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Weak Partial Differential Equations
(Homework 1)

Exercise 1 a) ($n = 1$) Let $u \in \mathbb{C}^1[0, 1]$ and $v \in \mathbb{C}^1[1, 2]$ and define the function

$$w(x) = \begin{cases} u(x), & x \in [0, 1], \\ v(x), & x \in (1, 2]. \end{cases}$$

Establish that w weakly differentiable in $[0, 2]$ if and only if w is continuous at 1.

b) ($n = 2$) Let $u \in \mathbb{C}^1([0, 1] \times [-1, 1])$ and $v \in \mathbb{C}^1([1, 2] \times [-1, 1])$ and define the function

$$w(x, y) = \begin{cases} u(x, y), & x \in [0, 1], y \in [-1, 1], \\ v(x, y), & x \in (1, 2], y \in [-1, 1]. \end{cases}$$

Establish that w weakly differentiable in $[0, 2] \times [-1, 1]$ if and only if w is continuous at $\{1\} \times [-1, 1]$.

Exercise 2 a) Let $\Omega \subset \mathbb{R}^n$ be open and connected and $u \in \mathbb{W}_{loc}^{1,1}(\Omega)$ satisfies

$$\nabla u(x) = 0, \text{ a.e. in } \Omega.$$

Establish that there exists $c \in \mathbb{R}$ such that $u(x) = c$, a.e. in Ω .

b) Suppose $\Omega \subseteq \mathbb{R}^2$ is open and connected. Find all functions $v \in \mathbb{W}_{loc}^{1,1}(\Omega)$, that satisfy

$$\frac{\partial v}{\partial x}(x, y) = 2xy + y^2, \text{ a.e. } (x, y) \in \Omega,$$

$$\frac{\partial v}{\partial y}(x, y) = x^2 + 2xy, \text{ a.e. } (x, y) \in \Omega.$$

Notation for the next Exercise. Denote

$\mathbb{C}_{00}^1([0, 1] \times [-1, 1])$ the set of $\mathbb{C}^1$ functions u that in addition satisfy $u(0, y) = 0$, $y \in [-1, 1]$,

$\mathbb{W}_{00}^{1,1}([0, 1] \times [-1, 1])$ the completion of $C_{00}^1([0, 1] \times [-1, 1])$ functions under the $\mathbb{W}^{1,1}([0, 1] \times [-1, 1])$ norm.

Similarly denote

$\mathbb{C}_{22}^1([1, 2] \times [-1, 1])$ the set of $\mathbb{C}^1$ functions u that in addition satisfy $u(2, y) = 0, y \in [-1, 1]$,

$\mathbb{W}_{22}^{1,1}([1, 2] \times [-1, 1])$ the completion of $C_{22}^1([1, 2] \times [-1, 1])$ functions under the $\mathbb{W}^{1,1}([1, 2] \times [-1, 1])$ norm.

$\mathbb{C}_{02}^1([0, 2] \times [-1, 1])$ the set of $\mathbb{C}^1$ functions u that in addition satisfy $u(0, y) = u(2, y) = 0, y \in [-1, 1]$,

$\mathbb{W}_{02}^{1,1}([0, 2] \times [-1, 1])$ the completion of $C_{02}^1([0, 2] \times [-1, 1])$ functions under the $\mathbb{W}^{1,1}([0, 2] \times [-1, 1])$ norm.

Exercise 3 a) Let $u \in \mathbb{W}_{00}^{1,1}([0, 1] \times [-1, 1])$. Establish the existence of L^1 trace at $\{1\} \times [-1, 1]$ by establishing the existence of a positive constant c such that $\mathbb{W}_{00}^{1,1}([0, 1] \times [-1, 1])$ there holds

$$\int_{-1}^1 |u(1, y)| dy \leq c \int_0^1 \int_{-1}^1 \left[|u(x, y)| + \sqrt{u_x^2(x, y) + u_y^2(x, y)} \right] dy dx.$$

b) Let $u \in \mathbb{W}_{00}^{1,1}([0, 1] \times [-1, 1])$, $v \in \mathbb{W}_{22}^{1,1}([1, 2] \times [-1, 1])$. Define

$$w(x, y) = \begin{cases} u(x, y), & x \in [0, 1], y \in [-1, 1], \\ v(x, y), & x \in (1, 2], y \in [-1, 1]. \end{cases}$$

Establish that $w \in \mathbb{W}_{02}^{1,1}([0, 2] \times [-1, 1])$ if and only if

$$tr u(1, y) = tr v(1, y), \text{ a.e. } y \in [-1, 1].$$

Exercise 4 Let $p > 1$. a) If $u \in \mathbb{C}_0^1([0, 1])$, establish the inequality

$$|u(x) - u(y)| \leq |x - y|^{1-\frac{1}{p}} \left(\int_0^1 |u'(t)|^p dt \right)^{\frac{1}{p}}, \quad x, y \in [0, 1].$$

b) Let $v \in \mathbb{W}_0^{1,p}([0, 1])$. Establish the existence of a Hölder continuous function w with $w = v$ a.e. in $\mathbb{W}_0^{1,p}([0, 1])$ that satisfies

$$|w(x) - w(y)| \leq |x - y|^{1-\frac{1}{p}} \left(\int_0^1 |w'(t)|^p dt \right)^{\frac{1}{p}}, \quad x, y \in [0, 1].$$

Exercise 5 a) Let $1 \leq p < n$, $2 \leq n$. Find exponent q that supports the existence of a positive constant $c = c_{n,p}$ such that the following inequality holds true:

$$\left(\int_{\mathbb{R}^{n-1}} |u(x', 0)|^q dx' \right)^{\frac{1}{q}} \leq c \left(\int_{\mathbb{R}_+^n} |\nabla u|^p dx \right)^{\frac{1}{p}}, \quad \forall u \in \mathbb{C}_c^\infty(\mathbb{R}^n).$$

Next establish the inequality for your choice in q .

b) For $u \in \mathbb{D}_0^{1,p}(\mathbb{R}^n)$, establish that $Tr u \in L^q(\mathbb{R}^{n-1})$ and actually the following inequality holds true:

$$\left(\int_{\mathbb{R}^{n-1}} |Tr u(x', 0)|^q dx' \right)^{\frac{1}{q}} \leq c \left(\int_{\mathbb{R}_+^n} |\nabla u|^p dx \right)^{\frac{1}{p}}, \quad \forall u \in \mathbb{D}_0^{1,p}(\mathbb{R}^n),$$

for your choice in q .