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On the fast diffusion with strong absorption

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In the present paper the initial boundary value problem for the fast diffusion equation with strong absorption is considered. An optimal condition guaranteeing the strict positivity of the solution is proposed. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4798792]

I. INTRODUCTION AND FORMULATION OF THE RESULT

According to the law of heat conduction (or Fourier's law), the heat flux density is equal to the product of heat conductivity $\kappa > 0$ and the negative gradient $-\nabla u$ of the absolute temperature u. The coefficient κ is generally a function of the temperature. The case of fast diffusion is characterized by the unbounded growth of κ when u approaches 0 and as a consequence heat propagates from the warm regions into the cold ones with extremely high speed. This fact usually is modeled by taking $\kappa = mu^{m-1}$ (up to constant multiplier) with $m \in (0, 1)$. The heat equation then takes the form

$$u_t - \Delta u^m = 0.$$

If in a medium a process with (nonlinear) absorption takes place than in the right hand side of the equation we should add the term responsible for this process, namely $-u^p$ multiplied by a positive constant (for more details see, for example, Refs. 5 and 7 and the references therein).

In the present paper we consider the following equation:

$$u_t - \Delta u^m = -\lambda u^p, \quad 0 < m < 1, \quad 0 < p < 1, \quad \lambda > 0, \quad \text{in } Q_T = (0, T) \times \Omega, \tag{1.1}$$

coupled with the boundary and initial conditions

$$u\Big|_{\Gamma_T} = \phi\Big|_{\Gamma_T} > 0, \quad \text{on } \Gamma_T = [0, T] \times \partial\Omega \cup \Omega, \tag{1.2}$$

here Ω is a bounded domain in $\mathbf{R^n}$ satisfying the exterior sphere condition, T is an arbitrary positive constant, and ϕ is a continuous on Γ_T function. Without loss of generality assume that Ω is lying in the strip $|x_1| \leq l$.

The solution of problems (1.1) and (1.2) with strong absorption ($0 < p < 1$) is known to develop a nonempty set $\{\mathbf{x} \in \Omega : u(t, \mathbf{x}) = 0\}$, the so called *dead core*, after finite time (see Refs. 1–5, and 7). For $m \geq 1$ (slow diffusion) and $0 < p < 1$ it was shown in Ref. 1 that for large λ the dead core formation takes place. On the contrary the solution remains positive if $p \geq 1$. The case $0 < m < 1$ (fast diffusion) for Eq. (1.1), with one spatial variable, was considered in Ref. 3 where it was shown that under certain assumptions on the initial-boundary conditions the dead core formation takes place for $0 < p < m < 1$.

Our goal is to give a sufficient condition guaranteeing the positivity of the solution of the above problem for the multidimensional case. We show that for $0 < m \leq p < 1$ the fast diffusion dominates the absorption so that even if the temperature on the boundary is arbitrary small the heat flow from the boundary does not allow the formation of a zone with zero temperature. If the absorption is "stronger" than diffusion, i.e., $0 < p < m < 1$ in order to avoid the dead core formation we need the boundary temperature to be big enough to provide sufficient heat flow from the boundary. Taking

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into account that for $0 < p < m < 1$ without restrictions on ϕ the dead core formation takes place, we assert that our condition is in some sense optimal.

As it was already mentioned, from the point of view of physics u is an absolute temperature. Since the laws of thermodynamics state that the absolute zero cannot be reached, u should remains strictly positive. Thus the result of this paper can be interpreted as the condition guarantying the legitimacy of Eq. (1.1) with condition (1.2) in the description of nonlinear heat processes with strong absorption and fast diffusion for any $T > 0$.

Let us formulate the result. Define two positive constants κ and $\tilde{\kappa}$

$$\kappa = \min\{\Big(\frac{2m(5/4)^{m-1} - m(1-m)}{4l^2\lambda(5/4)^p}\Big)^{\frac{1}{p-m}}, \frac{4}{5}\min_{\Gamma_T}\phi\}, \quad p > m \tag{1.3}$$

$$\tilde{\kappa} = \Big(\frac{4l^2\lambda(5/4)^p}{2m(5/4)^{m-1} - m(1-m)}\Big)^{\frac{1}{m-p}}, \quad p < m.$$

Theorem.

1. If $0 < m \le p < 1$ then there exists a unique strictly positive in $\overline{Q}_T$ smooth solution $u(t, \mathbf{x})$ of problem (1.1), (1.2). Moreover, for $0 < m < p < 1$

$$u(t, \mathbf{x}) \ge \kappa$$

and for $0 < m = p < 1$

$$u(t, \mathbf{x}) \ge e^{\mu x_1}\min_{\Gamma_T}(\phi\, e^{-\mu x_1}), \quad \mu^2 = \frac{\lambda}{m^2}.$$

2. If $0 < p < m < 1$ and $\min_{\Gamma_T}\phi \ge \frac{5}{4}\tilde{\kappa}$ then there exists a unique strictly positive in $\overline{Q}_T$ smooth solution $u(t, \mathbf{x})$ of problem (1.1), (1.2). Moreover,

$$u(t, \mathbf{x}) \ge \tilde{\kappa}.$$

II. PROOF OF THE THEOREM

Rewrite Eq. (1.1) in the following form:

$$u_t - mu^{m-1}\Delta u = m(m-1)u^{m-2}|\nabla u|^2 - \lambda u^p, \tag{2.1}$$

and consider the auxiliary equation

$$u_t - ma_{m-1}(u)\Delta u = m(m-1)a_{m-2}(u)|\nabla u|^2 - \lambda a_p(u), \tag{2.2}$$

where

$$a_q(z) = \begin{cases} z^q, & \text{for } z > \kappa \\ \kappa^q, & \text{for } z \le \kappa \end{cases}, \tag{2.3}$$

if $0 < m < p < 1$, and

$$a_q(z) = \begin{cases} z^q, & \text{for } z > \tilde{\kappa} \\ \tilde{\kappa}^q, & \text{for } z \le \tilde{\kappa} \end{cases}$$

if $0 < p < m < 1$. Our goal is to establish the *a priori* estimate $u \ge \kappa$ ($u \ge \tilde{\kappa}$) for any classical solution of problem (2.2), (1.2). If $u \ge \kappa$ ($u \ge \tilde{\kappa}$) then first, the global existence and the uniqueness of a classical solution of problem (2.2), (1.2) follows from the standard theory,[6] second equations (2.1) and (1.1) coincide. Note that the coefficient a_{m-1} remains strictly positive due to the estimate $u \le \max_{\Gamma_T}\phi$ which can be easily established by standard consideration based on maximum principle.

Define the linear operator L,

$$L \equiv \frac{\partial}{\partial t} - ma_{m-1}(u)\Delta.$$

Let $0 < m < p < 1$. Define the function $h(x_1)$

$$h(x_1) = \frac{\kappa}{4l^2}x_1^2 + \kappa, \quad |x_1| \le l.$$

Obviously

$$0 < \kappa \le h(x_1) \le \frac{5}{4}\kappa. \tag{2.4}$$

For $w = u - h$ we have

$$L\,w = L\,u - L\,h = m(m-1)a_{m-2}(u)|\nabla u|^2 - \lambda a_p(u) + m a_{m-1}(u)\frac{\kappa}{2l^2}. \tag{2.5}$$

Suppose that at a point $N(t_0, \mathbf{x_0})$, where $t_0 \in (0, T]$ and $\mathbf{x_0} = (x_{01}, \ldots, x_{0n}) \in \overline{\Omega} \setminus \partial\Omega$, the function w attains its negative minimum, then at this point we have

$$w < 0, \quad \nabla w = 0 \quad \Leftrightarrow \quad u < h, \quad u_{x_1} = h' = \frac{\kappa}{2l^2}x_{01}, \quad u_{x_i} = 0, \quad i = 2, \ldots, n.$$

Due to the fact that $\mathbf{x_0}$ is an internal point of Ω we have $u_{x_1}^2(N) < \kappa^2(2l)^{-2}$, hence, taking into account that $m - 1 < 0$, from (2.5) we obtain

$$L\,w\Big|_N > m(m-1)a_{m-2}(u)\frac{\kappa^2}{4l^2} - \lambda a_p(u) + m a_{m-1}(u)\frac{\kappa}{2l^2}\Big|_N. \tag{2.6}$$

There is two possibilities (see (2.4)):

$$\kappa < u(N) \le \frac{5}{4}\kappa \ \text{ and } \ u(N) \le \kappa.$$

In the first case we have (see (2.3))

$$a_{m-2}(u(N)) = u^{m-2}(N) < \kappa^{m-2}, \quad a_{m-1}(u(N)) = u^{m-1}(N) \ge \left(\frac{5}{4}\kappa\right)^{m-1}$$

and

$$-\lambda a_p(u(N)) = -\lambda u^p(N) \ge -\lambda\left(\frac{5}{4}\kappa\right)^p$$

hence from (2.6)

$$L\,w\Big|_N > m(m-1)\frac{1}{4l^2}\kappa^m - \lambda\kappa^p\Big(\frac{5}{4}\Big)^p + m\kappa^m\frac{1}{2l^2}\Big(\frac{5}{4}\Big)^{m-1} =$$

$$\kappa^m\frac{2m(5/4)^{m-1} - m(1-m) - 4l^2\lambda\big(5/4\big)^p\kappa^{p-m}}{4l^2},$$

and from (1.3)

$$L\,w\Big|_N > 0,$$

which contradicts the assumption that at point N the function w attains its negative minimum.

Now, if $u(N) \le \kappa$, then $a_q(u(\tilde{N})) = \kappa^q$ $\forall q$ and from (2.6) and (1.3) we conclude that

$$L\,w\Big|_N > m(m-1)\frac{1}{4l^2}\kappa^m + m\frac{1}{2l^2}\kappa^m - \lambda\kappa^p =$$

$$\kappa^m\frac{m + m^2 - \lambda 4l^2\kappa^{p-m}}{4l^2} > 0,$$

which (as it was mentioned above) is impossible. Note that the last inequality is a consequence of the following one

$$\kappa^{p-m} \le \frac{2m(5/4)^{m-1} - m(1-m)}{4l^2\lambda(5/4)^p} < \frac{m+m^2}{4l^2\lambda}.$$

Hence, the function w can not attain its negative minimum in $(0, T] \times \Omega$. On the parabolic boundary Γ_T due (2.4) we have

$$w\Big|_{\Gamma_T} = \phi - h\Big|_{\Gamma_T} \ge \min_{\Gamma_T} \phi - \frac{5}{4}\kappa \ge 0.$$

Consequently we obtain that $w(t, \mathbf{x}) \ge 0$ in $\overline{Q}_T$ which means that for the solution of problem (2.2), (1.2) the estimate

$$u(t, \mathbf{x}) \ge h(x_1) \ge \kappa \ \text{ in } \ \mathbf{x} \in \overline{Q}_T$$

holds. Taking into account that (2.2) and (2.1) coincide for $u \ge \kappa$ we prove the theorem for $0 < m < p < 1$.

Let us turn to the case $0 < p < m < 1$. Define the function $\tilde{h}(x_1)$:

$$\tilde{h}(x_1) = \frac{\tilde{\kappa}}{4l^2}x_1^2 + \tilde{\kappa}, \quad |x_1| \le l.$$

Similarly to the previous case we can show that at the point $\tilde{N} \in (0, T] \times \Omega$ of negative minimum of the function $\tilde{w} = u - \tilde{h}$ we have

$$L\,\tilde{w}\Big|_{\tilde{N}} > \tilde{\kappa}^m\Big[\frac{2m(5/4)^{m-1} - m(1-m)}{4l^2} - \lambda\tilde{\kappa}^{p-m}\Big(\frac{5}{4}\Big)^p\Big] = 0$$

or

$$L\,\tilde{w}\Big|_{\tilde{N}} > \tilde{\kappa}^m\Big[\frac{m+m^2}{4l^2} - \lambda\tilde{\kappa}^{p-m}\Big] > 0,$$

which contradicts the assumption that at the point $\tilde{N}$ the function $\tilde{w}$ attains its negative minimum. Hence, $\tilde{w}$ cannot attain its negative minimum in $(0, T] \times \Omega$. On the parabolic boundary Γ_T we have

$$\tilde{w}\Big|_{\Gamma_T} = \phi - h\Big|_{\Gamma_T} \ge \min_{\Gamma_T} \phi - \frac{5}{4}\tilde{\kappa} \ge 0.$$

Consequently we obtain that $\tilde{w}(t, \mathbf{x}) \ge 0$ in $\overline{Q}_T$ which means that for the solution of problem (2.2), (1.2) the estimate

$$u(t, \mathbf{x}) \ge h(x_1) \ge \tilde{\kappa} \ \text{ in } \ \overline{Q}_T$$

holds. Taking into account that (2.2) and (2.1) coincide for $u \ge \kappa$ we prove the theorem for $0 < m < p < 1$.

The last case is $0 < m = p < 1$. One can easily see that the substitution $u = ve^{\mu x_1}$, where $\mu^2 m^2 = \lambda$ reduce (2.1) to the equation:

$$v_t - mv^{m-1}e^{\mu(m-1)x_1}\Delta v = m(m-1)v^{m-2}e^{\mu(m-1)x_1}|\nabla v|^2 + 2\mu m^2 v^{m-1}e^{\mu(m-1)x_1}v_{x_1}$$

coupled with condition

$$v = \phi e^{-\mu x_1} \ \text{ on } \ \Gamma_T.$$

By standard arguments we conclude that $v \ge \min_{\Gamma_T}(\phi e^{-\mu x_1})$ in Q_T and consequently $u \ge e^{\mu x_1}\min_{\Gamma_T}(\phi\, e^{-\mu x_1}) > 0$. The theorem is proved.

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