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Existence of Lipschitz continuous solutions to the Cauchy–Dirichlet problem for anisotropic parabolic equations

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The Cauchy–Dirichlet and the Cauchy problem for the degenerate and singular quasilinear anisotropic parabolic equations are considered. We show that the time derivative u_t of a solution u belongs to L_∞ under a suitable assumption on the smoothness of the initial data. Moreover, if the domain satisfies some additional geometric restrictions, then the spatial derivatives u_{x_i} belong to L_∞ as well. In the singular case we show that the second derivatives $u_{x_ix_j}$ of a solution of the Cauchy problem belong to L_2.

1. Introduction and main results

Let Ω be a bounded domain in $\mathbf{R}^n$ satisfying the exterior sphere condition and $\Omega_T = (0,T) \times \Omega$ with an arbitrary $T \in (0,\infty)$. We denote by $x = (x_1, \dots, x_n)$ the points in

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Ω and by t the time variable that varies in the interval $[0, T]$. Consider the following quasilinear parabolic equation

$$u_t = \sum_{i=1}^{n} (|u_{x_i}|^{p_i} u_{x_i})_{x_i} \quad \text{in} \quad \Omega_T, \tag{1.1}$$

coupled with the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad [0, T] \times \partial\Omega \tag{1.2}$$

and the initial condition

$$u(0, x) = u_0(x) \quad \text{in} \quad \Omega, \quad u_0(x) \in C^2(\Omega) \quad \text{and} \quad u_0(x) = 0 \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

Here $p_i > -1$, $i = 1, \dots, n$. Without loss of generality, we assume that the p_i are ordered:

$$-1 < p_1 \leq p_2 \leq ... \leq p_n < +\infty.$$

Let $-1 < p_i < 0$ for $i = 1, ..., m$ and $p_i \geq 0$ for $i = m + 1, ..., n$ where $0 \leq m \leq n$.

This class of equations has received considerable attention in the last years and not only, see, for example, [1–5,10–13,22,23] and the references therein. Concerning the different aspects of the stationary case, see, for example, [6,7,17,22,23]. From [13] it follows that if $u_0 \in L_\infty(\Omega)$, then there exists a unique weak solution of problem (1.1)–(1.3) which is defined as a function

$$u \in L_\infty(\Omega_T) \cap V(\Omega_T) \cap C([0, T]; L_s(\Omega)) \quad \forall s \in [1, \infty), \quad u_t \in V^*(\Omega_T),$$

satisfying the integral identity

$$\int\limits_{\Omega_T} \Big(u\phi_t - \sum_{i=1}^{n} |u_{x_i}|^{p_i} u_{x_i} \phi_{x_i} \Big) dxdt = - \int\limits_{\Omega} u_0 \phi(0, x) dx$$

for an arbitrary smooth function $\phi(t, x)$ which is equal to zero for $x \in \partial\Omega$ and for $t = T$. Here $V^*(\Omega_T)$ is the adjoint space to $V(\Omega_T) = \cap_{i=1}^{n} L_{p_i+2}(0, T; U_i(\Omega))$ where $U_i(\Omega)$ is the closure of $C^0_\infty(\Omega)$ with respect to the norm $\|u\|_{U_i} = \|u\|_{L_2} + \|u_{x_i}\|_{L_{p_i+2}}$ (for more details see [13]).

The main goal of the present paper is to show that under the following assumption on the initial data u_0:

$$\sum_{i=1}^{n} \max_{\Omega} |(|u_{0x_i}|^{p_i} u_{0x_i})_{x_i}| < +\infty, \tag{1.4}$$

the derivative of a solution with respect to t is an L_∞ function. The proof is based on the idea of introducing a new time variable inspired by the idea of introducing a new spatial variable. The last was proposed by Kruzhkov [8] in his investigations devoted to the second order quasilinear parabolic equations with one spatial variable. Based on this idea he obtained the estimate of the low order spatial derivative of a solution of initial boundary value problems. Later the idea of introducing a new spatial variable was modified and applied to a wide class of multidimensional quasilinear parabolic and elliptic equations [14,15,17] (see also [16,18]).

It must be emphasized that the method we use does not require differentiation of the equation and can be applied to more general operator (for more details see remarks at the end of sections 2, 3). We restrict ourselves with equation (1.1) in order to avoid complicated assumptions and to make the idea as simple as possible.

Definition 1. We say that a function $u(t,x)$ is a weak solution of problem (1.1)–(1.3) if

$$u \in L_\infty(\Omega_T), \quad u_{x_i} \in L_{p_i+2}(\Omega_T), \quad u_t \in L_\infty(\Omega_T)$$

and the following integral identity

$$\int\limits_{\Omega_T} \Big(u_t\, \phi + \sum_{i=1}^n |u_{x_i}|^{p_i} u_{x_i}\, \phi_{x_i} \Big) dxdt = 0$$

holds for an arbitrary smooth function ϕ which vanishes on $(0,T) \times \partial\Omega$. The initial condition is satisfied in the classical sense. The boundary condition is satisfied in the sense of the trace of function.

Theorem 1. *Suppose that condition (1.4) is fulfilled and Ω satisfies the exterior sphere condition. Then for an arbitrary $T > 0$, there exists a unique weak solution of problem (1.1)–(1.3). Moreover*

$$\|u_t\|_{L_\infty(\Omega_T)} \le C_0 = \sum_{i=1}^n \max_{\Omega} |(|u_{0x_i}|^{p_i} u_{0x_i})_{x_i}|.$$

Let us turn to the gradient estimates. The case

$$0 \le \min_i\, p_i \le \max_i\, p_i < \min_i\, p_i + \frac{4}{n+2},$$

was considered in [3], where the existence result for the Cauchy–Dirichlet problem with inhomogeneous initial-boundary data was proved under the above restriction on the exponents p_i. The obtained weak solution possess locally Lipschitz gradient with $u_t \in L^{\frac{p}{q-1}}(0,T; W^{-1,\frac{p}{q-1}}(\Omega))$, where $p = \min p_i$, $q = \max p_i$. Moreover in [3] the following regularity result was established: if

$$0 \le \min_i\, p_i \le \max_i\, p_i < \min_i\, p_i + \frac{4}{n}$$

then any weak solution $u \in L^p(0,T;W^{1,p}(\Omega)) \cap L^q_{loc}(0,T;W^{1,q}_{loc}(\Omega))$ admits a locally bounded spatial gradient ∇u. Taking into account the fact that the equation under consideration is nonuniformly parabolic, we can not expect the global L_∞ gradient estimate for arbitrary domains, but we show that for some classes of domains Ω the global gradient estimate takes place for arbitrary $p_i > -1$, $i = 1, ..., n$.

Concerning the additional restriction on Ω we suppose that either it is orthogonal parallelepiped

$$\Omega = (-l_1, l_1) \times ... \times (-l_n, l_n)$$

or it satisfies the following (A) condition

Ω is convex and the parts of $\partial\Omega$ lying in the half spaces $x_i \le 0$ and $x_i \ge 0$ can be expressed as

$$x_i = F_i \quad \textit{and} \quad x_i = G_i, \quad i = 1, ..., n, \tag{A}$$

respectively, where the C^2-functions F_i and G_i do not depend on variable x_i.

For example

$$\Omega = \{x \in \mathbf{R}^n : \sum_{i=1}^n \frac{x_i^2}{\alpha_i^2} < 1\}, \quad \alpha_i \in \mathbf{R} \setminus \{0\}, \quad i = 1, ..., n.$$

Theorem 2. *Assume that condition (1.4) is fulfilled and Ω satisfies assumption (A) or $\Omega = (-l_1, l_1) \times ... \times (-l_n, l_n)$. Then for an arbitrary $T > 0$, there exists a unique weak solution of problem (1.1)–(1.3) such that $u_{x_i} \in L_\infty(\Omega_T)$. Moreover*

$$\|u_t\|_{L_\infty(\Omega_T)} \le C_0, \quad \|u_{x_i}\|_{L_\infty(\Omega_T)} \le C_i = \max_{\Omega} |u_{0x_i}|, \quad i = 1, ..., n.$$

(This implies that u is continuous and so conditions (1.2), (1.3) are satisfied in the classical sense.)

If $\Omega = (-l_1, l_1) \times ... \times (-l_n, l_n)$ and the equation is singular, i.e. $p_i \in (-1, 0)$ $\forall i$, then the solution of the Cauchy–Dirichlet problem is more regular than in Theorem 2.

Definition 2. We say that a Lipschitz continuous function $u(t,x)$ is a strong solution of problem (1.1)–(1.3) if $u_{x_i x_j} \in L_2(\Omega_T)$, $u(t,x)$ satisfies equation

$$u_t = \sum_{i=1}^n (1 + p_i)|u_{x_i}|^{p_i} u_{x_i x_i}$$

almost everywhere in Ω_T and the initial and boundary conditions are satisfied in the classical sense.

Theorem 3. *Assume that condition (1.4) is fulfilled, $\Omega = (-l_1, l_1) \times ... \times (-l_n, l_n)$ and $-1 < p_1 \leq p_2 \leq ... \leq p_n < 0$. Then for an arbitrary $T > 0$, there exists a unique strong solution of problem (1.1)–(1.3). Moreover*

$$\|u_t\|_{L_\infty(\Omega_T)} \leq C_0, \quad \|u_{x_i}\|_{L_\infty(\Omega_T)} \leq C_i,$$
$$\|u_{x_i x_j}\|^2_{L_2(\Omega_T)} \leq \frac{1}{2(p_i+1)} K_i^2 C_i^{-p_i}, \quad K_i = \|u_{0x_i}\|_{L_2(\Omega)}, \quad i,j = 1,...,n.$$

Now let us turn to the Cauchy problem: consider the equation

$$u_t = \sum_{i=1}^n (|u_{x_i}|^{p_i} u_{x_i})_{x_i} \ \text{ in } \ \Pi_T = (0,T) \times \mathbf{R}^n, \tag{1.5}$$

coupled with the initial condition

$$u(0,x) = u_0(x) \ \text{ in } \ \mathbf{R}^n. \tag{1.6}$$

Suppose that $u_0(x)$ has a compact support which we denote by Ω.

Definition 3. We say that a globally Lipschitz continuous function $u(t,x)$ is a weak solution of problem (1.5), (1.6) if $u(0,x) = u_0(x)$ and the following integral identity

$$\int\limits_{\Pi_T} \Big(u_t\, \phi + \sum_{i=1}^n |u_{x_i}|^{p_i} u_{x_i}\, \phi_{x_i} \Big) dxdt = 0$$

holds for an arbitrary smooth function ϕ with compact support.

Theorem 4. *Suppose that condition (1.4) is satisfied. Then for an arbitrary $T > 0$, there exists a unique weak solution of problem (1.5), (1.6). Moreover*

$$\|u_{x_i}\|_{L_\infty(\Pi_T)} \leq C_i = \max_\Omega |u_{0x_i}|, \quad i = 1,...,n,$$
$$\|u_t\|_{L_\infty(\Pi_T)} \leq C_0 = \sum_{i=1}^n \max_\Omega \big| \big(|u_{0x_i}|^{p_i} u_{0x_i}\big)_{x_i} \big|.$$

Definition 4. We say that a globally Lipschitz continuous function $u(t,x)$ is a strong solution of problem (1.5), (1.6) if $u_{x_i x_j} \in L_2(\Pi_T)$, $u(0,x) = u_0(x)$ and $u(t,x)$ satisfies equation

$$u_t = \sum_{i=1}^n (1+p_i)|u_{x_i}|^{p_i} u_{x_i x_i}$$

almost everywhere in Π_T.

Theorem 5. *Suppose that (1.4) is fulfilled and assume that* $-1 < p_1 \le p_2 \le ... \le p_n < 0$. *Then for an arbitrary* $T > 0$, *there exists a unique strong solution of problem (1.5), (1.6). Moreover*

$$\|u_{x_i}\|_{L_\infty(\Pi_T)} \le C_i, \quad \|u_t\|_{L_\infty(\Pi_T)} \le C_0$$

and

$$\|u_{x_i x_j}\|^2_{L_2(\Pi_T)} \le \frac{1}{2(p_i+1)} K_i^2 C_i^{-p_i}, \qquad K_i = \|u_{0x_i}\|_{L_2(\Omega)}, \quad i, j = 1, ..., n.$$

The paper is organized as follows. In sections 2, 3, 4 we prove Theorems 1, 2, 3 respectively, section 5 is devoted to Theorems 4, 5.

2. Proof of Theorem 1

2.1. Regularization

Regularize equation (1.1):

$$u_{\varepsilon t} = \sum_{i=1}^{n} ((u^\alpha_{\varepsilon x_i} + \varepsilon)^{p_i/\alpha} u_{\varepsilon x_i})_{x_i} \quad \text{in} \quad \Omega_T, \tag{2.1}$$

where constant $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 > 0$ is an arbitrary fixed number and $\alpha \in (0, 1)$ is a constant such that $\alpha = r/\rho$ with positive integers r and ρ, where $r < \rho$ and r is even.

Note that for such α

$$(z^\alpha)^{p_i/\alpha} = |z|^{p_i}.$$

If $0 < m \le n$ (i.e. there exists at least one negative p_i) then we impose an additional condition on α: α is small enough so that

$$-1 \le p_i - \alpha \quad \text{for} \quad i = 1, ..., m. \tag{2.2}$$

We will use this condition below in the proof of Lemma 2.1.

Introduce functions $a_{i\varepsilon}(z_i)$

$$a_{i\varepsilon}(z_i) = (z_i^\alpha + \varepsilon)^{\frac{p_i}{\alpha}-1}\big((p_i+1)z_i^\alpha + \varepsilon\big).$$

Taking into account that

$$((u^\alpha_{\varepsilon x_i} + \varepsilon)^{p_i/\alpha} u_{\varepsilon x_i})_{x_i} = (u^\alpha_{\varepsilon x_i} + \varepsilon)^{\frac{p_i}{\alpha}-1}\big((p_i+1)u^\alpha_{\varepsilon x_i} + \varepsilon\big) u_{\varepsilon x_i x_i},$$

we rewrite equation (2.1) in the following form

$$u_{\varepsilon t} = \sum_{i=1}^{n} a_{i\varepsilon}(u_{\varepsilon x_i}) u_{\varepsilon x_i x_i} \quad \text{in} \quad \Omega_T. \tag{2.3}$$

Consider equation (2.3) coupled with conditions

$$u_\varepsilon = 0 \quad \text{on} \quad [0,T] \times \partial\Omega, \quad u_\varepsilon(0,x) = u_0(x) \quad \text{in} \quad \Omega. \tag{2.4}$$

The existence of a classical solution u_ε of problem (2.3), (2.4) follows from [13] (see p. 3024).

2.2. A priori estimates

Our goal in this section is to obtain uniform with respect to ε estimates of solutions to (2.3), (2.4) which would enable us to pass to the limit as $\varepsilon \to 0$.

Denote by

$$C(\varepsilon_0) = \sum_{i=1}^{m} \max_{\Omega} \left|(|u_{0x_i}|^{p_i} u_{0x_i})_{x_i}\right| + \sum_{i=m+1}^{n} \max_{\Omega} \left|\left((u_{0x_i}^{\alpha} + \varepsilon_0)^{p_i/\alpha} u_{0x_i}\right)_{x_i}\right|.$$

For simplicity in the proofs we will omit the subindex ε.

Lemma 2.1. *For every $\varepsilon \in (0, \varepsilon_0]$ and $(t,x) \in \overline{Q}_T$ the following estimate*

$$|u_\varepsilon(t,x) - u_0(x)| \leq C(\varepsilon_0)\, t$$

takes place.

Proof. First let us show that

$$C(\varepsilon_0) \geq \left|\sum_{i=1}^{n} a_{i\varepsilon}(u_{0x_i}) u_{0x_i x_i}\right|. \tag{2.5}$$

To this end consider $|a_{i\varepsilon}(u_{0x_i}) u_{0x_i x_i}|$, $i = 1, ..., n$. We have

$$|a_{i\varepsilon}(u_{0x_i}) u_{0x_i x_i}| = (u_{0x_i}^{\alpha} + \varepsilon)^{\frac{p_i}{\alpha}-1}[(p_i+1)u_{0x_i}^{\alpha} + \varepsilon]|u_{0x_i x_i}| \equiv f(\varepsilon, x).$$

By direct calculations we obtain

$$\frac{\partial f(\varepsilon,x)}{\partial \varepsilon} = \left(u_{0x_i}^{\alpha} + \varepsilon\right)^{\frac{p_i}{\alpha}-2}\left(\frac{p_i}{\alpha}\varepsilon + p_i \frac{p_i + 1 - \alpha}{\alpha} u_{0x_i}^{\alpha}\right)|u_{0x_i x_i}|. \tag{2.6}$$

Consequently $\partial f/\partial\varepsilon \geq 0$ for $p_i \geq 0$. Thus for $p_i \geq 0$ the function $|a_{i\varepsilon}(u_{0x_i})u_{0x_ix_i}|$ is increasing with respect to ε for arbitrary x and as a consequence $\max_\Omega |a_{i\varepsilon}(u_{0x_i})u_{0x_ix_i}|$ is increasing with respect to ε as well. Hence

$$\begin{aligned}\Big|\sum_{i=m+1}^{n} a_{i\varepsilon}(u_{0x_i})u_{0x_ix_i}\Big| &\leq \sum_{i=m+1}^{n} \max_\Omega \big|\big((u_{0x_i}^\alpha + \varepsilon)^{p_i/\alpha} u_{0x_i}\big)_{x_i}\big| \\ &\leq \sum_{i=m+1}^{n} \max_\Omega \big|\big((u_{0x_i}^\alpha + \varepsilon_0)^{p_i/\alpha} u_{0x_i}\big)_{x_i}\big|.\end{aligned}$$

From (2.6) we have that $\partial f/\partial\varepsilon \leq 0$ for $-1 < p_i < 0$ (see (2.2)). So we conclude that for negative p_i the function $|a_{i\varepsilon}(u_{0x_i})u_{0x_ix_i}|$ is decreasing with respect to ε for arbitrary x an as a consequence $\max_\Omega |a_{i\varepsilon}(u_{0x_i})u_{0x_ix_i}|$ is decreasing with respect to ε as well. Thus

$$\Big|\sum_{i=1}^{m} a_{i\varepsilon}(u_{0x_i})u_{0x_ix_i}\Big| \leq \sum_{i=1}^{m} \max_\Omega \big|\big((u_{0x_i}^\alpha + \varepsilon)^{p_i/\alpha} u_{0x_i}\big)_{x_i}\big| \leq \sum_{i=1}^{m} \max_\Omega \big|\big(|u_{0x_i}|^{p_i} u_{0x_i}\big)_{x_i}\big|$$

and (2.5) is proved.

Introduce the function

$$h(t) = (C(\varepsilon_0) + \delta)t, \quad t \in [0, T],$$

where constant $\delta > 0$. Let us prove the following inequality

$$u(t,x) - u_0(x) \leq h(t). \tag{2.7}$$

Consider the linear parabolic operator

$$L \equiv \sum_{i=1}^{n} a_{i\varepsilon}(u_{0x_i}) \frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial t}.$$

Define the function $\phi^- \equiv u - [u_0(x) + h(t)]$. Obviously

$$\begin{aligned}L\phi^- &= \sum_{i=1}^{n} a_{i\varepsilon}(u_{0x_i})u_{x_ix_i} - u_t - \Big[\sum_{i=1}^{n} a_{i\varepsilon}(u_{0x_i})u_{0x_ix_i} - h'(t)\Big] \\ &= \sum_{i=1}^{n} a_{i\varepsilon}(u_{0x_i})u_{x_ix_i} - u_t - \sum_{i=1}^{n} a_{i\varepsilon}(u_{0x_i})u_{0x_ix_i} + C(\varepsilon_0) + \delta \\ &> \sum_{i=1}^{n} a_{i\varepsilon}(u_{0x_i})u_{x_ix_i} - u_t.\end{aligned} \tag{2.8}$$

The last inequality is due to (2.5) and the positivity of δ. Denote by Γ_T the parabolic boundary of Ω_T, i.e.

$$\Gamma_T = \partial\Omega_T \setminus \{(T,x) : x \in \Omega\}.$$

Suppose that at some point $N \in \overline{\Omega}_T \setminus \Gamma_T$ the function ϕ^- attains its maximum, then at this point we have

$$\nabla\phi^- = 0 \quad \Leftrightarrow \quad \nabla u = \nabla u_0 \quad \Rightarrow \quad a_{i\varepsilon}(u_{x_i}) = a_{i\varepsilon}(u_{0x_i}) \quad i = 1, ..., n, \quad \Rightarrow$$
$$\sum_{i=1}^n a_{i\varepsilon}(u_{0x_i})u_{x_ix_i} - u_t\Big|_N = \sum_{i=1}^n a_{i\varepsilon}(u_{x_i})u_{x_ix_i} - u_t\Big|_N = 0.$$

Hence, from (2.8),

$$L\phi^-\Big|_N > 0,$$

which contradicts the assumption that ϕ^- attains its maximum at N. Consider ϕ^- on Γ_T:

for $x \in \partial\Omega$, $t \in [0, T]$ we have $\phi^- = -h(t) \leq 0$;
for $t = 0$, $x \in \Omega$ we have $\phi^- = -h(0) = 0$.

Thus $\phi^- \leq 0$ and (2.7) is proved.
Let us show now that

$$u(t,x) - u_0(x) \geq -h(t). \tag{2.9}$$

Introduce the function $\phi^+ \equiv u - [u_0(x) - h(t)]$. Similarly to the previous case we obtain

$$\begin{aligned} L\phi^+ &= \sum_{i=1}^n a_{i\varepsilon}(u_{0x_i})u_{x_ix_i} - u_t - \Big[\sum_{i=1}^n a_{i\varepsilon}(u_{0x_i})u_{0x_ix_i} + h'(t)\Big] \\ &= \sum_{i=1}^n a_{i\varepsilon}(u_{0x_i})u_{x_ix_i} - u_t - \sum_{i=1}^n a_{i\varepsilon}(u_{0x_i})u_{0x_ix_i} - C(\varepsilon_0) - \delta \\ &< \sum_{i=1}^n a_{i\varepsilon}(u_{0x_i})u_{x_ix_i} - u_t. \end{aligned} \tag{2.10}$$

Suppose that at some point $N_1 \in \overline{\Omega}_T \setminus \Gamma_T$ the function ϕ^+ attains its minimum, then at this point we have

$$\nabla\phi^+ = 0 \quad \Leftrightarrow \quad \nabla u = \nabla u_0 \quad \Rightarrow \quad a_{i\varepsilon}(u_{0x_i}) = a_{i\varepsilon}(u_{x_i}), \quad i = 1, ..., n, \quad \Rightarrow$$
$$\sum_{i=1}^n a_{i\varepsilon}(u_{0x_i})u_{x_ix_i} - u_t\Big|_{N_1} = \sum_{i=1}^n a_{i\varepsilon}(u_{x_i})u_{x_ix_i} - u_t\Big|_{N_1} = 0.$$

Hence, from (2.10),

$$L\phi^+\Big|_{N_1} < 0,$$

which contradicts the assumption that ϕ^+ attains its minimum at N_1. Consider ϕ^+ on Γ_T:

for $x \in \partial\Omega$, $t \in [0, T]$ we have $\phi^+ = h(t) \geq 0$;
for $t = 0$, $x \in \Omega$ we have $\phi^+ = h(0) = 0$.

Thus $\phi^+ \geq 0$ and (2.9) is proved. From (2.7) and (2.9) we have

$$|u(t, x) - u_0(x)| \leq h(t).$$

Passing to the limit as $\delta \to 0$ we finish the prove of Lemma 2.1. □

Lemma 2.2. *For every $\varepsilon \in (0, \varepsilon_0]$ and $(t, x) \in \overline{Q}_T$ the inequality*

$$|u_{\varepsilon t}| \leq C(\varepsilon_0)$$

holds.

Proof. Consider equation (2.3) at two different points (t, x) and (τ, x):

$$\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i})u_{x_i x_i} - u_t = 0, \quad u = u(t, x), \tag{2.11}$$

$$\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i})u_{x_i x_i} - u_\tau = 0, \quad u = u(\tau, x). \tag{2.12}$$

Subtracting (2.12) from (2.11) for $v(t, \tau, x) \equiv u(t, x) - u(\tau, x)$, since

$$v_t = u_t(t, x), \quad v_\tau = -u_\tau(\tau, x), \quad v_{x_i x_i} = u_{x_i x_i}(t, x) - u_{x_i x_i}(\tau, x),$$

we obtain

$$\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}(t, x))v_{x_i x_i} - v_t - v_\tau$$

$$= \sum_{i=1}^{n} \left[a_{i\varepsilon}(u_{x_i}(\tau, x)) - a_{i\varepsilon}(u_{x_i}(t, x))\right] u_{x_i x_i}(\tau, x).$$

Consider the function

$$\mathrm{w} \equiv v - C(\varepsilon_0)(t-\tau) \equiv u(t,x) - u(\tau,x) - C(\varepsilon_0)(t-\tau)$$

in the domain

$$P = \{(t,\tau,x) : t \in (0,T), \tau \in (0,T), x \in \Omega, t > \tau\}.$$

Obviously for $\mathrm{w}(t,\tau,x)$ we have:

$$\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}(t,x))\mathrm{w}_{x_ix_i} - \mathrm{w}_t - \mathrm{w}_\tau = \sum_{i=1}^{n} \big[a_{i\varepsilon}(u_{x_i}(\tau,x)) - a_{i\varepsilon}(u_{x_i}(t,x))\big] u_{x_ix_i}(\tau,x).$$

Introduce the function

$$\omega \equiv \mathrm{w}\, e^{-\tau}$$

which satisfies in P the following linear ultraparabolic equation

$$\begin{aligned} L\,\omega &\equiv \sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}(t,x))\omega_{x_ix_i} - \omega_t - \omega_\tau - \omega \\ &= e^{-\tau} \sum_{i=1}^{n} \big[a_{i\varepsilon}(u_{x_i}(\tau,x)) - a_{i\varepsilon}(u_{x_i}(t,x))\big] u_{x_ix_i}(\tau,x). \end{aligned} \tag{2.13}$$

Let

$$\Gamma_\tau = \partial P \setminus \{(t,\tau,x) : t = T,\ 0 < \tau < T,\ x \in \Omega\}.$$

Suppose that the function ω attains its positive maximum at some point $N \in \overline{P} \setminus \Gamma_\tau$. At this point it should be

$$L\,\omega\Big|_N = \sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}(t,x))\omega_{x_ix_i} - \omega_t - \omega_\tau - \omega\Big|_N < 0,$$

since $\omega_{x_ix_i}(N) \le 0$, $-\omega_t(N) \le 0$, $-\omega_\tau(N) = 0$, $-\omega(N) < 0$. On the other hand at this point

$$\nabla\omega = 0 \quad \Leftrightarrow \quad \nabla u(t,x) = \nabla u(\tau,x)$$

and hence, from (2.13),

$$L\,\omega\Big|_N = 0.$$

From this contradiction we conclude that ω can not attain its positive maximum in $\overline{P} \setminus \Gamma_\tau$.

Consider ω on Γ_τ:

for $x \in \partial\Omega$, $t \in [0,T]$, $\tau \in [0,T]$, $t > \tau$ we have $\omega = -C(\varepsilon_0)(t-\tau)e^{-\tau} \leq 0$;
for $t = \tau$, $x \in \overline{\Omega}$, $t \in [0,T]$ we have $\omega = 0$;
for $\tau = 0$, $t \in [0,T]$, $x \in \overline{\Omega}$ we have $\omega = u(t,x) - u_0(x) - C(\varepsilon_0)t \leq 0$ due to Lemma 2.1.

Consequently $\omega \leq 0$ in $\overline{P}$, i.e.

$$u(t,x) - u(\tau,x) \leq C(\varepsilon_0)(t-\tau). \tag{2.14}$$

Now subtracting (2.11) from (2.12) for $\tilde{v}(t,\tau,x) \equiv u(\tau,x) - u(t,x)$ we obtain

$$\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}(\tau,x))\tilde{v}_{x_i x_i} - \tilde{v}_t - \tilde{v}_\tau = \sum_{i=1}^{n} \left[a_{i\varepsilon}(u_{x_i}(t,x)) - a_{i\varepsilon}(u_{x_i}(\tau,x))\right] u_{x_i x_i}(t,x).$$

Obviously the function $\tilde{\mathrm{w}} \equiv \tilde{v} - C(\varepsilon_0)(t-\tau)$ satisfies in P the following relation

$$\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}(\tau,x))\tilde{\mathrm{w}}_{x_i x_i} - \tilde{\mathrm{w}}_t - \tilde{\mathrm{w}}_\tau = \sum_{i=1}^{n} \left[a_{i\varepsilon}(u_{x_i}(t,x)) - a_{i\varepsilon}(u_{x_i}(\tau,x))\right] u_{x_i x_i}(t,x).$$

Introduce the function $\tilde{\omega} \equiv \tilde{\mathrm{w}}\, e^{-\tau}$ which satisfies in P the following ultraparabolic equation

$$\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}(\tau,x))\tilde{\omega}_{x_i x_i} - \tilde{\omega}_t - \tilde{\omega}_\tau - \tilde{\omega}$$
$$= e^{-\tau} \sum_{i=1}^{n} \left[a_{i\varepsilon}(u_{x_i}(t,x)) - a_{i\varepsilon}(u_{x_i}(\tau,x))\right] u_{x_i x_i}(t,x).$$

Similarly to the previous case we obtain that $\tilde{\omega}$ can not attain its positive maximum in $\overline{P} \setminus \Gamma_\tau$ and that $\tilde{\omega} \leq 0$ on Γ_τ. The only difference is that for $\tau = 0$, $t \in [0,T]$, $x \in \overline{\Omega}$ we have $\tilde{\omega} = u_0(x) - u(t,x) - C(\varepsilon_0)t$, which is also nonpositive, due to Lemma 2.1.

Consequently, $\tilde{\omega} \leq 0$ in $\overline{P}$, i.e.

$$u(\tau,x) - u(t,x) \leq C(\varepsilon_0)(t-\tau) \quad \text{in} \quad \overline{P}. \tag{2.15}$$

From (2.14) and (2.15) we conclude that

$$|u(t,x) - u(\tau,x)| \leq C(\varepsilon_0)(t-\tau) \quad \text{in} \quad \overline{P}. \tag{2.16}$$

Taking into account the symmetry of the variables t and τ, we similarly consider the case $t < \tau$ to obtain that

$$|u(\tau, x) - u(t, x)| \leq C(\varepsilon_0)(\tau - t) \quad \text{in} \quad \overline{P}_1, \tag{2.17}$$

where

$$P_1 = \{(t, \tau, x) : t \in (0, T),\ \tau \in (0, T), x \in \Omega, \tau > t\}.$$

Note that here instead of Γ_τ we should take

$$\Gamma_t = \partial P_1 \setminus \{(t, \tau, x) : \tau = T, 0 < t < T, x \in \Omega\}.$$

From (2.16) and (2.17) we conclude that in

$$\{(t, \tau, x) : t \in [0, T],\ \tau \in [0, T], x \in \overline{\Omega}\}$$

the inequality

$$|u(t, x) - u(\tau, x)| \leq C(\varepsilon_0)|t - \tau|$$

holds. The last implies the required estimate. □

Remark 2.1. Concerning the linear and nonlinear ultraparabolic equations see [19,21] and the references therein.

In order to pass to the limit in (2.3), (2.4) we also need the next estimates of the spatial derivatives of a solution.

Lemma 2.3. *There exists a constant C such that*

$$\int\limits_{\Omega_T} |u_{\varepsilon x_i}(x, t)|^{p_i+2}\, dxdt \leq C, \quad i = 1, \ldots, n,$$

for every $\varepsilon \in (0, \varepsilon_0]$.

The proof of Lemma 2.3 follows from [13] p. 3016. The proof of the next lemma follows from standard considerations based on the maximum principle.

Lemma 2.4. *For every $\varepsilon \in (0, \varepsilon_0]$ the following estimate takes place*

$$|u_\varepsilon(x, t)| \leq \max_{\Omega} |u_0(x)|.$$

Now we are ready to pass to the limit as $\varepsilon \to 0$.

2.3. Passage to the limit

We will obtain a weak solution to problem (1.1)–(1.3) as a limit of the approximate solutions u_ε constructed in Section 2.1.

Multiplying equation (2.1) by an arbitrary smooth function ϕ, which vanishes on $(0,T)\times\partial\Omega$ and integrating by parts, we obtain

$$\int\limits_{\Omega_T} u_{\varepsilon t}\,\phi\,dxdt + \int\limits_{\Omega_T}\sum_{i=1}^{n}(u^{\alpha}_{\varepsilon x_i}+\varepsilon)^{p_i/\alpha}\,u_{\varepsilon x_i}\,\phi_{x_i}\,dxdt = 0. \tag{2.18}$$

As it follows from Lemmas 2.2–2.4, there exists a sequence ε_k such that

$$\begin{gathered} u_{\varepsilon_k} \to u \quad \text{*-weakly in} \quad L_\infty(\Omega_T), \\ u_{\varepsilon_k x_i} \to u_{x_i} \quad \text{weakly in} \quad L_{p_i+2}(\Omega_T), \quad i=1,...,n, \\ u_{\varepsilon_k t} \to u_t \quad \text{*-weakly in} \quad L_\infty(\Omega_T) \end{gathered}$$

as $\varepsilon_k \to 0$. Thus, in order to pass to the limit in (2.18), we only have to prove that

$$\int\limits_{\Omega_T}\sum_{i=1}^{n}(u^{\alpha}_{\varepsilon_k x_i}+\varepsilon_k)^{p_i/\alpha}\,u_{\varepsilon_k x_i}\,\phi_{x_i}\,dxdt \to \int\limits_{\Omega_T}\sum_{i=1}^{n}|u_{x_i}|^{p_i}\,u_{x_i}\,\phi_{x_i}\,dxdt \quad \text{as} \quad \varepsilon_k \to 0.$$

This can be done exactly in the same way as in [13] (see [13] p. 3019 relation (2.25)).

Finally, passing to the limit as $\varepsilon_0 \to 0$ we obtain the needed estimate

$$\|u_t\|_{L_\infty(\Omega_T)} \le C_0.$$

The uniqueness of the weak solution can be proved by standard considerations taking into account the monotonicity of the elliptic part of the operator (see [13], p. 3019).

Theorem 1 is proved.

Remark 2.2. The a priori estimate on $u_{\varepsilon t}$ can be obtained for more general case. Namely, instead of equation (2.3) we can consider the following one

$$u_{\varepsilon t} = \sum_{i=1}^{n} a_{i\varepsilon}(x,\nabla u_\varepsilon)u_{\varepsilon x_i x_i} + \sum_{i=1}^{n} b_i(x,\nabla u_\varepsilon) + f(x,u_\varepsilon,\nabla u_\varepsilon)$$

for a wide class of functions $a_{i\varepsilon}$, b_i and f under assumptions similar to the one dimensional case [20]. The problem here is the passage to the limit which is an open question. In some cases the passage to the limit is simple, for example if $a_{i\varepsilon}$ are as in (2.3), b_i are linear with respect to ∇u and $f = f(x,u_\varepsilon)$ is a continuous non-increasing in u_ε function such that $f(0) = 0$.

3. Proof of Theorem 2

The proof of Theorem 2 is similar to the proof of Theorem 1. The only difference is that we should prove the estimate $\|u_{\varepsilon x_k}\|_{L_\infty(\Omega_T)} \le C_k$, $k = 1, ..., n$ for a solution of problem (2.3), (2.4) under the additional assumption (A) on the domain Ω or in the case when Ω is orthogonal parallelepiped. We will do this in two steps: first we obtain the boundary estimate and then the global estimate. As in the previous section we will omit index ε in u_ε in the proofs.

Lemma 3.1. *For every $\varepsilon \in (0, \varepsilon_0]$ the following inequalities hold:*

(i) *if Ω satisfies assumption (A) then*

$$|u_\varepsilon(t,x)| \le C_k(G_k - x_k), \quad |u_\varepsilon(t,x)| \le C_k(x_k - F_k), \quad k = 1, ..., n$$

(ii) *if $\Omega = (-l_1, l_1) \times ... \times (-l_n, l_n)$ then*

$$|u_\varepsilon(t,x)| \le C_k(l_k - x_k), \quad |u_\varepsilon(t,x)| \le C_k(l_k + x_k), \quad k = 1, ..., n.$$

Proof. (i) Assume that Ω satisfies assumption (A). Let $k = 1$, the cases $k = 2, 3, ..., n$ are considered similarly. Introduce the function

$$v(t,x) = u(t,x) - C_1\big(G_1(x_2, ..., x_n) - x_1\big).$$

Obviously

$$\sum_{i=1}^n a_{i\varepsilon}(u_{x_i})v_{x_i x_i} - v_t = -C_1 \sum_{i=2}^n a_{i\varepsilon}(u_{x_i})G_{1x_i x_i},$$

and for $\tilde{v} = ve^{-t}$

$$\sum_{i=1}^n a_{i\varepsilon}(u_{x_i})\tilde{v}_{x_i x_i} - \tilde{v}_t - \tilde{v} = -C_1 e^{-t} \sum_{i=2}^n a_{i\varepsilon}(u_{x_i})G_{1x_i x_i} \ge 0, \tag{3.1}$$

the last inequality is due to the convexity of Ω which implies the inequality $G_{1x_i x_i} \le 0$. From (3.1) it follows that the function $\tilde{v}$ cannot attain its positive maximum in $\overline{\Omega}_T \setminus \Gamma_T$ (Γ_T was defined in the proof of Lemma 2.1). On the parabolic boundary Γ_T we have

1. for $x_1 = G_1$, $t \in [0, T]$: $\tilde{v} = 0$;
2. for $x_1 = F_1$, $t \in [0, T]$: $\tilde{v} = e^{-t}C_1(F_1 - G_1) \le 0$;
3. for $t = 0$, $x \in \Omega$: $\tilde{v} = u_0(x) - C_1\big(G_1(x_2, ..., x_n) - x_1\big) \le 0$, because $u_0\Big|_{x_1 = G_1} = 0$ and $|u_{0x_1}| \le C_1$.

Consequently

$$v \le 0 \ \text{ in } \ \overline{Q}_T \quad \Leftrightarrow \quad u \le C_1\big(G_1(x_2, ..., x_n) - x_1\big) \ \text{ in } \ \overline{Q}_T.$$

Next we obtain a lower bound. Introduce the function

$$w(t,x) = u(t,x) + C_1\big(G_1(x_2, ..., x_n) - x_1\big).$$

Similarly to the previous case for $\tilde{w} = we^{-t}$ we obtain

$$\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i})\tilde{w}_{x_i x_i} - \tilde{w}_t - \tilde{w} = C_1 e^{-t} \sum_{i=2}^{n} a_{i\varepsilon}(u_{x_i}) G_{1 x_i x_i} \le 0,$$

hence the function $\tilde{w}$ cannot attain its negative minimum in $\overline{Q}_T \setminus \Gamma_T$. Taking into account that on the parabolic boundary Γ_T we have $\tilde{w} \ge 0$ we conclude that

$$w \ge 0 \ \text{ in } \ \overline{Q}_T \quad \Leftrightarrow \quad u \ge -C_1\big(G_1(x_2, ..., x_n) - x_1\big) \ \text{ in } \ \overline{Q}_T.$$

Thus

$$|u(t,x)| \le C_1(G_1 - x_1).$$

The proof of the second inequality (i.e. the inequality $|u(t,x)| \le C_1(x_1 - F_1)$) is similar. Instead of $v = u - C_1\big(G_1(x_2, ..., x_n) - x_1\big)$ and $w = u + C_1\big(G_1(x_2, ..., x_n) - x_1\big)$ we take $v = u - C_1\big(x_1 - F_1(x_2, ..., x_n)\big)$ and $w = u + C_1\big(x_1 - F_1(x_2, ..., x_n)\big)$ respectively and use the fact that the convexity of the domain Ω implies that $F_{1 x_i x_i} \ge 0$.

(ii) Assume that $\Omega = (-l_1, l_1) \times ... \times (-l_n, l_n)$. The proof is similar to the previous case. In the first estimate the only difference is that we take v_1 instead of v and w_1 instead of w where

$$v_1 = u - C_1(l_1 - x_1) \ \text{ and } \ w_1 = u + C_1(l_1 - x_1).$$

In order to obtain the second estimate we take

$$v_1 = u - C_1(l_1 + x_1) \ \text{ and } \ w_1 = u + C_1(l_1 + x_1). \qquad \square$$

Lemma 3.2. *If Ω satisfies assumption (A) or $\Omega = (-l_1, l_1) \times ... \times (-l_n, l_n)$, then for every $\varepsilon \in (0, \varepsilon_0]$ the following estimates take place*

$$|u_{\varepsilon x_k}(t,x)| \le C_k, \quad k = 1, ..., n.$$

Proof. 1). Suppose that Ω satisfies assumption (A). We will prove the estimate for $k = 1$, for $k = 2, ..., n$ the proof is similar. Consider the equations

$$a_{1\varepsilon}(u_{x_1})u_{x_1x_1} + \sum_{i=2}^{n} a_{i\varepsilon}(u_{x_i})u_{x_ix_i} - u_t = 0, \quad u = u(t,x) \tag{3.2}$$

and

$$a_{1\varepsilon}(u_\xi)u_{\xi\xi} + \sum_{i=2}^{n} a_{i\varepsilon}(u_{x_i})u_{x_ix_i} - u_t = 0, \quad u = u(t,\tilde{x}), \tag{3.3}$$

where $x = (x_1, x_2, ..., x_n)$, $\tilde{x} = (\xi, x_2, ..., x_n)$. Subtracting (3.3) from (3.2), for

$$v(t,\xi,x) = u(t,x) - u(t,\tilde{x}) - C_1(x_1 - \xi)$$

we obtain

$$a_{1\varepsilon}(u_{x_1}(t,x))v_{x_1x_1} + a_{1\varepsilon}(u_\xi(t,\tilde{x}))v_{\xi\xi} + \sum_{i=2}^{n} a_{i\varepsilon}(u_{x_i}(t,x))v_{x_ix_i} - v_t$$
$$= \sum_{i=2}^{n} \left[a_{i\varepsilon}(u_{x_i}(t,\tilde{x})) - a_{i\varepsilon}(u_{x_i}(t,x))\right]u_{x_ix_i}(t,\tilde{x}).$$

For $\tilde{v} = ve^{-t}$ we have

$$a_{1\varepsilon}(u_{x_1}(t,x))\tilde{v}_{x_1x_1} + a_{1\varepsilon}(u_\xi(t,\tilde{x}))\tilde{v}_{\xi\xi} + \sum_{i=2}^{n} a_{i\varepsilon}(u_{x_i}(t,x))\tilde{v}_{x_ix_i} - \tilde{v}_t - \tilde{v}$$
$$= e^{-t}\sum_{i=2}^{n} \left[a_{i\varepsilon}(u_{x_i}(t,\tilde{x})) - a_{i\varepsilon}(u_{x_i}(t,x))\right]u_{x_ix_i}(t,\tilde{x}). \tag{3.4}$$

Consider (3.4) in the domain

$$P_T = \{(t,\xi,x) : t \in (0,T), \xi \in (F_1,G_1), x_1 \in (F_1,G_1), x_1 > \xi, (x_2,...,x_n) \in \Omega_1\},$$

where Ω_1 is a projection of Ω on the hyperplane $x_1 = 0$ (recall that $F_1 = F_1(x_2,...,x_n)$, $G_1 = G_1(x_2,...,x_n)$). Denote by Γ the parabolic boundary of P_T i.e.

$$\Gamma = \partial P_T \setminus \{(T,\xi,x) : \xi \in (F_1,G_1), x_1 \in (F_1,G_1), x_1 > \xi, (x_2,...,x_n) \in \Omega_1\}.$$

Suppose that at some point $N \in \overline{P}_T \setminus \Gamma$ the function $\tilde{v}$ attains its positive maximum. On the one hand we have

$$a_{1\varepsilon}(u_{x_1}(t,x))\tilde{v}_{x_1x_1} + a_{1\varepsilon}(u_\xi(t,\tilde{x}))\tilde{v}_{\xi\xi} + \sum_{i=2}^{n} a_{i\varepsilon}(u_{x_i}(t,x))\tilde{v}_{x_ix_i} - \tilde{v}_t - \tilde{v}\Big|_N < 0,$$

on the other

$$\nabla\tilde{v}\Big|_N = 0 \quad \Leftrightarrow \quad u_{x_i}(t,x) - u_{x_i}(t,\tilde{x})\Big|_N = 0, \quad i = 2,...,n \quad \Rightarrow$$
$$a_{i\varepsilon}(u_{x_i}(t,x)) - a_{i\varepsilon}(u_{x_i}(t,\tilde{x}))\Big|_N = 0,$$

and thus, from (3.4) we obtain

$$a_{1\varepsilon}(u_{x_1}(t,x))\tilde{v}_{x_1x_1} + a_{1\varepsilon}(u_\xi(t,\tilde{x}))\tilde{v}_{\xi\xi} + \sum_{i=2}^{n} a_{i\varepsilon}(u_{x_i}(t,x))\tilde{v}_{x_ix_i} - \tilde{v}_t - \tilde{v}\Big|_N = 0.$$

Hence $\tilde{v}$ cannot attain its positive maximum in $\overline{P}_T \setminus \Gamma$. Consider Γ which consists of four parts:

1. $\xi = F_1$, $x_1 \in [F_1, G_1]$, $(x_2, ..., x_n) \in \overline{\Omega}_1$, $t \in [0, T]$;
2. $x_1 = G_1$, $\xi \in [F_1, G_1]$, $(x_2, ..., x_n) \in \overline{\Omega}_1$, $t \in [0, T]$;
3. $x_1 = \xi \in [F_1, G_1]$, $(x_2, ..., x_n) \in \overline{\Omega}_1$, $t \in [0, T]$;
4. $t = 0$, x_1, $\xi \in [F_1, G_1]$, $(x_2, ..., x_n) \in \overline{\Omega}_1$.

According to the previous lemma, on the first and second parts we have

$$\tilde{v} = e^{-t}\big(u(t,x) - C_1(x_1 - F_1)\big) \le 0,$$
$$\tilde{v} = e^{-t}\big(-u(t,\tilde{x}) - C_1(G_1 - \xi)\big) \le 0,$$

respectively. For $x_1 = \xi$ we have $\tilde{v} = 0$ and for $t = 0$:

$$\tilde{v} = u_0(x) - u_0(\tilde{x}) - C_1(x_1 - \xi) \le 0.$$

The last is due to the inequality $|u_{0x_1}| \le C_1$. Thus we conclude that $\tilde{v} \le 0$ in $\overline{P}_T$ and consequently

$$u(t,x) - u(t,\tilde{x}) \le C_1(x_1 - \xi) \quad \text{in} \quad \overline{P}_T. \tag{3.5}$$

Similarly, subtracting (3.2) from (3.3) and considering the function

$$\tilde{v}_1 = e^{-t}\big(u(t,\tilde{x}) - u(t,x) - C_1(x_1 - \xi)\big),$$

instead of $\tilde{v}$, we obtain

$$u(t,\tilde{x}) - u(t,x) \le C_1(x_1 - \xi) \quad \text{in} \quad \overline{P}_T.$$

From this inequality and inequality (3.5) we conclude that

$$|u(t,x) - u(t,\tilde{x})| \le C_1(x_1 - \xi) \quad \text{in} \quad \overline{P}_T.$$

Due to the symmetry of the variables x_1 and ξ the case $x_1 < \xi$ can be considered in the same way. Thus for

$$x_1 \in [F_1, G_1], \quad \xi \in [F_1, G_1], \quad (x_2, ...x_n) \in \Omega_1, \quad t \in [0, T]$$

the inequality

$$|u(t, x) - u(t, \tilde{x})| \leq C_1 |x_1 - \xi|$$

holds, implying the needed estimate.

2). The case $\Omega = (-l_1, l_1) \times ... \times (-l_n, l_n)$ is treated similarly. The only difference is in the construction of the domain P_T, here we should take

$$P_T = \{(t, \xi, x) : t \in (0, T), \xi \in (-l_1, l_1), x_1 \in (-l_1, l_1), x_1 > \xi, |x_i| < l_i, i = 2, ..., n)\}. \qquad \square$$

Remark 3.1. The a priori estimate on ∇u_ε can be obtained for more general case. Namely, instead of equation (2.3) we can take the following one

$$u_{\varepsilon t} = \sum_{i=1}^{n} a_{i\varepsilon}(t, \nabla u_\varepsilon) u_{\varepsilon x_i x_i} + \sum_{i=1}^{n} b_i(t, \nabla u_\varepsilon) + f(t, x, u_\varepsilon, \nabla u_\varepsilon)$$

for a wide class of functions $a_{i\varepsilon}$, b_i and f under assumptions similar to [14,16]. However the passage to the limit in this case is an open question. In some cases the passage to the limit is simple, for example if $a_{i\varepsilon}$ are as in (2.3), b_i are linear with respect to ∇u and $f = f(t, u_\varepsilon)$ is a continuous non-increasing in u_ε function such that $f(t, 0) = 0$.

4. Proof of Theorem 3

In order to prove Theorem 3 we first obtain the estimates of $\|u_{\varepsilon x_i x_j}\|_{L_2(\Omega_T)}$ (independent of ε) and second, pass to the limit in the nonlinear term $a_{i\varepsilon}(u_{\varepsilon x_i}) u_{\varepsilon x_i x_i}$.

In this section we assume that $p_i \in (-1, 0)$ for all i and Ω is an orthogonal parallelepiped.

Lemma 4.1. *For every $\varepsilon \in (0, \varepsilon_0]$ the following estimates take place*

$$\int\limits_{\Omega_T} \left(\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}\right)^2 dtdx \leq \frac{1}{2(p_i + 1)} K_i^2 (C_i^\alpha + \varepsilon_0)^{-p_i/\alpha}, \quad i, j = 1, ..., n.$$

Proof. We restrict ourselves with $i = 1$ $(j = 1, ..., n)$, the considerations for $i = 2, ..., n$ are similar. Multiply equation (2.3) by $u_{x_1 x_1}$ and integrate by parts with respect to x_1 to obtain

$$-\frac{1}{2}\frac{\partial}{\partial t}\int_{-l_1}^{l_1} u_{x_1}^2 dx_1 = \int_{-l_1}^{l_1} a_{1\varepsilon}(u_{x_1})u_{x_1x_1}^2 dx_1 - \sum_{j=2}^{n}\int_{-l_1}^{l_1} a_{j\varepsilon}(u_{x_j})u_{x_jx_jx_1}u_{x_1}dx_1 - \sum_{j=2}^{n}\int_{-l_1}^{l_1} a'_{j\varepsilon}(u_{x_j})u_{x_jx_1}u_{x_jx_j}u_{x_1}dx_1, \tag{4.1}$$

we use here the fact that $u_{x_jx_j}\Big|_{x_1=\pm l_1} = 0$ for $j = 2, ..., n$ and that $u_t\Big|_{x_1=\pm l_1} = 0$. Integrate (4.1) by parts with respect to $x_2, ..., x_n$ to obtain

$$-\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega} u_{x_1}^2 dx = \int_{\Omega} a_{1\varepsilon}(u_{x_1})u_{x_1x_1}^2 dx + \sum_{j=2}^{n}\int_{\Omega} a_{j\varepsilon}(u_{x_j})u_{x_jx_1}^2 dx.$$

We use here the fact that $u_{x_1}\Big|_{x_j=\pm l_j} = 0$ for $j = 2, ..., n$.

Integrate the last relation with respect to t to obtain

$$\int_{\Omega_T} a_{j\varepsilon}(u_{x_j})u_{x_jx_1}^2 dxdt \le \frac{1}{2}\int_{\Omega} u_{0x_1}^2 dx, \quad j = 1, ..., n.$$

Taking into account that

$$a_{j\varepsilon}(u_{x_j}) = (u_{x_j}^\alpha + \varepsilon)^{\frac{p_j}{\alpha}-1}\big((p_j+1)u_{x_j}^\alpha + \varepsilon\big) \ge (p_j+1)(u_{x_j}^\alpha + \varepsilon)^{\frac{p_j}{\alpha}} \ge (p_j+1)(C_j^\alpha + \varepsilon_0)^{\frac{p_j}{\alpha}}$$

we obtain the needed estimates. □

Recall that for the solution of problem (2.3), (2.4) the estimates of the previous sections hold as well. Thus we have that there exists a sequence ε_k such that

$$u_{\varepsilon_k} \to u \quad \text{uniformly},$$
$$\frac{\partial u_{\varepsilon_k}}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad \text{*-weakly in } \ L_\infty(\Omega_T), \quad i = 1, ..., n,$$
$$\frac{u_{\varepsilon_k}}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{*-weakly in } \ L_\infty(\Omega_T),$$
$$\frac{\partial^2 u_{\varepsilon_k}}{\partial x_i \partial x_j} \to \frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{weakly in } \ L_2(\Omega_T), \quad i = 1, ..., n,$$

as $\varepsilon_k \to 0$.

Multiply equation (2.3) by an arbitrary smooth function ϕ and integrate to obtain

$$\int_{\Omega_T} \big[u_{\varepsilon t} - \sum_{i=1}^{n} a_{i\varepsilon}(u_{\varepsilon x_i}) u_{\varepsilon x_i x_i}\big] \phi \, dxdt = 0. \tag{4.2}$$

In order to pass to the limit in (4.2) we show that in some sense

$$a_{i\varepsilon}(u_{\varepsilon_k x_i}) \to a(u_{x_i}) = (1+p_i)|u_{x_i}|^{p_i}.$$

To this end we will show that

$$\frac{\partial u_{\varepsilon_k}}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad \text{in} \quad L_2(\Omega_T), \quad i = 1, ..., n. \tag{4.3}$$

In fact, consider

$$\mathcal{W} \equiv \{u : u \in L_2(0,T; H^2(\Omega)), \, u_t \in L_2(0,T; L_2(\Omega))\}$$

(actually $u_t \in L_\infty(0,T; L_\infty(\Omega))$). From the compactness lemma (see, for example, [10], Ch. 1, Section 5) it follows that the embedding $\mathcal{W} \subset L_2(0,T; H^1(\Omega))$ is compact and consequently (4.3) holds.

Thus we can pass to the limit in (4.2) and obtain strong solution (according to Definition 2).

The last step in the proof of Theorem 3 is to pass to the limit $\varepsilon_0 \to 0$ in order to obtain the declared estimate $\|u_{x_i x_j}\|^2_{L_2(\Omega_T)} \le \frac{1}{2(p_i+1)} K_i^2 C_i^{-p_i}$.

5. Proof of Theorems 4 and 5

In order to prove Theorems 4 and 5 consider the problem (1.1)–(1.3) taking $\Omega = \Omega_l = (-l, l)^n$ and choosing $l > 0$ so that the support of $u_0(x)$ lies in Ω_l. Denote the solution of this problem by u_l. Note that all estimates obtained in the previous sections are independent of the size of the domain Ω i.e. of l in this case. The solution of the Cauchy problem is obtained as a limit of a sequence of solutions u_l of problem (1.1)–(1.3) under an unlimited dilatation of the domain Ω_l when $l \to \infty$ applying the diagonal process (see, for example, [9]).

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