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Singular and degenerate anisotropic parabolic equations with a nonlinear source

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ABSTRACT

The Dirichlet problem in arbitrary domain for degenerate and singular anisotropic parabolic equations with a nonlinear source term is considered. We state conditions that guarantee the existence and uniqueness of a global weak solution to the problem. A similar result is proved for the parabolic p-Laplace equation.

1. Introduction and main results

Let Ω be a bounded domain in $\mathbb{R}^n$ and $Q_T = \Omega \times (0, T)$ with an arbitrary $T \in (0, \infty)$. By $\mathbf{x} = (x_1, \dots, x_n)$ we denote points in Ω and by t the time variable that varies in the interval $[0, T]$. The goal of the present paper is to prove the solvability of the following quasilinear parabolic equation

$$u_t - \sum_{i=1}^{n} (|u_{x_i}|^{p_i} u_{x_i})_{x_i} = \lambda g(u) + f \quad \text{in } Q_T, \tag{1.1}$$

coupled with the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega \times (0, T) \tag{1.2}$$

and the initial condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}) \quad \text{in } \Omega. \tag{1.3}$$

Here $u = u(\mathbf{x}, t)$ is the unknown function that has to be found, the function $f = f(\mathbf{x}, t)$ is prescribed, $p_i > -1$, $i = 1, \dots, n$ and λ are constants, the function g will be discussed later.

The unique solvability of problem (1.1)–(1.3) with $g \equiv 0$ was proved in [1, Ch. 2]. It is well known that the solution of this problem can blow-up in finite time if $g \neq 0$ (see [2] and the references there for the case $p_i = 0$, $i = 1, \dots, n$ and [3,4]

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for the general case). Anisotropic elliptic and parabolic equations have received much attention in recent years, see [5–16]. The goal of the present paper is to obtain sufficient conditions guaranteeing the global solvability of problem (1.1)–(1.3) in general case. We will look for bounded weak solutions. Before we give the definition of the weak solution let us introduce some notations. Denote

$$p^* = \max\{p_1, p_2, \dots, p_n\}, \qquad p_* = \min\{p_1, p_2, \dots, p_n\}.$$

Without loss of generality we may suppose that $p_1 = p^*$. Let l be a positive constant such that $\Omega \subset \{\mathbf{x} \in \mathbb{R}^n \mid -l \le x_1 \le l\}$.

Besides the usual Lebesgue and Sobolev function spaces, we will use spaces that are specific for our problem. For $i = 1, \dots, n$, we introduce the Banach space $U_i(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{U_i(\Omega)} = \|u\|_{L^2(\Omega)} + \|u_{x_i}\|_{L^{p_i+2}(\Omega)}$. Define the space

$$U(\Omega) = \bigcap_{i=1}^{n} U_i(\Omega)$$

with the norm $\|u\|_{U(\Omega)} = \max_{i=1,\dots,n} \|u\|_{U_i(\Omega)}$. Clearly, the trace of function from $U(\Omega)$ at $\partial\Omega$ exists and is equal to zero. We denote by $U_i^*(\Omega)$ the space adjoint (topologically) to $U_i(\Omega)$. Notice that $U^*(\Omega) = \sum_{i=1}^n U_i^*(\Omega)$ (the sum in the space of distributions $C_0^\infty(\Omega)^*$).

We shall meet spaces of functions depending on t. Let us introduce the following Banach space

$$V(Q_T) = \bigcap_{i=1}^{n} L^{p_i+2}(0, T; U_i(\Omega)).$$

Its adjoint is

$$V^*(Q_T) = \sum_{i=1}^{n} L^{q_i+2}(0, T; U_i^*(\Omega)),$$

where q_i is such that $(p_i + 2)^{-1} + (q_i + 2)^{-1} = 1$. It is not difficult to see that

$$L^{p^*+2}(0, T; U(\Omega)) \subset V(Q_T) \subset L^{p_*+2}(0, T; U(\Omega))$$

and, by duality,

$$L^{q_*+2}(0, T; U^*(\Omega)) \subset V^*(Q_T) \subset L^{q^*+2}(0, T; U^*(\Omega)),$$

where q_* and q^* are such that $(q_* + 2)^{-1} + (p_* + 2)^{-1} = 1$ and $(q^* + 2)^{-1} + (p^* + 2)^{-1} = 1$.

Definition 1. We say that a function $u : Q_T \to \mathbb{R}$ is a weak solution of problem (1.1)–(1.3) if

$$u \in L^\infty(Q_T) \cap V(Q_T) \cap C\big([0, T]; L^s(\Omega)\big) \quad \text{for all } s \in [1, \infty), \ u_t \in V^*(Q_T)$$

and the following integral identity

$$\int_{Q_T} \Big(u\phi_t - \sum_{i=1}^{n} |u_{x_i}|^{p_i} u_{x_i}\phi_{x_i} + \lambda g(u)\phi + f\phi\Big)\mathrm{d}\mathbf{x}\mathrm{d}t = -\int_\Omega u_0\phi_0 \mathrm{d}\mathbf{x}$$

holds for an arbitrary smooth function $\phi : Q_T \to \mathbb{R}$ which is equal to zero for $\mathbf{x} \in \partial\Omega$ and for $t = T$ (here $\phi_0(\mathbf{x}) = \phi(\mathbf{x}, 0)$).

Let us denote

$$m = \|u_0\|_{L^\infty(\Omega)}, \qquad f^* = \|f\|_{L^\infty(Q_T)},$$

$$\mathfrak{M} = \left\{ M \in (0, \infty) \mid f^* + |\lambda| g(M + m) < (p^* + 1)\left(\frac{2M}{3l^2 + 2l}\right)^{p^*+1} \right\},$$

$$M_* = \inf \mathfrak{M}.$$

We shall prove the existence of a weak solution to the problem under the following assumptions on g:

$$\text{the function } g : \mathbb{R} \to \mathbb{R} \text{ is continuous;} \tag{1.4}$$

$$|g(\xi)| \le g(\eta) \quad \text{for all } \xi \text{ and } \eta \text{ such that } |\xi| \le \eta; \tag{1.5}$$

$$\mathfrak{M} \ne \varnothing. \tag{1.6}$$

For example, functions $g(u) = \log(|u| + 1)$, $g(u) = |u|^{q-1}u$ (or $g(u) = u^q$ if defined) with an arbitrary $q \ge 0$, $g(u) = |u|^q$, and $g(u) = \mathrm{e}^u$ satisfy (1.4) and (1.5). Condition (1.6) will be discussed in examples below.

Theorem 1. *Suppose that Ω satisfies the exterior sphere condition, $u_0 \in L^\infty(\Omega)$, $f \in L^\infty(Q_T)$ and conditions (1.4)–(1.6) are fulfilled. Then for an arbitrary $T > 0$, there exists a weak solution of problem (1.1)–(1.3) such that*

$$\|u\|_{L^\infty(Q_T)} \le M_* + m. \tag{1.7}$$

If, in addition, g is Lipschitz continuous function, then the solution is unique.

Now let us give several examples concerning assumption (1.6).

Example 1. It is not difficult to see that for $g(u) = \ln(|u| + 1)$ as well as for the functions $g(u) = |u|^{q-1}u$, $g(u) = |u|^q$ and $g(u) = u^q$ (if defined) with $0 \le q < p^* + 1$ one can always find a positive M that belongs to $\mathfrak{M}$. That is $\mathfrak{M} \neq \varnothing$. As a consequence, Theorem 1 guarantees the existence of a global (i.e., for an arbitrary $T > 0$) weak solution to problem (1.1)–(1.3) with such functions g and arbitrary bounded u_0 and f.

Example 2. Suppose that $f \equiv 0$, $\lambda \neq 0$, $g(u) = |u|^{q-1}u$ and $q = p^* + 1$. In this case, the set $\mathfrak{M}$ consists of numbers M that satisfy the following inequality

$$\frac{M+m}{M} < \frac{2}{3l^2+2l}\left(\frac{q}{|\lambda|}\right)^{1/q}$$

which is equivalent to

$$\frac{m}{M} < \frac{2}{3l^2+2l}\left(\frac{q}{|\lambda|}\right)^{1/q} - 1.$$

Obviously, if the right-hand side is strictly positive, then $\mathfrak{M} \neq \varnothing$. Thus, if

$$3l^2 + 2l < 2\left(\frac{q}{|\lambda|}\right)^{1/q}, \tag{1.8}$$

then there exists a global weak solution of the problem. Condition (1.8) is the restriction on the size of the domain Ω but only in one direction (in the direction of x_1). Notice that this condition does not depend on $\|u_0\|_{L^\infty(\Omega)}$.

Instead of $g(u) = |u|^{q-1}u$, we can take $g(u) = |u|^q$ (or $g(u) = u^q$ if defined).

Example 3. Let us take $g(u) = u^4$, $f \equiv 0$, $\lambda = 1$, and $p_i = 1$ for all $i = 1, \ldots, n$. The set $\mathfrak{M}$ is described by the following inequality

$$(M+m)^4 < \frac{8M^2}{(3l^2+2l)^2}$$

that can be rewritten as

$$M^2 + 2\left(m - \frac{\sqrt{2}}{3l^2+2l}\right)M + m^2 < 0.$$

Therefore, $\mathfrak{M} \neq \varnothing$ if

$$\|u_0\|_{L^\infty(\Omega)} = m < \frac{1}{\sqrt{2}(3l^2+2l)}. \tag{1.9}$$

This is the smallness condition which guarantees the existence of a global weak solution of problem (1.1)–(1.3).

Example 4. Similarly to the previous example, direct calculations show that condition (1.9) guarantees the existence of a global weak solution of problem (1.1)–(1.3) with $g(u) = u$, $f \equiv 0$, $\lambda = 2^{-5/4}$, and $p_i = -1/2$ for all $i = 1, \ldots, n$.

Example 5. Let us take $g(u) = e^u$, $f \equiv 0$, $\lambda = 1$, and $p_i > -1$ for all $i = 1, \ldots, n$. The set $\mathfrak{M}$ consists of numbers M such that

$$e^{M+m} < (p^*+1)\left(\frac{2M}{3l^2+2l}\right)^{p^*+1}.$$

This condition can be rewritten as follows

$$e^M < C\,M^{p^*+1}, \qquad C = \frac{(p^*+1)2^{p^*+1}}{e^m(3l^2+2l)^{p^*+1}}.$$

If the constant C is sufficiently large, then there exists $M > 0$ that satisfies the last inequality, i.e., $\mathfrak{M} \neq \varnothing$. Actually, this is the smallness condition.

Example 6. If $\lambda = 0$, then

$$\mathfrak{M} = \left\{ M \in (0, \infty) \mid f^* < (p^*+1)\left(\frac{2M}{3l^2+2l}\right)^{p^*+1} \right\}.$$

This set is always nonempty and

$$M_* = \frac{3l^2+2l}{2}\left(\frac{f^*}{p^*+1}\right)^{1/(p^*+1)}.$$

The approach developed in this paper can be easily extended to the p-Laplace equation. Consider the following parabolic equation

$$u_t - \operatorname{div}(|\nabla u|^p \nabla u) = \lambda g(u) + f(\mathbf{x}, t), \tag{1.10}$$

in $Q_T = \Omega \times (0, T)$, coupled with conditions (1.2) and (1.3).

Definition 2. We say that a function $u : Q_T \to \mathbb{R}$ is a weak solution of problem (1.10), (1.2) and (1.3) if

$$u \in L^\infty(Q_T) \cap L^{p+2}\big(0, T; W_0^{1,p+2}(\Omega)\big) \cap C\big(0, T; L^2(\Omega)\big),$$
$$u_t \in L^{q+2}\big(0, T; W^{-1,q+2}(\Omega)\big), \qquad (p+2)^{-1} + (q+2)^{-1} = 1,$$

and

$$\int_{Q_T} \big(u\phi_t - |\nabla u|^p \nabla u \cdot \nabla\phi + \lambda g(u)\phi + f\phi\big)\mathrm{d}\mathbf{x}\mathrm{d}t = -\int_\Omega u_0\phi_0 \mathrm{d}\mathbf{x}$$

for an arbitrary smooth function $\phi : Q_T \to \mathbb{R}$ which is equal to zero for $\mathbf{x} \in \partial\Omega$ and for $t = T$ ($\phi_0 = \phi(\mathbf{x}, 0)$).

Let us introduce the following set

$$\mathfrak{N} = \left\{ M \in (0, \infty) \mid f^* + |\lambda| g(M+m) < (p+1)\left(\frac{2M}{3l_*^2 + 2l_*}\right)^{p+1} \right\},$$

where m and f^* are the same as in the definition of the set $\mathfrak{M}$, $l_* = \min\{l_1, \dots, l_n\}$ and numbers $l_1, \dots, l_n$ are such that $\Omega \subset \{\mathbf{x} : |x_i| \le l_i\}$. Denote $N_* = \mathit{inf}\ \mathfrak{N}$.

Theorem 2. *Suppose that Ω, u_0 and f are the same as in Theorem 1, $\mathfrak{N} \neq \varnothing$ and the function g satisfies conditions (1.4) and (1.5). Then for an arbitrary $T > 0$ there exists a weak solution of problem (1.10), (1.2) and (1.3) such that*

$$\|u\|_{L^\infty(Q_T)} \le N_* + m.$$

If, in addition, g is Lipschitz continuous function then the solution is unique.

Similarly to problem (1.1)–(1.3) (see Examples 1–5), we can show that Theorem 2 guarantees the existence of a weak solution of problem (1.10), (1.2) and (1.3) for arbitrary $T > 0$ in the following cases:

1. $g(u) = \ln(|u|+1)$ and $g(u) = |u|^{q-1}u$ or $g(u) = |u|^q$ or $g(u) = u^q$ (if defined) with $0 \le q < p+1$ and arbitrary bounded u_0, f;
2. $f \equiv 0, \lambda \neq 0, g(u) = |u|^{q-1}u$ (or u^q if defined) with $q = p+1$ and
$$3l_*^2 + 2l_* < 2\left(\frac{q}{|\lambda|}\right)^{\frac{1}{q}}$$
with arbitrary bounded u_0;
3. $f \equiv 0, g(u) = u^4, \lambda = 1, p = 1$ (or $f \equiv 0, g(u) = u, \lambda = 2^{-5/4}, p = -1/2$) and
$$\|u_0\|_{L^\infty(\Omega)} = m < \frac{1}{\sqrt{2}(3l_*^2 + 2l_*)};$$
4. $f \equiv 0, g(u) = \mathrm{e}^u, \lambda = 1, p > -1$ and
$$\mathrm{e}^M < \frac{(p+1)2^{p+1}}{\mathrm{e}^m(3l_*^2 + 2l_*)^{p+1}} M^{p+1}$$

for some $N \in \mathfrak{N}$.

Let us compare the assertion of Theorem 2 with known results concerning the global solvability of problem (1.10), (1.2) and (1.3). From [17] it follows that if

$$|\lambda g(u) + f| \le C_0(1 + |u|^q)$$

with $q < p+1$ then the global weak solution exists. The same conclusion follows from Theorem 2 (see case 1.). If the measure of the domain is sufficiently small then [17] guarantees the existence of the global weak solution for $q = p+1$ (the critical case). For the critical case, Theorem 2 guarantees the existence of the global weak solution under the assumption that the domain is small enough only in one direction (see case 2.). In [3,4], the problem was considered with the source $g(u) = |u|^{q-1}u$ or $g(u) = u^q$. It was shown there that the global weak solution exists if $q < p+1$. For $q > p+1$ the global existence was established for small initial data. Theorem 2 guarantees the global weak solvability of the problem with small initial data for a wide class of nonlinear sources which grow up faster than u^{p+1} (see cases 3. and 4.).

For large initial data and $q > p+1$, the finite time blow-up of the solution was proved in [3,4]. The finite blow-up of the solution for $q = p+1$ in arbitrary domain (without smallness restriction on the measure of the domain) was demonstrated in [17]. The case $q = p+1$ was also considered in [18]. It was shown that global solutions exist only if $\lambda \le \lambda_1$, where λ_1 is the first eigenvalue of the problem

$$\operatorname{div}(|\nabla\psi|^p\nabla\psi) + \lambda|\psi|^p\psi = 0 \quad \text{in } \Omega, \ \ \psi|_{\partial\Omega} = 0.$$

Obviously, for λ small enough Theorem 2 guarantees the existence of global weak solution to problem (1.10), (1.2) and (1.3).

The paper is organized as follows. Sections 2 and 3 are devoted to the proof of Theorems 1 and 2, respectively. In both cases, we first obtain a priori estimates for the regularized problem and then pass to the limit in order to obtain the required result. Finally, in Appendix we prove some auxiliary statements used in the proofs.

2. Proof of Theorem 1

2.1. Regularized problem

We regularize the functions f and u_0 by sequences of smooth functions $f_\varepsilon : \overline{Q}_T \to \mathbb{R}$ and $u_{0\varepsilon} : \overline{\Omega} \to \mathbb{R}$ such that $f_\varepsilon(\mathbf{x}, t) \to f(\mathbf{x}, t)$ as $\varepsilon \to 0$ for almost all $(\mathbf{x}, t) \in Q_T$ and $u_{0\varepsilon}(\mathbf{x}) \to u_0(\mathbf{x})$ as $\varepsilon \to 0$ for almost all $\mathbf{x} \in \Omega$. Besides, we suppose that $\max_{(\mathbf{x},t)\in\overline{Q}_T} |f_\varepsilon(\mathbf{x}, t)| \le f^*$ and $\max_{\mathbf{x}\in\overline{\Omega}} |u_{0\varepsilon}(\mathbf{x})| \le m$ for all ε. It is not difficult to prove the existence of such sequences. Indeed, extend the function u_0 by zero outside of Ω. Clearly $u_{0\epsilon} \to u_0$ in $L_s(\Omega)$ for any $s \in [1, +\infty)$ where $u_{0\epsilon}$ is a standard mollification of u_0. Hence, there exists a subsequence $u_{0\epsilon_k} \to u_0$ almost everywhere in Ω and $\max_{\mathbf{x}\in\overline{\Omega}} |u_{0\epsilon_k}(\mathbf{x})| \le m$ for all ϵ_k. The function f can be treated in the same way.

Let $\{g_\varepsilon\}$ be a sequence of continuously differentiable functions such that

(i_ε) $g_\varepsilon \to g$ as $\varepsilon \to 0$ uniformly on every compact subset of $\mathbb{R}$;
(ii_ε) for every $\varepsilon > 0$, function g_ε satisfies (1.5): $|g_\varepsilon(\xi)| \le g_\varepsilon(\eta)$ for all real ξ and η such that $|\xi| \le \eta$.

We prove the existence of such a sequence $\{g_\varepsilon\}$ in Appendix A.1.

Lemma 2.1. *For every compact subset $\mathfrak{M}_c$ of $\mathfrak{M}$ there exists $\varepsilon_0 > 0$ such that*

$$f^* + |\lambda| g_\varepsilon(M+m) < (p^*+1)\left(\frac{2M}{3l^2+2l}\right)^{p^*+1} \quad \text{for all } M \in \mathfrak{M}_c \text{ and } \varepsilon \in (0, \varepsilon_0).$$

Proof. Let us introduce the following functions

$$G(\xi) = f^* + |\lambda| g(\xi+m) - (p^*+1)\left(\frac{2\xi}{3l^2+2l}\right)^{p^*+1},$$

$$G_\varepsilon(\xi) = f^* + |\lambda| g_\varepsilon(\xi+m) - (p^*+1)\left(\frac{2\xi}{3l^2+2l}\right)^{p^*+1}.$$

Then $\mathfrak{M} = \{\xi \mid G(\xi) < 0\}$. For every compact $\mathfrak{M}_c \subset \mathfrak{M}$, there exists $\delta > 0$ such that $G(\xi) \le -\delta$ for $\xi \in \mathfrak{M}_c$. According to (i_ε), there exists $\varepsilon_0 > 0$ such that $|G(\xi) - G_\varepsilon(\xi)| < \delta$ and, as a consequence, $G_\varepsilon(\xi) < 0$ for all $\xi \in \mathfrak{M}_c$ and $\varepsilon \in (0, \varepsilon_0)$. The lemma is proved. □

Let us make two remarks concerning the previous lemma. First, if we have found ε_0 corresponding a compact set $\mathfrak{M}_c$, we can take a smaller ε_0 in the formulation of the lemma. Second, since $\mathfrak{M}$ is the union of all its compact subsets,

$$M_* = \inf_{\mathfrak{M}_c \subset \mathfrak{M}} \min\{M \mid M \in \mathfrak{M}_c\}. \tag{2.1}$$

Let us fix an arbitrary compact set $\mathfrak{M}_c \subset \mathfrak{M}$ and a positive number ε_0 that corresponds to $\mathfrak{M}_c$ according to Lemma 2.1. For all $\varepsilon \in (0, \varepsilon_0)$ and for arbitrary $M \in \mathfrak{M}_c$, we introduce the following cut-off function

$$\bar{g}_\varepsilon(z) = \begin{cases} g_\varepsilon(-M-m), & z < -M-m, \\ g_\varepsilon(z), & |z| \le M+m, \\ g_\varepsilon(M+m), & z > M+m, \end{cases}$$

and consider the following regularized problem

$$u_{\varepsilon t} - \sum_{i=1}^{n}((u_{\varepsilon x_i}^{\alpha} + \varepsilon)^{p_i/\alpha} u_{\varepsilon x_i})_{x_i} = \lambda \bar{g}_{\varepsilon}(u_{\varepsilon}) + f_{\varepsilon} \quad \text{in } Q_T, \tag{2.2}$$

$$u_{\varepsilon} = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{2.3}$$

$$u_{\varepsilon}(\mathbf{x}, 0) = u_{0\varepsilon}(\mathbf{x}) \quad \text{in } \Omega, \tag{2.4}$$

where $\alpha \in (0, 1)$ is a constant such that $\alpha = r/m$ with positive integers r and m, $r < m$, and r is even. In the singular case $p^* \in (-1, 0)$, we additionally require that α satisfies the following conditions:

$$\alpha > p^* + 1 \quad \text{and} \quad (\alpha - p^* + 1)\left(\frac{2M}{3l^2 + 2l}\right)^{\alpha} \geq \varepsilon_0. \tag{2.5}$$

For example, if $p^* \geq 0$, one can take $\alpha = 2/3$; if $p^* = -1/2$, one can also take $\alpha = 2/3$; if $p^* = -1/3$, one can put $\alpha = 4/5$. Clearly, condition (2.5) can always be satisfied, if one takes ε_0 small enough.

These conditions on α are motivated by two reasons. The first one is that for such α

$$(z^{\alpha})^{p_i/\alpha} = |z|^{p_i} \quad \text{and} \quad z^{\alpha} = (-z)^{\alpha},$$

since α is a rational number with an even numerator. The second reason is that, for such α, the function $a_1(\varepsilon, z)$ which is defined below is nondecreasing with respect to ε on the interval $(0, \varepsilon_0)$. This property of this function will be used in the proof of Lemma 2.2.

Concerning the existence of a classical solution u_{ε} of problem (2.2)–(2.4) see Appendix A.2. Our goal in this section is to obtain uniform estimates on this solution which would enable us to pass to the limit as $\varepsilon \to 0$. First we obtain a uniform estimate on u_{ε} in L^{∞} norm.

Lemma 2.2. *For every compact set $\mathfrak{M}_c \subset \mathfrak{M}$ and for every $\varepsilon \in (0, \varepsilon_0)$, the classical solution of problem (2.2)–(2.4) satisfies the following estimate:*

$$\|u_{\varepsilon}\|_{L^{\infty}(Q_T)} \leq m + M_c,$$

where $M_c = \min\{M \mid M \in \mathfrak{M}_c\}$ and ε_0 is the positive number corresponding to $\mathfrak{M}_c$ according to Lemma 2.1.

Proof. In order to simplify the notation, we shall write u instead of u_{ε}. Let us take an arbitrary $M \in \mathfrak{M}_c$ and rewrite Eq. (2.2) in the non-divergent form:

$$u_t - \sum_{i=1}^{n} a_i(u_{x_i}, \varepsilon) u_{x_i x_i} = \lambda \bar{g}_{\varepsilon}(u) + f_{\varepsilon} \tag{2.6}$$

with

$$a_i(z, \varepsilon) = (z^{\alpha} + \varepsilon)^{\frac{p_i}{\alpha} - 1}((p_i + 1)z^{\alpha} + \varepsilon).$$

Define a function $h = h(x_1)$ as follows:

$$h(x_1) = \widetilde{M}\left(\frac{l^2 - x_1^2}{2} + (1 + l)(l + x_1)\right) + m,$$

where

$$\widetilde{M} = \frac{2M}{3l^2 + 2l}.$$

Let us introduce the following nonlinear differential operator

$$Lu = u_t - \sum_{i=1}^{n} a_i(u_{x_i}, \varepsilon) u_{x_i x_i}.$$

With this notation, Eq. (2.2) can be rewritten as

$$Lu = \lambda \bar{g}_{\varepsilon}(u) + f_{\varepsilon}. \tag{2.7}$$

Besides that, since h depends only on x_1, (recall that $p^* = p_1$)

$$Lh = h_t - \sum_{i=1}^{n} a_i(h_{x_i}, \varepsilon) h_{x_i x_i} - a_1(h', \varepsilon) h'' = \widetilde{M} a_1(h', \varepsilon). \tag{2.8}$$

Consider the function $a_1(z, \varepsilon)$. By direct calculations, we obtain

$$\frac{\partial a_1}{\partial \varepsilon}(z, \varepsilon) = \frac{p^*}{\alpha}(z^\alpha + \varepsilon)^{\frac{p^*}{\alpha}-2}\big(z^\alpha(p^* + 1 - \alpha) + \varepsilon\big).$$

Due to the assumptions on α, we find that

$$\frac{\partial a_1}{\partial \varepsilon}(z, \varepsilon) \geq 0 \quad \text{for } \varepsilon \in (0, \varepsilon_0) \text{ and } z \geq \widetilde{M}.$$

Since $h' \geq \widetilde{M}$, (2.8) implies that

$$Lh \geq \widetilde{M} a_1(h', 0) = \widetilde{M}(p^* + 1)h'^{p^*} \geq (p^* + 1)\widetilde{M}^{p^*+1}. \tag{2.9}$$

Let us denote $v(\mathbf{x}, t) = u(\mathbf{x}, t) - h(x_1)$. It is not difficult to deduce that

$$\begin{aligned} Lu - Lh &= u_t - \sum_{i=1}^n a_i(u_{x_i}, \varepsilon)u_{x_i x_i} - h_t + \sum_{i=1}^n a_i(h_{x_i}, \varepsilon)h_{x_i x_i} \\ &= v_t - \sum_{i=1}^n a_i(u_{x_i}, \varepsilon)v_{x_i x_i} + \big(a_1(h', \varepsilon) - a_1(u_{x_1}, \varepsilon)\big)h''. \end{aligned}$$

On the other hand, due to (2.7) and (2.9), we have

$$Lu - Lh = \lambda \bar{g}_\varepsilon(u) + f_\varepsilon - Lh \leq \lambda \bar{g}_\varepsilon(u) + f_\varepsilon - (p^* + 1)\widetilde{M}^{p^*+1}.$$

Hence,

$$v_t - \sum_{i=1}^n a_i(u_{x_i}, \varepsilon)v_{x_i x_i} \leq \big(a_1(u_{x_1}, \varepsilon) - a_1(h', \varepsilon)\big)h'' + \lambda \bar{g}_\varepsilon(u) + f_\varepsilon - (p^* + 1)\widetilde{M}^{p^*+1}. \tag{2.10}$$

Denote by Γ_T the parabolic boundary of Q_T, i.e., $\Gamma_T = \{(\mathbf{x}, t) \in \partial Q_T \mid t \neq T\}$. Suppose that the function v attains its positive maximum at the point $N \in \overline{Q}_T \setminus \Gamma_T$. At this point, $v > 0$ and $v_{x_i} = 0$ or, in other words, $u > h \geq m$ and $u_{x_1} = h'$. Thus, $a_1(u_{x_1}, \varepsilon) = a_1(h', \varepsilon)$ and consequently, due to (2.10), at the point N, we have

$$\begin{aligned} v_t - \sum_{i=1}^n a_i(u_{x_i}, \varepsilon)v_{x_i x_i} &\leq \lambda \bar{g}_\varepsilon(u) + f_\varepsilon - (p^* + 1)\widetilde{M}^{p^*+1} \\ &\leq |\lambda| \bar{g}_\varepsilon(M + m) + f^* - (p^* + 1)\widetilde{M}^{p^*+1} \\ &= |\lambda| g_\varepsilon(M + m) + f^* - (p^* + 1)\left(\frac{2M}{3l^2 + 2l}\right)^{p^*+1}. \end{aligned}$$

We used the fact that $0 \leq \bar{g}_\varepsilon(z) \leq g_\varepsilon(M + m)$ for positive z. Therefore, as it follows from Lemma 2.1,

$$v_t - \sum_{i=1}^n a_i(u_{x_i}, \varepsilon)v_{x_i x_i} < 0$$

at the point N. This inequality contradicts the assumption that v attains its maximum at N.

Due to (2.3), $v = -h \leq 0$ on $\partial\Omega \times (0, T)$. Besides, $v(\mathbf{x}, 0) = u_{0\varepsilon}(\mathbf{x}) - h(x_1) \leq u_{0\varepsilon}(\mathbf{x}) - m \leq 0$. Thus, $v \leq 0$ on Γ_T. This enables us to conclude that $v \leq 0$ in $\overline{Q}_T$, i.e.,

$$u(\mathbf{x}, t) \leq h(x_1) \quad \text{for } (\mathbf{x}, t) \in \overline{Q}_T.$$

In order to obtain a similar estimate from below, let us introduce the function $w(\mathbf{x}, t) = u(\mathbf{x}, t) + h(x_1)$. We have

$$\begin{aligned} Lu + Lh &= u_t - \sum_{i=1}^n a_i(u_{x_i}, \varepsilon)u_{x_i x_i} + h_t - \sum_{i=1}^n a_i(h_{x_i}, \varepsilon)h_{x_i x_i} \\ &= w_t - \sum_{i=1}^n a_i(u_{x_i}, \varepsilon)w_{x_i x_i} - \big(a_1(h', \varepsilon) - a_1(u_{x_1}, \varepsilon)\big)h''. \end{aligned}$$

On the other hand, due to (2.7) and (2.9),

$$Lu + Lh = \lambda \bar{g}_\varepsilon(u) + f_\varepsilon + Lh \geq \lambda \bar{g}_\varepsilon(u) + f_\varepsilon + (p^* + 1)\widetilde{M}^{p^*+1}.$$

Thus,

$$w_t - \sum_{i=1}^{n} a_i(u_{x_i}, \varepsilon) w_{x_i x_i} \geq \big(a_1(h', \varepsilon) - a_1(u_{x_1}, \varepsilon)\big)h'' + \lambda \bar{g}_\varepsilon(u) + f_\varepsilon + (p^* + 1)\widetilde{M}^{p^*+1}.$$

Suppose that at the point $N_1 \in \overline{Q}_T \setminus \Gamma_T$ the function w attains its negative minimum. At this point we have $w < 0$ and $w_{x_i} = 0$, i.e., $u < -h \leq 0$ and $u_{x_1} = -h'$. Therefore, $a_1(u_{x_1}, \varepsilon) = a_1(h', \varepsilon)$ (recall that, due to the choice of α, $a_i(z, \varepsilon) = a_i(-z, \varepsilon)$) and at the point N_1,

$$\begin{aligned} w_t - \sum_{i=1}^{n} a_i(u_{x_i}, \varepsilon) w_{x_i x_i} &\geq \lambda \bar{g}_\varepsilon(u) + f_\varepsilon + (p^* + 1)\widetilde{M}^{p^*+1} \\ &\geq -|\lambda| g_\varepsilon(M + m) - f^* + (p^* + 1)\left(\frac{2M}{(3l^2 + 2l)}\right)^{p^*+1}. \end{aligned} \tag{2.11}$$

Here, we used the inequality

$$\lambda \bar{g}_\varepsilon(u(N_1)) \geq -|\lambda| g_\varepsilon(M + m)$$

which is true due to the following arguments. If $\lambda \geq 0$, then this inequality follows from the fact that $\bar{g}_\varepsilon(u) \geq -g_\varepsilon(M + m)$ (see assumption (ii$_\varepsilon$)). If $\lambda < 0$, then the inequality follows from the fact that $\bar{g}_\varepsilon(u) \leq g_\varepsilon(M + m)$. Inequality (2.11) and Lemma 2.1 imply that

$$w_t - \sum_{i=1}^{n} a_i(u_{x_i}, \varepsilon) w_{x_i x_i} > 0$$

at the point N_1. This contradicts the assumption that w attains its minimum at N_1.

Due to (2.3) and (2.4), $w = h \geq 0$ on $\partial\Omega \times (0, T)$. Moreover $w(\mathbf{x}, 0) = u_{0\varepsilon}(\mathbf{x}) + h(x_1) \geq u_{0\varepsilon}(\mathbf{x}) + m \geq 0$. Taking into account that w cannot attain its positive maximum in $\overline{Q}_T \setminus \Gamma_T$, we conclude that $w \geq 0$ in $\overline{Q}_T$ and, as a consequence,

$$u(\mathbf{x}, t) \geq -h(x_1) \quad \text{for } (\mathbf{x}, t) \in \overline{Q}_T.$$

Thus,

$$-h(x_1) \leq u(\mathbf{x}, t) \leq h(x_1) \quad \text{for } (\mathbf{x}, t) \in \overline{Q}_T. \tag{2.12}$$

Let us introduce the function $\tilde{h}(x_1) = h(-x_1)$. Obviously $\tilde{h}' \leq -\widetilde{M}$ and $\tilde{h}'' = -\widetilde{M}$. Moreover, since $-\tilde{h}' \geq \widetilde{M} \geq 0$, due to the choice of α, we have $\tilde{h}'^{\alpha} \geq \widetilde{M}^{\alpha}$. Hence,

$$L\tilde{h} \geq (p^* + 1)\widetilde{M}^{p^*+1}.$$

Therefore, similarly to (2.12), we obtain

$$-\tilde{h}(x_1) \leq u(\mathbf{x}, t) \leq \tilde{h}(x_1) \quad \text{for } (\mathbf{x}, t) \in \overline{Q}_T.$$

This estimate together with (2.12) implies that

$$-\min\{h(x_1), \tilde{h}(x_1)\} \leq u(\mathbf{x}, t) \leq \min\{h(x_1), \tilde{h}(x_1)\} \quad \text{for } (\mathbf{x}, t) \in \overline{Q}_T.$$

Since $\min\{h(x_1), \tilde{h}(x_1)\} \leq h(0)$, we have

$$|u(\mathbf{x}, t)| \leq h(0) = \widetilde{M}\frac{3l^2 + 2l}{2} + m = M + m \quad \text{for } (\mathbf{x}, t) \in \overline{Q}_T.$$

The assertion of the lemma follows now from the fact that M is an arbitrary number from the set $\mathfrak{M}_c$. Lemma 2.2 is proved. □

Remark 1. The estimate in Lemma 2.2 is weaker that the similar one in Theorem 1. The latter will be obtained in the limit as $\varepsilon \to 0$. Nevertheless, Lemma 2.2 gives us a uniform estimate of $\|u_\varepsilon\|_{L^\infty(Q_T)}$ and this is enough to obtain other estimates and to perform the passage to the limit. Besides that, this estimate enables us to replace $\bar{g}_\varepsilon$ by g_ε in (2.2).

Let us estimate the derivatives of u_ε.

Lemma 2.3. *Let $\mathfrak{M}_c$ and ε_0 be as in Lemma 2.2. There exists a constant C such that*

$$\int_{Q_T} |u_{\varepsilon x_i}(\mathbf{x}, t)|^{p_i+2} \mathrm{d}\mathbf{x}\mathrm{d}t \leq C, \quad i = 1, \ldots, n,$$

for every $\varepsilon \in (0, \varepsilon_0)$.

Proof. As in the proof of the previous lemma, we shall write u instead of u_ε. Multiplying Eq. (2.2) by u and integrating by parts we obtain

$$\int_\Omega u^2(\mathbf{x},t)\mathrm{d}\mathbf{x} + 2\sum_{i=1}^n \int_{Q_T} (u_{x_i}^\alpha + \varepsilon)^{p_i/\alpha} u_{x_i}^2 \mathrm{d}\mathbf{x}\mathrm{d}t = 2\int_{Q_T} \big(\lambda \bar{g}_\varepsilon(u) + f\big) u \mathrm{d}\mathbf{x}\mathrm{d}t + \int_\Omega u_0^2 \mathrm{d}\mathbf{x}.$$

Due to Lemma 2.2, this equality implies the following estimate

$$\int_{Q_T} (u_{x_i}^\alpha + \varepsilon)^{p_i/\alpha} u_{x_i}^2 \mathrm{d}\mathbf{x}\mathrm{d}t \le C_1, \quad i = 1, \dots, n, \tag{2.13}$$

with a constant C_1 that does not depend on ε.

First, we suppose that $p_i \ge 0$. In this case, $\big(z^\alpha + \varepsilon\big)^{p_i/\alpha} z^2 \ge |z|^{p_i+2}$ and consequently, from (2.13), we conclude that

$$\int_{Q_T} |u_{x_i}|^{p_i+2} \mathrm{d}\mathbf{x}\mathrm{d}t \le C_1, \quad i = 1, \dots, n.$$

Now, consider the case $p_i \in (-1, 0)$. Let $(Q_T)^\varepsilon$ be the subset of Q_T where $u_{x_i}^\alpha > \varepsilon$ and $(Q_T)_\varepsilon = Q_T \setminus (Q_T)^\varepsilon$. Obviously, $u_{x_i}^\alpha + \varepsilon < 2u_{x_i}^\alpha$ in $(Q_T)^\varepsilon$ and, since $p_i < 0$,

$$C_1 \ge \int_{(Q_T)^\varepsilon} (u_{x_i}^\alpha + \varepsilon)^{p_i/\alpha} |u_{x_i}|^2 \mathrm{d}\mathbf{x}\mathrm{d}t > 2^{p_i/\alpha} \int_{(Q_T)^\varepsilon} |u_{x_i}|^{p_i+2} \mathrm{d}\mathbf{x}\mathrm{d}t.$$

At the same time,

$$\int_{(Q_T)_\varepsilon} |u_{x_i}|^{p_i+2} \mathrm{d}\mathbf{x}\mathrm{d}t \le T\varepsilon^{p_i+2} \operatorname{mes} \Omega.$$

Thus,

$$\int_{Q_T} |u_{x_i}|^{p_i+2} \mathrm{d}\mathbf{x}\mathrm{d}t \le 2^{-p_i/\alpha} C_1 + T\varepsilon^{p_i+2} \operatorname{mes} \Omega.$$

The lemma is proved. □

Let us prove the following auxiliary lemma.

Lemma 2.4. *For any $v \in L^\infty(\Omega) \cap U_i(\Omega)$, $i \in \{1, \dots, n\}$ there exists a constant C (depending on $\|v_{x_i}\|_{L^{p_i+2}(\Omega)}$) such that*

$$\left| \int_\Omega (v_{x_i}^\alpha + \varepsilon)^{p_i/\alpha} v_{x_i} \phi_{x_i} \mathrm{d}\mathbf{x} \right| \le C \|\phi_{x_i}\|_{L^{p_i+2}(\Omega)}, \quad i = 1, \dots, n,$$

for every function $\phi \in U_i(\Omega)$ and for all $\varepsilon \in (0, 1)$.

Proof. For a fixed $\varepsilon \in (0, 1)$, denote by Ω^ε the subset of Ω, where $v_{x_i}^\alpha > \varepsilon$, and by Ω_ε the subset of Ω, where $v_{x_i}^\alpha \le \varepsilon$. Recall that $v_{x_i}^\alpha \ge 0$, since α has an even numerator. In Ω^ε, we have

$$(v_{x_i}^\alpha + \varepsilon)^{p_i/\alpha} |v_{x_i}| < 2^{p_i/\alpha} |v_{x_i}|^{p_i+1}.$$

Therefore,

$$\begin{aligned}\left| \int_{\Omega^\varepsilon} (v_{x_i}^\alpha + \varepsilon)^{p_i/\alpha} v_{x_i} \phi_{x_i} \mathrm{d}\mathbf{x}\mathrm{d}t \right| &< 2^{p_i/\alpha} \int_{\Omega^\varepsilon} |v_{x_i}|^{p_i+1} |\phi_{x_i}| \mathrm{d}\mathbf{x} \\ &\le 2^{p_i/\alpha} \int_\Omega |v_{x_i}|^{p_i+1} |\phi_{x_i}| \mathrm{d}\mathbf{x} \le 2^{p_i/\alpha} \|v_{x_i}\|_{L^{p_i+2}(\Omega)}^{p_i+1} \|\phi_{x_i}\|_{L^{p_i+2}(\Omega)}.\end{aligned}$$

The integral over Ω_ε can be estimated as follows

$$\left| \int_{\Omega_\varepsilon} (v_{x_i}^\alpha + \varepsilon)^{p_i/\alpha} v_{x_i} \phi_{x_i} \mathrm{d}\mathbf{x} \right| \le (2\varepsilon)^{p_i/\alpha} \varepsilon^{1/\alpha} \int_{\Omega_\varepsilon} |\phi_{x_i}| \mathrm{d}\mathbf{x}\mathrm{d}t \le 2^{p_i/\alpha} \varepsilon^{(p_i+1)/\alpha} \|\phi_{x_i}\|_{L^1(\Omega)}.$$

The assertion of the lemma follows from these estimates. □

Lemma 2.5. *Let $\mathfrak{M}_c$ and ε_0 be as in Lemma 2.2. There exists a constant C such that*

$$\left| \int_{Q_T} u_{\varepsilon t} \phi \mathrm{d}\mathbf{x}\mathrm{d}t \right| \le C \|\phi\|_{V(Q_T)}$$

for every function $\phi \in V(Q_T)$ and for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. As earlier, we shall write u instead of u_ε. Multiplying (2.2) by an arbitrary function $\phi \in V(Q_T)$ and integrating over Q_T, we obtain that

$$\left|\int_{Q_T} u_t\phi \mathrm{d}\mathbf{x}\mathrm{d}t\right| \le \sum_{i=1}^{n}\left|\int_{Q_T}(u_{x_i}^{\alpha}+\varepsilon)^{p_i/\alpha}u_{x_i}\phi_{x_i}\mathrm{d}\mathbf{x}\mathrm{d}t\right| + \int_{Q_T}\left(|\lambda\bar{g}_\varepsilon(u)|+|f|\right)|\phi|\mathrm{d}\mathbf{x}\mathrm{d}t. \tag{2.14}$$

From Lemmas 2.1 and 2.2 it follows that there exists a constant C_1 such that

$$\int_{Q_T}\left(|\lambda\,\bar{g}_\varepsilon(u)|+|f|\right)|\phi|\mathrm{d}\mathbf{x}\mathrm{d}t \le C_1\|\phi\|_{L^1(Q_T)}. \tag{2.15}$$

Consider the first term on the right-hand side of (2.14). Due to Lemmas 2.3 and 2.4, there exists a constant C_2 such that

$$\left|\int_{Q_T}(u_{x_i}^{\alpha}+\varepsilon)^{p_i/\alpha}u_{x_i}\phi_{x_i}\mathrm{d}\mathbf{x}\mathrm{d}t\right| \le C_2\|\phi_{x_i}\|_{L^{p_i+2}(Q_T)}, \quad i=1,\dots,n. \tag{2.16}$$

The assertion of the lemma follows now from (2.14)–(2.16). □

2.2. Passage to the limit

We obtain a weak solution to problem (1.1)–(1.3) as a limit of the approximate solutions u_ε constructed in the previous section. Let $\{\varepsilon_k\}$ be a monotone sequence of positive numbers which tends to zero as $k \to \infty$. In order to simplify the notations, we shall omit the index k and shall write ε and $\varepsilon \to 0$.

The generalized formulation of the regularized problem (2.2)–(2.4) reads as follows

$$\int_{Q_T}\left(u_\varepsilon\phi_t - \sum_{i=1}^{n}(u_{\varepsilon x_i}^{\alpha}+\varepsilon)^{p_i/\alpha}u_{\varepsilon x_i}\phi_{x_i} + \lambda g_\varepsilon(u_\varepsilon)\phi + f_\varepsilon\phi\right)\mathrm{d}\mathbf{x}\mathrm{d}t = -\int_\Omega u_{0\varepsilon}\phi_0\mathrm{d}\mathbf{x}, \tag{2.17}$$

where $\phi_0 = \phi(\mathbf{x},0)$ and ϕ is an arbitrary smooth function which vanishes on $\partial\Omega\times(0,T)$ and on $\Omega\times\{T\}$. Notice that we replaced here $\bar{g}_\varepsilon$ by g_ε, which is explained in Remark 1 in the previous section.

First of all, by the definition of the sequences $\{f_\varepsilon\}$ and $\{u_{0\varepsilon}\}$ we have that

$$f_\varepsilon \to f \quad \text{in } L^s(Q_T), \qquad u_{0\varepsilon}\to u_0 \quad \text{in } L^s(\Omega)$$

as $\varepsilon \to 0$ for all $s \in [6,\infty)$. This implies that

$$\int_{Q_T} f_\varepsilon\phi\mathrm{d}\mathbf{x}\mathrm{d}t \to \int_{Q_T} f\phi\mathrm{d}\mathbf{x}\mathrm{d}t \quad\text{and}\quad \int_\Omega u_{0\varepsilon}\phi_0\mathrm{d}\mathbf{x} \to \int_\Omega u_0\phi_0\mathrm{d}\mathbf{x} \tag{2.18}$$

as $\varepsilon \to 0$.

Let us investigate the convergence of the sequence $\{u_\varepsilon\}$. As it follows from Lemmas 2.2, 2.3 and 2.5, this sequence has a subsequence which will be denoted again by $\{u_\varepsilon\}$ such that

$$u_\varepsilon \to u \quad *\text{-weakly in } L^\infty(Q_T), \tag{2.19}$$

$$u_{\varepsilon x_i} \to u_{x_i} \quad \text{weakly in } L^{p_i+2}(Q_T), \quad i=1,\dots,n, \tag{2.20}$$

$$u_{\varepsilon t} \to u_t \quad *\text{-weakly in } V^*(Q_T) \tag{2.21}$$

as $\varepsilon \to 0$. Obviously (2.19) and (2.21) imply that

$$\int_{Q_T} u_\varepsilon\phi_t\mathrm{d}\mathbf{x}\mathrm{d}t \to \int_{Q_T} u\phi_t\mathrm{d}\mathbf{x}\mathrm{d}t \quad \text{as } \varepsilon\to 0. \tag{2.22}$$

Our next step is the proof of the strong convergence of $\{u_\varepsilon\}$ in $L^s(Q_T)$ for all $s \in [1,\infty)$. We shall do this by the Aubin–Lions compactness theorem (see [1, Ch. 1.5] and [19]). Lemmas 2.2, 2.3 and 2.5 imply that the sequences $\{u_\varepsilon\}$ and $\{u_{\varepsilon t}\}$ are bounded in the spaces $L^{p_*+2}(0,T;W_0^{1,p_*+2}(\Omega))$ and $L^{q^*+2}(0,T;W^{-1,q^*+2}(\Omega))$, respectively. Here, q^* is such that $(q^*+2)^{-1}+(p^*+2)^{-1}=1$ and $W^{-1,q^*+2}(\Omega)$ is the space adjoint to $W_0^{1,p^*+2}(\Omega)$. The space $W_0^{1,p^*+2}(\Omega)$ is compactly and densely embedded into $L^{p^*+2}(\Omega)$, but it can occur that $L^{p^*+2}(\Omega)$ is not embedded into $W^{-1,q^*+2}(\Omega)$. However, we can take a larger space instead of $W^{-1,q^*+2}(\Omega)$. For instance, $W^{-k,q^*+2}(\Omega)$ with $k > \max\{n/(p^*+2),1\}$ is suitable. As if follows from the embedding theorems, $L^{p^*+2}(\Omega) \subset W^{-k,q^*+2}(\Omega)$ and $W^{-1,q^*+2}(\Omega) \subset W^{-k,q^*+2}(\Omega)$. Therefore, the Aubin–Lions theorem implies that the sequence $\{u_\varepsilon\}$ is compact in $L^{p_*+2}(Q_T)$. Thus, there exists a subsequence $\{u_\varepsilon\}$ such that

$$u_\varepsilon \to u \quad \text{almost everywhere in } Q_T. \tag{2.23}$$

Moreover, since the sequence $\{u_\varepsilon\}$ is bounded in $L^\infty(Q_T)$, the Lebesgue dominated convergence theorem yields that

$$u_\varepsilon \to u \quad \text{in } L^s(Q_T) \text{ for all } s \in [1, \infty).$$

From (2.23), we can easily deduce that

$$\int_{Q_T} \lambda g_\varepsilon(u_\varepsilon)\phi \mathrm{d}\mathbf{x}\mathrm{d}t \to \int_{Q_T} \lambda g(u)\phi \mathrm{d}\mathbf{x}\mathrm{d}t \quad \text{as } \varepsilon \to 0. \tag{2.24}$$

Indeed, there exists a constant M such that $|u_\varepsilon(\mathbf{x}, t)| \le M$ for all $(\mathbf{x}, t) \in Q_T$ and for all $\varepsilon \in (0, \varepsilon_0)$. At the same time, property (i_ε) of the sequence $\{g_\varepsilon\}$ implies that for every $\gamma > 0$ there exists ε_γ such that $\max_{\xi\in[-M,M]} |g_\varepsilon(\xi) - g(\xi)| < \gamma$ for $\varepsilon \in (0, \varepsilon_\gamma)$. Therefore, $\max_{(\mathbf{x},t)\in\overline{Q}_T} |g_\varepsilon(u_\varepsilon(\mathbf{x}, t)) - g(u_\varepsilon(\mathbf{x}, t))| < \gamma$ for $\varepsilon \in (0, \varepsilon_\gamma)$ and, as a consequence,

$$\begin{aligned}\left|\int_{Q_T} \lambda\big(g_\varepsilon(u_\varepsilon) - g(u)\big)\phi \mathrm{d}\mathbf{x}\mathrm{d}t\right| &\le \int_{Q_T} |g_\varepsilon(u_\varepsilon) - g(u_\varepsilon)|\, |\lambda\phi| \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{Q_T} |g(u_\varepsilon) - g(u)|\, |\lambda\phi| \mathrm{d}\mathbf{x}\mathrm{d}t \\ &\le \gamma \operatorname{mes} Q_T \max_{(\mathbf{x},t)\in Q_T} |\lambda\phi(\mathbf{x}, t)| + \int_{Q_T} |g(u_\varepsilon) - g(u)|\, |\lambda\phi| \mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned}$$

Due to (2.23) and the Lebesgue dominated convergence theorem, the integral term on the right-hand side of the last inequality tends to zero as $\varepsilon \to 0$. Since γ can be taken arbitrary small, (2.24) holds true.

Thus, in order to complete the passage to the limit in (2.17), we only have to prove that

$$\int_{Q_T} \sum_{i=1}^n (u_{\varepsilon x_i}^\alpha + \varepsilon)^{p_i/\alpha} u_{\varepsilon x_i} \phi_{x_i} \mathrm{d}\mathbf{x}\mathrm{d}t \to \int_{Q_T} \sum_{i=1}^n |u_{x_i}|^{p_i} u_{x_i} \phi_{x_i} \mathrm{d}\mathbf{x}\mathrm{d}t \quad \text{as } \varepsilon \to 0. \tag{2.25}$$

For this purpose, we employ the monotonicity method described in [1, Ch. 2]. Notice that we will slightly modify this method. Namely, unlike in [1], we shall use the monotonicity of the approximate operator.

For $i = 1, \dots, n$, define nonlinear operators $A_{i\varepsilon}$ and A_i which act from $U_i(\Omega)$ to $U_i^*(\Omega)$. For arbitrary functions v and w from the space $U_i(\Omega)$, we put

$$\langle A_{i\varepsilon}(v), w\rangle = \sum_{i=1}^n \int_\Omega (v_{x_i}^\alpha + \varepsilon)^{p_i/\alpha} v_{x_i} w_{x_i} \mathrm{d}\mathbf{x},$$

$$\langle A_i(v), w\rangle = \sum_{i=1}^n \int_\Omega |v_{x_i}|^{p_i} v_{x_i} w_{x_i} \mathrm{d}\mathbf{x},$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $U_i^*(\Omega)$ and $U_i(\Omega)$. It is not difficult to see that the operators $A_{i\varepsilon}$ and A_i indeed map $U_i(\Omega)$ into $U_i^*(\Omega)$. For the operator $A_{i\varepsilon}$ it follows from Lemma 2.4, whereas for the operator A_i, this fact is the direct consequence of the Hölder inequality:

$$\int_\Omega |v_{x_i}|^{p_i} v_{x_i} w_{x_i} \mathrm{d}\mathbf{x} \le \left(\int_\Omega |v_{x_i}|^{p_i+2} \mathrm{d}\mathbf{x}\right)^{(p_i+1)/(p_i+2)} \left(\int_\Omega |w_{x_i}|^{p_i+2} \mathrm{d}\mathbf{x}\right)^{1/(p_i+2)}.$$

For every $v \in U(\Omega)$ we denote

$$A_\varepsilon(v) = \sum_{i=1}^n A_{i\varepsilon}(v), \qquad A(v) = \sum_{i=1}^n A_i(v).$$

Clearly, operators A_ε and A act from $U(\Omega)$ to $U^*(\Omega)$.

Thus, (2.25) is equivalent to the following assertion

$$\int_0^T \langle A_\varepsilon(u_\varepsilon), \phi\rangle \mathrm{d}t \to \int_0^T \langle A(u), \phi\rangle \mathrm{d}t \quad \text{as } \varepsilon \to 0,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $U^*(\Omega)$ and $U(\Omega)$.

Notice that, since the sequence $\{A_\varepsilon(u_\varepsilon)\}$ is bounded in $V^*(Q_T)$ (Lemma 2.4), there exists $\chi \in V^*(Q_T)$ such that up to a subsequence

$$A_\varepsilon(u_\varepsilon) \to \chi \quad *\text{-weakly in } V^*(Q_T). \tag{2.26}$$

Our goal is to prove that $\chi = A(u)$.

Relations (2.18), (2.21), (2.24) and (2.26) imply that the limit function u satisfies the equation:

$$u_t + \chi = \lambda g(u) + f \quad \text{in } V^*(Q_T).$$

Obviously $U(\Omega) \subset L^2(\Omega) \subset U^*(\Omega)$ and these embeddings are dense. Therefore, exactly as in [1] we obtain that

$$u \in C(0, T; L^2(\Omega)) \tag{2.27}$$

and

$$\int_{Q_T} \big(\lambda g(u) + f\big) u \mathrm{d}\mathbf{x}\mathrm{d}t + \frac{1}{2}\int_\Omega u_0^2 \mathrm{d}\mathbf{x} - \frac{1}{2}\int_\Omega u^2(\mathbf{x}, T)\mathrm{d}\mathbf{x} = \int_0^T \langle \chi, u\rangle \mathrm{d}t. \tag{2.28}$$

Multiplying Eq. (2.2) by u_ε and integrating over Q_T, we obtain

$$\int_0^T \langle A_\varepsilon(u_\varepsilon), u_\varepsilon\rangle \mathrm{d}t = \int_{Q_T} \big(\lambda g_\varepsilon(u_\varepsilon) + f_\varepsilon\big) u_\varepsilon \mathrm{d}\mathbf{x}\mathrm{d}t + \frac{1}{2}\int_\Omega u_{0\varepsilon}^2 \mathrm{d}\mathbf{x} - \frac{1}{2}\int_\Omega u_\varepsilon^2(\mathbf{x}, T)\mathrm{d}\mathbf{x}. \tag{2.29}$$

Operator A_ε is monotone (see Appendix A.3), i.e.,

$$\langle A_\varepsilon(u_\varepsilon) - A_\varepsilon(v), u_\varepsilon - v\rangle \geq 0$$

for an arbitrary $v \in U(\Omega)$ and for all $t \in [0, T]$. This means that

$$\int_0^T \langle A_\varepsilon(u_\varepsilon), u_\varepsilon\rangle \mathrm{d}t \geq \int_0^T \langle A_\varepsilon(u_\varepsilon), \phi\rangle \mathrm{d}t + \int_0^T \langle A_\varepsilon(\phi), u_\varepsilon - \phi\rangle \mathrm{d}t$$

for an arbitrary $\phi \in V(Q_T)$. Therefore, (2.29) yields

$$\int_{Q_T} \big(\lambda g_\varepsilon(u_\varepsilon) + f_\varepsilon\big) u_\varepsilon \mathrm{d}\mathbf{x}\mathrm{d}t + \frac{1}{2}\int_\Omega u_{0\varepsilon}^2 \mathrm{d}\mathbf{x} - \frac{1}{2}\int_\Omega u_\varepsilon^2(\mathbf{x}, T)\mathrm{d}\mathbf{x} - \int_0^T \langle A_\varepsilon(u_\varepsilon), \phi\rangle \mathrm{d}t - \int_0^T \langle A_\varepsilon(\phi), u_\varepsilon - \phi\rangle \mathrm{d}t \geq 0.$$

The passage to the limit as $\varepsilon \to 0$ in this inequality together with (2.28) implies that

$$\int_0^T \langle \chi - A(\phi), u - \phi\rangle \mathrm{d}t \geq 0. \tag{2.30}$$

Here, we used the facts that (see Appendix A.4)

$$\int_\Omega u^2(\mathbf{x}, T)\mathrm{d}\mathbf{x} \leq \liminf_{\varepsilon\to 0} \int_\Omega u_\varepsilon^2(\mathbf{x}, T)\mathrm{d}\mathbf{x} \tag{2.31}$$

and that

$$\lim_{\varepsilon\to 0} \int_0^T \langle A_\varepsilon(\phi), u_\varepsilon - \phi\rangle \mathrm{d}t = \int_0^T \langle A(\phi), u - \phi\rangle \mathrm{d}t.$$

From (2.30) by standard arguments ([1, Ch. 2]), we conclude that

$$\chi = A(u).$$

The existence of a weak solution to the problem is proved.

As it follows from (2.21), for the constructed solution u we have $u_t \in V^*(Q_T)$. Inclusion $u \in C([0, T]; L^s(\Omega))$ follows from (2.27) and (1.7) due to the Lebesgue dominated convergence theorem. Let us prove (1.7). Lemma 2.2 implies that $\|u\|_{L^\infty(Q_T)} \leq m + M_c$ for every compact set $\mathfrak{M}_c \subset \mathfrak{M}$. Taking now infimum in this inequality over all $\mathfrak{M}_c \subset \mathfrak{M}$, we obtain (1.7) thanks to (2.1).

In order to complete the proof of the theorem, we have only to establish the uniqueness of the weak solution. Suppose that there exist two solutions u_1 and u_2. Then the function $w = u_1 - u_2$ satisfies the following equation

$${}_{V^*(Q_{t_0})}\langle w_t, w\rangle_{V(Q_{t_0})} + \int_0^{t_0} {}_{U^*}\langle A(u_1) - A(u_2), u_1 - u_2\rangle_U \mathrm{d}t = \lambda \int_{Q_{t_0}} \big(g(u_1) - g(u_2)\big)(u_1 - u_2)\mathrm{d}\mathbf{x}\mathrm{d}t$$

with an arbitrary $t_0 \in (0, T]$. Taking into account the monotonicity of A, the Lipschitz continuity of g and the fact that $w(\mathbf{x}, 0) = 0$, we obtain the following inequality

$$\frac{1}{2}\|w(t_0)\|_{L_2(\Omega)}^2 = {}_{V^*(Q_{t_0})}\langle w_t, w\rangle_{V(Q_{t_0})} \leq 2|\lambda| C \int_0^{t_0} \frac{1}{2}\|w(t_0)\|_{L_2(\Omega)}^2 \mathrm{d}t,$$

where the constant C is such that $|g(u_1) - g(u_2)| \leq C|u_1 - u_2|$. Due to the Gronwall lemma this implies that $u_1 = u_2$.

Theorem 1 is proved.

3. Proof of Theorem 2

The proof of Theorem 2 is similar to that of Theorem 1. We regularize the problem, obtain uniform estimates and pass to the limit. Consider the regularized equation

$$u_{\varepsilon t} - \operatorname{div}\big((|\nabla u_\varepsilon|^\alpha + \varepsilon)^{p/\alpha}\nabla u_\varepsilon\big) = \lambda \bar{g}(u_\varepsilon) + f_\varepsilon, \tag{3.1}$$

with boundary conditions (2.3) and (2.4). The functions g_ε, $\bar{g}_\varepsilon$ and f_ε are the same as in Section 2.1. If $p \geq 1$ then we take arbitrary positive ε and $\alpha = 2$. If $p \leq 0$ then we again take $\alpha = 2$ but we require ε to be small enough, namely

$$0 < \varepsilon < (1-p)\left(\frac{2M}{3l_*^2 + 2l_*}\right)^2.$$

If $p \in (0, 1)$ we require that, as in Section 2, $\alpha = r/m$ with positive integers r and m such that $r < m$ and r is even.

Notice that for every compact subset $\mathfrak{N}_c$ of $\mathfrak{N}$ there exists $\varepsilon_0 > 0$ such that

$$f^* + |\lambda| g_\varepsilon(M+m) < (p+1)\left(\frac{2M}{3l_*^2 + 2l_*}\right)^{p+1}$$

for all $M \in \mathfrak{N}$ and $\varepsilon \in (0, \varepsilon_0)$. This fact can be proved similar to Lemma 2.1. We shall say that ε_0 corresponds to $\mathfrak{N}_c$.

Let us estimate a classical solution u_ε of problem (3.1), (2.3) and (2.4) in L^∞ norm. To simplify the notation, in the proofs of Lemmas 3.1–3.4 we will omit the subscript ε.

Lemma 3.1. *For every compact set $\mathfrak{N}_c \subset \mathfrak{N}$ and for every $\varepsilon \in (0, \varepsilon_0)$, the classical solution of problem (3.1), (2.3) and (2.4) satisfies the following estimate:*

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq m + M_c,$$

where $M_c = \min\{M \mid M \in \mathfrak{N}_c\}$ and ε_0 is the positive number corresponding to $\mathfrak{N}_c$.

Proof. Without loss of generality suppose that $l_* = l_1$. Let us take an arbitrary $M \in \mathfrak{M}_c$. Define the function $h(x_1)$ by the following:

$$h(x_1) = \tilde{M}\left(\frac{l_1^2 - x_1^2}{2} + (1+l_1)(l_1+x_1)\right) + m,$$

where

$$\tilde{M} = \frac{M}{d_1}, \qquad d_1 = \frac{3l_1^2 + 2l_1}{2}.$$

Define the operator L:

$$Lu \equiv u_t - \operatorname{div}\big((|\nabla u|^\alpha + \varepsilon)^{p/\alpha}\nabla u\big).$$

We have

$$Lu = |\lambda| \bar{g}(u) + f(\mathbf{x}, t)$$

and

$$Lh = h_t - \operatorname{div}\big((|\nabla h|^\alpha + \varepsilon)^{p/\alpha}\nabla \mathrm{h}\big) - (h'^\alpha + \varepsilon)(\varepsilon + (1+p)h'^\alpha)h'' = (h'^\alpha + \varepsilon)(\varepsilon + (1+p)h'^\alpha)\frac{M}{d_1}.$$

Consider function

$$E(\varepsilon) \equiv (h'^\alpha + \varepsilon)^{(p-\alpha)/\alpha}(h'^\alpha(1+p) + \varepsilon).$$

Due to the assumptions on α and ε we conclude that

$$E'(\varepsilon) = \frac{p}{\alpha}(h'^\alpha + \varepsilon)^{\frac{p-2\alpha}{\alpha}}(h'^\alpha(1+p-\alpha) + \varepsilon) \geq 0.$$

This inequality implies that

$$E(\varepsilon) \geq E(0)$$

and

$$Lh \geq |h'|^p(1+p)\frac{M}{d_1} \geq \left(1 + p\left(\frac{M}{d_1}\right)^{p+1}\right).$$

Now, similarly to the proof of Lemma 2.2, we deduce that

$$-h(x_1) \le u(\mathbf{x}, t) \le h(x_1) \quad \text{in } \bar{Q}_T$$

and

$$-h(-x_1) \le u(\mathbf{x}, t) \le h(-x_1) \quad \text{in } \bar{Q}_T.$$

As a consequence we conclude that

$$\|u(\mathbf{x}, t)\|_{L^\infty(Q_T)} \le h(0) = M + m.$$

The assertion of the lemma follows now from the fact that M is an arbitrary number from $\mathfrak{N}_c$. $\square$

Remark 2. Due to the estimates in the previous lemmas, we can take $g(u)$ instead of $\bar{g}(u)$ in Eq. (3.1).

Let us turn now to integral estimates.

Lemma 3.2. *Suppose that conditions of Lemma 3.1 are fulfilled, then for any classical solution of problem (3.1), (2.3) and (2.4) the following estimate is valid*

$$\int_{Q_T} |\nabla u_\varepsilon(t, \mathbf{x})|^{p+2} \mathrm{d}\mathbf{x}\mathrm{d}t \le C,$$

where the constant C is independent of ε.

Proof. Multiply Eq. (3.1) by u, integrating by parts and taking into account that $|u| \le M + m$, we find that

$$\int_{Q_T} \left(|\nabla u|^\alpha + \varepsilon\right)^{p/\alpha} |\nabla u|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \le C_1 \tag{3.2}$$

for some independent of ε constant C_1.

Suppose that $p \ge 0$. In this case $(z^\alpha + \varepsilon)^{p/\alpha} z^2 \ge |z|^{p+2}$ and consequently (3.2) implies that

$$\int_{Q_T} |\nabla u|^{p+2} \mathrm{d}\mathbf{x}\mathrm{d}t \le C_1.$$

Consider the case $p \in (-1, 0)$. Let $\left(Q_T\right)^\varepsilon$ be the subset of Q_T where $|\nabla u|^\alpha > \varepsilon$ and $\left(Q_T\right)_\varepsilon = Q_T \setminus \left(Q_T\right)^\varepsilon$. Obviously, $|\nabla u|^\alpha + \varepsilon < 2|\nabla u|^\alpha$ in $\left(Q_T\right)^\varepsilon$ and, since p is negative,

$$C_1 \ge \int_{\left(Q_T\right)^\varepsilon} \left(|\nabla u|^\alpha + \varepsilon\right)^{p/\alpha} |\nabla u|^2 \mathrm{d}\mathbf{x}\mathrm{d}t > 2^{p/\alpha} \int_{\left(Q_T\right)^\varepsilon} |\nabla u|^{p+2} \mathrm{d}\mathbf{x}\mathrm{d}t.$$

At the same time

$$\int_{\left(Q_T\right)_\varepsilon} |\nabla u|^{p+2} \mathrm{d}\mathbf{x}\mathrm{d}t \le T\varepsilon^{p+2} \text{ mes } \Omega.$$

Thus

$$\int_{Q_T} |\nabla u|^{p+2} \mathrm{d}\mathbf{x}\mathrm{d}t \le 2^{-p/\alpha} C_1 + T\varepsilon^{p+2} \text{ mes } \Omega.$$

Without loss of generality we can assume that $\varepsilon \le 1$, hence the lemma is proved. $\square$

Let us prove the following auxiliary lemma.

Lemma 3.3. *Let functions w and ϕ be in $W_0^{1,p+2}(\Omega)$, $p > -1$. Then*

$$\left| \int_\Omega (|\nabla w|^\alpha + \varepsilon)^{p/\alpha} \nabla w \cdot \nabla \phi \mathrm{d}\mathbf{x} \right| \le 2^{p/\alpha} \left(\|\nabla w\|_{L^{p+2}(\Omega)}^{p+1} + \varepsilon^{(p+1)/\alpha} \right) \|\nabla \phi\|_{L^{p+2}(\Omega)}$$

for all $\varepsilon \in (0, 1)$.

Proof. For a fixed $\varepsilon \in (0, 1)$, denote by Ω^ε the subset of Ω, where $|\nabla w|^\alpha > \varepsilon$, and by Ω_ε the subset of Ω, where $|\nabla w|^\alpha \le \varepsilon$. In Ω^ε, we have

$$(|\nabla w|^\alpha + \varepsilon)^{p/\alpha} |\nabla w| < 2^{p/\alpha} |\nabla w|^{p+1}.$$

Therefore,

$$\left|\int_{\Omega^\varepsilon} (|\nabla w|^\alpha + \varepsilon)^{p/\alpha} \nabla w \cdot \nabla\phi \mathrm{d}\mathbf{x}\mathrm{d}t\right| < 2^{p/\alpha} \int_{\Omega^\varepsilon} |\nabla w|^{p+1} |\nabla\phi| \mathrm{d}\mathbf{x} \le 2^{p/\alpha} \|\nabla w\|^{p+1}_{L^{p+2}(\Omega)} \|\nabla\phi\|_{L^{p+2}(\Omega)}.$$

The integral over Ω_ε can be estimated as follows

$$\left|\int_{\Omega_\varepsilon} (|\nabla w|^\alpha + \varepsilon)^{p/\alpha} \nabla w \cdot \nabla\phi \mathrm{d}\mathbf{x}\right| \le (2\varepsilon)^{p/\alpha} \varepsilon^{1/\alpha} \int_{\Omega_\varepsilon} |\nabla\phi| \mathrm{d}\mathbf{x}\mathrm{d}t \le 2^{p/\alpha} \varepsilon^{(p+1)/\alpha} \|\nabla\phi\|_{L^1(\Omega)}.$$

The assertion of the lemma follows from these estimates. □

Lemma 3.4. *There exists a constant C such that*

$$\left|\int_{Q_T} u_{\varepsilon t} \phi \mathrm{d}\mathbf{x}\mathrm{d}t\right| \le C\|\phi\|_{L^{p+2}(0,T;W_0^{1,p+2}(\Omega))}$$

for every function $\phi \in L^{p+2}(0, T; W_0^{1,p+2}(\Omega)) \cap L^{(p+2)/(p+1)}(Q_T)$ and for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. As earlier, we shall write u instead of u_ε. Multiplying (3.1) by an arbitrary smooth function $\phi(t, \mathbf{x})$ that is equal to zero on $\partial\Omega$ and integrating over Q_T, we find that

$$\left|\int_{Q_T} u_t \phi \mathrm{d}\mathbf{x}\mathrm{d}t\right| \le \left|\int_{Q_T} (|\nabla u|^\alpha + \varepsilon)^{p/\alpha} \nabla u \nabla\phi \mathrm{d}\mathbf{x}\mathrm{d}t\right| + \int_{Q_T} |\lambda \bar{g}_\varepsilon(u) + f_\varepsilon||\phi| \mathrm{d}\mathbf{x}\mathrm{d}t. \tag{3.3}$$

From Lemma 3.1, it follows that there exists a constant C_1 such that

$$\int_{Q_T} |\bar{g}_\varepsilon(u) + f||\phi| \mathrm{d}\mathbf{x}\mathrm{d}t \le C_2\|\phi\|_{L^1(Q_T)} \le C_3\|\phi\|_{L^{p+2}(0,T;W_0^{1,p+2}(\Omega))}. \tag{3.4}$$

Consider the first term on the right-hand side of (3.3). Due to Lemma 3.3, there exists a constant C_2 such that

$$\left|\int_{Q_T} (|\nabla u|^\alpha + \varepsilon)^{p/\alpha} \nabla u \cdot \nabla\phi \mathrm{d}\mathbf{x}\mathrm{d}t\right| \le C_2\|\nabla\phi\|_{L^{p+2}(Q_T)}. \tag{3.5}$$

The assertion of the lemma follows now from (3.4) and (3.5). □

We have obtained all estimates needed to pass to the limit as $\varepsilon \to 0$ in Eqs. (3.1), (2.3) and (2.4). It can be done absolutely in the same way as in Section 2.2. The only difference is in the definition of operators A_ε and A. We define operators acting from $W_0^{1,p+2}(\Omega)$ to $W^{-1,q+2}(\Omega)$ with $(p+2)^{-1} + (q+2)^{-1} = 1$. For arbitrary functions v and w from the space $W_0^{1,p+2}(\Omega)$, we put

$$\langle \mathcal{A}_\varepsilon(v), w\rangle = \int_\Omega (|\nabla v|^\alpha + \varepsilon)^{p/\alpha} \nabla v \cdot \nabla w \mathrm{d}\mathbf{x},$$

$$\langle \mathcal{A}(v), w\rangle = \int_\Omega |\nabla v|^p \nabla v \cdot \nabla w \mathrm{d}\mathbf{x},$$

where $\langle \cdot, \cdot\rangle$ is the duality pairing between $W^{-1,q+2}(\Omega)$ and $W_0^{1,p+2}(\Omega)$.

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Appendix

A.1. Approximation of the function g

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|g(\xi)| \le g(\eta)$ whenever $|\xi| \le \eta$. We want to prove that there exists a sequence of functions $g_\varepsilon : \mathbb{R} \to \mathbb{R}$ that satisfy the following conditions:

(1) they are continuously differentiable on every compact subset of $\mathbb{R}$;
(2) $|g_\varepsilon(\xi)| \le g_\varepsilon(\eta)$ for all real ξ and η such that $|\xi| \le \eta$;
(3) the sequence $\{g_\varepsilon\}$ converges to the function g as $\varepsilon \to 0$ uniformly on every compact subset of $\mathbb{R}$.

Let us define

$$g_\varepsilon(\xi) = \frac{1}{2\varepsilon}\int_{\xi-\varepsilon}^{\xi+\varepsilon} g(\eta)\mathrm{d}\eta$$

and verify the properties formulated above. The first and the third properties are obvious. Moreover, g_ε is continuously differentiable since the function g is continuous. Let us turn to the proof of the second property.

Note that the function g is nondecreasing and nonnegative on $[0,\infty)$. Let us prove that the function g_ε has the same properties, which implies property (2) for $\xi \ge 0$. Indeed, $g_\varepsilon'(\xi) = (2\varepsilon)^{-1}\big(g(\xi+\varepsilon) - g(\xi-\varepsilon)\big)$. If $\xi \ge \varepsilon$, then $g_\varepsilon'(\xi) \ge 0$. If $\xi \in [0,\varepsilon]$, then $|g(\xi-\varepsilon)| \le g(|\xi-\varepsilon|) \le g(\xi+\varepsilon)$ and we have again $g_\varepsilon'(\xi) \ge 0$. In order to prove that $g_\varepsilon \ge 0$ on $[0,\infty)$, it is enough to show that $g_\varepsilon(0) \ge 0$. This fact is an obvious consequence of the following relations

$$g_\varepsilon(0) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} g(\eta)\mathrm{d}\eta = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{0} g(\eta)\mathrm{d}\eta + \frac{1}{2\varepsilon}\int_{0}^{\varepsilon} g(\eta)\mathrm{d}\eta,$$
$$\left|\int_{-\varepsilon}^{0} g(\eta)\mathrm{d}\eta\right| \le \int_{-\varepsilon}^{0} |g(\eta)|\mathrm{d}\eta \le \int_{-\varepsilon}^{0} g(-\eta)\mathrm{d}\eta = \int_{0}^{\varepsilon} g(\eta)\mathrm{d}\eta.$$

Thus, $g_\varepsilon(\xi)$ satisfies (2) for $\xi \ge 0$.

For negative ξ, due to the monotonicity of the function g_ε on $[0,\infty)$, we have to prove that $|g_\varepsilon(\xi)| \le g_\varepsilon(|\xi|) = g_\varepsilon(-\xi)$. If $\xi \in (-\infty, -\varepsilon]$, then

$$\begin{aligned}|g_\varepsilon(\xi)| &\le \frac{1}{2\varepsilon}\int_{\xi-\varepsilon}^{\xi+\varepsilon} |g(\eta)|\mathrm{d}\eta \le \frac{1}{2\varepsilon}\int_{\xi-\varepsilon}^{\xi+\varepsilon} g(-\eta)\mathrm{d}\eta \\ &= -\frac{1}{2\varepsilon}\int_{-\xi-\varepsilon}^{-\xi+\varepsilon} g(\eta)\mathrm{d}\eta = \frac{1}{2\varepsilon}\int_{|\xi|-\varepsilon}^{|\xi|+\varepsilon} g(\eta)\mathrm{d}\eta = g_\varepsilon(|\xi|).\end{aligned}$$

Finally, suppose that $\xi \in (-\varepsilon, 0)$. Since $\int_{-\xi-\varepsilon}^{\xi+\varepsilon} g(\eta)\mathrm{d}\eta \ge 0$, we have

$$\begin{aligned}|g_\varepsilon(\xi)| &= \left|\frac{1}{2\varepsilon}\int_{\xi-\varepsilon}^{\xi+\varepsilon} g(\eta)\mathrm{d}\eta\right| \le \frac{1}{2\varepsilon}\int_{\xi-\varepsilon}^{-\xi-\varepsilon} |g(\eta)|\mathrm{d}\eta + \frac{1}{2\varepsilon}\int_{-\xi-\varepsilon}^{\xi+\varepsilon} g(\eta)\mathrm{d}\eta \\ &\le \frac{1}{2\varepsilon}\int_{\xi-\varepsilon}^{-\xi-\varepsilon} g(-\eta)\mathrm{d}\eta + \frac{1}{2\varepsilon}\int_{-\xi-\varepsilon}^{\xi+\varepsilon} g(\eta)\mathrm{d}\eta = \frac{1}{2\varepsilon}\int_{\xi+\varepsilon}^{-\xi+\varepsilon} g(\eta)\mathrm{d}\eta + \frac{1}{2\varepsilon}\int_{-\xi-\varepsilon}^{\xi+\varepsilon} g(\eta)\mathrm{d}\eta \\ &= \frac{1}{2\varepsilon}\int_{-\xi-\varepsilon}^{-\xi+\varepsilon} g(\eta)\mathrm{d}\eta = g_\varepsilon(-\xi) = g_\varepsilon(|\xi|).\end{aligned}$$

Thus, property (2) is entirely proved.

A.2. Proof of a classical solvability of the regularized problems

Regularized equation (3.1) is uniformly parabolic equation and the global solvability of problem (3.1), (2.3) and (2.4) follows, for example, from [20].

Regularized equation (2.2) is strictly parabolic but not uniformly parabolic equation. In order to prove the global classical solvability of problem (2.2)–(2.4) it is sufficient to prove the a priori estimate of $|\nabla u|$. After this, Eq. (2.2) can be considered as uniformly parabolic and the required existence follows from [20]. In order to prove this estimate we will apply the classical Bernstein method (see [21,20]), which involves differentiation of Eq. (2.2) with respect to x_i, $i = 1, \ldots, n$ followed by multiplication by u_{x_i} and summation over i. The maximum principle is then applied to the resulting equation in the function $v = |\nabla u|^2$. For v we obtain

$$\sum_{i=1}^n a_i(u_{x_i},\varepsilon)v_{x_ix_i} - v_t + \sum_{i=1}^n \frac{\partial a_i(u_{x_i},\varepsilon)}{\partial u_{x_i}} u_{x_ix_i}v_{x_i} + 2\left(\lambda g_\varepsilon'(u) + \frac{\sum_{i=1}^n f_{x_i}u_{x_i}}{|\nabla u|^2}\right) v \ge 0.$$

If $v \ge 1$ then there exists a constant K depending on $g(M)$ and $\max|f_{x_i}|$ such that

$$2\left(\lambda g_\varepsilon'(u) + \frac{\sum_{i=1}^n f_{x_i}u_{x_i}}{|\nabla u|^2}\right) < K.$$

For $\omega = v\mathrm{e}^{-Kt}$ we obtain

$$\sum_{i=1}^{n} a_i(u_{x_i},\varepsilon)\omega_{x_ix_i} - \omega_t + \sum_{i=1}^{n} \frac{\partial a_i(u_{x_i},\varepsilon)}{\partial u_{x_i}} u_{x_ix_i}\omega_{x_i} + \left(2\left(\lambda g_\varepsilon'(u) + \frac{\sum_{i=1}^{n} f_{x_i}u_{x_i}}{v}\right) - K\right)\omega \geq 0,$$

from the classical maximum principle, we conclude that $\omega \leq \max_{\Gamma_T}\omega$ where Γ_T is the parabolic boundary of the domain, and hence

$$v \leq \max\{\mathrm{e}^{KT}\max_{\Gamma_T} v, 1\}.$$

Thus, it is sufficient to estimate $\max v$ on the set $\partial\Omega\times(0,T]$. Recall that Ω satisfies an exterior sphere condition at a point $\mathbf{x^0}\in\partial\Omega$ so that there exists a ball $B_R(\mathbf{y^0})$ (with center at a point $\mathbf{y}^0\notin\Omega$ and radius R) such that $\mathbf{x^0} = \partial\Omega\cap\bar{B}_R(\mathbf{y^0})$. Consider the distance function $r(\mathbf{x}) \equiv |\mathbf{x}-\mathbf{y^0}| - R$. Let $w = w(r)$ be a smooth function such that $w'(r)\geq 1$. We have

$$\begin{aligned} Lw(r) &\equiv \sum_{i=1}^{n} a_i(w_{x_i},\varepsilon)w_{x_ix_i} + \lambda g(u) + f - w_t \\ &= w'\sum_{i=1}^{n} a_i(w_{x_i},\varepsilon)r_{x_ix_i} + \frac{w''}{w'^2}\sum_{i=1}^{n} a_i(w_{x_i},\varepsilon)r_{x_i}^2 w'^2 + \lambda g(u) + f \\ &\leq w'\frac{n-1}{R}\sum_{i=1}^{n} a_i(w_{x_i},\varepsilon) + \frac{w''}{w'^2}\sum_{i=1}^{n} a_i(w_{x_i},\varepsilon)w_{x_i}^2 + |\lambda|g(M) + f^*. \end{aligned}$$

Since $|\nabla w| = |w'| = w' \geq 1$, for sufficiently large μ we have

$$|\nabla w|\sum_{i=1}^{n} a_i(w_{x_i},\varepsilon) \leq \mu\sum_{i=1}^{n} a_i(w_{x_i},\varepsilon)w_{x_i}^2,$$

and

$$|\lambda|g(M) + f^* \leq \mu\sum_{i=1}^{n} a_i(w_{x_i},\varepsilon)w_{x_i}^2.$$

Hence

$$Lw(r) \leq \left(\frac{w''}{w'^2} + \nu\right)\sum_{i=1}^{n} a_i(w_{x_i},\varepsilon)w_{x_i}^2,$$

where $\nu = (1 + (n-1)R^{-1})\mu$.

Consider the cylinder

$$Q_{\mathbf{x_0},d} = \{(\mathbf{x},t) : 0 < r(\mathbf{x}) < d, 0 < t \leq T\}\cap Q_T,$$

where the quantity d will be defined below. Let $w(r)$ be a solution of the problem

$$w'' + \nu w'^2 = 0, \qquad w(0) = 0, \qquad w(d) = M$$

such that $w' \geq 1$. Obviously,

$$w(r) = \frac{1}{\nu}\ln\left(1 + \frac{\mathrm{e}^{M\nu-1}}{d}r\right).$$

In order to satisfy condition $w' \geq 1$ we select d small enough (such that $\mathrm{e}^{M\nu}(1-d\nu)\geq 1$).

In $Q_{\mathbf{x_0},d}$ we have

$$Lw \leq 0.$$

On the part of parabolic boundary of $Q_{\mathbf{x_0},d}$ belonging to $\partial\Omega\times(0,T]$ we have $u - w = -w < 0$, on the part of parabolic boundary of $Q_{\mathbf{x_0},d}$ which is lying inside of the domain Q_T $(t > 0)$ we have $u - w = u - M \leq 0$. For $t = 0$ we obtain the inequality $u_0 \leq w$ by selecting d small enough (and as a consequence w' will be big enough). So, due to the maximum principle we obtain

$$u(\mathbf{x},t) \leq w(r(\mathbf{x})) \quad \text{in } \bar{Q}_{\mathbf{x_0},d}.$$

Similarly we prove that

$$u(\mathbf{x},t) \geq -w(r(\mathbf{x})) \quad \text{in } \bar{Q}_{\mathbf{x_0},d},$$

from where we obtain the needed estimate

$$v(\mathbf{x_0},t) = |\nabla u(\mathbf{x_0},t)|^2 \leq w'^2(0)n.$$

A.3. Monotonicity of operators A_ε and A

In this section, we prove that the operators A_i and $A_{i\varepsilon}$ are monotone. This will imply that the operators A and A_ε are also monotone. As it follows from Proposition 1.1 in [1, Ch. 2], it is sufficient to prove that these operators are Gateau derivatives of convex functionals. Let us define functionals Φ_i and $\Psi_i : U_i(\Omega) \to \mathbb{R}$ as follows

$$\Phi_i(u) = \int_\Omega |u_{x_i}|^{p_i+2} \mathrm{d}\mathbf{x}, \qquad \Psi_i(u) = \int_\Omega F_i\big(|u_{x_i}|^2\big) \mathrm{d}\mathbf{x},$$

where

$$F_i(\eta) = \int_0^\eta (\xi^{\alpha/2} + \varepsilon)^{p_i/\alpha} \mathrm{d}\xi.$$

It is not difficult to see that $\Phi_i'(u) = A(u)$ and $\Psi_i'(u) = A_\varepsilon(u)$. Convexity of the functional Φ_i is obvious. Convexity of Ψ_i will be proved if we establish convexity of the function $F_i(|\xi|^2)$ with respect to ξ. However this fact is a direct consequence of the following properties of the function F_i:

$$F_i'(\eta) \ge 0 \quad \text{and} \quad F_i''(\eta) \ge 0 \quad \text{for } \eta \ge 0.$$

Similarly we can prove here that the operators $\mathcal{A}$ and $\mathcal{A}_\varepsilon$ are monotone. We have to prove that these operators are Gateau derivatives of convex functionals. Let us define functionals Φ and $\Psi : W_0^{1,p+2}(\Omega) \to \mathbb{R}$ as follows

$$\Phi(u) = \int_\Omega |\nabla u|^{p+2} \mathrm{d}\mathbf{x}, \qquad \Psi(u) = \int_\Omega F\big(|\nabla u|^2\big) \mathrm{d}\mathbf{x},$$

where

$$F(\eta) = \int_0^\eta (\xi^{\alpha/2} + \varepsilon)^{p/\alpha} \mathrm{d}\xi.$$

It is not difficult to see that $\Phi'(u) = \mathcal{A}(u)$ and $\Psi'(u) = \mathcal{A}_\varepsilon(u)$. Convexity of the functional Φ is obvious. Convexity of Ψ will be proved if we establish convexity of the function $\xi \mapsto F(|\xi|^2)$. But this fact is a direct consequence of the following properties of the function F:

$$F'(\eta) \ge 0 \quad \text{and} \quad F''(\eta) \ge 0 \quad \text{for } \eta \ge 0.$$

A.4. Proof of relation (2.31)

In order to prove (2.31), we consider the following equality

$$\int_\Omega u_\varepsilon(\mathbf{x}, T) v(\mathbf{x}, T) \mathrm{d}\mathbf{x} = \int_0^T \int_\Omega \big(u_{\varepsilon t} v + u_\varepsilon v_t\big) \mathrm{d}\mathbf{x}\mathrm{d}t + \int_\Omega u_{0\varepsilon}(\mathbf{x}) v(\mathbf{x}, 0) \mathrm{d}\mathbf{x}$$

with a smooth function v. Passing to the limit, we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(\mathbf{x}, T) v(\mathbf{x}, T) \mathrm{d}\mathbf{x} &= {}_{V^*(Q_T)}\langle u_t, v\rangle_{V(Q_T)} + {}_{V^*(Q_T)}\langle u, v_t\rangle_{V(Q_T)} + \int_\Omega u_0(\mathbf{x}) v(\mathbf{x}, 0) \mathrm{d}\mathbf{x} \\ &= \int_\Omega u(\mathbf{x}, T) v(\mathbf{x}, T) \mathrm{d}\mathbf{x}. \end{aligned}$$

Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, $u_\varepsilon(T, \mathbf{x}) \to u(T, \mathbf{x})$ weakly in $L^2(\Omega)$. Thus, due to the weak lower semicontinuity of the norm in $L^2(\Omega)$, we obtain (2.31).

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