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# THE DIRICHLET PROBLEM FOR SECOND ORDER SEMILINEAR ELLIPTIC AND PARABOLIC EQUATIONS

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*Abstract.* In the present paper the Dirichlet problem for semilinear elliptic and parabolic equations in general form is considered. New condition guaranteeing the global solvability of this problem for a wide class of superlinear sources, including $e^u$ and $|u|^{p-1}u$, $p>1$, is formulated. For sublinear case (for example $\ln(1+|u|)$ or $|u|^{p-1}u$, $p<1$) this condition is automatically fulfilled. Our approach gives new a priori estimate of the solution for superlinear, sublinear and linear case as well.

## §0. **Introduction and Main Results**

### I. Elliptic case.

Consider semilinear strictly elliptic equation

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)g(u) = f(x) \ \text{ in } \ \Omega, \tag{0.1}$$

coupled with boundary condition

$$u(x)\Big|_{\partial\Omega} = \phi(s). \tag{0.2}$$

Here $\Omega$ is a bounded domain in $\mathbf{R}^n$. Concerning the nonlinear term $g$ we assume that

$$|g(\xi)| \leqslant g(\eta) \ \text{ for all } \ \xi \ \text{ and } \ \eta \ \text{ such that } \ |\xi| \leqslant \eta. \tag{0.3}$$

For example, functions $g(u) = |u|^q$ with $q \in (0,1)$, $g(u) = \ln(1+|u|)$, $g(u) = u^p$ for $p \geqslant 1$ integer, or $g(u) = |u|^{p-1}u$ for arbitrary $p \geqslant 0$ as well as $g(u) = e^u$ satisfy condition (0.3).

Recall that (0.1) is strictly elliptic if

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \varepsilon|\xi|^2 \ \text{ for any } \ x \in \bar{\Omega}, \ \ \xi \in \mathbf{R}^n, \tag{0.4$_1$}$$

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where $\varepsilon$ is a positive constant. We are interested in the existence of classical solution of problem (0.1), (0.2), i.e. the solution from $C^{2+\alpha}(\Omega)\cap C^0(\bar{\Omega})$ for some $\alpha\in(0,1)$. It is well known that the classical solvability of problem (0.1), (0.2) with smooth coefficients follows from the a priori estimate of $\max|u(\mathbf{x})|$. There are a lot of different sufficient conditions guaranteeing the needed a priori estimate (see [2], [4]). In the present paper we develop a new approach in order to obtain a new condition in which we take into account the influence of all coefficients of the equation. This approach gives new results for sublinear, superlinear as well as linear case.

Without loss of generality suppose that

$$\Omega\subset\{(x_1,\dots,x_n)\in\mathbf{R}^n : |x_i|\leqslant d_i, i=1,\dots,n\}$$

where $d_i$ are arbitrarily given positive numbers. Define the quantities $\alpha_i$ by the following:

$$0<\alpha_i<\min_{\bar{\Omega}} a_{ii},\ \text{ for }\ i=1,\dots,n. \tag{0.4$_2$}$$

Consider the coefficients $b_i$. Assume that $b_i$ are strictly positive functions for $i=1,\dots,m$, strictly negative functions for $i=m+1,\dots,k$ and $b_i\equiv 0$ for $i=k+1,\dots,l$, where $0\leqslant m\leqslant k\leqslant l\leqslant n$. Define $\beta_i$ by the following:

$$\beta_i=\min_{\bar{\Omega}} b_i>0\ \text{ for }\ i=1,\dots,m, \tag{0.5$_1$}$$

$$-\beta_i=\max_{\bar{\Omega}} b_i<0\ \text{ for }\ i=m+1,\dots,k, \tag{0.5$_2$}$$

$$\beta_i=b_i\equiv 0,\ \text{ for }\ i=k+1,\dots,l, \tag{0.5$_3$}$$

$$\beta_i=\max_{\bar{\Omega}}|b_i|>0\ \text{ for }\ i=l+1,\dots,n. \tag{0.5$_4$}$$

Here $m=0$ means that there are no strictly positive $b_i$, as a consequence $(0.5_1)$ is absent and in $(0.5_2)$ $i=1,\dots,k$ ($(0.5_3)$, $(0.5_4)$ remain the same);
$k=m$ means that there are no strictly negative $b_i$, i. e. $(0.5_2)$ is absent, and in $(0.5_3)$ $i=m+1,\dots,l$ ($(0.5_1)$, $(0.5_4)$ remain the same);
$l=k$ means that there are no identically equal to zero $b_i$, in this case $(0.5_3)$ is absent, in $(0.5_4)$ $i=k+1,\dots,n$ ($(0.5_1)$, $(0.5_2)$ remain the same);
finally, $n=l$ means that there are no changing sign coefficients $b_i$.

Put $\lambda=\max\{\lambda^-,\lambda^+\}$ where

$$\lambda^+=\begin{cases}\max_{\bar{\Omega}} c(x), & \text{if }\ \max_{\bar{\Omega}} c(x)\geqslant 0\\ 0, & \text{if }\ \max_{\bar{\Omega}} c(x)<0,\end{cases}$$

$$\lambda^-=\begin{cases}0, & \text{if }\ \max_{\bar{\Omega}} c(x)\geqslant 0\\ -\max_{\bar{\Omega}} c(x), & \text{if }\ \max_{\bar{\Omega}} c(x)<0.\end{cases}$$

Obviously

$$-\lambda^-\leqslant c(x)\leqslant\lambda^+,\ \ |c(x)|\leqslant\lambda. \tag{0.6}$$

Before we pass to the strict formulation of the result let us briefly describe it. The key role plays the constant $K$ which depends on $\alpha_i,\beta_i$ and $d_i$. This constant we find in the explicit form (see (1.1) in Section 1), here we only mention that $K \to +\infty$ when at least for one value of $i$ (say $i_0$) $\alpha_{i_0} \to +\infty$ or $\beta_{i_0} \to +\infty$ or $d_{i_0} \to 0$. Consider the relation $KM - \lambda g(M + \max_{\partial\Omega}|\phi|)$. For nondecreasing $g(u)$ we take $\lambda^+$ instead of $\lambda$. If there exists a constant $M$ such that

$$KM - \lambda g(M + \max_{\partial\Omega}|\phi|) \geqslant \max_{\Omega}|f|$$

then (under smoothness assumptions) we prove the existence of a classical solution of problem (0.1), (0.2) such that $|u| \leqslant M$. Obviously in sublinear case such $M$ always exists. In superlinear case we need some additional restrictions. Let us give two examples (for details see Examples 2 and 4 from Section 3). Consider the case $g(u) = |u|^{p-1}u$, $p > 1$ (or $g(u) = u^p$ if defined) and $\phi \equiv 0$. If

$$\max_{\Omega}|f(x)| \leqslant \left(\frac{K}{p\lambda}\right)^{\frac{1}{p-1}} K\frac{p-1}{p}$$

then there exists at least one solution for which the estimate

$$|u(x)| \leqslant \left(\frac{K}{p\lambda}\right)^{\frac{1}{p-1}}$$

takes place. Let us take $g(u) = e^u$ and $\phi \equiv 0$. If $c(x) \leqslant 0$ (i.e. $\lambda^+ = 0$) then for arbitrary (smooth) $f(x)$ we prove the existence of at least one solution and for this solution the estimate

$$|u(x)| \leqslant \frac{\max_{\Omega}|f(x)|}{K}$$

holds. If $K > \lambda^+ > 0$ then we prove the existence of at least one solution under the assumption

$$\max_{\Omega}|f(x)| \leqslant K\ln\frac{K}{\lambda^+} - K + \lambda^+,$$

moreover this solution satisfies the estimate

$$|u(x)| \leqslant \ln\frac{K}{\lambda^+}.$$

In particular for the equation

$$\Delta u + e^u = 0$$

with zero boundary conditions we guarantee (for details see Section 3 Example 3) the existence of at least one solution if

$$d_1 \leqslant \sqrt{\frac{2}{e}},$$

this solution satisfies the inequality

$$|u(x)| \leqslant 1.$$

Let us now formulate the result. Denote

$$f_0=\max_{\bar\Omega}|f(x)|,\quad \phi_0=\max_{\partial\Omega}|\phi(s)|.$$

THEOREM I. **1.** *Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ satisfying an exterior sphere condition at each point of the boundary. Assume that $a_{ij}(x)$, $b_i(x)$, $c(x)$, $f(x)\in C^{\alpha}(\bar\Omega)$, $g(z)\in C^{\alpha}_{loc}$ and $\phi(s)\in C^0(\partial\Omega)$ for some $\alpha\in(0,1)$. Suppose that conditions $(0.3)$, $(0.4_1)$ hold and there exists a positive constant $M$ such that*

$$KM-\lambda g(M+\phi_0)\geqslant f_0, \tag{0.7}$$

*then there exists a classical solution of problem (0.1), (0.2) such that*

$$\max_{\bar\Omega}|u(x)|\leqslant M+\phi_0.$$

*If function $g(u)$ is nondecreasing, then condition (0.7) can be substituted by the following one*

$$KM-\lambda^{+}g(M+\phi_0)\geqslant f_0. \tag{0.8}$$

**2.** *If $c(x)<0$ and $g(u)$ is a strictly increasing function then the solution is unique.*

COROLLARY. *Assume that conditions of the Theorem concerning the smoothness of $a_{ij}$, $b_i$, $c$, $g$, $\partial\Omega$, $\phi$ are fulfilled. Suppose that conditions $(0.3)$, $(0.4_1)$ hold. If $g$ is nondecreasing function and $c\leqslant 0$, then for any $f(x)\in C^{\alpha}(\bar\Omega)$ there exists a classical solution of problem (0.1), (0.2) and*

$$\max_{\Omega}|u(x)|\leqslant\frac{f_0}{K}.$$

This Corollary immediately follows from (0.8), since $\lambda^{+}=0$ if $c(x)\leqslant 0$.

**II. Parabolic case.**

This approach can be easily extended to the parabolic case. Consider the following equation

$$\sum_{i,j=1}^{n}\mathrm{a}_{ij}(t,x)u_{x_ix_j}+\sum_{i=1}^{n}\mathrm{b}_i(t,x)u_{x_i}+\mathrm{c}(t,x)g(u)-u_t=\mathrm{f}(t,x)\ \text{ in }\ Q_T=(0,T)\times\Omega, \tag{0.9}$$

coupled with conditions

$$u(0,x)=u_0(x)\ \text{ for }\ x\in\Omega,\quad u(t,x)\Big|_{S_T}=\phi(t,s), \tag{0.10}$$

here $S_T=(0,T)\times\partial\Omega$. Similarly to the elliptic case we suppose that

$$\sum_{i,j=1}^{n}\mathrm{a}_{ij}(t,x)\xi_i\xi_j\geqslant\varepsilon|\xi|^2\ \text{ for any }\ (t,x)\in\bar Q_T,\quad \xi\in\mathbf{R}^n, \tag{0.11$_1$}$$

$$0 < \alpha_i < \min_{\bar{Q}_T} \mathrm{a}_{ii} \ \text{ for } \ i = 1,\dots,n. \tag{0.11$_2$}$$

Assume that $\mathrm{b}_i$ are strictly positive functions for $i = 1,\dots,m$, strictly negative functions for $i = m+1,\dots,k$ and $\mathrm{b}_i \equiv 0$ for $i = k+1,\dots,l$,

$$\beta_i = \min_{\bar{Q}_T} \mathrm{b}_i > 0 \ \text{ for } \ i = 1,\dots,m, \tag{0.12$_1$}$$

$$-\beta_i = \max_{\bar{Q}_T} \mathrm{b}_i < 0 \ \text{ for } \ i = m+1,\dots,k, \tag{0.12$_2$}$$

$$-\beta_i = \mathrm{b}_i \equiv 0 \ \text{ for } \ i = k+1,\dots,l. \tag{0.12$_3$}$$

$$-\beta_i = \max_{\bar{Q}_T} |\mathrm{b}_i| > 0 \ \text{ for } \ i = l+1,\dots,n. \tag{0.12$_4$}$$

Concerning $m$, $k$ and $l$ see the elliptic case.

Put $\lambda = \max\{\lambda^-, \lambda^+\}$ where

$$\lambda^+ = \begin{cases} \max_{\bar{Q}_T} \mathrm{c}(t,x), & \text{if } \ \max_{\bar{Q}_T} \mathrm{c}(t,x) \geqslant 0 \\ 0, & \text{if } \ \max_{\bar{Q}_T} \mathrm{c}(t,x) < 0, \end{cases}$$

$$\lambda^- = \begin{cases} 0, & \text{if } \ \max_{\bar{Q}_T} \mathrm{c}(t,x) \geqslant 0 \\ -\max_{\bar{Q}_T} \mathrm{c}(t,x), & \text{if } \ \max_{\bar{Q}_T} \mathrm{c}(t,x) < 0. \end{cases}$$

Similarly to the elliptic case, the global solvability of problem (0.9), (0.10) follows from the a priori estimate of $\max |u(t,x)|$ (see [3]). It is well known that the phenomenon of blowing up of the solution may occur (see [6]), i.e. $|u(t,x^*)| \to +\infty$ when $t \to t^*$ at least for one $x \in \Omega$. The preventive effect of the linear diffusion and of the convection was investigated in [8] (see also the references therein). The Theorem below extends these results to the equation in general form, moreover here we take into account the effect of the diffusion and of the convection in all directions simultaneously. Concerning the preventive effect of the nonlinear diffusion see [7] and [9].

Let us formulate the result. Denote

$$\mathrm{f}_0 = \max_{\bar{Q}_T} |\mathrm{f}(t,x)|, \ \ \phi_0 = \max_{S_T} |\phi(t,s)|, \ \ m = \max_{\Omega} |u_0(x)|.$$

THEOREM II. **1.** *Let $\Omega$ and $g(z)$ are as in Theorem I. Assume that $a_{ij}(t,x)$, $b_i(t,\mathbf{x})$, $c(t,x)$, $f(t,x) \in C^{\alpha/2,\alpha}_{t,x}(\bar{Q}_T)$, $g(z) \in C^{\alpha}_{loc}$ and $\phi(t,s) \in C^0(\partial\Omega \times (0,T))$, $u_0(x) \in C^0(\bar{\Omega})$ for some $\alpha \in (0,1)$. Suppose that conditions $(0.3)$, $(0.11_1)$ hold and there exists a positive constant $M$ such that*

$$KM - \lambda g(M + m + \phi_0) \geqslant f_0, \tag{0.13}$$

*then there exists a classical solution of problem (0.9), (0.10) such that*

$$\max_{\bar{Q}_T} |u(t,x)| \leqslant M + m + \phi_0.$$

*If function $g(u)$ is nondecreasing, then condition (0.13) can be substituted by the following one*

$$KM-\lambda^{+}g(M+m+\phi_0)\geqslant f_0. \tag{0.14}$$

**2.** *If $g(u)$ is Lipschitz continuous function then the solution is unique.*

By classical solution we mean function from $C^{1+\alpha/2,2+\alpha}_{t,x}(Q_T)\cap C^0(\bar{Q}_T)$.

REMARK. The estimate of the solution is independent of $T$.
Consider the following equation

$$u_t+\mathbf{b}\cdot\nabla u=\alpha\Delta u+|u|^{p-1}u \ \text{ in } \ (0,T)\times\{|x|<d_0\} \tag{0.15}$$

where $\alpha>0$, $\mathbf{b}=(b,\dots,b)$ and the constant $b\neq 0$ and $p>1$, with initial boundary conditions

$$u(0,x)=u_0(x) \ \text{ for } \ x\in\Omega, \ \ u=0 \ \text{ on } \ S_T. \tag{0.16}$$

From Theorem II it follows (for details see Example 7 from Section 3) that if

$$\frac{\alpha+b}{2d_0(1+d_0)}\geqslant\frac{p^p}{(p-1)^{p-1}}m^{p-1},$$

then there exists a global (i.e. for all $t>0$) solution of problem (0.15), (0.16) and

$$\max_{Q_T}|u(t,x)|\leqslant\frac{p}{p-1}m.$$

Actually this means that for the given initial data and domain the solution of problem (0.15), (0.16) can not blow up if convection or diffusion or the sum of convection and diffusion are big enough.

In the first section we define in the explicit form the constant $K$, in the second we prove Theorems I and II and in the last section we give several examples demonstrating the results of the paper for linear, sublinear and superlinear equations.

## §1. **Definition of the constant** $K(\alpha_i,\beta_i,d_i)$

Suppose that $k\geqslant 1$, we put

$$K=\begin{cases}\max_{r_i}\frac{\sum_{i=1}^n r_i(\alpha_i+\beta_i)}{\sum_{i=1}^n r_i 2d_i(1+d_i)} \ \text{ if } \ n=l=k,\\ \max_{r_i}\frac{\sum_{i=1}^k r_i(\alpha_i+\beta_i)+\sum_{i=k+1}^n r_i\alpha_i}{\sum_{i=1}^k r_i 2d_i(1+d_i)+\sum_{i=k+1}^n r_i d_i^2/2} \ \text{ if } \ n=l>k,\\ \max_{r_i}\frac{\sum_{i=1}^k r_i(\alpha_i+\beta_i)+\sum_{i=k+1}^n r_i\alpha_i}{\sum_{i=1}^k r_i 2d_i(1+d_i)+\sum_{i=k+1}^n \frac{r_i}{\gamma_i}(e^{2\gamma_i d_i}-2\gamma_i d_i-1)} \ \text{ if } \ n>l=k,\\ \max_{r_i}\frac{\sum_{i=1}^k r_i(\alpha_i+\beta_i)+\sum_{i=k+1}^l r_i\alpha_i+\sum_{i=l+1}^n r_i\alpha_i}{\sum_{i=1}^k r_i 2d_i(1+d_i)+\sum_{i=k+1}^l r_i d_i^2/2+\sum_{i=l+1}^n \frac{r_i}{\gamma_i}(e^{2\gamma_i d_i}-2\gamma_i d_i-1)} \ \text{ if } \ n>l>k,\end{cases} \tag{1.1$_1$}$$

for $k=0$ we put

$$K=\begin{cases} \max_{r_i} \frac{\sum_{i=1}^{n} r_i\alpha_i}{\sum_{i=k+1}^{n} r_i d_i^2/2} & \text{if } n=l>0, \\ \max_{r_i} \frac{\sum_{i=1}^{n} r_i\alpha_i}{\sum_{i=1}^{n} \frac{r_i}{\gamma_i}(e^{2\gamma_i d_i}-2\gamma_i d_i-1)} & \text{if } n>l=0, \\ \max_{r_i} \frac{\sum_{i=1}^{l} r_i\alpha_i+\sum_{i=l+1}^{n} r_i\alpha_i}{\sum_{i=1}^{l} r_i d_i^2/2+\sum_{i=l+1}^{n} \frac{r_i}{\gamma_i}(e^{2\gamma_i d_i}-2\gamma_i d_i-1)} & \text{if } n>l>0, \end{cases} \tag{1.1$_2$}$$

here $\gamma_i=\frac{\beta_i}{\alpha_i}$ for $i=l+1,\dots,n$ and $r_i=0$ or $1$ for $i=1,\dots,n$. Denote

$$d=\sum_{i=1}^{k} r_i 2d_i(1+d_i)+\sum_{i=k+1}^{l} r_i d_i^2/2+\sum_{i=l+1}^{n} \frac{r_i}{\gamma_i}(e^{2\gamma_i d_i}-2\gamma_i d_i-1), \tag{1.2}$$

where $r_i$ ($0$ or $1$) are defined from (1.1). If the definition of $K$ gives several sets of numbers $\{r_1,\dots,r_n\}$ for which the maximum in (1.1) is obtained, then we select the set which gives minimum for $d$. Consider several examples in order to explain (1.1), (1.2).

$i$) Suppose that $d_i=d_0$, $\alpha_i=\alpha_0$, $\beta_i=\beta_0$ for $i=1,\dots,n$, i. e. $n=l=k\geqslant 1$, then

$$K=\max_{r_i}\frac{\sum_{i=1}^{n} r_i(\alpha_0+\beta_0)}{\sum_{i=1}^{n} r_i 2d_0(1+d_0)}=\frac{\alpha_0+\beta_0}{2d_0(1+d_0)}$$

and

$$d=2d_0(1+d_0).$$

$ii$) Consider the equation:

$$\sum_{i,j=1}^{3} a_{ij}(x)u_{x_i x_j}+c(x)g(u)=f(x).$$

In this case $k=0$, $n=l$, i.e.

$$K=\max_{r_i}\frac{2\sum_{i=1}^{3} r_i\alpha_i}{\sum_{i=1}^{3} r_i d_i^2}=$$

$$2\max\left\{\frac{\alpha_1}{d_1^2},\frac{\alpha_2}{d_2^2},\frac{\alpha_3}{d_3^2},\frac{\alpha_1+\alpha_2}{d_1^2+d_2^2},\frac{\alpha_1+\alpha_3}{d_1^2+d_3^2},\frac{\alpha_2+\alpha_3}{d_2^2+d_2^2},\frac{\alpha_1+\alpha_2+\alpha_3}{d_1^2+d_2^2+d_3^2}\right\}.$$

For example,

$$K=2\frac{\alpha_1+\alpha_2+\alpha_3}{d_1^2+d_2^2+d_3^2}$$

means that maximum is obtained when $r_1=r_2=r_3=1$ and we take

$$d=d_1^2+d_2^2+d_3^2.$$

If

$$2\frac{\alpha_1+\alpha_2+\alpha_3}{d_1^2+d_2^2+d_3^2}=2\frac{\alpha_1+\alpha_2}{d_1^2+d_2^2}=2\frac{\alpha_2}{d_2^2},$$

i.e. maximum is obtained when $r_1 = r_2 = r_3 = 1$, as well as when $r_1 = r_2 = 1$, $r_3 = 0$ or $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, then we take

$$K = 2\frac{\alpha_2}{d_2^2}, \;\; d = d_2^2,$$

i.e. $r_1 = 0$, $r_2 = 1$, $r_3 = 0$.

Suppose that $d_1 = min\{d_1, d_2, d_3\}$ and $\alpha_1 = \alpha_2 = \alpha_3$. In this case we have

$$K = \max_{r_i} \frac{2\sum_{i=1}^3 r_i\alpha_i}{\sum_{i=1}^3 r_i d_i^2} = \frac{2\alpha_1}{d_1^2}, \;\; d = d_1^2.$$

*iii*) Consider the equation:

$$\sum_{i,j=1}^3 a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^3 b_i(x)u_{x_i} + c(x)g(u) = f(x) \;\text{ in }\; \Omega \subset \mathbf{R}^3.$$

Assume that $b_1 \geqslant \beta_1 > 0$, $b_2 \leqslant -\beta_2 < 0$, $\max|b_3| > 0$. In this case $n > l = k$ ($n = 3$, $k = 2$) i. e.

$$K = \max_{r_i} \frac{r_1(\alpha_1+\beta_1) + r_2(\alpha_2+\beta_2) + r_3\alpha_3}{r_1 2d_1(1+d_1) + r_2 2d_2(1+d_2) + \frac{r_3}{\gamma_3}(e^{2\gamma_3 d_3} - 2\gamma_3 d_3 - 1)}$$

$$= \max\Big\{\frac{\alpha_1+\beta_1}{2d_1(1+d_1)}, \frac{\alpha_2+\beta_2}{2d_2(1+d_2)}, \frac{\alpha_3\gamma_3}{(e^{2\gamma_3 d_3} - 2\gamma_3 d_3 - 1)},$$

$$\frac{\alpha_1+\alpha_2+\beta_1+\beta_2}{2d_1(1+d_1)+2d_2(1+d_2)}, \frac{\alpha_1+\beta_1+\alpha_3}{2d_1(1+d_1)+\frac{e^{2\gamma_3 d_3}-2\gamma_3 d_3-1}{\gamma_3}}, \frac{\alpha_2+\beta_2+\alpha_3}{2d_2(1+d_2)+\frac{e^{2\gamma_3 d_3}-2\gamma_3 d_3-1}{\gamma_3}},$$

$$\frac{\alpha_1+\alpha_2+\alpha_3+\beta_1+\beta_2}{2d_1(1+d_1)+2d_2(1+d_2)+\frac{e^{2\gamma_3 d_3}-2\gamma_3 d_3-1}{\gamma_3}}\Big\}.$$

For example,

$$K = \frac{\alpha_1+\beta_1}{2d_1(1+d_1)}$$

means that maximum is obtained when $r_1 = 1$, $r_2 = r_3 = 0$ and we put

$$d = 2d_1(1+d_1).$$

Suppose that

$$\frac{\alpha_1+\beta_1}{2d_1(1+d_1)} = \frac{\alpha_2+\beta_2}{2d_2(1+d_2)},$$

i.e. maximum is obtained when $r_1 = 1$, $r_2 = r_3 = 0$ and $r_1 = 0$, $r_2 = 1$, $r_3 = 0$. If $d_1 > d_2$ we take

$$K = \frac{\alpha_2+\beta_2}{2d_2(1+d_2)}, \;\; d = 2d_2(1+d_2),$$

otherwise

$$K = \frac{\alpha_1+\beta_1}{2d_1(1+d_1)}, \;\; d = 2d_1(1+d_1).$$

## §2. Proof of Theorems I and II

Consider the auxiliary equation

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)g_M(u) = f(x) \text{ in } \Omega \subset \mathbf{R}^n, \tag{2.1}$$

where

$$g_M(u) = \begin{cases} g(u), & \text{for } |u| \leqslant M + \phi_0 \\ g(M+\phi_0), & \text{for } u \geqslant M + \phi_0 \\ g(-M-\phi_0), & \text{for } u \leqslant -M - \phi_0. \end{cases} \tag{2.2}$$

For $|u| \leqslant M + \phi_0$ equations (2.1) and (0.1) coincide. Our goal is to establish the a priori estimate $|u| \leqslant M + \phi_0$ for the solution of problem (2.1), (0.2) and by this to reduce the solvability of problem (0.1), (0.2) to the solvability of problem (2.1), (0.2). If $\partial\Omega \in C^{2+\alpha}$ and $\phi \in C^{2+\alpha}(\partial\Omega)$ then the estimate $|u| \leqslant M + \phi_0$ of the solution of problem (2.1), (0.2) implies the estimate of the solution of this problem in $C^{2+\alpha}(\bar{\Omega})$ (due to the Schauder estimates) depending only on the data of the problem. Hence the classical solvability of problem (2.1), (0.2) follows from the Leray-Schauder theorem (see [3, Chapter 11]). For the continuous boundary value and the domain satisfying only the exterior sphere condition the basic procedure is to approximate the function $\phi$ by smooth functions and the domain by smooth domains (for more details see [3, Sections 15.5, 15.16]).

LEMMA 1. *Assume that all conditions of Theorem I except of the condition on monotonicity of $g(u)$ are fulfilled. Then for any classical solution of problem (2.1), (0.2) the following estimate holds*

$$|u(x)| \leqslant M + \phi_0.$$

*The constant $M$ is defined in (0.7).*

*Proof.* Denote

$$\tilde{M} = \frac{M}{d},$$

$d$ was defined in (1.2). Introduce nonnegative functions $h_i(x_i)$ by the following: for $i = 1, \dots, m$ we put

$$h_i(x_i) = \tilde{M}\left(\frac{d_i^2 - x_i^2}{2} - (1+d_i)(x_i - d_i)\right), \quad \left(h_i'' = -\tilde{M}, \ h_i' \leqslant -\tilde{M}\right);$$

for $i = m+1, \dots, k$

$$h_i(x_i) = \tilde{M}\left(\frac{d_i^2 - x_i^2}{2} + (1+d_i)(x_i + d_i)\right), \quad \left(h_i'' = -\tilde{M}, \ h_i' \geqslant \tilde{M}\right);$$

for $i = k+1, \dots, l$

$$h_i(x_i) = \tilde{M}\frac{d_i^2 - x_i^2}{2}, \quad \left(h_i'' = -\tilde{M}\right),$$

and for $i = l+1, \dots, n$ we take

$$h_i(x_i) = \frac{\tilde{M}}{\gamma_i^2}\left(e^{2\gamma_i d_i} - e^{\gamma_i(d_i - x_i)} - \gamma_i(x_i + d_i)\right), \ \left(h_i'' = -\tilde{M} - \gamma_i h_i', \ h_i' \geqslant 0\right).$$

Put

$$v(x) \equiv u(x) - h(x), \ h(x) \equiv \sum_{i=1}^{n} r_i h_i(x_i) + \phi_0 \geqslant \phi_0,$$

$r_i$ were defined in (1.1), (1.2). For

$$L \equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$$

we have

$$Lu = f(x) - \sum_{i=1}^{n} b_i(x) u_{x_i} - c(x) g_M(u);$$

$$Lh = \sum_{i=1}^{n} a_{ii}(x) r_i h_i''(x_i) = -\tilde{M}\sum_{i=1}^{l} r_i a_{ii}(x) - \sum_{i=l+1}^{n} r_i a_{ii}(x)(\tilde{M} + \gamma_i h_i'(x_i)).$$

It is clear (due to $(0.4_2)$) that

$$Lv = f(x) - \sum_{i=1}^{n} b_i(x) u_{x_i} - c(x) g_M(u) + \tilde{M}\sum_{i=1}^{n} r_i a_{ii}(x) + \sum_{i=l+1}^{n} r_i a_{ii}(x)\gamma_i h_i'(x_i)$$

$$> f(x) - \sum_{i=1}^{n} b_i(x) u_{x_i} - c(x) g_M(u) + \tilde{M}\sum_{i=1}^{n} r_i \alpha_i + \sum_{i=l+1}^{n} r_i \alpha_i \gamma_i h_i'.$$

Suppose that at the point $N = (N_1, \dots, N_n) \in \bar{\Omega} \setminus \partial\Omega$ function $v(x)$ attains its positive maximum. At this point we have $v > 0$, $\nabla v = 0$ or:

$$u(N) > h(N) \geqslant \phi_0, \ u_{x_i}(N) = r_i h_i'(N_i) \ \ i = 1, \dots, n.$$

Taking into account conditions (0.5) and the fact that $h_i' \leqslant -\tilde{M}$ for $i = 1, \dots, m$, $h_i' \geqslant \tilde{M}$ for $i = m+1, \dots, k$ and $h_i'(x_i) \geqslant 0$, $\alpha_i \gamma_i = \beta_i \geqslant |b_i|$ for $i = l+1, \dots, n$, we conclude

$$Lv\Big|_N > f(N) + \tilde{M}\sum_{i=1}^{k} r_i \beta_i + \sum_{i=l+1}^{n} r_i h_i'(N_i)(\alpha_i\gamma_i - b_i(N)) - c(N) g_M(u(N)) + \tilde{M}\sum_{i=1}^{n} r_i \alpha_i$$

$$\geqslant f(N) + \tilde{M}\sum_{i=1}^{k} r_i \beta_i + \tilde{M}\sum_{i=1}^{n} r_i \alpha_i - c(N) g_M(u(N))$$

$$= f(N) + MK - c(N) g_M(u(N)). \tag{2.3}$$

Recall that $c \leqslant \lambda^+$ (see 0.6), now due to the fact that $g_M(u) \geqslant 0$ for $u \geqslant 0$, we obtain

$$c(N) g_M(u(N)) \leqslant \lambda^+ g_M(u(N)).$$

Moreover $g_M(u) \leqslant g(M+\phi_0)$ for $u \geqslant 0$. In fact, if $u > M+\phi_0$, then $g_M(u) = g(M+\phi_0)$, if $0 \leqslant u \leqslant M+\phi_0$, then $g_M(u) = g(u) \leqslant g(M+\phi_0)$. Thus, due to the fact that $g(u)$ is nondecreasing function for $u \geqslant 0$,

$$\lambda^+ g_M(u(N)) \leqslant \lambda^+ g(M+\phi_0)$$

and from (2.3)

$$\begin{aligned} Lv\Big|_N &> f(N) + MK - c(N)g_M(u(N)) \geqslant f(N) + MK - \lambda^+ g(M+\phi_0) \\ &\geqslant f(N) + MK - \lambda g(M+\phi_0) \geqslant 0. \end{aligned} \tag{2.4}$$

This contradicts the assumption that $v(x)$ attains its positive maximum at $N$. From inequality $u(x) - h(x) \leqslant 0$ on $\partial\Omega$ we conclude that

$$u(x) - h(x) \leqslant 0 \ \text{ in } \ \bar{\Omega}.$$

Obviously,

$$\begin{aligned} h(x) &= \sum_{i=1}^{n} r_i h_i(x_i) + \phi_0 \\ &\leqslant \sum_{i=1}^{m} r_i h_i(-d_i) + \sum_{i=m+1}^{k} r_i h_i(d_i) + \sum_{i=k+1}^{l} r_i h_i(0) + \sum_{i=l+1}^{n} r_i h_i(d_i) + \phi_0 \\ &= \tilde{M}\Big[\sum_{i=1}^{k} r_i 2d_i(1+d_i) + \sum_{i=k+1}^{l} r_i d_i^2/2 + \sum_{i=l+1}^{n} r_i \frac{1}{\gamma_i^2}(e^{2\gamma_i d_i} - 2\gamma_i d_i - 1)\Big] + \phi_0 = M + \phi_0. \end{aligned}$$

Finally,

$$u(x) \leqslant M + \phi_0 \text{ in } \bar{\Omega}.$$

Now let us obtain the estimate $u(x) \geqslant -M - \phi_0$. Consider function

$$w(x) \equiv u(x) + h(x).$$

Due to $(0.4_2)$ we have

$$\begin{aligned} Lw &= f(x) - \sum_{i=1}^{n} b_i(x) u_{x_i} - c(x) g_M(u) - \tilde{M} \sum_{i=1}^{n} r_i a_{ii}(x) - \sum_{i=l+1}^{n} r_i a_{ii}(x) \gamma_i h_i'(x_i) \\ &< f(x) - \sum_{i=1}^{n} b_i(x) u_{x_i} - c(x) g_M(u) - \tilde{M} \sum_{i=1}^{n} r_i \alpha_i - \sum_{i=l+1}^{n} r_i \alpha_i \gamma_i h_i'(x_i). \end{aligned}$$

Suppose that at the point $N' = (N_1', \dots, N_n') \in \bar{\Omega} \setminus \partial\Omega$ function $w(x)$ attains its negative minimum. At this point we have $w < 0$, $\nabla w = 0$ or:

$$u(N') < -h(N') \leqslant -\phi_0 \leqslant 0, \ \ u_{x_i}(N') = -r_i h_i'(N_i'), \ \ i = 1, \dots, n.$$

Similarly to the previous case we obtain

$$Lw\Big|_{N'} < f(N') - \tilde{M}\sum_{i=1}^{k} r_i\beta_i - \sum_{i=l+1}^{n} r_i h_i'(N_i')(\alpha_i\gamma_i - b_i(N')) - c(x)g_M(u(N')) - \tilde{M}\sum_{i=1}^{n} r_i\alpha_i$$

$$\leqslant f(N') - MK - c(N')g_M(u(N')). \tag{2.5}$$

We have

$$g_M(u(N')) \geqslant -g(M+\phi_0).$$

In fact, if $-M-\phi_0 \leqslant u(N') < 0$, then, due to (2.2) and (0.3),

$$g_M(u(N')) = g(u(N')) \geqslant -g(M+\phi_0),$$

if $u(N') \leqslant -M-\phi_0$, then

$$g_M(u(N')) = g(-M-\phi_0) \geqslant -g(M+\phi_0).$$

Assume that $c(N') \geqslant 0$, then $c(N')g_M(u(N')) \geqslant -c(N')g(M+\phi_0)$ and

$$MK + c(N')g_M(u(N')) \geqslant MK - c(N')g(M+\phi_0)$$

$$\geqslant MK - \lambda^+ g(M+\phi_0) \geqslant MK - \lambda g(M+\phi_0).$$

If $c(N') \leqslant 0$, then $c(N')g_M(u) \geqslant c(N')g(M+\phi_0)$ (because $g_M(u) \leqslant g(M+\phi_0)$) and

$$MK + c(N')g_M(u(N')) \geqslant MK + c(N')g(M+\phi_0)$$

$$\geqslant MK - \lambda^- g(M+\phi_0) \geqslant MK - \lambda g(M+\phi_0).$$

Hence from (2.5) we obtain

$$Lw\Big|_{N'} < f(N') - MK + \lambda g(M+\phi_0) \leqslant 0.$$

This contradicts the assumption that $w(x)$ attains its negative minimum at $N'$. Taking into account that $u+h \geqslant 0$ on $\partial\Omega$ we conclude that

$$u(x) \geqslant -h(x) \geqslant -M-\phi_0 \ \text{ in } \ \bar{\Omega}.$$

Finally

$$|u(x)| \leqslant M+\phi_0.$$

Lemma is proved.

LEMMA 2. *Suppose that conditions of Lemma 1 are fulfilled and $g(u)$ is nondecreasing function, then for any classical solution of problem (2.1), (0.2) the following estimate holds*

$$|u(x)| \leqslant M+\phi_0.$$

*The constant $M$ is defined in (0.8).*

*Proof.* Similarly to (2.4) we obtain

$$Lv\Big|_N > f(N) + MK - \lambda^+ g(M+\phi_0) \geqslant 0$$

and as a consequence

$$u(x) \leqslant M + \phi_0.$$

Let us obtain the estimate from the below. Similarly to (2.5) we have

$$Lw\Big|_{N'} < f(N') - MK - c(N')g_M(u(N')).$$

If $c(N') \geqslant 0$, then (as in the proof of Lemma 1)

$$MK + c(N')g_M(u(N')) \geqslant MK - \lambda^+ g(M+\phi_0).$$

Suppose now that $c(N') \leqslant 0$. Due to (0.3) if $g(u)$ is nondecreasing then $g(u) \leqslant 0$ for $u \leqslant 0$, hence

$$MK + c(N')g_M(u(N')) \geqslant MK.$$

Hence

$$Lw\Big|_{N'} < 0.$$

As a consequence we conclude that

$$-M - \phi_0 \leqslant u(x).$$

Lemma 2 is proved.

Thus we conclude that any solution of problem (2.1), (0.2) is at the same time the solution of problem (0.1), (0.2).

The uniqueness can be proved by standard arguments based on the maximum principle. Theorem I is proved.

Similarly we can prove Theorem II. The only difference here is that in the regularized equation

$$\sum_{i,j=1}^{n} \mathrm{a}_{ij}(t,x)u_{x_i x_j} + \sum_{i=1}^{n} \mathrm{b}_i(t,x)u_{x_i} + \mathrm{c}(t,x)g_M(u) - u_t = \mathrm{f}(t,x) \;\text{ in }\; Q_T = (0,T)\times\Omega$$

we take

$$g_M(u) = \begin{cases} g(u) & \text{for } |u| \leqslant M+m+\phi_0, \\ g(M+m+\phi_0) & \text{for } u \geqslant M+m+\phi_0, \\ g(-M-m-\phi_0) & \text{for } u \leqslant -M-m-\phi_0. \end{cases}$$

and in the definition of $h(x)$ we put

$$h(x) \equiv \sum_{i=1}^{n} r_i h_i(x_i) + m + \phi_0.$$

Similarly to the elliptic case we can prove that $u(t,x)-h(x)$ can not attain its positive maximum and $u(t,x)+h(x)$ its negative minimum in $\bar{Q}_T \setminus \Gamma_T$, where $\Gamma_T = \Omega \cup S_T$. Taking into account that

$$u-h \leqslant 0, \quad u+h \geqslant 0 \ \text{ on } \ \Gamma_T$$

we conclude that

$$-h(x) \leqslant u(t,x) \leqslant h(x)$$

and hence

$$|u(t,x)| \leqslant M+m+\phi_0.$$

## §3. **Examples**

Let us give several examples demonstrating Theorem I and Theorem II. We start with elliptic case. Suppose that conditions of Theorem I are fulfilled.

EXAMPLE 1. Sublinear cases.

Consider equation

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)|u|^q = f(x) \ \ 0<q<1. \tag{3.1}$$

Condition (0.7) takes the form: there exists $M>0$ such that

$$KM - \lambda(M+\phi_0)^q - f_0 \geqslant 0.$$

Obviously, for $q \in (0,1)$ we can always select such constant $M$.

For the equation

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)\ln(1+|u|) = f(x) \tag{3.2}$$

condition (0.7) takes the form: there exists $M>0$ such that

$$KM - \lambda\ln(1+M+\phi_0) - f_0 \geqslant 0.$$

Here we can also select such constant $M$.

Thus the classical solvability of problem (3.1), (0.2) and (3.2), (0.2) for any Hölder continuous $f(x)$ follows immediately from Theorem I.

EXAMPLE 2. Superlinear case $g(u) = e^u$.

Consider equation

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)e^u = f(x). \tag{3.3}$$

For simplicity we take here homogeneous boundary conditions

$$u\Big|_{\partial\Omega} = 0. \tag{3.4}$$

Condition (0.8) for (3.3) takes the form: there exists $M > 0$ such that

$$MK - \lambda^+ e^M \geqslant f_0.$$

Suppose that $K > \lambda^+ > 0$. Function $F(z) = Kz - \lambda^+ e^z$ for $z > 0$ attains its maximal value at the point

$$z_0 = \ln \frac{K}{\lambda^+}.$$

Put $M = z_0$. Obviously

$$F(M) = K \ln \frac{K}{\lambda^+} - K.$$

Thus if

$$K > \lambda^+ > 0 \text{ and } f_0 \leqslant K \ln \frac{K}{\lambda^+} - K, \tag{3.5}$$

then Theorem I guarantees the existence of a classical solution satisfying

$$|u(x)| \leqslant \ln \frac{K}{\lambda^+}.$$

If $c(x) \leqslant 0$, i.e. $\lambda^+ = 0$, then condition (0.8) takes the form: there exists $M > 0$ such that

$$MK \geqslant f_0.$$

Thus taking $M = f_0 K^{-1}$ we conclude that there exists a classical solution for any $f(x) \in C^\alpha(\bar{\Omega})$ and

$$|u(x)| \leqslant \frac{f_0}{K}.$$

EXAMPLE 3. Consider a special case of equation (3.3):

$$\Delta u + e^u = 0. \tag{3.6}$$

In [1] it was shown that there exists a positive number $\kappa$ depending on the dimension $n$, such that if the diameter of $\Omega$ is less than $\kappa$, then there exists at least one solution of problem (3.6), (3.4). Let us apply Theorem I to equation (3.6). Here $\lambda^+ = 1$, $f_0 = 0$. Condition (3.5) takes the form $K > 1$, $0 \leqslant K(\ln K - 1)$ or

$$K \geqslant e.$$

One can easily see from (1.1) (Section 1) that for (3.6)

$$K = 2 \max_{r_i} \frac{\sum_{i=1}^n r_i}{\sum_{i=1}^n d_i^2},$$

i.e.

$$K = 2\max\left\{\frac{1}{d_1^2},\dots,\frac{1}{d_n^2},\frac{2}{d_1^2+d_2^2},\dots,\frac{2}{d_1^2+d_n^2},\dots,\frac{n}{d_1^2+\dots+d_n^2}\right\}.$$

Without loss of generality we can assume that $d_1 = \min_i\{d_i\}$ and hence

$$K = \frac{2}{d_1^2}.$$

Thus we see, that if the size of the domain is small enough even in one direction, namely

$$d_1 \leqslant \sqrt{\frac{2}{e}},$$

then (independently of the dimension of the domain) there exists at least one solution of problem (3.6), (3.4). Moreover, for this solution the a priori estimate $|u(x)| \leqslant \ln K$ holds. We can take $K = e$ to obtain the estimate

$$|u(x)| \leqslant 1.$$

EXAMPLE 4. Superlinear case $g(u) = |u|^{p-1}u$ and $g(u) = u^p$, $p > 1$. Consider equation

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)|u|^{p-1}u = f(x). \tag{3.7}$$

Condition (0.7) for (3.7) takes the form

$$KM - \lambda M^p \geqslant f_0.$$

Function $F(z) = Kz - \lambda z^p$ for positive $z$ attains its maximal value at the point

$$z_0 = \left(\frac{K}{p\lambda}\right)^{\frac{1}{p-1}}.$$

Put $M = z_0$, obviously

$$F(M) = \left(\frac{K}{p\lambda}\right)^{\frac{1}{p-1}} K\frac{p-1}{p}.$$

We see that Theorem I guarantees the existence of the classical solution for Hölder continuous $f(x)$ satisfying the additional condition

$$f_0 \leqslant \left(\frac{K}{p\lambda}\right)^{\frac{1}{p-1}} K\frac{p-1}{p}$$

and the solution in this case satisfies the inequality

$$|u(x)| \leqslant M = \left(\frac{K}{p\lambda}\right)^{\frac{1}{p-1}}.$$

Obviously, instead of $g(u) = |u|^{p-1}u$ we can take $g(u) = u^p$ if defined.

EXAMPLE 5. Consider a special case of equation (3.7):

$$\Delta u + \mu u^2 = f(x). \tag{3.8}$$

In [5] it was shown that problem (3.8), (3.4) with $f \equiv 0$ has nontrivial solution. Let us formulate the condition guaranteeing the solvability of problem (3.8), (3.4) with $f \not\equiv 0$. As in Example 3

$$K = \frac{2}{d_1^2},$$

where $d_1 = \min_i\{d_i\}$. Thus if

$$f_0 \leqslant \frac{\sqrt{2}}{|\mu| d_1^3},$$

then there exists at least one solution of problem (3.8), (3.4) and for this solution we have the estimate

$$|u(x)| \leqslant \frac{1}{d_1\sqrt{|\mu|}}.$$

EXAMPLE 6. Linear case.

Consider the equation

$$Lu + c(x)u = f(x), \tag{3.9}$$

where

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i}.$$

From Theorem I it follows that if

$$c(x) \leqslant \lambda^+ < K,$$

then there exists a solution of problem (3.9), (0.2) for an arbitrary (Hölder continuous) $f(x)$. Thus we have the estimate from the below on the first eigenvalue of the operator $-L$. If $\lambda_1$ is the first eigenvalue of $-L$, i.e. there exists a nontrivial solution of the problem

$$-Lu = \lambda_1 u \ \text{ in } \ \Omega, \ \ u = 0 \ \text{ on } \ \partial\Omega,$$

then

$$\lambda_1 \geqslant K.$$

Condition (0.8) for (3.9) takes the form

$$(K - \lambda^+)M \geqslant f_0 + \lambda^+ \phi_0.$$

From Theorem I it immediately follows that if

$$K - \lambda^+ > 0,$$

then for an arbitrary Hölder continuous $f(x)$ there exists a classical solution of problem (3.8), (0.2) such that

$$\max_{\Omega}|u(x)| \leqslant \frac{f_0+\lambda^+\phi_0}{K-\lambda^+}+\phi_0.$$

This estimate is more precise than standard estimates for the linear elliptic equations (see, for example, [2, Sec. 3.3] or [4, Chapter III, §1]).

Let us turn to the parabolic case. Suppose that conditions of Theorem II are fulfilled.

EXAMPLE 7. Consider the following equation

$$u_t+\mathbf{b}\cdot\nabla u=\alpha\Delta u+g(u) \text{ in } (0,T)\times\{|x|<d_0\} \tag{3.10}$$

where $\alpha>0$, $\mathbf{b}=(b,\dots,b)$ and the constant $b\neq 0$ with initial and boundary conditions

$$u(0,x)=u_0(x) \text{ and } u=0 \text{ on } S_T. \tag{3.11}$$

One can easily see (see Section 1) that in this case

$$K=\frac{\alpha+b}{2d_0(1+d_0)}.$$

Condition (0.13) for $g(u)=|u|^{p-1}u$, $p>1$ (or $g(u)=u^p$ if defined) takes the form

$$KM-(M+m)^p \geqslant 0.$$

Function $F(z)\equiv(z+m)^p z^{-1}$ obtains its minimum at the point $z=m(p-1)^{-1}$. Let us take

$$M=\frac{m}{p-1}.$$

From Theorem II we have that if

$$\frac{\alpha+b}{2d_0(1+d_0)} \geqslant \frac{p^p}{(p-1)^{p-1}}m^{p-1},$$

then there exists a global solution of problem (3.10), (3.11) and

$$\max_{Q_T}|u(t,x)| \leqslant \frac{p}{p-1}m.$$

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