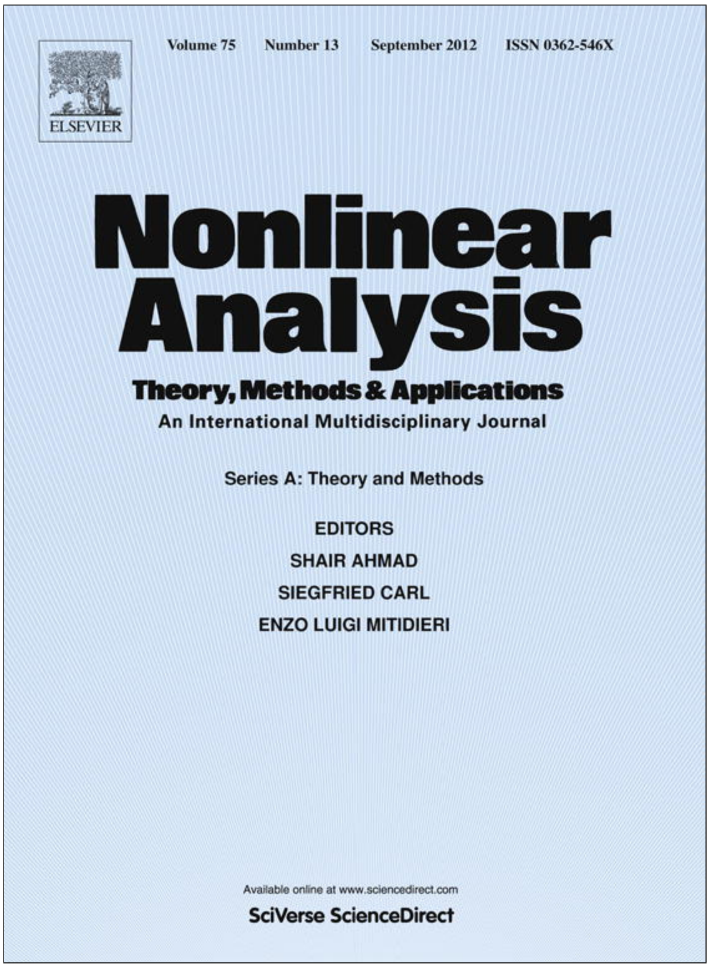
Volume 75 Number 13 September 2012 ISSN 0362-546X
ELSEVIER
Nonlinear Analysis
Theory, Methods & Applications
An International Multidisciplinary Journal
Series A: Theory and Methods
EDITORS
SHAIR AHMAD
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Available online at www.sciencedirect.com
SciVerse ScienceDirect

Nonlinear Analysis 75 (2012) 5119–5122

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

A condition guaranteeing the absence of the blow-up phenomenon for the generalized Burgers equation

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ARTICLE INFO

Article history:
Received 24 October 2011
Accepted 13 April 2012
Communicated by Enzo Mitidieri

MSC:
35K55
35B45

Keywords:
Boundary value problems
Generalized Burgers equation
Nonlinear source
A priori estimates
Global solvability

ABSTRACT

The initial boundary value problem for the generalized Burgers equation with nonlinear sources is considered. We formulate a condition guaranteeing the absence of the blow-up of a solution and discuss the optimality of this condition.

1. Introduction and main results

Consider the following equation

$$u_t + \mathbf{g}(t, \mathbf{x}, u) \cdot \nabla u = \varepsilon \Delta u + \lambda f(u) \quad \text{in } Q_T = (0, T) \times \Omega, \Omega \subset \mathbf{R}^n \tag{1.1}$$

coupled with initial and boundary conditions

$$u = \phi \quad \text{for } (t, \mathbf{x}) \in \Gamma_T = \partial \overline{Q}_T \setminus \{(T, \mathbf{x}) : \mathbf{x} \in \Omega\}. \tag{1.2}$$

Here $\varepsilon > 0$ and λ are constants, $\mathbf{g} = (g_1, \dots, g_n)$, $g_i = g_i(t, \mathbf{x}, u)$, $i = 1, \dots, n$. Assume that

$$g_1 = a(t, \mathbf{x})u^q + b(t, \mathbf{x}) \quad \text{and} \quad \min_{\overline{Q}_T} |a(t, x)| = a_0 > 0,$$

where $a, b \in C^0(\overline{Q}_T)$ and the positive constant q is such that $y^q \in \mathbf{R}$ for any $y \in \mathbf{R}$. If the solution is nonnegative the last assumption is unnecessary. Concerning the nonlinear source $f(u)$ we assume that

$$|f(\xi)| \le f(\eta) \quad \text{for all } \xi \text{ and } \eta \text{ such that } |\xi| \le \eta. \tag{1.3}$$

In particular, functions $f(u) = |u|^{p-1}u$, $p \ge 0$ (or $f(u) = u^p$ if defined) and $f(u) = e^u$ satisfy this condition.

Eq. (1.1) with $f(u) = u^p$ arises in many applications (see, for example, [1, Eq. (136)], [2,3] and the references in [4]).

There is an enormous number of papers devoted to this problem if $\mathbf{g} \equiv 0$ (see, for example [5] and the references therein) and it is well known that if $f(u)$ is superlinear the phenomenon of the solution blowing up may occur. Our goal is

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0362-546X/$ – see front matter
doi:10.1016/j.na.2012.04.027

to investigate the preventive influence of the convective term $(a\,u^q + b)\,u_{x_1}$. Concerning the preventive effect of the linear gradient term (i.e. $q = 0$ or $a \equiv 0$), see [6,7]. Due to the fact that the parameter ε is often small we will not take into account the helpful influence of the diffusing term $\varepsilon\Delta u$; concerning the preventive effect of linear and nonlinear diffusion, see [7,8].

Let us formulate our result. Denote

$$b_1 = \max_{Q_T} |b(t,x)|, \qquad m = \max_{\Gamma_T} |u|,$$

here Γ_T is the parabolic boundary of Q_T i.e. $\Gamma_T = \partial\overline{Q}_T \setminus \{(T,\mathbf{x}) : \mathbf{x} \in \Omega\}$. Without loss of generality suppose that the domain Ω is lying in the strip $-l_1 < x_1 < l_1$. We will prove the global (i.e. for arbitrary $T > 0$) classical solvability of problem (1.1), (1.2) under the following assumption:

$$\text{there exists a constant } M \geq m \quad \text{such that } 2l_1\,|\lambda| f(2M) + b_1 M \leq a_0\, M^{q+1}. \tag{1.4}$$

One can easily see that this condition does not depend on ε.

Theorem. *Assume that $\phi \in C^0(\Gamma_T)$, $g_i(t,\mathbf{x},u) \in C^1(Q_T \times (-2M,2M))$, $f(u) \in C^1(-2M,2M)$. If conditions (1.3) and (1.4) are satisfied then for an arbitrary $T > 0$, there exists a unique classical solution of problem (1.1), (1.2) such that*

$$|u(t,\mathbf{x})| \leq 2M. \tag{1.5}$$

Remark. In the one dimensional case in order to prove the existence it is sufficient to impose the following (less restrictive) smoothness assumptions on the coefficients: $g_i(t,\mathbf{x},u) \in C^\beta(Q_T \times (-2M,2M))$, $f(u) \in C^\beta(-2M,2M)$ for some $\beta \in (0,1)$.

The next example demonstrates the optimality of condition (1.4). For simplicity we restrict ourselves with the one dimensional case which can be easily extended to the multidimensional one.

Example. Consider the most typical case (we omit subscript 1 in l_1 and x_1): $a(t,x) \equiv \alpha \neq 0$, $b(t,x) \equiv 0$, $f(u) = u^p$:

$$u_t + \alpha\, u^q\, u_x = \varepsilon\, u_{xx} + \lambda\, u^p \quad \text{in } Q_T = (0,T) \times (-l,l). \tag{1.6}$$

We assume here that u^p is defined, otherwise we take $|u|^{p-1}u$. It is known (see [9–13]) that there exists a global solution of problem (1.6), (1.2) without smallness restrictions on initial data for $p \leq q+1$, $\phi \geq 0$ and zero boundary conditions, moreover if $p > q+1$ then a finite time blow up occurs if the initial data is sufficiently large. Let us apply our Theorem to Eq. (1.6). Condition (1.4) takes the form

$$\exists M \geq m \quad \text{such that } 2^{p+1} l\,|\lambda| M^p \leq |\alpha|\, M^{q+1}. \tag{1.7}$$

If $p < q+1$ then condition (1.7) is always fulfilled with

$$M = \max\left\{ m, \left(\frac{2^{p+1}|\lambda| l}{|\alpha|}\right)^{\frac{1}{q+1-p}} \right\}$$

and as a consequence for arbitrary data there exists a global solution satisfying the estimate $|u| \leq 2M$.

Suppose that $p > q+1$. Condition (1.7) becomes a smallness restriction. In fact, rewrite (1.7) in the form

$$\exists M \geq m \quad \text{such that } M^{p-q-1} \leq \frac{|\alpha|}{2^{p+1} l\,|\lambda|}.$$

Obviously if

$$m \leq \left(\frac{|\alpha|}{2^{p+1} l\,|\lambda|}\right)^{\frac{1}{p-q-1}}$$

then (1.7) is fulfilled with $M = m$ and there exists a global solution such that $|u| \leq 2\,m$.

Consider the critical case $p = q+1$:

$$u_t + \alpha\, u^q\, u_x = \varepsilon\, u_{xx} + \lambda\, u^{q+1}. \tag{1.8}$$

For the function $v = u\, e^{-\mu x}$ with $\mu = \lambda/\alpha$ we have

$$\begin{aligned} &v_t + (\alpha\, v^q e^{q\mu x} - 2\mu\,\varepsilon) v_x = \varepsilon\, v_{xx} + \varepsilon\mu^2\, v, \\ &v = \phi e^{-\mu x} \quad \text{for } (t,x) \in \Gamma_T = \{t = 0, |x| \leq l\} \cup \{0 < t \leq T, x = \pm l\}. \end{aligned} \tag{1.9}$$

Condition (1.4) for Eq. (1.9) takes the form

$$\exists M \geq m\, e^{|\mu| l} \quad \text{such that } 4l\varepsilon\mu^2 M + 2\varepsilon|\mu| M \leq |\alpha| e^{-q|\mu| l} M^{q+1}.$$

Obviously this condition is fulfilled with

$$M = \max\left\{ me^{|\mu|l}, \left(\frac{4\varepsilon\, l\mu^2 + 2\varepsilon|\mu|}{|\alpha|}\right)^{1/q} e^{|\mu|l} \right\}, \quad \mu = \frac{\lambda}{\alpha}$$

and the estimate

$$|v(t,x)| \le 2M$$

is obtained. As a consequence the Theorem guarantees the global solvability of problem (1.8), (1.2) and the solution satisfies the estimate

$$|u(t,x)| \le 2Me^{\mu x}.$$

2. Proof of the Theorem

Introduce the following cut-off function

$$\bar{f}_M(z) = \begin{cases} f(2M), & z > 2M, \\ f(z), & |z| \le 2M, \\ f(-2M), & z < -2M. \end{cases}$$

Obviously, due to (1.3),

$$-f(2M) \le \bar{f}_M(u) \le f(2M). \tag{2.1}$$

Consider the auxiliary equation

$$u_t + \mathbf{g}\cdot\nabla u = \varepsilon\,\Delta u + \lambda \bar{f}_M(u) \quad \text{in } Q_T = (0,T)\times\Omega. \tag{2.2}$$

The classical solvability of problem (2.2), (1.2) follows, for example, from [14]. In the one dimensional case (see Remark in the Introduction) it is sufficient to require only Hölder continuity of the coefficients; see, for example, [15].

Our goal is to obtain the a priori estimate $|u(t,\mathbf{x})| \le 2M$ for a solution of problem (2.2), (1.2) and by this to show that Eqs. (2.2) and (1.1) coincide.

We start from the case $a(t,x) \ge a_0 > 0$.

Put

$$h(x_1) \equiv \frac{M}{2l_1}(l_1 + x_1) + M, \qquad w \equiv u - h, \qquad L \equiv \frac{\partial}{\partial t} - \varepsilon\Delta.$$

Obviously

$$Lw = -\mathbf{g}\cdot\nabla u + \lambda\bar{f}_M(u).$$

Suppose that at a point $N \in \overline{Q}_T \setminus \Gamma_T$ the function w attains its positive maximum. Then at this point we have

$$w > 0, \quad \nabla w = 0 \quad \text{so } u > h \ge M \ (u^q > M^q) \quad \text{and} \quad u_{x_1} = h' = M/2l_1, \qquad u_{x_i} = 0, \quad i = 2,\ldots,n$$

thus

$$Lw\Big|_N = -(a\,u^q + b)\frac{M}{2l_1} + \lambda\bar{f}_M(u)\Big|_N < -a_0\frac{M^{q+1}}{2l_1} + b_1\frac{M}{2l_1} + |\lambda| f(2M) \le 0$$

which is impossible (the last inequality is due to (1.4)). Taking into account that $w \le 0$ on Γ_T we conclude that $w \le 0$ in Q_T, hence

$$u \le h \le 2M \quad \text{in } Q_T.$$

Let us obtain now the estimate from the below.

(i) Assume that $(-1)^q = -1$. Consider the function

$$\omega \equiv u + h(-x_1) = u + \frac{M}{2l_1}(l_1 - x) + M.$$

Suppose that at a point $N_1 \in \overline{Q}_T \setminus \Gamma_T$ function ω attains its negative minimum. Then at this point we have

$$\omega < 0, \quad \nabla\omega = 0 \quad \text{i.e. } u < -M \quad (\text{and} \quad u^q < -M^q), \qquad u_{x_1} = \frac{M}{2l_1}, \qquad u_{x_i} = 0,\ i = 2,\ldots,n.$$

Thus (due to (2.1))

$$L\,\omega\Big|_{N_1} = -(a\,u^q + b)\frac{M}{2l_1} + \lambda \bar{f}_M(u)\Big|_{N_1} > a_0\frac{M^{q+1}}{2l_1} - b_1\frac{M}{2l_1} - |\lambda|\bar{f}_M(u)\Big|_{N_1}$$
$$\geq a_0\frac{M^{q+1}}{2l_1} - b_1\frac{M}{2l_1} - |\lambda| f(2M) \geq 0$$

which is impossible. Taking into account that $\omega \geq 0$ on Γ_T we conclude that $\omega \geq 0$ in Q_T and

$$u \geq -\frac{M}{2l_1}(l_1 - x_1) - M \geq -2M.$$

(ii) Now consider the case when $(-1)^q = 1$. Instead of ω we take the function

$$\tilde{\omega} \equiv u + h(x_1).$$

Suppose that at a point $N_2 \in \bar{Q}_T \setminus \Gamma_T$ function $\tilde{\omega}$ attains its negative minimum. Then at this point we have

$$\tilde{\omega} < 0, \quad \tilde{\omega}_x = 0 \quad \text{i.e.}\ u < -M \quad (\text{and} \quad u^q > M^q), \qquad u_{x_1} = -\frac{M}{2l_1}, \qquad u_{x_i} = 0, \quad i = 2, \ldots, n.$$

Thus (due to (2.1))

$$L\tilde{\omega}\Big|_{N_2} = (a\,u^q + b)\frac{M}{2l_1} + \lambda \bar{f}_M(u)\Big|_{N_2} > a_0\frac{M^{q+1}}{2l_1} - b_1\frac{M}{2l_1} - |\lambda| f(2M) \geq 0.$$

Similarly to the previous case we conclude that $\tilde{\omega} \geq 0$ in Q_T or

$$u \geq -h(x_1) = -\frac{M}{2l_1}(l_1 + x_1) - M \geq -2M.$$

Finally

$$|u(t, \mathbf{x})| \leq 2\,M.$$

Let us turn to the case $a(t, x) \leq -a_0 < 0$. Here in order to obtain the estimate $u \leq 2M$ instead of $w = u - h(x_1)$ we take

$$w \equiv u - h(-x_1)$$

and repeat the same procedure as in the previous case.

For the establishment of the estimate from the below $u \geq -2M$ for q such that $(-1)^q = -1$ we take

$$\omega \equiv u + h(x_1)$$

instead of $\omega = u + h(-x_1)$ and for q satisfying $(-1)^q = 1$ we consider

$$\tilde{\omega} = u + h(-x_1)$$

instead of $\tilde{\omega} = u + h(x_1)$ and then repeat the same approach.

The Theorem is proved.

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