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Weak Partial Differential Equations (Homework 2)

Exercise 1 Establish the following Poincare inequality :

$$\int_0^1 \left(u(t) - \int_0^1 u \, dt \right)^2 dt \leq \frac{1}{2} \int_0^1 (u'(t))^2 dt, \quad \forall u \in \mathbf{H}^1[0, 1].$$

Suggestion: If $u \in \mathbf{C}[0, 1]$ then there exists $\rho \in [0, 1]$ such that $u(\rho) - \int_0^1 u(t) \, dt = 0$. (why?)

Exercise 2 Let Ω be a bounded domain in $\mathbf{R}^n$.

a) Establish the existence of a positive constant $c = c(\Omega)$ such that:

$$\int_{\Omega} |u|^2 \, dx \leq c \int_{\Omega} |\nabla u|^2 \, dx, \quad \forall u \in \mathbf{H}_0^1(\Omega).$$

b) If c_{Ω} is the best constant, in the above inequality, establish the existence of a nontrivial function $\phi \in \mathbf{H}_0^1(\Omega)$ that satisfy

$$\int_{\Omega} |\phi|^2 \, dx = c_{\Omega} \int_{\Omega} |\nabla \phi|^2 \, dx$$

What is the equation that ϕ solves?

Exercise 3 Let Ω be a bounded domain in $\mathbf{R}^n$ with $\mathbf{C}^1$ boundary. Let also $E \subseteq \Omega$ measurable satisfying

$$0 < |E| < |\Omega|.$$

Establish the existence of a positive constant $c = c(\Omega, E)$ such that:

$$\int_{\Omega} |u - u_E|^2 \, dx \leq c \int_{\Omega} |\nabla u|^2 \, dx, \quad \forall u \in \mathbf{H}^1(\Omega),$$

where

$$u_E = \oint_E u \, dx.$$

Exercise 4 Let Ω be a bounded domain in $\mathbf{R}^3$, the vector field $\mathbf{b} \in (\mathbf{L}^3(\Omega))^3$ and $f \in \mathbf{L}^2(\Omega)$. We study the existence of $\mathbf{H}_0^1(\Omega)$ weak solutions to the BVP

$$-\Delta u(x) + \lambda \mathbf{b}(x) \cdot \nabla u(x) = f(x), \quad x \in \Omega,$$

$$u(x) = 0, \quad x \in \partial\Omega.$$

a) Establish the existence of $\lambda_0 > 0$, such that if $|\lambda| < \lambda_0$, the BVP has a weak solution.

b) If in addition $\mathbf{b} \in (\mathbf{W}^{1, \frac{3}{2}}(\Omega))^3$ with

$$-\nabla \cdot \mathbf{b}(x) \geq 0, \quad x \in \Omega,$$

then BVP has a weak solution for all $\lambda \geq 0$.

Exercise 5 Let Ω be a bounded domain in $\mathbf{R}^3$ with $\mathbf{C}^1$ boundary and $f \in \mathbf{L}^2(\Omega)$. We study the functional

$$I[u] = \int_{\Omega} \frac{1}{2} |\nabla u|^2 \, dx + \int_{\Omega} |u|^4 \, dx + \int_{\Omega} f u \, dx \quad u \in \mathbf{H}^1(\Omega).$$

Establish that:

a) The functional is well defined for $u \in \mathbf{H}^1(\Omega)$.

b)

$$-\infty < \inf_{v \in \mathbf{H}^1(\Omega)} I[v].$$

c) There exists a function $u \in \mathbf{H}^1(\Omega)$ with :

$$I[u] = \inf_{v \in \mathbf{H}^1(\Omega)} I[v].$$