R} \times \mathbb{R}^n$*,* $\varphi(|\mathbf{x}|) \in \mathbb{C}^{1+\alpha}(|\mathbf{x}| \leq R)$ *for some* $\alpha \in (0,1)$*, and suppose conditions* (1.2), (1.9)-(1.11) *hold. In addition assume that* $\varphi(R) = 0$*,* $\widetilde{f}(t,r,u,p) \geq 0$ *when* $u < 0$*,* $p > 0$ *and* $\widetilde{f}(t,r,u,p) \leq 0$ *when* $u > 0$*,* $p < 0$*. Then for any* $T \in (0, +\infty)$ *there exists a solution of problem* (1.1), (1.4), (1.7) *which belongs to* $\mathbb{C}^{1+\beta/2, 2+\beta}_{t,\mathbf{x}}(K_R) \cap \mathbb{C}^{1/2, 1+\beta}_{t,\mathbf{x}}(\bar{K}_R)$ *for some* $\beta \in (0,1)$.

Theorem 1.3. *Suppose that the function* $F(t, \mathbf{x}, u, \mathbf{p})$ *is Hölder continuous in* $\mathbf{x}$*,* u*,* $\mathbf{p}$ *with exponent* α *and Hölder continuous in* t *with exponent* $\alpha/2$ *in the domain* $K_R \times \mathbb{R} \times \mathbb{R}^n$*,* $\varphi(|\mathbf{x}|) \in \mathbb{C}^{1+\alpha}(|\mathbf{x}| \leq R)$ *for some* $\alpha \in (0,1)$*. Suppose conditions* (1.2), (1.9)-(1.11) *hold. In addition assume that* $\varphi'(R) = 0$*. Then for any* $T \in (0, +\infty)$ *there exists a solution of problem* (1.1), (1.5), (1.7) *which belongs to* $\mathbb{C}^{1+\beta/2, 2+\beta}_{t,\mathbf{x}}(K_R) \cap \mathbb{C}^{1/2, 1+\beta}_{t,\mathbf{x}}(\bar{K}_R)$ *for some* $\beta \in (0,1)$.

Theorem 1.4. *Suppose that the function* $F(t, \mathbf{x}, u, \mathbf{p})$ *is Hölder continuous in* $\mathbf{x}$*,* u*,* $\mathbf{p}$ *with exponent* α *and Hölder continuous in* t *with exponent* $\alpha/2$ *in the domain* $\Pi_T \times \mathbb{R} \times \mathbb{R}^n$ *for some* $\alpha \in (0,1)$*, where* $\Pi_T = (0,T) \times \mathbb{R}^n$*. Assume that the function* $\varphi(|\mathbf{x}|)$ *belongs to* $\mathbb{C}^{1+\alpha}(\mathbb{R})$ *and vanishes with its first derivative when* $|\mathbf{x}| \to +\infty$*. Suppose conditions* (1.2), (1.9)-(1.11) *hold. Then for any* $T \in (0, +\infty)$ *there exists a bounded solution of problem* (1.1), (1.7) *which belongs to* $\mathbb{C}^{1+\beta/2, 2+\beta}_{t,\mathbf{x}}(\Pi_T) \cap \mathbb{C}^0(\bar{\Pi}_T)$ *for some* $\beta \in (0,1)$.

Uniquenses Theorem. *If the function* $F(t, \mathbf{x}, u, \mathbf{p})$ *and its partial derivative with respect to* u *are bounded for* $(t, \mathbf{x}) \in K_R$ *(or* Π_T*) and finite* u*,* p*, then the solutions in Theorems* 1.*1*-1.*4 are unique.*

The proofs of the Theorems are given in Section 3. In Section 2 we prove the global gradient estimate (independent of ε) which is the principal step in order to prove the existence theorems. In deriving this apriori estimate we use Kruzhkov's idea of introducing a new spatial variable [4] (see also [10]-[13]).

Now let us present some examples of functions not satysfying the growth restrictions of Bernstein's type but for which the indicated theorems take place. Let m be an arbitrary positive integer and $g(t, \mathbf{x}, u, \nabla u)$ is an arbitrary function which satisfies condition (1.10) and (1.2). There exists a global solution of the first, second, and the third boundary value problems for the following equation:

$$u_t = \varepsilon \Delta u - u|\nabla u|^m + g(t, \mathbf{x}, u, \nabla u),$$

and a global solution for the second and third boundary value problems for the equations

$$u_t = \varepsilon \Delta u - (\mathbf{x} \cdot \nabla u)^{2m+1} + g(t, \mathbf{x}, u, \nabla u),$$
$$u_t = \varepsilon \Delta u - u e^{|\nabla u|} + g(t, \mathbf{x}, u, \nabla u).$$

In order to simplify the notation below we will omit the symbol "tilde" over the functions. Denote by $Q_R = \{(t,r) \mid 0 < t < T,\ 0 < r < R\}$. Equation (1.1) in the (t,r) variables and the conditions corresponding to (1.4), (1.5), (1.6) and (1.7) appears in the following form:

$$u_t = \varepsilon u_{rr} + \frac{\varepsilon(n-1)}{r} u_r + f(t,r,u,u_r) + g(t,r,u,u_r) \quad \text{in } Q_R, \tag{1.12}$$

$$u_r(t,r)\big|_{r=0} = 0, \quad u(t,r)\big|_{r=R} = 0, \tag{1.13}$$

$$u_r(t,r)\big|_{r=0} = 0, \quad u_r(t,r)\big|_{r=R} = 0, \tag{1.14}$$

$$u_r(t,r)\big|_{r=0} = 0, \quad u_r(t,r) + \sigma(t, u(t,r))\big|_{r=R} = 0, \tag{1.15}$$

$$u(t,r)\big|_{t=0} = \varphi(r). \tag{1.16}$$

Note that the left condition in (1.13)-(1.15) appears due to the fact that we seek smooth radially-symmetric solutions of the indicated problems (see Section 3).

2. The gradient estimates

Lemma 2.1. *Let $u(t,r)$ be a classical solution of problem* (1.12), (1.14), (1.16). *Suppose that for $(t,r) \in Q_R$, $|u(t,r)| \le M$ and arbitrary p*

$$|g(t,r,u,p)| \le \varepsilon \psi(|p|), \qquad \int^{+\infty} \frac{z\,dz}{\psi(z)} = +\infty, \tag{2.1}$$

where $\psi(z) \ge 1$ is a smooth function. Suppose that the function f satisfies the following conditions:

$$\begin{cases} f(t,r_1,u_1,p) - f(t,r_2,u_2,p) \ge 0, \\ f(t,r_2,u_1,-p) - f(t,r_1,u_2,-p) \ge 0 \end{cases} \tag{2.2}$$

for $r_2 > r_1$, $u_2 > u_1$, $p > 0$ and $\varphi(r)$ is a Lipschitz continuous function:

$$|\varphi(r_1) - \varphi(r_2)| \le K|r_1 - r_2|. \tag{2.3}$$

Then in $\bar{Q}_R$ the inequality $|u_r(t,r)| \le C_1$ holds, where the constant C_1 depends only on M, K, ψ.

Proof. Write equation (1.12) at a point (t,ρ) where $r \neq \rho$:

$$u_t = \varepsilon u_{\rho\rho} + \frac{\varepsilon(n-1)}{\rho} u_\rho + f(t,\rho,u,u_\rho) + g(t,\rho,u,u_\rho), \quad u = u(t,\rho).$$

Introduce the function $v(t,r,\rho) = u(t,r) - u(t,\rho)$. Then $v(t,r,\rho)$ in $\Omega = \{(t,r,\rho) \mid 0 < t < T,\ \rho < r,\ 0 < \rho, r < R\}$ satisfies the following equation:

$$v_t = \varepsilon v_{rr} + \varepsilon v_{\rho\rho} + f^{(r)} - f^{(\rho)} + g^{(r)} - g^{(\rho)} + \frac{\varepsilon(n-1)}{r} u_r(t,r) - \frac{\varepsilon(n-1)}{\rho} u_\rho(t,\rho),$$

where

$$f^{(\lambda)} = f(t,\lambda,u(t,\lambda),u_\lambda(t,\lambda)),$$
$$g^{(\lambda)} = g(t,\lambda,u(t,\lambda),u_\lambda(t,\lambda)).$$

Define the following operator

$$L(v) \equiv -v_t + \varepsilon[v_{rr} + \psi(|v_r|)] + \varepsilon[v_{\rho\rho} + \psi(|v_\rho|)].$$

From (2.1) it follows that

$$L(v) \geq f^{(\rho)} - f^{(r)} + \frac{\varepsilon(n-1)}{\rho} u_\rho - \frac{\varepsilon(n-1)}{r} u_r.$$

Let the function $h(\tau)$ be a solution of the following ordinary differential equation:

$$h''(\tau) + \psi(|h'(\tau)|) = 0 \tag{2.4}$$

on the interval $[0,\tau_0]$, where $\tau_0 > 0$ will be selected below, and satisfies the boundary conditions:

$$h(0) = 0, \quad h(\tau_0) = 2M. \tag{2.5}$$

Represent the solution of (2.4), (2.5) in parametrical form (using the substitution $h'(\tau) = q(h)$, $dq/d\tau = q\,dq/dh$):

$$h(q) = \int_q^{q_1} \frac{z\,dz}{\psi(z)}, \quad \tau(q) = \int_q^{q_1} \frac{dz}{\psi(z)},$$

where the parameter q varies in the interval $[q_0,q_1]$ and q_0, q_1 are selected so that $q_1 > q_0 = K$ and

$$h(q_0) = \int_{q_0}^{q_1} \frac{z\,dz}{\psi(z)} = 2M. \tag{2.6}$$

This is possible due to (2.1). Put $\tau_0 \equiv \tau(q_0)$. Consider the function $w(t,r,\rho) = v(t,r,\rho) - h(r-\rho)$ in $P = \{(t,r,\rho) \mid 0 < t < T,\ 0 < r-\rho < \tau_0,\ 0 < r,\ \rho < R\}$. Due to the fact that $h(\tau)$ satisfies (2.4) we have $L(h(r-\rho)) = 0$ and

$$\begin{aligned} L(w) &\equiv L(v) - L(h) \equiv -w_t + \varepsilon[w_{rr} + \alpha_1 w_r] + \varepsilon[w_{\rho\rho} + \alpha_2 w_\rho] \\ &\geq f^{(\rho)} - f^{(r)} + \frac{\varepsilon(n-1)}{\rho} u_\rho - \frac{\varepsilon(n-1)}{r} u_r, \end{aligned}$$

where $|\alpha_i| < +\infty$, $i = 1,2$ by virtue of the fact that ψ is a smooth function and u is a classical solution of (1.12), (1.14), (1.16). Denote by Γ the parabolic boundary of P (i.e., $\Gamma = \partial P \setminus \{(t,r,\rho) \mid t = T,\ 0 < r-\rho < \tau_0,\ 0 < r,\ \rho < R\}$. Let the function w attains its positive maximum at the point $(t_0, r_0, \rho_0) \in \bar{P} \setminus \Gamma$. Then at this point we have

$$w > 0, \quad w_r = w_\rho = 0, \quad w_{rr} \leq 0, \quad w_{\rho\rho} \leq 0, \quad w_t \geq 0. \tag{2.7}$$

From (2.7) it immediatelly follows that $\widetilde{L}(w) \leq 0$. On the other hand from (2.7) we obtain:

$$\begin{aligned} w(t_0,r_0,\rho_0) &= u(t_0,r_0) - u(t_0,\rho_0) - h(r_0-\rho_0) > 0, \\ w_r(t_0,r_0,\rho_0) &= u_r(t_0,r_0) - h'(r_0-\rho_0) = 0, \\ w_\rho(t_0,r_0,\rho_0) &= -u_\rho(t_0,\rho_0) + h'(r_0-\rho_0) = 0, \end{aligned}$$

and as a consequence

$$u(t_0,r_0) > u(t_0,\rho_0), \quad u_r(t_0,r_0) = u_\rho(t_0,\rho_0) = h'(r_0-\rho_0) > 0.$$

Thus we have that

$$\begin{aligned} &\varepsilon\left(\frac{n-1}{\rho_0} u_\rho(t_0,\rho_0) - \frac{n-1}{r_0} u_r(t_0,r_0)\right) \\ &\qquad = \varepsilon h'(r_0-\rho_0)\left(\frac{n-1}{\rho_0} - \frac{n-1}{r_0}\right) > 0. \end{aligned}$$

From (2.2) it follows that

$$f(t,\rho_0,u(t_0,\rho_0),h'(r_0-\rho_0)) - f(t,r_0,u(t_0,r_0),h'(r_0-\rho_0)) \geq 0$$

and finally $\widetilde{L}(w) > 0$. From this contradiction we conclude that w cannot attain positive maximum in $\bar{P} \setminus \Gamma$.

Consider w on Γ. Let $\tau_0 \leq R$. When $t = 0$:

$$\begin{aligned} w(0,r,\rho) &= \varphi(r) - \varphi(\rho) - h(r-\rho) \\ &\leq K(r-\rho) - (h(r-\rho) - h(0)) \\ &\leq K(r-\rho) - q_0(r-\rho) = 0, \end{aligned}$$

due to the selection of q_0 ($q_0 = K$). Further, $w(t,r,r) = 0$ and when $r - \rho = \tau_0$ we have $w = u(t,r) - u(t,\rho) - h(\tau_0) \le 0$ due to (2.5). Denote by $Q_1 = \{(t,r) \mid 0 < t \le T,\ 0 < r < \tau_0,\ \rho = 0\}$, $Q_2 = \{t,\rho) \mid 0 < t \le T,\ R - \tau_0 < \rho < R,\ r = R\}$. Taking into account (1.14) and the fact that $h' \ge q_0 > 0$, we conclude that

$$-w_\rho(t,r,0) = u_\rho(t,0) - h'(r) = -h'(r) < 0,$$
$$w_r(t,R,\rho) = u_r(t,R) - h'(R-\rho) = -h'(R-\rho) < 0.$$

Thus the function $w(t,r,\rho)$ cannot attain its positive maximum neither on Q_1 nor on Q_2. The case when $\tau_0 > R$ can be treated similarly. The only difference is the absence of the boundary $r - \rho = \tau_0$ and we put $Q_1 = \{(t,r) \mid 0 < t \le T,\ 0 < r \le R,\ \rho = 0\}$, $Q_2 = \{t,\rho) \mid 0 < t \le T,\ 0 \le \rho < R,\ r = R\}$.

So we obtain that $w(t,r,\rho) \le 0$ in $\bar{P}$ and $u(t,r) - u(t,\rho) \le h(r-\rho)$ in $\bar{P}$. Treating analogously the function $\tilde{v}(t,r,\rho) = u(t,\rho) - u(t,r)$ one can easily see that for $\widetilde{w}(t,r,\rho) = \tilde{v} - h(r-\rho)$ we obtain

$$\widetilde{L}(\widetilde{w}) \ge f^{(r)} - f^{(\rho)} + \frac{\varepsilon(n-1)}{r} u_r - \frac{\varepsilon(n-1)}{\rho} u_\rho \quad \text{in } P.$$

Suppose that the function $\widetilde{w}$ attains its positive maximum at $(\tilde{t}_0, \tilde{r}_0, \tilde{\rho}_0) \in \bar{P} \setminus \Gamma$. In the same way as for w we conclude that $\widetilde{L}(\widetilde{w}) \le 0$. On the other hand, we obtain that

$$u(\tilde{t}_0, \tilde{\rho}_0) > u(\tilde{t}_0, \tilde{r}_0), \quad u_r(\tilde{t}_0, \tilde{r}_0) = u_\rho(\tilde{t}_0, \tilde{\rho}_0) = -h'(\tilde{r}_0 - \tilde{\rho}_0) < 0.$$

Using the second inequality from (2.2) and the fact that

$$\varepsilon\left(\frac{n-1}{\tilde{r}_0} u_r(\tilde{t}_0, \tilde{r}_0) - \frac{n-1}{\tilde{\rho}_0} u_\rho(\tilde{t}_0, \tilde{\rho}_0)\right) = -\varepsilon h'(\tilde{r}_0 - \tilde{\rho}_0)\left(\frac{n-1}{\tilde{r}_0} - \frac{n-1}{\tilde{\rho}_0}\right) > 0,$$

finally we obtain that $\widetilde{L}(\widetilde{w}) > 0$. From this contradiction we conclude that $\widetilde{w}$ cannot attain positive maximum in $\bar{P} \setminus \Gamma$.

Consider $\widetilde{w}$ on Γ. One can easily see that all considerations concerning the estimate of the function w on the boundary Γ can be done without any changes in estimate of the function $\widetilde{w}$. Thus we have that $u(t,\rho) - u(t,r) \le h(r-\rho)$ in $\bar{P}$.

In view of the symmetry of the variables r, ρ, in the same manner we examine the case $\rho > r$. As a result we have that for $0 \le t \le T$, $0 \le r, \rho \le R$, $0 < |r - \rho| \le \tau_0$, the inequality

$$\left| \frac{u(t,r) - u(t,\rho)}{r - \rho} \right| \le \frac{h(|r-\rho|) - h(0)}{|r - \rho|}$$

holds, which implies $|u_r(t,r)| \le h'(0) = q_1 = C_1$. Lemma is proved. $\square$

Remark 1. Let us mention here that if we select $q_0 = \max[K, 2M/R]$ then $\tau_0 \le R$. In fact,

$$\tau_0 \equiv \tau(q_0) = \int_{q_0}^{q_1} \frac{dz}{\psi(z)} \le \frac{1}{q_0} \int_{q_0}^{q_1} \frac{z\,dz}{\psi(z)} = \frac{2M}{q_0} \le R.$$

Consider problem (1.12), (1.15), (1.16).

Lemma 2.2. *Let $u(t,r)$ be a classical solution of* (1.12), (1.15), (1.16), *and all conditions of Lemma* 2.1 *be fulfilled. Then in $\bar{Q}_R$ the inequality*

$$|u_r(t,r)| \le C_2$$

holds, where the constant C_2 depends only on M, K, N, and ψ, where $N = \sup|\sigma|$ (the supremum is taken over the set $[0,T] \times [-M,M]$).

Proof. The proof of Lemma 2.2 differs from the proof of the previous one only in selection of q_0 and in analyzing of the behavior of $w(t,r,\rho)$ on the bound Q_2. We select q_0 so that $q_0 > \max[K,N]$. Taking into account the right boundary condition in (1.15), we obtain that on the bound Q_2:

$$\begin{aligned} w_r\big|_{r=R} &= u_r(t,R) - h'(R-\rho) \\ &= -\sigma(t,u)\big|_{r=R} - h'(R-\rho) \le N - q_0 < 0. \end{aligned}$$

Hence on Q_2 the function $w(t,r,\rho)$ cannot attain its positive maximum.

Consider the function $\widetilde{w} = u(t,\rho) - u(t,r) - h(r-\rho)$. On Q_2 we have:

$$\begin{aligned} \widetilde{w}_r\big|_{r=R} &= -u_r(t,R) - h'(R-\rho) \\ &= \sigma(t,u)\big|_{r=R} - h'(R-\rho) \le N - q_0 < 0, \end{aligned}$$

from where it follows that $\widetilde{w}$ also cannot attain its positive maximum on Q_2. By using a similar arguments as in Lemma 2.1, we complete the proof. $\square$

Consider problem (1.12), (1.13), (1.16).

Lemma 2.3. *Let $u(t,r)$ be a classical solution of* (1.12), (1.13), (1.16), *and all conditions of Lemma* 2.1 *be fulfilled. Suppose in addition that for $|u(t,r)| \le M$ and arbitrary p*

$$\begin{cases} f(t,r,u,p) \ge 0 & \text{when } u < 0,\ p > 0, \\ f(t,r,u,p) \le 0 & \text{when } u > 0,\ p < 0, \\ \varphi(R) = 0. \end{cases} \tag{2.8}$$

Then in $\bar{Q}_R$ the inequality $|u_r(t,r)| \le C_3$ holds, where the constant C_3 depends only on M, K, and ψ.

Proof. The proof of Lemma 2.3 differs from the proof of Lemma 2.1 only in the way we analyze the behavior of $w(t,r,\rho)$ on Q_2. Let us show that $w(t,r,\rho) \leq 0$ on $\bar{Q}_2$. When $r = R$ we have

$$\begin{aligned} w(t,R,\rho) &= u(t,R) - u(t,\rho) - h(R-\rho) \\ &= -u(t,\rho) - h(R-\rho) = -w_2(t,\rho). \end{aligned}$$

Define the following operator $L_0(u) \equiv -u_t + \varepsilon u_{\rho\rho}$; obviously

$$L_0(u) = -\frac{\varepsilon(n-1)}{\rho} u_\rho - f^{(\rho)} - g^{(\rho)},$$

and

$$\begin{aligned} L_0(h(R-\rho)) &= \varepsilon h_{\rho\rho} = -\varepsilon\psi(|h'|), \\ L_0(w_2) &= -\frac{\varepsilon(n-1)}{\rho} u_\rho - f^{(\rho)} - g^{(\rho)} - \varepsilon\psi(|h'|). \end{aligned}$$

We will show that $w_2(t,\rho) \geq 0$ on $\bar{Q}_2$. If the function $w_2(t,\rho)$ attains its negative minimum at the point $(t_0,\rho_0) \in Q_2$, then at this point $L_0(w_2) \equiv -w_{2t} + \varepsilon w_{2\rho\rho} \geq 0$. At the same time, due to (1.10) and taking into account that $u_\rho(t_0,\rho_0) = h'(R-\rho_0)$, we have $\varepsilon(-g(t_0,\rho_0,u(t_0,\rho_0),h') - \psi(|h'|)) \leq 0$. Moreover, by virtue of the fact that

$$u_\rho(t_0,\rho_0) = h'(R-\rho_0) > 0, \quad u(t_0,\rho_0) < 0,$$

using the first inequality (2.8) we obtain

$$-\frac{\varepsilon(n-1)}{\rho} h'(R-\rho_0) - f(t_0,\rho_0,u(t_0,\rho_0),h'(R-\rho_0)) < 0.$$

As a consequence, $L_0(w_2) < 0$. From this contradiction we conclude that w_2 cannot attain its negative minimum on Q_2. Let us show that $w_2 \geq 0$ on the parabolic boundary of Q_2. Due to the fact that $h' \geq q_0 = K$ and $\varphi(R) = 0$, we have

$$\begin{aligned} w_2(0,\rho) &= u(0,\rho) + h(R-\rho) = \varphi(\rho) + h(R-\rho) \\ &= \varphi(\rho) - \varphi(R) + h(R-\rho) - h(0) \\ &\geq -K(R-\rho) + q_0(R-\rho) = 0. \end{aligned}$$

If $\tau_0 \leq R$, then $w_2(t,R-\tau_0) = u(t,R-\tau_0) + h(\tau_0) > 0$ (recall that $h(\tau_0) = 2M$.) If $\tau_0 > R$, then $-w_{2\rho}(t,0) = -u_\rho(t,0) + h'(R) > 0$ and hence w_2 cannot attain minimum when $\rho = 0$. As a consequence we obtain that $w(t,r,\rho) \leq 0$ on $\bar{Q}_2$.

Consider the function $\widetilde{w} = u(t,\rho) - u(t,r) - h(r-\rho)$. For $r = R$ we have

$$\widetilde{w} = u(t,\rho) - h(R-\rho) = \widetilde{w}_2(t,\rho).$$

Consider the function $\widetilde{w}_2(t,\rho)$ on Q_2. Let us show that $\widetilde{w}_2 \leq 0$ on $\bar{Q}_2$.

Define operator $L_1(u) \equiv -u_t + \varepsilon[u_{\rho\rho} + \psi(|u_\rho|)]$. Obviously

$$L_1(u) \geq -\frac{\varepsilon(n-1)}{\rho} u_\rho - f(t,\rho,u,u_\rho), \quad L_1(h) = 0.$$

we conclude

$$\widetilde{L}_1(\widetilde{w}_2) \geq -\frac{\varepsilon(n-1)}{\rho} u_\rho - f(t,\rho,u,u_\rho).$$

If $\widetilde{w}_2(t,\rho)$ attains its positive maximum at the point $(t_1,\rho_1) \in Q_2$, then

$$u_\rho(t_1,\rho_1) = -h'(R-\rho_1) < 0, \quad u(t_1,\rho_1) > 0,$$

and hence due to the second inequality (2.8)

$$\frac{\varepsilon(n-1)}{\rho} h'(R-\rho_1) - f(t_1,\rho_1,u(t_1,\rho_1),-h'(R-\rho_1)) > 0.$$

Thus we obtain that $\widetilde{L}_1(\widetilde{w}_2) > 0$, which in turn contradicts the assumption that $\widetilde{w}_2$ attains its positive maximum. One can easily obtain that $\widetilde{w}_2 \leq 0$ on the parabolic boundary of Q_2. Note that in the case $\tau_0 > R$ we have $-\widetilde{w}_{2\rho}(t,0) = -u_\rho(t,0) - h'(R) = -h'(R) < 0$ and hence $\widetilde{w}_2$ cannot obtain maximum when $\rho = 0$. Whence it immediatelly follows that $\widetilde{w}(t,R,\rho) \leq 0$ on $\bar{Q}_2$. The Lemma is proved. $\square$

In the previous Lemmas we obtained the apriori estimates of $|u_r(t,r)|$ which depend on $M = \max|u|$. In the following lemma we show that, substituting condition (2.1) by a more restrictive one, we can obtain similar estimates independent of M. We consider problem (1.12), (1.14), (1.16); problems (1.12), (1.13), (1.16) and (1.12), (1.15), (1.16) can be treated similarly.

Lemma 2.4. *Let $u(t,r)$ be a classical solution of* (1.12), (1.14), (1.16) *and all conditions of Lemma* 2.1 *except condition* (2.1) *be fulfilled. Instead of* (2.1) *we assume that for $(t,x) \in Q_R$ and arbitrary (u,p) the function g satisfies the following relation*

$$|g(t,r,u,p)| \leq \varepsilon\psi(|p|), \quad \int^\infty \frac{dz}{\psi(z)} = +\infty. \tag{2.9}$$

Then in $\bar{Q}_R$ the inequality $|u_r(t,r)| \leq C_4$, holds, where the constant C_4 depends only on K and ψ.

Proof. The proof of Lemma 2.4 differs from the proof of Lemma 2.1 in construction of the barrier function $h(\tau)$. Again represent the solution of (2.4) in parametrical form:

$$h(q) = \int_q^{q_1} \frac{z\,dz}{\psi(z)}, \quad \tau(q) = \int_q^{q_1} \frac{dz}{\psi(z)},$$

where the parameter q varies in $[q_0, q_1]$, and $q_1 > q_0 = K$. From (2.9) it follows that one can select q_1 so as $\tau_0 \equiv \tau(q_0) = R$. Put

$$h(q_0) = \int_{q_0}^{q_1} \frac{z\,dz}{\psi(z)} \equiv H.$$

Thus we have that the function $h(\tau)$ satisfies equation (2.4) and the following boundary conditions:

$$h(0) = 0, \quad h(R) = H. \tag{2.10}$$

We prove that functions $w(t,r,\rho) = u(t,r) - u(t,\rho) - h(r-\rho)$ and $\widetilde{w} = u(t,\rho) - u(t,r) - h(r-\rho)$ cannot attain their positive maximum in $\bar{P} = \{(t,r,\rho) \mid 0 \le t \le T,\ 0 \le r-\rho,\ 0 \le r,\ \rho \le R\} \setminus \Gamma$, using the same considerations as in Lemma 2.1. □

Remark 2. Our assumptions on the functions $f(t,r,u,p)$ and $g(t,r,u,p)$ that appear in Lemmas 2.1, 2.2, 2.3, 2.4 can be somehow weakened. One can easily see that, in order to prove the above mentioned Lemmas, those assumptions must be fulfilled only for $p \in [-q_1, -q_0] \cup [q_0, q_1]$.

Remark 3. All estimates obtained in this section (i.e., C_1, C_2, C_3, C_4) do not depend on ε.

Examples.

1. Suppose that all assumptions of Lemma 2.1 are fulfilled and in addition $g(t,r,u,u_r) \equiv 0$. In that case for the classical solution of problem (1.12), (1.14), (1.16) we have $|u_r(t,r)| \le K$, where K is a constant in condition (2.3). In fact, when $g \equiv 0$ we can take $\psi \equiv 0$. So the barrier function is the solution of the equation $h'' = 0$. It is not difficult to show that $|u(t,r) - u(t,\rho)| \le K|r-\rho|$ in $\bar{P} = \{(t,r,\rho) \mid 0 \le t \le T,\ 0 \le r-\rho,\ 0 \le r,\ \rho \le R\}$. Hence $h = K\tau$ is the desired barrier function and $|u_r| \le h'(0) = K$.

2. Suppose that all assumptions of Lemma 2.2 are fulfilled and in addition $g \equiv 0$. Analoguosly to the previous example we obtain that for the classical solution of problem (1.12), (1.15), (1.16) we have $|u_r(t,r)| \le \max\{K,N\}$.

3. Suppose that all assumptions of Lemma 2.3 are fulfilled and in addition $g \equiv 0$. Analoguosly to the previous examples, we obtain that for the classical solution of problem (1.12), (1.13), (1.16) we have $|u_r(t,r)| \le K$.

3. THE EXISTENCE THEOREMS

In order to prove the existence theorems we need some supplementary apriori estimates for the classical solution. Consider equation (1.12) as a linear equation

$$u_t = \varepsilon u_{rr} + \frac{\varepsilon(n-1)}{r} u_r + \widetilde{F}(t,r), \tag{3.1}$$

where $\widetilde{F}(t,r) = F(t,r,u,u_r)$ is a Hölder continuous function. Equation (1.12) degenerate when $r \to 0$, the left boundary condition $u_r\big|_{r=0} = 0$ actually is developed by the solution of this equation. In order to estimate the $|u_r|$ in the Hölder space one have to show that:

$$\lim_{r\to 0} \left| \frac{u_r}{r} \right| < +\infty. \tag{3.2}$$

Using the method of separation of variables we obtain the following eigenvalue problem:

$$U''(r) + \frac{n-1}{r} U'(r) + \frac{\nu^2}{\varepsilon} U(r) = 0, \tag{3.3}$$

with boundary conditions

$$U'\big|_{r=0} = 0, \quad U\big|_{r=R} = 0. \tag{3.4}$$

One can easily see that the solution of (3.3) is the function $U = (\lambda r)^{1-\kappa} J_{\kappa-1}(\lambda r)$, where $\lambda = \nu/\sqrt{\varepsilon}$, $2\kappa = n$, and $J_{\kappa-1}(\lambda r)$ is a Bessel function of the type:

$$J_{\kappa-1}(y) = y^{\kappa-1} \sum_{m=0}^{\infty} \frac{(-1)^m y^{2m}}{m!\Gamma(m+1)2^{2m}}. \tag{3.5}$$

For problem (3.3), (3.4) we have $J_{\kappa-1}(\mu) = 0$, where $\mu = \lambda R$. It is well-known [14] that the equation $J_{\kappa-1}(\mu) = 0$, has a denumerable set of roots $\mu_s > 0$, $s = 0$, 1, 2, ..., and Bessel functions make up an orthogonal basis in L_2. So we can represent the solution of problem (1.12), (1.13), (1.16) in the following form:

$$u(t,r) = \sum_{s=0}^{\infty} u_s(t)(\lambda_s r)^{1-\kappa} J_{\kappa-1}(\lambda_s r), \tag{3.6}$$

where $\lambda_s = \mu_s/R$. We can also write the representation in terms of Bessel functions for $\widetilde{F}(t,r)$ and $\varphi(r)$:

$$\widetilde{F}(t,r) = \sum_{s=0}^{\infty} F_s(t)(\lambda_s r)^{1-\kappa} J_{\kappa-1}(\lambda_s r), \tag{3.7}$$

$$\varphi(r) = \sum_{s=0}^{\infty} C_s^{\varphi} (\lambda_s r)^{1-\kappa} J_{\kappa-1}(\lambda_s r). \tag{3.8}$$

Substituting (3.6) in (1.12), (1.13), (1.16) and using (3.7) and (3.8) we obtain the following problem for $u_s(t)$:

$$u_s'(t) + \varepsilon\lambda_s^2 u_s(t) = F_s(t), \quad u_s(0) = C_s^{\varphi}.$$

Hence

$$u(t,r) = \sum_{s=0}^{\infty} (\lambda_s r)^{1-\kappa} J_{\kappa-1}(\lambda_s r)\left(\int_0^t e^{-\varepsilon\lambda_s^2(t-\tau)} F_s(\tau)d\tau + C_s^{\varphi}\right).$$

Using the representation of Bessel functions (3.5) it is easy to show that (3.2) holds. Relation (3.2) can be obtained in an analogous way for problems (1.12), (1.14), (1.16) and (1.12), (1.15), (1.16). For more details see [14], [15].

Now we can treat equation (1.12) as

$$u_t = \varepsilon u_{rr} + \widetilde{F}_1(t,r), \tag{3.9}$$

where $\widetilde{F}_1(t,r)$ is a bounded function. From [4] it follows that

$$|u(t_1,r) - u(t_2,r)| \le C_5|t_1 - t_2|^{1/2},$$

the constant C_5 depends only on $\max|u_r|$ and $\max|\widetilde{F}_1|$ and

$$|u_r(t_1,r_1) - u_r(t_2,r_2)| \le C_6(|r_1 - r_2|^{\gamma} + |t_1 - t_2|^{\gamma/2}),$$

where C_6 and $\gamma < \alpha$ depend only on ε, $\max|u_r|$, and $\max|\widetilde{F}_1|$. The apriori estimate for $|u|$ for problem (1.12), (1.13), (1.16) follows from (1.2), for problem (1.12), (1.15), (1.16) from (1.2) and condition $u\sigma(t,u)\big|_{|\mathbf{x}|=R} > 0$ for $u \ne 0$. The apriori estimate for $|u|$ for problem (1.12), (1.14), (1.16) follows from [2] (see Theorem 13.1 and also Exercise 13.1).

The solvability of the above mentioned problems follows from the obtained apriori estimates and the Leray-Shauder theorem (see [16, Theorem 11.3]). In fact, consider $V \equiv C_{t,r}^{\delta,1+\delta}(\bar{Q}_R)$, where $0 < \delta < \min\{\frac{1}{2},\gamma\}$. If $v(t,r) \in V$, then $F_0(t,r) \in C_{t,r}^{\beta/2,\beta}(Q_R)$ for some $\beta \in (0,\frac{1}{2})$. Here $F_0 \equiv F(t,r,v,v_r)$. Construct operator $T : v(t,r) \in V \to u(t,r)$ where $u(t,r)$ is a solution of the linear problem

$$u_t = \varepsilon u_{rr} + \lambda\left(F_0(t,r) + \varepsilon\frac{n-1}{r}u_r\right),$$
$$u(0,r) = \lambda\varphi(r), \quad u_r(t,0) = u(t,R) = 0, \quad \lambda \in [0,1].$$

It is clear that $u(t,r) \in C_{t,r}^{1+\beta/2,2+\beta}(Q_R) \cap C_{t,r}^{1/2,1+\gamma}(\bar{Q}_R) \equiv U$. So we have that the mapping $T : V \to U$ is bounded, and the mapping $T : V \to V$ is compact.

Hence there exists a fixed point $u = Tu$. Concerning the proof of Theorem 1.4, we note that the solution of the Cauchy problem can be obtained as the limit of a sequence of the solutions of the third boundary value problems in K_R, under unlimited dilatation of the domain B_R [1].

Finally, we want to mention that the Uniqueness Theorem follows immediately from [12].

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